Notes on Linear Programming: Part I
THE GENERALIZED SIMPLEX METHOD
for
MINIMIZING A LINEAR FORM UNDER
LINEAR INEQUALITY RESTRAINTS

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SUMMARY

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THE GENERALIZED SIMPLEX METHOD FOR MINIMIZING A LINEAR FORM UNDER LINEAR INEQUALITY RESTRAINTS

by

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Background and Summary

The determination of "optimum" solutions to systems of linear inequalities is assuming increasing importance as a tool for mathematical analysis of certain problems in economics, logistics, and the theory of games [1],[5]. The solution of large systems is becoming more feasible with the advent of high-speed digital computers; however, as in the related problem of inversion of large matrices, there are difficulties which remain to be resolved connected with rank. This paper develops a theory for avoiding assumptions regarding rank of underlying matrices which has import in applications where little or nothing is known about the rank of the linear inequality system under consideration.

The simplex procedure is a finite iterative method which deals with problems involving linear inequalities in a manner closely analogous to the solution of linear equations or matrix inversion by Gaussian elimination. Like the latter it is useful in proving fundamental theorems on linear algebraic systems. For example, one form of the fundamental duality theorem associated with linear inequalities is easily shown as a direct consequence of solving the main problem. Other forms can be obtained by trivial manipulations (for a fuller discussion of these interrelations see [13]); in particular,

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the duality theorem [8], [10], [11], [12] leads directly to the Minmax theorem for zero-sum two-person games [1d] and to a computational method (pointed out informally by Herman Rubin and demonstrated by Robert Dorfman, Ref. [1a]) which simultaneously yields optimal strategies for both players and the value of the game.

The term "simplex" evolved from an early geometrical version in which (like in game theory) the variables were non-negative and summed to unity. In that formulation a class of "solutions" was considered which lay in a simplex.

The generalized method given here was outlined earlier by the first of the authors (Dantzig) in a short footnote, Ref. [1b] and then discussed somewhat more fully at the Symposium of Linear Inequalities in 1951. Its purpose, as we have already indicated, is to remove the restrictive assumptions regarding the rank of the matrix of coefficients and constant elements without which a condition called "degeneracy" can occur.

Under degeneracy it is possible for the value of the solution to remain unchanged from one iteration to the next using the original simplex method. This causes the proof that no basis can be repeated, to break down. In fact, for certain examples Alan Hoffman [14] and one of the authors (Wolfe) have shown that it was possible to repeat the basis and thus cycle forever with the value of the solution remaining unchanged and greater than the desired minimum. On the other hand, it is interesting to note that while most problems that arise from practical sources

(in the author's experience) have been degenerate, none have ever cycled, [9].

The essential scheme for avoiding the assumptions on rank is to replace the original problem by a "perturbation" that satisfies these conditions. That such perturbations exist is, of course, intuitively evident but the question remained to show how to do it in a simple way. For the special case of the transportation problem a simple method of producing a perturbation is found in Ref. [10]. The second of the authors has considered several types of perturbations for the general case. A. Charnes has extensively investigated this approach and his writing represents the best available published material in this regard, Ref. [2], [3], [4].

It was noticed early in the development of these methods that the limit concept in which a set of perturbations tends in the limit to one of the solutions to the original problem was not essential to the proof. Accordingly, the third author (Wolfe) considered a purely algebraic approach which imbeds the original problem as a component of a generalized matrix problem and replaces the original non-negative real variables by lexicographically ordered vectors. Because this approach gives a simple presentation of the theory, we adopt it here.

