A New Mathematical Formulation for Strapdown Inertial Navigation

JOHN E. BORTZ, Member, IEEE
The Analytic Sciences Corporation
Reading, Mass. 01867

Abstract

A differential equation is developed for the orientation vector relating the body frame to a chosen reference frame. The time derivative of this vector is the sum of the inertially measurable angular velocity vector and of the inertially nonmeasurable noncommutativity rate vector.

It is precisely this noncommutativity rate vector that causes the computational problems when numerically integrating the direction cosine matrix. The orientation vector formulation allows the noncommutativity contribution to be isolated and, therefore, treated separately and advantageously.

An orientation vector mechanization is presented for a strapdown inertial system. Further, an example is given of the application of this formulation to a typical rigid body rotation problem.

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I. Introduction

The conventional method for updating the coordinate transformation matrix in a strapdown inertial navigation system is to integrate the matrix differential equation

$$\dot{C}^{RB} = C^{RB}\,[\underline{\omega}_{RB}\times] \tag{1}$$

where C^{RB} is the direction cosine matrix that transforms a vector body frame representation to a reference frame representation and

$$[\underline{\omega}_{RB}\times] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

is a skew symmetric matrix (representing the cross product operation) formed from the elements of $\underline{\omega}_{RB}$, the angular velocity of the body frame with respect to the reference frame. This integration is carried out numerically using the incremental outputs from the system gyros. The major problem in this method is the well known phenomenon of noncommutativity of finite rotations. The two conventional ways of combatting errors due to this effect are 1) to update the direction cosine matrix at or near the gyro rebalance frequency using a simple update algorithm or 2) to update the direction cosine matrix after many rebalance cycles using a more sophisticated algorithm. Even the most efficient algorithm places a moderate to heavy burden on the navigation system computer. If the update process is slowed down to ease the computational load, system bandwidth and accuracy are sacrificed.

This paper introduces a new[1] concept for examining rigid body motion leading to the vector differential equation (to be derived in Section III)

$$\dot{\underline{\varphi}}_{RB}(t) = \underline{\omega}_{RB}(t) + \dot{\underline{\sigma}}_{RB}(t) \tag{2}$$

where

$\underline{\varphi}_{RB}(t)$ is an orientation vector such that if at time t the chosen reference frame were rotated about an axis pointing in the direction of $\underline{\varphi}_{RB}(t)$ through an angle equal to the magnitude of $\underline{\varphi}_{RB}(t)$, it would be brought into coincidence with the body frame (Fig. 1)

$\underline{\omega}_{RB}(t)$ is that component of $\dot{\underline{\varphi}}_{RB}(t)$ due to inertially measurable angular motion (angular velocity vector)

[1]The mathematical theory presented here was actually introduced by J. Halcombe Laning, Jr. [1] in 1949 as a mathematical tool for use in fire control analysis. Unfortunately, at the time there was no sustaining external interest in this work and the results never became widely known. Laning's complete and elegant treatment of finite angles and rotations was presented in rather abstract terms. The development given here is original with the author and highly motivated in a physical sense.

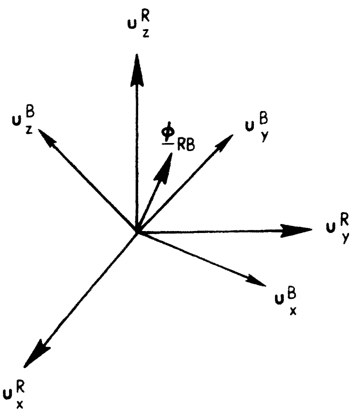

Fig. 1. The orientation vector. **u** indicates a unit vector in superscript frame and subscript direction.

$\dot{\underline{\sigma}}_{RB}(t)$ is that component of $\dot{\underline{\varphi}}_{RB}(t)$ due to noninertially measurable angular motion (noncommutativity rate vector).

