STAT 5361 - Homework 3

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Problem 1

$$n(y_i - x_i^T \beta_j^{(k)}; 0, \sigma^{2(k)}) \tag{1}$$

$$= \frac{(2\pi)^{-d/2}}{\sqrt{|\sigma^{2k}|}} \exp\{(-\frac{1}{2}(y_i - x_i^T \beta_j^{(k)})(\sigma^2)_k^{-1}(y_i - x_i^T \beta_j^{(k)})\} \tag{2}$$

$$Q(\Psi|\Psi^{(k)}) \tag{3}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{k+1} \ln[(2\pi)^{-d/2}\pi_j^{(k)}] - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{k+1} \ln|\sigma^{2k}| - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{k+1}(y_i - x_i^T \beta_j^{(k)})(\sigma^2)_k^{-1}(y_i - x_i^T \beta_j^{(k)}) \tag{4}$$

$$= I_1 - \frac{I_2}{2} - \frac{I_3}{2} \tag{5}$$

$$I_1 = \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{k+1} \ln[(2\pi)^{-d/2}\pi_j^{(k)}] \tag{6}$$

$$I_2 = \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{k+1} \ln|\sigma^{2k}| \tag{7}$$

$$I_3 = \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{k+1}(y_i - x_i^T \beta_j^{(k)})(\sigma^2)_k^{-1}(y_i - x_i^T \beta_j^{(k)}) \tag{8}$$

$$\beta_j^{(k+1)} = \frac{\sum_{i=1}^{n} \mathrm{x}_i y_i p_{ij}^{(k+1)}}{\sum_{i=1}^{n} \mathrm{x}_i \mathrm{x}_i^{(T)} p_{ij}^{(k+1)}}, j = 1, ..., m \tag{9}$$

$$S_k = \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{k+1} \ln|\sigma^{2k}| + \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{k+1}(y_i - x_i^T \beta_j^{(k)})(\sigma^2)_k^{-1}(y_i - x_i^T \beta_j^{(k)}) \tag{10}$$

$$\sigma^{2(k+1)} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{k+1}(y_i - \mathrm{x}_i^T \beta_j^{(k+1)})^2}{n} \tag{11}$$

$$I_1 = \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{k+1} \ln[(2\pi)^{-d/2}\pi_j^{(k)}] \tag{12}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{k+1} \ln[(2\pi)^{-d/2}] + \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{k+1} \ln[\pi_j^{(k)}] \tag{13}$$

$$= -(d/2)\ln(2\pi)\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{k+1} + \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{k+1} \ln[\pi_j^{(k)}] \tag{14}$$

$$\pi_j^{(k+1)} = \frac{\sum_{n}^{i=1} p_{ij}^{(k+1)}}{n} \tag{15}$$

$$\tag{16}$$

Problem 2

Problem 2(a)

We want $C\int_0^\infty (2x^{\theta-1} + x^{\theta-\frac{1}{2}}) * e^{-x}\mathrm{d}x = 1$ And we know that$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} * e^{-x}\mathrm{d}x$. Therefore,

$$C[\int_0^\infty 2x^{\theta-1} * e^{-x}\mathrm{d}x + \int_0^\infty x^{\theta-\frac{1}{2}} * e^{-x}\mathrm{d}x] = C(2 * \Gamma(\theta) + \Gamma(\theta + \frac{1}{2})) = 1 \tag{17}$$

$$C = \frac{1}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})} \tag{18}$$

$$g(x) = \frac{1}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})}(2x^{\theta-1} * e^{-x} + x^{\theta-\frac{1}{2}} * e^{-x}) \tag{19}$$

$$= \frac{1}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})}(2\Gamma(\theta) * f(x;\theta,1) + \Gamma(\theta + \frac{1}{2}) * f(x;\theta + \frac{1}{2}, 1)) \tag{20}$$

$$= \frac{2\Gamma(\theta)}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})} * f(x;\theta,1) + \frac{\Gamma(\theta + \frac{1}{2})}{2\Gamma(\theta) + \Gamma(\theta + \frac{1}{2})} * f(x;\theta + \frac{1}{2}, 1) \tag{21}$$

$$\tag{22}$$

f(x:θ,1) is a gamma distribution with $\alpha = \theta; \beta = 1$;

f(x:$\theta + \frac{1}{2}$,1) is a gammadistribution with $\alpha = \theta + \frac{1}{2}; \beta = 1$.

So, $g(x)$ is a mixture of two Gamma distributions. The weights of these two distributions are shown in the fuction above.