SECTION I

THE GENERALIZED SIMPLEX METHOD

As is well known, a system of linear inequalities by trivial substitution and augmentation of the variables can be replaced by an equivalent _system of linear equations in non-negative variables_; hence, with no loss of generality, we shall consider the basic problem in the latter form throughout this paper. One may easily associate with such a system another system in which the constant terms are replaced by ℓ-component constant row vectors and the real variables are replaced by real ℓ-component variable row vectors. In the original system the real variables are non-negative; in the generalized system we shall mean by a vector variable $\bar{x} > 0$ (in the _lexicographic_ sense) that it has non-zero components, the first of which is positive and by $\bar{x} > \bar{y}$ that $\bar{x} - \bar{y} > 0$. It is easy to see that the first components of the vector variables of the generalized system satisfy a linear system in non-negative variables in which the constant terms are the 1st components of the constant vectors.

Let $P = [P_0, P_1, \cdots, P_n]$ be a given matrix whose j^{th} column, P_j, is a vector of $(m+1)$ components. Let M be a fixed matrix of _rank_ $m+1$ consisting of $m+1$ ℓ-component row vectors. The generalized matrix problem is concerned with finding a matrix X satisfying

(1) $$P\tilde{X} = \sum_0^n P_j \bar{x}_j = M$$

where $\bar{x}_j$ (the jth row of $\widetilde{X}$) is a row vector of ℓ-components satisfying the conditions, in the lexicographic sense,

(2) $\bar{x}_j \geq 0$ $(j = 1, 2, \cdots, n)$

(3) $\bar{x}_0 = \text{Max}$.

Any set X of "variables" $(\bar{x}_0; \bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ satisfying (1) and (2) in the above lexicographic sense will be referred to as a "feasible solution" (or more simply as a "solution") — a term derived from practical applications in which such a solution represents a situation which is physically realizable but not necessarily optimal. The first variable, $\bar{x}_0$, which will be called the "value" of the solution, is to be maximized; it is not constrained like the others to be non-negative. In certain applications (as in Section II) it may happen that some of the other variables also are not restricted to be non-negative. This leads to a slight variation in the method (see the discussion following Theorem V).

Among the class of feasible solutions, the simplex method is particularly concerned with those called "basic". These have the properties which we mention in passing (1) that whenever any solution exists a basic solution also exists, (Theorem VIII) and (2) that whenever a maximizing solution exists and is unique it is a basic solution, and whenever a maximizing solution is not unique there is a basic solution that has the same maximizing value, (Theorem VI). A basic solution is one in which only m+1 variables (including $\bar{x}_0$) are considered in (1), the remainder being set equal

to zero; i.e., it is of the form

$$(4) \qquad BV = P_0 \overline{v}_0 + \sum_{i=1}^{m} P_{J_i} \overline{v}_i = M , \qquad (\overline{v}_i \geq 0, \ J_i \neq 0)$$

where $B = \begin{bmatrix} P_0, & P_{J_1}, & \cdots, & P_{J_m} \end{bmatrix}$ is an (m+1) <u>square</u> <u>matrix</u> and V is a matrix of m + 1 rows and $\mathscr{l}$—columns whose i^{th} row is denoted by $\overline{v}_i$, (i = 0, 1, $\cdots$, m).

It is clear from (4) that since M is of rank (m+1) so are B and V. From this it readily follows that the (m+1) columns of B constitute a <u>basis</u> in the space of vectors P_J, and the solution V is uniquely determined. Moreover, since the rank of V is (m+1), none of the (m+1) rows of V can vanish; i.e., it is not possible that $\overline{v}_i = 0$. <u>Thus in a basic solution all variables associated with the vectors in the basis (except possibly $\overline{v}_0$) are positive; all others are zero.</u> The condition in (4) can now be strengthened to strict inequality

$$(5) \qquad \overline{v}_i > 0 \qquad (i = 1, \cdots, m) .$$

Let β_i denote the ith <u>row</u> of B inverse

$$(6) \qquad B^{-1} = \begin{bmatrix} P_0, & P_{J_1}, & P_{J_2}, & \cdots, & P_{J_m} \end{bmatrix}^{-1} = \begin{bmatrix} \beta_0', & \beta_1', & \cdots, & \beta_m' \end{bmatrix}'$$

where primed letters stand for transpose.