To obtain the direction cosine matrix, (2) is solved for $\underline{\varphi}_{RB}(t)$ and then C^{RB} is simply evaluated as a matrix function of the vector argument $\underline{\varphi}_{RB}(t)$. In Section II it is shown that

$$C^{RB}(\underline{\varphi}) = \frac{\underline{\varphi}\underline{\varphi}^T}{\varphi^2}(1 - \cos\varphi) + I\cos\varphi + \frac{\sin\varphi}{\varphi}\,[\underline{\varphi}\times] \quad (3)$$

where

$$\varphi = |\underline{\varphi}|$$

$$I = 3 \times 3 \text{ identity matrix}$$

$$[\underline{\varphi}\times] = \begin{bmatrix} 0 & -\varphi_z & \varphi_y \\ \varphi_z & 0 & -\varphi_x \\ -\varphi_y & \varphi_x & 0 \end{bmatrix} \quad (4)$$

and the subscripts RB have been omitted for convenience. The argument of $\underline{\varphi}$ will be understood to be the time t. The advantages of the relationship given by (3) are:

1) it is an exact relationship
2) it generates an orthogonal matrix
3) it need be evaluated only when required to transform a vector or compute Euler angles.

Equation (2) is derived and discussed in Section III. A hybrid system for solving (2) is presented in Section IV. This hybrid method is more efficient and has higher accuracy and bandwidth than any digital algorithm for solving (1) when high-frequency angular vibrations are present.

An interesting example of the power of the new formulation is given in Section V.

II. The Direction Cosine Matrix

C^{RB} is derived in this section as a unique matrix function of the orientation (rotation) vector $\underline{\varphi}_{RB}$. This exact derivation relies entirely upon geometrical arguments.

Let $\mathbf{r}^B$ be an arbitrary vector fixed in the body frame. Suppose that at $t = t_0$, $C^{RB} = I$. Then

$$\mathbf{r}_0{}^R = \mathbf{r}_0{}^B$$

where the superscript indicates the frame in which the vector is coordinatized. Then let the body frame rotate with respect to the reference frame through the angle φ about a body frame axis $\mathbf{u}$. At time t, a new vector is defined by

$$\mathbf{r}^R(t) = C^{RB}(t)\,\mathbf{r}^B(t)$$

or, since $\mathbf{r}^B(t) = \mathbf{r}_0{}^B$ is fixed in the body frame, $\mathbf{r}^R(t)$ is given by

$$\mathbf{r}^R(t) = C^{RB}(t)\,\mathbf{r}_0{}^B\ . \quad (5)$$

These relationships are shown in Fig. 2.

The vector relationships used in constructing the figure are

$$\begin{aligned} &\mathbf{u}\cdot\mathbf{u} = 1 \\ &\mathbf{u}\cdot\mathbf{b} = 0 \\ &\mathbf{u}\cdot\mathbf{c} = 0 \\ &\mathbf{b}\cdot\mathbf{c} = 0 \\ &\mathbf{a} = (\mathbf{r}_0{}^B\cdot\mathbf{u})\mathbf{u} \\ &\mathbf{b} = \mathbf{u}\times\mathbf{r}_0{}^B \\ &\mathbf{c} = \mathbf{b}\times\mathbf{u} = \mathbf{u}\times(\mathbf{r}_0{}^B\times\mathbf{u}). \end{aligned}$$

In matrix notation, **a**, **b**, and **c** become

$$\begin{aligned} &\mathbf{a} = \mathbf{u}\mathbf{u}^T\mathbf{r}_0{}^B \\ &\mathbf{b} = [\mathbf{u}\times]\,\mathbf{r}_0{}^B \\ &\mathbf{c} = \mathbf{r}_0{}^B - \mathbf{u}\mathbf{u}^T\mathbf{r}_0{}^B \end{aligned} \quad (6)$$

where $[\cdot\times]$ is the matrix equivalent to the vector cross product operation and is defined in similar manner to (4). From Fig. 2 it is seen that

$$\mathbf{r}^R(t) = \mathbf{a} + \mathbf{b}\sin\varphi + \mathbf{c}\cos\varphi\ . \quad (7)$$

Equations (5) through (7) can be combined to get

$$C^{RB}\mathbf{r}_0{}^B = \left\{\mathbf{u}\mathbf{u}^T(1-\cos\varphi) + I\cos\varphi + [\mathbf{u}\times]\sin\varphi\right\}\mathbf{r}_0{}^B\ .$$