Problem 2(b)

Frist, we set $\theta = 3$. The weights for$f(x;\theta,1)$ and $f(x;\theta + \frac{1}{2}, 1)$ are 0.712155 and 0.287845 respectively.

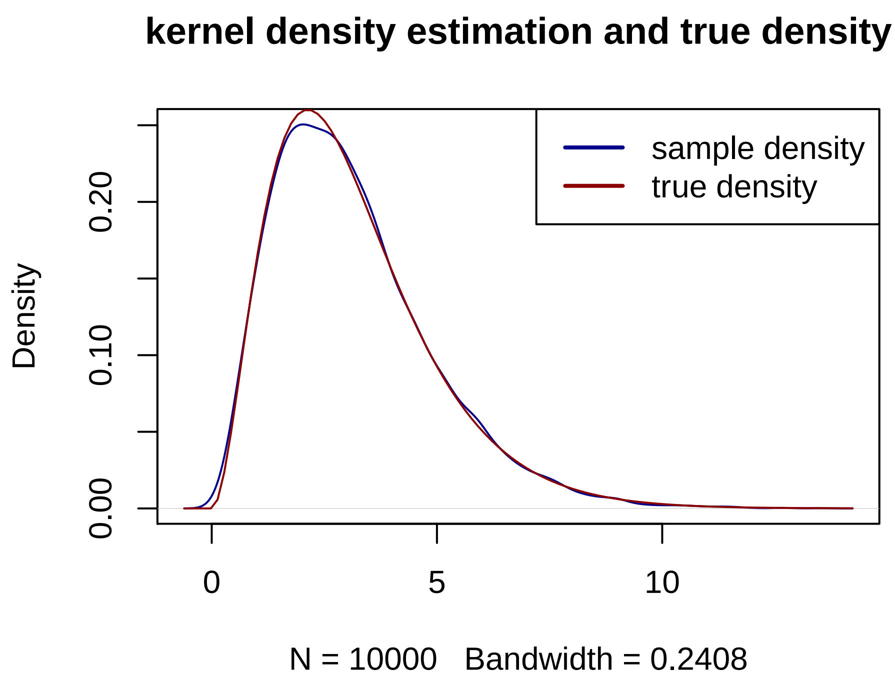
```
integrand1 <- function(x) {2*x^2*exp(-x)}
integrand2 <- function(x) {x^(5/2)*exp(-x)}
in1 <- (integrate(integrand1, lower = 0, upper = Inf))
in2 <- (integrate(integrand2, lower = 0, upper = Inf))
print(in1)
```

```
## 4 with absolute error < 0.00014
```

```
print(in2)
```

```
## 3.323351 with absolute error < 0.00012
```

```
N <- 10000
components <- sample(1:2, prob=c(0.712155,0.287845),size=N,replace=TRUE)
sps <- c(3, 7/2)
samples <- rgamma(n=N,shape = sps[components])
d1 <- density(samples)
plot(d1,col="darkblue", main = "kernel density estimation and true density")
curve((0.712155 * dgamma(x, 3) + 0.287845 * dgamma(x, 7/2)), col = "darkred", add = TRUE
legend('topright', c('sample density','true density'), lwd = 2, col = c('darkblue','dark
```