Theorem I: <u>A</u> <u>necessary</u> <u>and</u> <u>sufficient</u> <u>condition</u> <u>that</u> <u>a</u>
<u>basic</u> <u>solution</u> <u>be</u> <u>a</u> <u>maximizing</u> <u>solution</u> <u>is</u>

$$(7) \qquad \beta_0 P_j \geq 0 \qquad (j = 1, \cdots, n) .$$

Theorem II: <u>If</u> <u>a</u> <u>basic</u> <u>solution</u> <u>is</u> <u>optimal</u>, <u>then</u> <u>any</u> <u>other</u>
<u>solution</u> (<u>basic</u> <u>or</u> <u>not</u>) <u>with</u> <u>the</u> <u>property</u> <u>that</u> $\overline{x}_j = 0$ <u>whenever</u>
$(\beta_0 P_j) > 0$ <u>is</u> <u>also</u> <u>optimal</u>; <u>any</u> <u>solution</u> <u>with</u> $\overline{x}_j > 0$ <u>for</u> <u>some</u>
$(\beta_0 P_j) > 0$ <u>is</u> <u>not</u> <u>optimal</u>.

Proofs: Let $\widetilde{X}$ represent <u>any</u> solution to (1) and V a basic
solution with basis B; then multiplying both (1) and (4) through by
β_0 and equating, one obtains, after noting from (6) that $\beta_0 P_0 = 1$
and $\beta_0 P_{j_1} = 0$,

$$(8) \qquad \overline{x}_0 + \sum_1^n (\beta_0 P_j)\overline{x}_j = \overline{v}_0 \qquad ;$$

whence, assuming $\beta_0 P_j \geq 0$, then $\overline{x}_0 \leq \overline{v}_0$ (which establishes the
sufficiency of Theorem 1); moreover the condition $\overline{x}_j = 0$ whenever
$\beta_0 P_j > 0$ $(j \neq 0)$, implies the summation term of (8) vanishes and
$\overline{x}_0 = \overline{v}_0$; whereas denial of this condition implies the summation
term is positive (establishing Theorem II if Theorem I is true).

In order to establish the necessity of (7) for Theorem 1,
let Y_s be a column vector which expresses a vector P_s as a linear
combination of the vectors in the basis:

(9) $\qquad P_s = B(B^{-1} P_s) = BY_s = \sum_{i=0}^{m} P_{j_i} y_{is} \qquad (P_0 = P_{j_0})$

where it is evident from (6) that, by definition,

(10) $\qquad y_{is} = \beta_i P_s \qquad (i = 0, 1, \cdots, m)$.

Consider a _class of solutions_ which may be formed from (4) and (9) of the form

(11) $\qquad B[\overline{V} - Y_s \overline{\theta}] + P_s \overline{\theta} = M$

or more explicitly

(12) $\qquad P_0[\overline{v}_0 - y_{0s}\overline{\theta}] + \sum_{i=1}^{m} P_{j_i}[\overline{v}_i - y_{is}\overline{\theta}] + P_s \overline{\theta} = M$.

It is clear that since $\overline{v}_i > 0$ for $i \geq 1$ has been established earlier, (5), a class of solutions with $\overline{\theta} > 0$ (i.e. with $\overline{\theta}$ strictly positive) always exists such that the variables associated with P_s and P_{j_i} in (12) are non-negative, hence admissible as a solution to (1). If $y_{0s} < 0$, then the values of these solutions are

(13) $\qquad \overline{v}_0 - y_{0s}\overline{\theta} > \overline{v}_0 \qquad (y_{0s} < 0, \overline{\theta} > 0)$.

For a given increase in $\overline{\theta}$ the greatest increase in the value of the solution (i.e. _direction of steepest ascent_) is obtained by choosing $s = j$ such that

(14) $\qquad \beta_0 P_s = \underset{j}{\text{Min}} (\beta_0 P_j) < 0$

This establishes Theorem III (below) which is clearly only a restatement of the necessity of condition (7) of Theorem I.