But since $\mathbf{r}_0{}^B$ is an arbitrary vector, it follows that

$$C^{RB} = \mathbf{u}\mathbf{u}^T(1-\cos\varphi) + I\cos\varphi + [\mathbf{u}\times]\sin\varphi\ . \quad (8)$$

Now let

$$\mathbf{u} = \frac{\underline{\varphi}}{\varphi}\ .$$

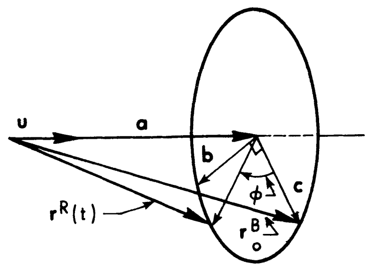

Fig. 2. The geometry of rotation.

Then (8) becomes

$$C^{RB} = \frac{\underline{\varphi}\underline{\varphi}^T}{\varphi^2}(1-\cos\varphi) + I\cos\varphi + \frac{\sin\varphi}{\varphi}[\underline{\varphi}\times] \quad (9)$$

which is (3). Equation (9) shows C^{RB} to be a well behaved function of $\underline{\varphi}$ with no singularities or indeterminacies.

III. The Orientation Vector Differential Equation

Using the matrix identity

$$\frac{\underline{\varphi}\underline{\varphi}^T}{\varphi^2} = \frac{[\underline{\varphi}\times][\underline{\varphi}\times]}{\varphi^2} + I$$

where $[\underline{\varphi}\times]$ is defined[2] in (4), (9) can be rewritten as

$$C = I + \frac{\sin\varphi}{\varphi}[\underline{\varphi}\times] + \frac{1-\cos\varphi}{\varphi^2}[\underline{\varphi}\times]^2 \ . \quad (10)$$

We will now obtain the desired vector differential equation

$$\dot{\underline{\varphi}} = \underline{\omega} + \dot{\underline{\sigma}} \quad (11)$$

given earlier as (2) where the form of the noncommutativity rate vector $\dot{\underline{\sigma}}$ is anticipated to be

$$\dot{\underline{\sigma}} = \tfrac{1}{2}\underline{\varphi}\times\underline{\omega} + \frac{1}{\varphi^2}\left(1 - \frac{\varphi\sin\varphi}{2(1-\cos\varphi)}\right)\underline{\varphi}\times(\underline{\varphi}\times\underline{\omega}) \ . \quad (12)$$

To do this it will be necessary to employ the following steps:

1) differentiate (10) with respect to time to get $\dot{C}$
2) write $\dot{C}$ also as $C[\underline{\omega}\times]$ where C is given by (10)
3) form a vector $\mathbf{s}_1$ from the results of step 1 and form a vector $\mathbf{s}_2$ from the results of step 2
4) manipulate the equation $\mathbf{s}_1 = \mathbf{s}_2$ to obtain (11) and (12).

[2]This notation has the flexibility and the economy of symbols to conveniently recast vector equations in matrix form. For example, the vector equation

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = -(\mathbf{b}\times\mathbf{c})\times\mathbf{a}$$

becomes in matrix form

$$[\mathbf{a}\times][\mathbf{b}\times]\mathbf{c} = -[\{[\mathbf{b}\times]\mathbf{c}\}\times]\mathbf{a} \ .$$

Step 1: In order to differentiate (10), two derivatives are obtained first. They are

$$f_1(\varphi) = \frac{\sin\varphi}{\varphi} \quad (13)$$

$$\dot{f}_1(\varphi) = [\cos\varphi - f_1(\varphi)]\frac{\underline{\varphi}\cdot\dot{\underline{\varphi}}}{\varphi^2} \quad (14)$$

$$f_2(\varphi) = \frac{1-\cos\varphi}{\varphi^2}$$

$$\dot{f}_2(\varphi) = [f_1(\varphi) - 2f_2(\varphi)]\frac{\underline{\varphi}\cdot\dot{\underline{\varphi}}}{\varphi^2} \ . \quad (15)$$