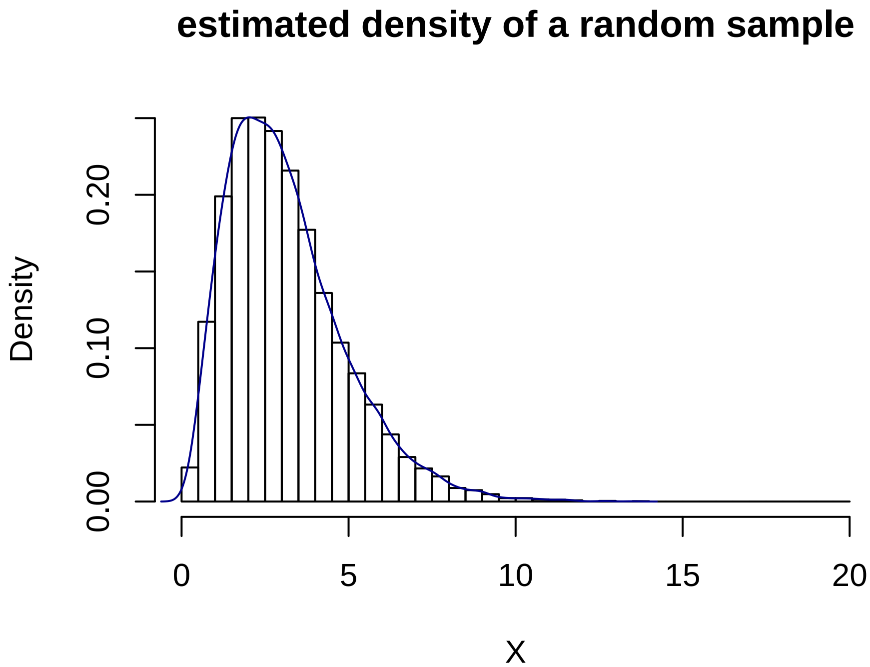
Problem 2(c)

```
# set theta = 3

sample2 <- samples
envelope2 <- function(x){
  (0.712155 * dgamma(x, 3) + 0.287845 * dgamma(x, 7/2))
}

target2 <- function (x)
{
  (4 + x)^(1/2) * x^2* exp(-x)
}
accept = c()
rejection_sampling <- function(sample2, envelope2, target2){
  for(i in 1:length(sample2)){
```

```r
    U = runif(1,0,1)
    if(envelope2(sample2[i])*U*1<= target2(sample2[i])) {
      accept[i] = 'Yes'}
    else {
      accept[i] = 'No'
    }
  }
}
T2 <- data.frame(sample2, accept = 'Yes')
hist(T2[,1][T2$accept=='Yes'], freq = FALSE, breaks = seq(0,20,0.5), main = 'estimated d
lines(density(sample2), col = "darkblue")
```



Problem 3

Problem 3(a)

Let U~Unif(0,1), for any $\theta, \beta > 0$, $U^{1/\theta} \sim \text{Beta}(\theta, 1)$, $1 - U^{1/\beta} \sim \text{Beta}(1, \beta)$

The density of the distributions are $g_\theta(x) = \theta x^{\theta-1}$, $g_\beta(x) = \beta(1-x)^{\beta-1}$

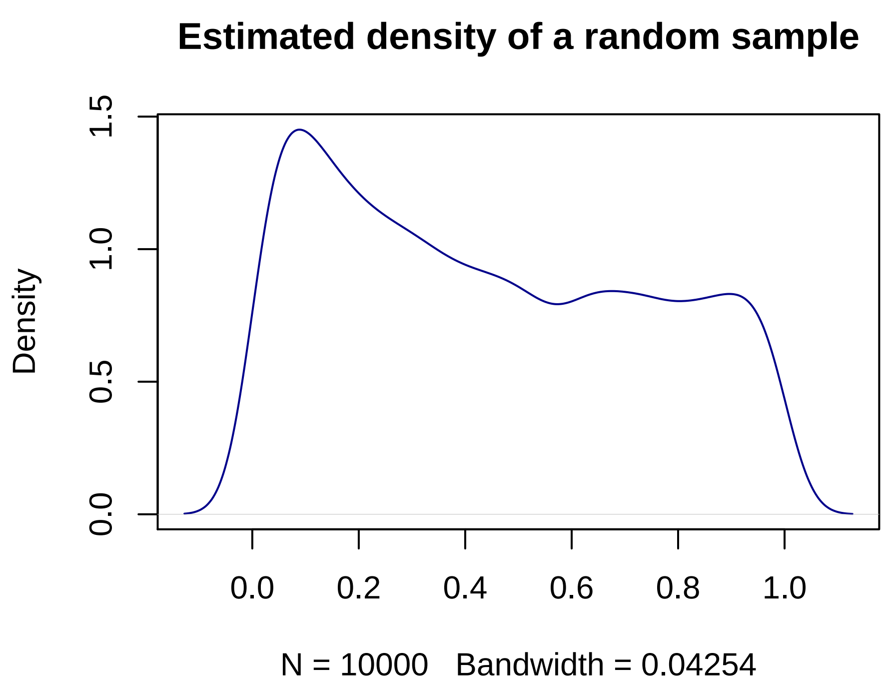
Therefore, if $g(x)$ is a mixture of $g_\theta(x)$ and $g_\beta(x)$, $g(x) = p_1 g_\theta(x) + p_2 g_\beta(x)$,

$p_1 = \dfrac{1}{1+x^2}\text{B}(\theta, 1)$,

$p_2 = \sqrt{2+x^2}\text{B}(1, \beta)$.