Theorem III: <u>There</u> <u>exists</u> <u>a</u> <u>class</u> <u>of</u> solutions <u>with</u> <u>values</u> $\bar{x}_0 > \bar{v}_0$, <u>if</u> <u>for</u> <u>some</u> $j = s$

$$(15) \qquad y_{0s} = \beta_0 P_s < 0$$

Theorem IV: <u>There</u> <u>exists</u> <u>a</u> <u>class</u> <u>of</u> solutions <u>with</u> <u>no</u> <u>upper</u> <u>bound</u> <u>for</u> <u>values</u> $\bar{x}_0$ <u>if</u> <u>for</u> <u>some</u> s , $y_{0s} < 0$ <u>and</u> $y_{is} \leq 0$ <u>for</u> <u>all</u> i .

Theorem V: <u>There</u> <u>exists</u> <u>a</u> <u>new</u> <u>basic</u> solution <u>with</u> <u>value</u> $\bar{x}_0 > \bar{v}_0$, (<u>obtained</u> <u>by</u> introducing P_s <u>into</u> <u>the</u> <u>basis</u> <u>and</u> <u>dropping</u> <u>a</u> <u>unique</u> P_{j_r}), <u>if</u> <u>for</u> <u>some</u> s , $y_{0s} < 0$ <u>and</u> <u>for</u> <u>some</u> i, $y_{is} > 0$.

From (12) if $y_{is} \leq 0$ for all i, then $\bar{\theta}$ can be arbitrarily large (i.e., its first component can tend to $+ \infty$) and the coefficients of P_{j_i} will remain non-negative. The value of these solutions (13) will also be arbitrarily large providing $y_{0s} < 0$ (establishing Theorem IV). In the event that some $y_{is} > 0$, the maximum value of $\bar{\theta}$ becomes

$$(16) \qquad \text{Max } \bar{\theta} = (1/y_{rs})\bar{v}_r = \min_{y_{is} > 0} (1/y_{is})\bar{v}_i > 0 \quad , \quad (y_{rs} > 0, i \neq 0)$$

where the minimum of the vectors (taken in the lexicographic sense) occurs for a <u>unique</u> i = r (since the rank of V is m+1 , no two rows of V can be proportional, whereas the assumption of non-uniqueness in (16) would imply two rows of V to be so — a contradiction). Setting $\hat{\theta} = \text{Max } \bar{\theta}$ in (12) yields a new basic solution since the coefficient of P_{j_r} vanishes. Thus a new basis has been formed

consisting of $[P_0, P_{j_1}, \cdots, P_s, \cdots, P_{j_m}]$ where P_{j_r} is omitted

and P_s is put in instead (Theorem V).

The next section considers an application of the generalized simplex procedure in which the restriction $\bar{x}_j \geq 0$ is not imposed on all variables $(j = 1, 2, \cdots, n)$. This leads to a slight modification of procedure: first, for all j for which $\bar{x}_j \geq 0$ is not required, both P_j and $- P_j$ should be considered as columns of P; second, if P_{j_i} is in the basis and the restriction $\bar{v}_i > 0$ is not required, then this term cannot impose a bound on $\bar{\theta}$; hence the corresponding i should be omitted from (16) in forming the minimum.

Starting with any basis $B = B^{(k)}$, one can determine a new basis $B^{(k+1)}$ by first determining the vector P_s to introduce into the basis by (14). If there exists no $\beta_0 P_s < 0$, then by Theorem I, the solution is optimal and $B^{(k)}$ is the __final basis__. If a P_s exists, then one forms $y_{is} = (\beta_i P_s)$ and determines the vector P_{j_r} to drop from the basis by (16) providing there are $y_{is} > 0$. If there exist no $y_{is} > 0$, then, by Theorem IV, a class of solutions is obtained from (12) with no upper bound for $\bar{v}_0$ for arbitrary $\bar{\theta} > 0$. If P_{j_r} can be determined, then a new basis $B^{(k+1)}$ is formed dropping P_{j_r} and replacing it by P_s; by (13) the value, $\bar{v}_0$, of this solution is strictly greater for $B^{(k+1)}$ than for $B^{(k)}$ since $\bar{\theta} > 0$ is chosen by (16). Thus one may proceed iteratively starting with the assumed initial basis and forming $k = 0, 1, 2, \cdots$ until the process stops because (a) an optimal solution has been

obtained or (b) because a class of solutions with no finite upper bound has been obtained.