These are obtained by making use of $\varphi = (\underline{\varphi}\cdot\underline{\varphi})^{1/2}$. Then differentiate (10) term by term to get

$$\dot{C} = \dot{f}_1(\varphi)[\underline{\varphi}\times] + f_1(\varphi)[\dot{\underline{\varphi}}\times] + \dot{f}_2(\varphi)[\underline{\varphi}\times]^2 + f_2(\varphi)[\dot{\underline{\varphi}}\times][\underline{\varphi}\times] + f_2(\varphi)[\underline{\varphi}\times][\dot{\underline{\varphi}}\times] \ . \quad (16)$$

Step 2: Combine (1), (10), (13), and (15) to get

$$\dot{C} = [\underline{\omega}\times] + f_1(\varphi)[\underline{\varphi}\times][\underline{\omega}\times] + f_2(\varphi)[\underline{\varphi}\times]^2[\underline{\omega}\times] \ . \quad (17)$$

Step 3: Let $\mathbf{r}$ be an arbitrary vector. Define the vector $\mathbf{s}_1$ to be

$$\mathbf{s}_1 = \tfrac{1}{2}(\dot{C} - \dot{C}^T)\mathbf{r} \ . \quad (18)$$

Using (16) and the skew-symmetric property

$$[\underline{\varphi}\times]^T = -[\underline{\varphi}\times]$$

(18) can be reduced to

$$\mathbf{s}_1 = \dot{f}_1(\varphi)[\underline{\varphi}\times]\mathbf{r} + f_1(\varphi)[\dot{\underline{\varphi}}\times]\mathbf{r}$$

or, in vector format,

$$\mathbf{s}_1 = \dot{f}_1(\varphi)\underline{\varphi}\times\mathbf{r} + f_1(\varphi)\dot{\underline{\varphi}}\times\mathbf{r} \ . \quad (19)$$

Define the vector $\mathbf{s}_2$ to be

$$\mathbf{s}_2 = \tfrac{1}{2}(\dot{C} - \dot{C}^T)\mathbf{r} \ . \quad (20)$$

Obviously, from (18) and (20),

$$\mathbf{s}_1 \equiv \mathbf{s}_2 \ . \quad (21)$$

Using (17) in (20) gives, in vector format,

$$\mathbf{s}_2 = \underline{\omega}\times\mathbf{r} + \tfrac{1}{2}f_1(\varphi)[\underline{\varphi}\times(\underline{\omega}\times\mathbf{r}) + \underline{\omega}\times(\underline{\varphi}\times\mathbf{r})] + \tfrac{1}{2}f_2(\varphi)[\underline{\varphi}\times(\underline{\varphi}\times(\underline{\omega}\times\mathbf{r})) + \underline{\omega}\times(\underline{\varphi}\times(\underline{\varphi}\times\mathbf{r}))] \ .$$

The terms in brackets can be manipulated slightly to get

$$\mathbf{s}_2 = \underline{\omega}\times\mathbf{r} + \tfrac{1}{2}f_1(\varphi)(\underline{\varphi}\times\underline{\omega})\times\mathbf{r} - \tfrac{1}{2}f_2(\varphi)[\underline{\varphi}\cdot\underline{\omega}(\underline{\varphi}\times\mathbf{r}) + \varphi^2(\underline{\omega}\times\mathbf{r})] \ . \quad (22)$$

Step 4: Substitution of (19) and (22) into (21) gives

$$\dot{f}_1(\varphi)\underline{\varphi}\times\mathbf{r} + f_1(\varphi)\dot{\underline{\varphi}}\times\mathbf{r} = \underline{\omega}\times\mathbf{r} + \tfrac{1}{2}f_1(\varphi)(\underline{\varphi}\times\underline{\omega})\times\mathbf{r}$$
$$- \tfrac{1}{2}f_2(\varphi)[\underline{\varphi}\cdot\underline{\omega}(\underline{\varphi}\times\mathbf{r}) + \varphi^2\,\underline{\omega}\times\mathbf{r}] \ .$$

But since $\mathbf{r}$ is an arbitrary vector, it follows that

$$\dot{f}_1(\varphi)\underline{\varphi} + f_1(\varphi)\dot{\underline{\varphi}} = \underline{\omega} + \tfrac{1}{2}f_1(\varphi)\underline{\varphi}\times\underline{\omega}$$
$$- \tfrac{1}{2}f_2(\varphi)[(\underline{\varphi}\cdot\underline{\omega})\underline{\varphi} + \varphi^2\underline{\omega}] \ . \qquad (23)$$