Assume $\theta = 2$, $\beta = 3$:

```r
p1 <- beta(2,1)
p2 <- beta(1,3)*3^(1/2)
N3 <- 10000
components3 <- sample(1:2, prob=c(p1,p2),size=N3,replace=TRUE)
sps1 <- c(2, 1)
sps2 <- c(1, 3)
sample3 <- rbeta(n=N3,shape1=sps1[components3], shape2 = sps2[components3])
envelope3 <- function(x){
  dbeta(x,shape1=sps1[components3], shape2 = sps2[components3])
}
target3 <- function(x){
  x/(1+x^2)+(2+x^2)^(1/2)*(1-x)^2
}
accept = c()
rejection_sampling <- function(sample3, envelope3, target3){
  for(i in 1:length(sample3)){
    U = runif(1,0,1)
    if(envelope3(sample3[i])*U <= target3(sample3[i])) {
      accept[i] = 'Yes'}
    else {
      accept[i] = 'No'
    }
  }
}
T3 = data.frame(sample3, accept = 'Yes')
plot(density(sample3),main="Estimated density of a random sample",col = "darkblue")
```



Problem 3(b)

$p_1 = \frac{1}{1+x^2}\mathrm{B}(\theta, 1),$

$p_2 = \sqrt{2+x^2}\mathrm{B}(1, \beta).$

Assume $\theta = 2$, $\beta = 3$:

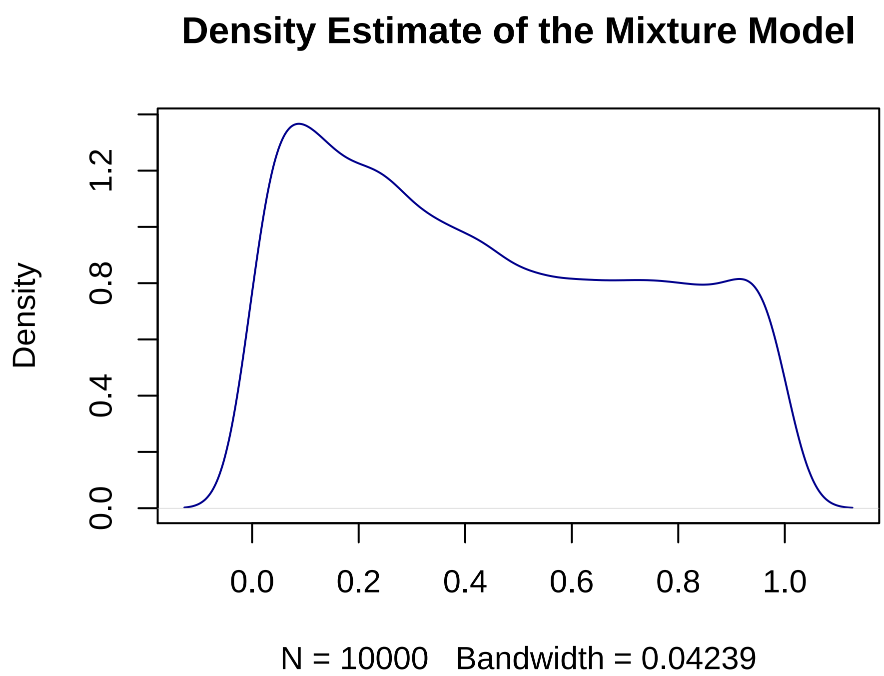
```
p1 <- beta(2,1)
p2 <- beta(1,3)*3^(1/2)
p <- p1 + p2
N <- 10000

#Sample N random uniforms U
U <- runif(N)

#Variable to store the samples from the mixture distribution
rand.samples <- rep(NA,N)

#Sampling from the mixture
for(i in 1:N){
    if (U[i] < p1/p) {
        rand.samples[i] <- rbeta(1,2,1)
    }else{
        rand.samples[i] <- rbeta(1,1,3)
    }
}
#rejection-sampling
rejection_sampling <- function(rand.samples, envelope3, target3){
  for(i in 1:length(rand.samples)){
    U = runif(1,0,1)
    if(envelope3(rand.samples[i])*U <= target3(rand.samples[i])) {
      accept[i] = 'Yes'}
    else {
      accept[i] = 'No'
    }
  }
}
T4 <- data.frame(rand.samples, accept = 'Yes')
#Density plot of the random samples

plot(density(rand.samples),main="Density Estimate of the Mixture Model",col = "darkblue'
```



Density Estimate of the Mixture Model
Density
0.0
0.4
0.8
1.2
0.0
0.2
0.4
0.6
0.8
1.0
N = 10000 Bandwidth = 0.04239