The number of different bases is finite, not exceeding the number of combinations of n things taken m at a time; associated with each basis B is a unique basic solution $V = B^{-1}M$ — hence the number of distinct basic solutions is finite; finally, no basis can be repeated by the iterative procedure because contrariwise this would imply a repetition of the value $\bar{v}_0$ whereas by (13) the values for successive basic solutions are strictly monotonically increasing — hence the number of iterations is finite.

The $k + 1^{st}$ iterate is closely related to the kth by simple transformations that constitute the computational algorithm [6], based on the method: thus for $i = 0, 1, \cdots, m$, $(i \neq r)$,

$$(17.0) \qquad \bar{v}_i^{k+1} = \bar{v}_i^k + \eta_i \bar{v}_r^k \qquad\qquad \bar{v}_r^{k+1} = \eta_r \bar{v}_r^k \qquad ;$$

$$(17.1) \qquad \beta_i^{k+1} = \beta_i^k + \eta_i \beta_r^k \qquad ; \qquad \beta_r^{k+1} = \eta_r \beta_r^k \qquad ;$$

where the superscripts $k + 1$ and k are introduced here to distinguish the successive solutions and bases, and where η_i are constants

$$(18) \qquad \eta_i = -y_{is}/y_{rs} = -(\beta_i P_s)/(\beta_r P_s), \qquad\qquad (i \neq r)$$

$$\eta_r = 1/y_{rs} = 1/(\beta_r P_s)$$

Relation (17.0) is a consequence

of (12) and (16); it is easy to verify that the matrix whose rows are defined by (17) satisfies the proper orthogonality properties for the inverse when multiplied on the right by the $k + 1^{st}$ basis $[P_0, P_{j_1}, \cdots, P_s, \cdots, P_{j_m}]$. As a consequence of the iterative procedure we have established two theorems.

Theorem VI: <u>If solutions exist and their values have a finite upper bound, then a maximizing solution exists which is a basic solution with the properties</u>

$$(19) \quad B V = \sum_{i=0}^{m} P_{j_i} \overline{V}_i = M \qquad (P_{j_0} = P_0, \overline{V}_i > 0, i = 1, \cdots, m)$$

$$\beta_0 P_0 = 1, \quad \beta_0 P_{j_i} = 0, \quad \beta_0 P_j \geq 0 \qquad (j = 1, 2, \cdots, n)$$

$$\overline{V}_0 = \beta_0 M = Max \ \overline{x}_0$$

<u>where β_0 is the 1st row of B^{-1}.</u>

Theorem VII: <u>If solutions exist and their values have no finite upper bound, then a basis B and a vector P_s exist with the properties</u>

$$(20) \quad B V = \sum_{i=0}^{m} P_{j_i} \overline{V}_i = M \qquad (P_{j_0} = P_0, \overline{V}_i > 0, i = 1, \cdots, m)$$

$$\beta_0 P_s < 0, \quad \beta_1 P_s \leq 0$$

$$\sum P_{j_i} \left[\overline{V}_i - (\beta_i P_s)\overline{\theta} \right] + P_s \overline{\theta} = M$$

where the latter, with $\bar\theta \geq 0$ arbitrary, forms a class of solutions with unbounded values (β_i is the $i + 1^{st}$ row of B^{-1}).

Closely related to the methods of the next section, a constructive proof will now be given to

Theorem VIII: If any solution exists, then a basic solution exists.