Now, introduce (13), (14), and (15) into (23) to get

$$\cos\varphi\,\frac{\underline{\varphi}\cdot\dot{\underline{\varphi}}}{\varphi^2}\,\underline{\varphi} - \frac{\sin\varphi}{\varphi}\,\frac{\underline{\varphi}\cdot\dot{\underline{\varphi}}}{\varphi^2}\,\underline{\varphi} + \frac{\sin\varphi}{\varphi}\,\dot{\underline{\varphi}}$$
$$= \underline{\omega} + \tfrac{1}{2}\,\frac{\sin\varphi}{\varphi}\,\underline{\varphi}\times\underline{\omega} - \tfrac{1}{2}\,\frac{\underline{\varphi}\cdot\underline{\omega}}{\varphi^2}\,\underline{\varphi}$$
$$+ \frac{\cos\varphi}{2}\,\frac{\underline{\varphi}\cdot\underline{\omega}}{\varphi^2}\,\underline{\varphi} - \tfrac{1}{2}\underline{\omega} + \frac{\cos\varphi}{2}\,\underline{\omega} \ . \qquad (24)$$

Next take the dot product of $\underline{\varphi}$ into each term of (24) and collect terms obtaining

$$\cos\varphi\,\underline{\varphi}\cdot\dot{\underline{\varphi}} = \cos\varphi\,\underline{\varphi}\cdot\underline{\omega}$$

so for $|\underline{\varphi}|<\pi$, we have the interesting relationship

$$\underline{\varphi}\cdot\dot{\underline{\varphi}} \equiv \underline{\varphi}\cdot\underline{\omega} \qquad (25)$$

which shows that the component of the orientation rate vector parallel to the orientation vector is the same as the component of angular velocity parallel to the orientation vector. Consequently, as can be seen from (2) or (11), and as can be proved independently from (12),

$$\dot{\underline{\sigma}}\cdot\underline{\varphi} \equiv 0 \ .$$

Thus the noncommutativity rate vector $\dot{\underline{\sigma}}$ is always perpendicular to the orientation vector $\underline{\varphi}$.

Employing (25) in (24) and collecting terms gives

$$\frac{\sin\varphi}{\varphi}\dot{\underline{\varphi}} = \tfrac{1}{2}\underline{\omega} + \tfrac{1}{2}\,\frac{\sin\varphi}{\varphi}\,\underline{\varphi}\times\underline{\omega}$$
$$+ \left\{\frac{\sin\varphi}{\varphi} - \frac{\cos\varphi}{2} - \tfrac{1}{2}\right\}\frac{\underline{\varphi}\cdot\underline{\omega}}{\varphi^2}\,\underline{\varphi} + \frac{\cos\varphi}{2}\,\underline{\omega} \ .$$

But

$$\frac{\underline{\varphi}\cdot\underline{\omega}}{\varphi^2}\,\underline{\varphi} \equiv \frac{\underline{\varphi}\times(\underline{\varphi}\times\underline{\omega})}{\varphi^2} + \underline{\omega}$$

so the above equation becomes

$$\frac{\sin\varphi}{\varphi}\,\dot{\underline{\varphi}} = \frac{\sin\varphi}{\varphi}\,\underline{\omega} + \frac{\sin\varphi}{\varphi}\,\tfrac{1}{2}\underline{\varphi}\times\underline{\omega}$$
$$+ \frac{\sin\varphi}{\varphi}\,\frac{1}{\varphi^2}\left(1 - \frac{\varphi(\cos\varphi + 1)}{2\sin\varphi}\right)\underline{\varphi}\times(\underline{\varphi}\times\underline{\omega}) \ .$$

Now, multiply each terms by $\varphi/\sin\varphi$ (which exists and is unique for $|\underline{\varphi}|<\pi$) and recognize that

$$\frac{\varphi(\cos\varphi + 1)}{2\sin\varphi} = \frac{\varphi\sin\varphi}{2(1 - \cos\varphi)}$$

to get the final result

$$\dot{\underline{\varphi}} = \underline{\omega} + \tfrac{1}{2}\underline{\varphi}\times\underline{\omega} + \frac{1}{\varphi^2}\left(1 - \frac{\varphi\sin\varphi}{2(1 - \cos\varphi)}\right)\underline{\varphi}\times(\underline{\varphi}\times\underline{\omega})$$
$$\varphi \neq n\pi,\ n = \pm 1, \pm 2, \pm 3, \ldots \qquad (26)$$

which is the same as (11) and (12).