For this purpose, adjust M so that the first non–zero component of each row is positive and consider the augmented system

$$(20.1) \qquad \sum_{j=0}^{n} \begin{bmatrix} P_j \\ 0 \end{bmatrix} \begin{bmatrix} \bar x_j' \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} U_i \\ 1 \end{bmatrix} \bar x_{n+i}' + \begin{bmatrix} \cdot \\ 1 \end{bmatrix} \bar x_{n+m+1} = \begin{bmatrix} M & \cdot \\ \cdot & 1 \end{bmatrix} \qquad ,$$

$$(\bar x_j' \geq 0 , \qquad j = 1, \ldots, n+m) \qquad ,$$

where $\bar x_j'$ has one more component than $\bar x_j$ and $\cdot$ represents the null vector. Noting neither $\bar x_0'$ nor $\bar x_{n+m+1}'$ is required to be positive, an obvious basic solution is obtained using the variables $\begin{bmatrix} \bar x_0', & \bar x_{n+1}', & \ldots, & \bar x_{n+m+1}' \end{bmatrix}$. It will be noted that the hypothesis of the theorem permits construction of a solution for which $\bar x_{n+i}' = 0$ ($i=1,2,\ldots,m$). Indeed, for $j \leq n$ set $\bar x_j' = (\bar x_j, 0) > 0$. However, it will be noted also that $\sum \bar x_{n+i}' = \begin{bmatrix} \cdot & 1 \end{bmatrix}$ so that Max $\bar x_{n+i} = \begin{bmatrix} \cdot & 1 \end{bmatrix}$. Accordingly, one may start with the basic solution for the augmented system, keeping the vectors corresponding to x_0' and x_{n+m+1}' always in the basis use the simplex algorithm to Max x_{n+m+1}'. Since at the maximum $\bar x_{n+i}' = 0$, ($i \neq m+1$), the corresponding vectors are not in the basis any longer, see (5). By dropping the last component of this basic solution and by dropping x_{n+m+1}', one is left with a basic solution to the original system, (Q.E.D.).

SECTION II

MINIMIZING A LINEAR FORM

The application of the generalized simplex method to the problem

of minimizing a linear form subject to linear inequality restraints

consists in bordering the matrix of coefficients and constant terms

of the given system by appropriate vectors. This can be done in

many ways — the one selected is one which identifies the inverse

of the basis as the additional components in a generalized matrix

problem so that computationally no additional labor is required

when the inverse is known.

The fundamental problem which we wish now to solve is to

find a set $x = (x_0, x_1, \cdots, x_n)$ of real numbers satisfying the

equations

$$(21) \qquad x_0 + \sum_1^n a_{0j} x_j = 0$$

$$\sum_1^n a_{kj} x_j = b_k \qquad (b_k \geq 0), \quad (k = 2, 3, \cdots, m)$$

such that

$$(22) \qquad x_j \geq 0$$

$$(23) \qquad x_0 = \text{Max}$$

where without loss of generality one may assume $b_k \geq 0$. It will be

noted that the subscript $k = 1$ has been omitted from (21). After

some experimentation it has been found convenient[*] to augment the

equations of (21) by a redundant equation formed by taking the

negative sum of equations $k = 2, \cdots, m$.   Thus

$$(24) \qquad \sum_1^n a_{1j}x_j = b_1 \qquad\qquad (a_{1j} = -\sum_{k=2}^m a_{kj}, \quad b_1 = -\sum_2^m b_k) \; .$$