The physical significance of the restriction $\varphi \neq n\pi$, $n = \pm 1, \pm 2, \pm 3, \ldots$ is that a rotation through an angle of π radians about some axis $\mathbf{u}$ yields the same orientation as a rotation through an angle of π radians about the axis $-\mathbf{u}$. Also, a rotation through an angle of 2π radians about a particular axis results in the same orientation as about any other axis.

Note that intuition is satisfied by the expression for the noncommutativity rate vector

$$\dot{\underline{\sigma}} = \tfrac{1}{2}\underline{\varphi}\times\underline{\omega} + \frac{1}{\varphi^2}\left(1 - \frac{\varphi\sin\varphi}{2(1 - \cos\varphi)}\right)\underline{\varphi}\times(\underline{\varphi}\times\underline{\omega}) \ . \qquad (12)$$

We know that the effect of the noncommutativity phenomenon is to cause a final orientation after a series of rotations that depends not only on the individual rotations that were executed, but also on the order in which they were taken. Equation (12) shows how the history of previous rotations $\underline{\varphi}$ interacts with the current angular velocity $\underline{\omega}$ to produce the rate $\dot{\underline{\sigma}}$ at which the noncommutativity effect accumulates.

IV. Strapdown System Mechanization

In a strapdown inertial navigation system the conventional method for obtaining the coordinate transformation is to integrate the matrix differential equation

$$\dot{C}^{RB} = C^{RB}\,[\underline{\omega}_{RB}\times] \qquad (1)$$

in the system digital computer.

With a strapdown system configured as shown in Fig. 3, the computational burden of the all-digital solution for C^{RB} can be reduced by a factor of as much as 20 to 1 at no sacrifice in direction cosine matrix accuracy and at a computational bandwidth limited only by the gyro time constants. The cost of the hybrid solution is the requirement for the additional analog circuitry.

The hybrid system shown in Fig. 3 was built and tested by the author at NASA Electronics Research Center. Test results and error analyses are included in [2]. The experimental results were in close agreement with the theoretical predictions. A very similar technique is also presented in [2] for processing the strapdown accelerometer measurements, allowing the rate at which the specific force increments are transformed from the body to the reference frame to be slowed down substantially without sacrificing accuracy and at the same time improving velocity measurement bandwidth.

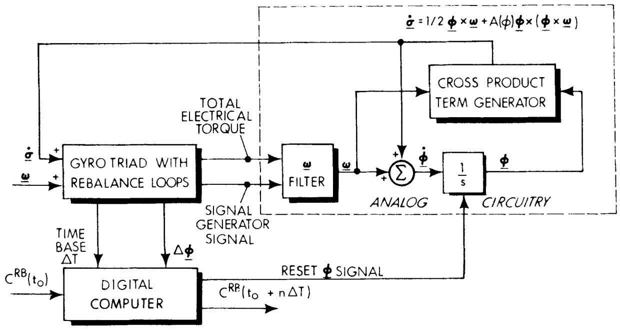

Fig. 3. Symbolic hybrid system diagram.

The basic principle involved is to generate a set of signals $\dot{\sigma}_x$, $\dot{\sigma}_y$, and $\dot{\sigma}_z$ representing the components of the noncommutativity rate vector $\dot{\underline{\sigma}}$. It is shown in [2] that under certain reasonable conditions and system design choices,

$$|\dot{\underline{\sigma}}| << \omega_{\max}$$

where $\omega_{\max}$ is the maximum angular velocity capability of the gyro. Thus, since the noncommutativity rate vector is quite small compared with the maximum angular velocity magnitude, well designed analog circuitry has sufficient accuracy for generating the noncommutativity rate correction signals. These signals are applied as electrically generated torques to the torque summing members of the gyro triad (shown symbolically as $\dot{\underline{\sigma}}$ in Fig. 3).