Consider the generalized problem of finding a set of vector

"variables" (in the sense of Section I) $(\bar{x}_0, \bar{x}_1, \cdots, \bar{x}_n)$ and

auxiliary variables $(\bar{x}_{n+1}, \bar{x}_{n+2}, \cdots, \bar{x}_{n+m})$ satisfying the matrix

equations

$$(25) \quad \bar{x}_0 + \sum_1^n a_{0j}\bar{x}_j = (0, \; 1, 0, \qquad \cdots, 0)$$

$$\bar{x}_{n+k} + \sum_1^n a_{kj}\bar{x}_j = (b_k, 0, 0, \cdots, 1, \cdots, 0) \quad (b_1 \leq 0; \; b_k \geq 0, \; k=2, \cdots, m)$$

where the constant vectors have $\ell = m + 2$ components with unity in

position $k + 2$, $\bar{x}_0$ and $\bar{x}_{n+1}$ are unrestricted as to sign and, for

all other j,

$$(26) \qquad \bar{x}_j \geq 0 \qquad\qquad (j = 1, \cdots, n, n+2, \cdots, n+m) \; .$$

Adding equations $k = 1, \cdots, m$ in (25) and noting the definitions

of a_{1j} and b_1 given in (24)

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[*] Based on a recent suggestion of W. Orchard—Hays.

(27) $\qquad \sum\limits_{1}^{m} \overline{x}_{n+k} = (0, 0, 1, 1, \cdots, 1)$.

There is a close relationship between the solutions of (25) and those of (21) when $\overline{x}_{n+1} \geq 0$, for then the first components of $\overline{x}_j$ for $j = 0, \cdots, n$ satisfy (21). Indeed by (27), if all $\overline{x}_{n+k} \geq 0$, the first component of all $\overline{x}_{n+k}$ must <u>vanish</u>, but the 1st component of the vector equations (25) reduces to (21) when the terms involving $\overline{x}_{n+k}$ are dropped. This proves the sufficiency of Theorem IX (below).

Theorem IX: <u>A</u> <u>necessary</u> <u>and</u> <u>sufficient</u> <u>condition</u> <u>for</u> <u>a</u> <u>solution</u> <u>to</u> (21) <u>to</u> <u>exist</u> <u>is</u> <u>a</u> <u>solution</u> <u>to</u> (25) <u>to</u> <u>exist</u> <u>with</u> $\overline{x}_{n+1} \geq 0$.

Theorem X: <u>Maximizing</u> <u>solutions</u> (<u>or</u> <u>a</u> <u>class</u> <u>of</u> <u>solutions</u> <u>with</u> <u>unbounded</u> <u>values</u>) <u>to</u> (21) <u>are</u> <u>obtained</u> <u>from</u> <u>the</u> 1st <u>components</u> <u>of</u> $(\overline{x}_0, \cdots, \overline{x}_n)$ <u>of</u> <u>the</u> <u>corresponding</u> <u>type</u> <u>solution</u> <u>to</u> (25) <u>with</u> $\overline{x}_{n+1} \geq 0$.

To prove necessity in IX, assume $(x_0, \cdots, x_n)$ satisfies (21); then

(28) $\qquad \overline{x}_0 = (x_0, 1, 0, \quad \cdots, 0)$

$\qquad\qquad \overline{x}_j = (x_j, 0, 0, \quad \cdots, 0) \qquad\qquad 1 \leq j \leq n$

$\qquad\qquad \overline{x}_{n+k} = (0, 0, \cdots, 1, \cdots, 0) \geq 0 \qquad 1 \leq k \leq m$

(where unity occurs in position k + 2) satisfies (25). Because of the possibility of forming solutions of the type (28) from solutions to

(21), it is easy to show that 1st components of maximizing solutions to (25) must be maximizing solutions to (28) (Theorem X).

It will be noted that (25) satisfies the requirements for the generalized simplex process: first the right-hand side considered as a matrix is of form $M = [Q, U_0, U_1, \cdots, U_m]$ where U_k is a unit column vector with unity in component $k + 1$ and is of rank $m + 1$ (the number of equations); second, an initial basic solution is available. Indeed set $\overline{x}_0, \overline{x}_{n+1}, \overline{x}_{n+2}, \cdots, \overline{x}_{n+m}$ equal to the corresponding constant vectors in (25) where $\overline{x}_{n+k} \geq 0$ for $k = 2, \cdots, m$ because $b_k \geq 0$.