Then the gyro integrates and quantizes the orientation rate vector $\dot{\underline{\varphi}}$ at full inertial quality gyro accuracy. The digital computer accumulates these increments over extended periods of time and evaluates the direction cosine matrix using (3) only as often as necessary. Thus the digital computer's role has been reduced from integrating a matrix differential equation at high repetition rates to evaluating a matrix function at relatively low rates.

The $\underline{\omega}$ filter extracts, component by component, a high bandwidth analog representation of angular velocity. It does this by first removing from the signal generator signal the effects of all torques that were electrically applied to the gyro torque summing member and then treating the resultant signal as if it were the output of a float that had responded to input axis angular velocity alone.

Whenever $|\underline{\varphi}|$ reaches $\varphi_{\max}$ (where $\varphi_{\max}$ is chosen to be, say, 0.1 radian in a practical situation), $\underline{\varphi}$ is reset to **0**, the time of reset is labeled t_0, and the initial condition matrix $C^{RB}(t_0)$ is updated to reflect the value of $\underline{\varphi}$ just prior to reset. This avoids the problem of the singularity in the $\underline{\varphi}$ equation at $|\underline{\varphi}| = \pi$ radians. It also affords some great simplifications in the analog circuitry. These considerations are developed at length in [2].

High overall system bandwidth is achieved by using high bandwidth analog circuitry to track the noncommutativity rate vector. System accuracy is maintained by using the digital computer to generate the coordinate transformation matrix from the $\Delta\underline{\varphi}$ accumulated from the gyro triad.

An all-digital implementation of the orientation vector formulation is developed in [3].

V. An Example

It is shown in [2] that the equation

$$\underline{\omega} = \dot{\underline{\varphi}} - \frac{1 - \cos\varphi}{\varphi^2}\,\underline{\varphi}\times\dot{\underline{\varphi}} + \frac{1}{\varphi^2}\left(1 - \frac{\sin\varphi}{\varphi}\right)\underline{\varphi}\times(\underline{\varphi}\times\dot{\underline{\varphi}}) \tag{27}$$

is entirely equivalent to (26) in that either can be uniquely derived from the other. This form allows us to answer the question, "Given the time history of the orientation of a rigid body, what was the angular velocity that generated that specified time history?" This is an important result in the error propagation analysis of digital algorithms for updating the coordinate transformation. It allows the analyst to propose his own exact truth model for $\underline{\varphi}_{RB}(t)$ (and, therefore, for C^{RB}) and then to exercise any desired algorithm with the exact angular velocity $\underline{\omega}_{RB}$ that generates $\underline{\varphi}(t)$ to see how that algorithm performs. He does this by comparing the direction cosine matrix generated by that algorithm (as excited by $\underline{\omega}_{RB}$) with the exact C^{RB} truth model. Almost any type of angular velocity can be generated by thoughtful choice of $\underline{\varphi}(t)$.

For example, suppose the time history

$$\underline{\varphi}(t) = \begin{bmatrix} \theta \sin \omega t \\ \theta \cos \omega t \\ 0 \end{bmatrix} \tag{28}$$

is given. This is the classical coning motion. To find the angular velocity $\underline{\omega}(t)$ that generates this orientation, note that

$$\varphi \equiv |\underline{\varphi}(t)| = \theta \tag{29}$$

and that by differentiating (28)

$$\dot{\underline{\varphi}}(t) = \begin{bmatrix} \omega\theta \cos \omega t \\ -\omega\theta \sin \omega t \\ 0 \end{bmatrix}. \tag{30}$$

Using (28) through (30) in (27) gives

$$\underline{\omega} = \begin{bmatrix} \omega\theta \cos \omega t \\ -\omega\theta \cos \omega t \\ 0 \end{bmatrix} - \frac{1 - \cos\theta}{\theta^2}\begin{bmatrix} 0 \\ 0 \\ -\theta^2 \omega \end{bmatrix}$$

$$+ \frac{1}{\theta^2}\left(1 - \frac{\sin\theta}{\theta}\right)\begin{bmatrix} -\theta^3 \omega \cos \omega t \\ \theta^3 \omega \sin \omega t \\ 0 \end{bmatrix}$$

or, after collecting terms,

$$\underline{\omega} = \begin{bmatrix} \omega \sin\theta \ \cos \omega t \\ -\omega \sin\theta \ \sin \omega t \\ \omega(1 - \cos\theta) \end{bmatrix} \tag{31}$$

which is the well known description for coning motion angular velocity.