In applying the generalized simplex procedure, however, both $\overline{x}_0$ and $\overline{x}_{n+1}$ are not restricted to be non-negative. Since $\overline{x}_{n+k} \geq 0$ for $(k = 2, \cdots, m)$, it follows that the values of the solutions, $\overline{x}_{n+1}$, to (27) have the right-hand side of (27) as an upper bound.

To obtain a maximizing solution to (25), the first phase is to apply the generalized simplex procedure to maximize the variable $\overline{x}_{n+1}$ (with no restriction on $\overline{x}_0$). Since $\overline{x}_{n+1}$ has a finite upper bound, a basic solution will be produced after a finite number of changes of basis in which $\overline{x}_{n+1} \geq 0$, providing Max $\overline{x}_{n+1} \geq 0$. If during the first phase $\overline{x}_{n+1}$ reaches a maximum less than zero, then, of course, by Theorem IX there is no solution to (21) and the process terminates. If, in the iterative process, $\overline{x}_{n+1}$ becomes positive (even though not maximum), the first phase, which is the search for a solution to (21), is completed and the second phase, which is the search for

an optimal solution, begins. Using the final basis of the first

phase in the second phase, $\overline{x}_0$ is maximized under the additional

restraint $\overline{x}_{n+1} \geq 0$.

Since the basic set of variables are taken in the initial

order $(\overline{x}_0, \overline{x}_{n+1}, \cdots, \overline{x}_{n+m})$ and in the <u>first</u> <u>phase</u> the variable

$\overline{x}_{n+1}$ is maximized, the <u>second</u> <u>row</u> <u>of</u> <u>the</u> <u>inverse</u> <u>of</u> <u>the</u> <u>basis</u>,

β_1, is used to "select" the candidate P_s to introduce into the

basis in order to increase $\overline{x}_{n+1}$, see (14); hence s is determined

such that

$$(29) \qquad \beta_1 P_s = \underset{j}{\text{Min}} \ (\beta_1 P_j) < 0 \ .$$

However, in the <u>second</u> <u>phase</u>, since the variable to be maximized

is $\overline{x}_0$ and the order of the basic set of variables is $(\overline{x}_0, \overline{x}_{n+1}, \cdots)$,

then the <u>first</u> <u>row</u> <u>of</u> <u>the</u> <u>inverse</u> <u>of</u> <u>the</u> <u>basis</u>, β_0, is used; i.e.,

one reverts back to (14). Application of the generalized simplex

procedure in the second phase yields, after a finite number of

changes in basis, either a solution with Max $\overline{x}_0$ or a class of

solutions of form (12) with no upper bound for $\overline{x}_0$. By Theorem X

the first components of $\overline{x}_0, \overline{x}_1, \cdots, \overline{x}_n$ form the corresponding

solutions to the real variable problem.

The computational convenience of this setup is apparent.

In the first place (as noted earlier), the right-hand side of (21)

considered as a matrix is of form $M = [Q, U_0, U_1, \cdots, U_m]$ where U_i

is a unit column vector with unity in component $k + 1$. In this case,

by (4), the basic solution $V = B^{-1} M = [B^{-1}Q; B^{-1}]$. This means

(in this case) that of the $\ell = m + 2$ components of the vector $\bar{v}_i$ the last $m + 1$ components of the vector variables $\bar{v}_i$ in the basic solution are identical with β_i, the corresponding row of the inverse. In applications this fact is important because the last $m + 1$ components of $\bar{v}_i$ are artificial in the sense that they belong to the perturbation and not to the original problem and it is desirable to obtain them with as little effort as possible. In the event that M has the special form above, no additional computational effort is required when the inverse of the basis is known. Moreover, the columns of (25) corresponding to the $(m+1)$ variables $(\bar{x}_0, \bar{x}_{m+1}, \cdots, \bar{x}_{n+m})$ form the initial identity basis $(U_0, U_1, \cdots, U_m)$, so that the inverse of the initial basis is readily available as the identity matrix to initiate the first iteration.

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