The orientation resulting from coning motion is found by using (28) in (3) to get

$$C = \begin{bmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{bmatrix} + (1 - \cos\theta) \begin{bmatrix} \sin^2 \omega t & \tfrac{1}{2} \sin 2\omega t & 0 \\ \tfrac{1}{2} \sin 2\omega t & \cos^2 \omega t & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \sin\theta \cos \omega t \\ 0 & 0 & -\sin\theta \sin \omega t \\ -\sin\theta \cos \omega t & \sin\theta \sin \omega t & 0 \end{bmatrix}$$

or

$$C = \begin{bmatrix} \cos\theta + (1 - \cos\theta)\sin^2 \omega t & \tfrac{1}{2}(1 - \cos\theta) \sin 2\omega t & \sin\theta \cos \omega t \\ \tfrac{1}{2}(1 - \cos\theta)\sin 2\omega t & \cos\theta + (1 - \cos\theta)\cos^2 \omega t & -\sin\theta \sin \omega t \\ -\sin\theta \cos \omega t & \sin\theta \sin \omega t & \cos\theta \end{bmatrix}. \quad (32)$$

Thus there are three equivalent ways to represent coning motion. Equation (28) gives the time history of the orientation vector by means of which we can readily *visualize* coning motion, (31) describes the angular velocity vector that *generates* coning motion, and (32) is the evaluation of the direction cosine matrix by means of which we *account for the effects* of coning motion.

VI. Conclusions

A new exact formulation for rigid body angular motion has been developed. This is a vector formulation especially well suited to the strapdown inertial navigation computation problem. Use of this formulation leads to significant reduction in digital computer time and memory requirements and an improvement in computational bandwidth at no sacrifice in accuracy.

The significant feature of the vector orientation concept is that it allows the noncommutativity of finite rotations effect to be separated from other effects. Since it is precisely this effect that causes the computational problems associated with finite rotations, this formulation allows advantageous treatment of the principle source of strapdown computational difficulty.

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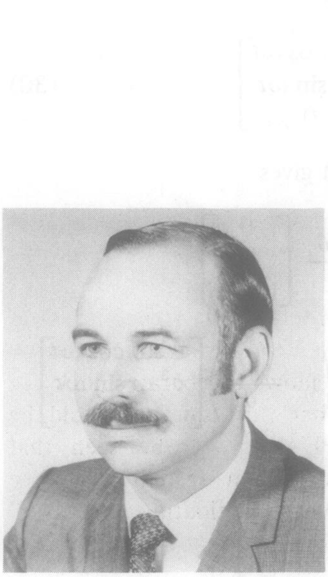

John E. Bortz (M'63) was born in Reading, Pa., on March 12, 1935. He received the B.S. degree from the U.S. Naval Academy, Annapolis, Md., in 1957, and the S.M. and Sc.D. degrees from the Massachusetts Institute of Technology, Cambridge, in 1965 and 1969, respectively.

He was a navigator in the Strategic Air Command of the U.S. Air Force from 1957 to 1961. From 1961 to 1963 he was employed by Bell Aerospace Corp., Buffalo, N.Y., as an electronic circuit designer. From 1963 to 1965 he was a member of the staff of the M.I.T. Instrumentation Laboratory in the Gimballess Inertial Guidance Group, and from 1966 to 1970 he was a member of the Guidance and Control Directorate of the NASA Electronics Research Center in Cambridge, Mass. He is now a member of the technical staff of The Analytic Sciences Corporation, Reading, Mass.

Dr. Bortz is a member of the American Institute of Aeronautics and Astronautics and is a navigator in the USAF Reserve.