Review of Linear Algebra and Vector Calculus

Adopted from notes by Andrew Rosenberg of CUNY

Linear Algebra Basics

- What is a vector?
- What is a matrix?
- Transposition
- Adding matrices and vectors
- Multiplying matrices.

Definitions

- A vector is a one dimensional array.
- We denote vectors as boldface lower case letter **x**
- If we don't specify otherwise assume **x** is a column vector

$$\mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ \dots \\ x_{n-1} \end{pmatrix}$$

Transposition

Transposing a matrix or vector swaps rows and columns.

A column-vector becomes a row-vector

$$\mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ \dots \\ x_{n-1} \end{pmatrix}$$

$$\mathbf{x}^T = \begin{pmatrix} x_0 & x_1 & \dots & x_{n-1} \end{pmatrix}$$

Inner and Outer Product

- Inner product between two equal-length vectors x, y:

$$x \cdot y = x^T y = x_0\, y_0 + x_1\, y_1 + \ldots + x_{n-1}\, y_{n-1}$$

- Outer product between two equal-length vectors x, y:

$$xy^T = \begin{bmatrix} x_0 y_0 & x_0 y_1 & \cdots & x_0 y_{n-1} \\ \cdots & \cdots & & \cdots \\ x_{n-1} y_0 & x_{n-1} y_1 & \cdots & x_{n-1} y_{n-1} \end{bmatrix}$$

Definition

A matrix is a higher dimensional array.
We typically denote matrices as capital letters e.g., A.
If A is an n-by-m matrix, it has the following structure

$$A = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,m-1} \\ a_{1,0} & a_{1,1} & & a_{1,m-1} \\ \vdots & & \ddots & \vdots \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,m-1} \end{pmatrix}$$

Transposing a matrix or vector swaps rows and columns.

A column-vector becomes a row-vector

$$A = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,m-1} \\ a_{1,0} & a_{1,1} & & a_{1,m-1} \\ \vdots & & \ddots & \vdots \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,m-1} \end{pmatrix}$$

$$A^T = \begin{pmatrix} a_{0,0} & a_{1,0} & \cdots & a_{n-1,0} \\ a_{0,1} & a_{1,1} & & a_{1,m-1} \\ \vdots & & \ddots & \vdots \\ a_{0,m-1} & a_{1,m-1} & \cdots & a_{n-1,m-1} \end{pmatrix}$$

If A is n-by-m, then A^T is m-by-n.

Matrices can only be added if they have the same dimension.

$$A+B = \begin{pmatrix} a_{0,0} + b_{0,0} & a_{0,1} + b_{0,1} & \cdots & a_{0,m-1} + b_{0,m-1} \\ a_{1,0} + b_{1,0} & a_{1,1} + b_{1,1} & & a_{1,m-1} + b_{1,m-1} \\ \vdots & & \ddots & \vdots \\ a_{n-1,0} + b_{n-1,0} & a_{n-1,1} + b_{n-1,1} & \cdots & a_{n-1,m-1} + b_{n-1,m-1} \end{pmatrix}$$

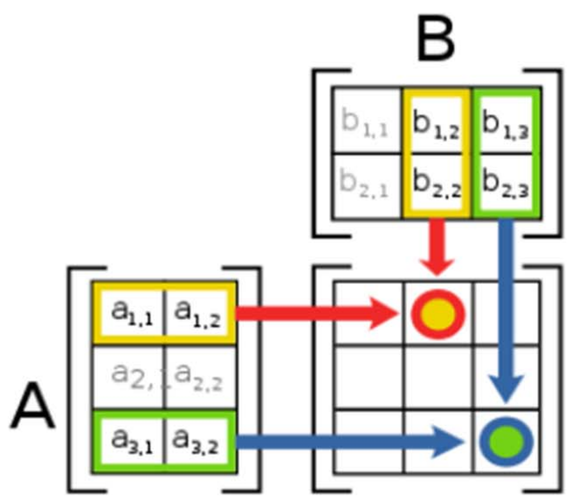
To multiply two matrices, the *inner dimensions* must match.

- An n-by-m can be multiplied by an n'-by-m' matrix iff $m = n'$.

$$AB = C$$

$$c_{ij} = \sum_{k=0}^{m} a_{ik} * b_{kj}$$

That is, multiply the i-th row by the j-th column.



Useful matrix operations

- Inversion
- Norm
- Eigenvector decomposition

Matrix Inversion

The inverse of an n-by-m matrix A is denoted A^{-1}, and has the following property.

$$AA^{-1} = I$$

Where I is the **identity matrix**, an n-by-n matrix where $I_{ij} = 1$ iff $i = j$ and 0 otherwise.
If A is a **square** matrix (iff $n = m$) then,

$$A^{-1}A = I$$

What is the inverse of a vector? $\mathbf{x}^{-1} =?$

Some useful Matrix Inversion Properties

$$(A^{-1})^{-1} = A$$

$$(kA)^{-1} = k^{-1}A^{-1}$$

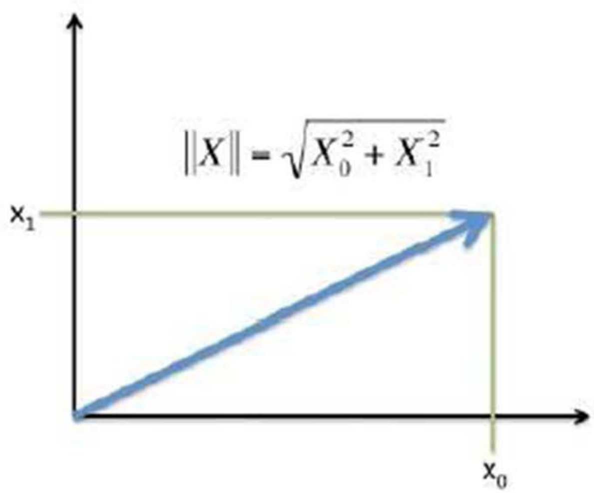
$$(A^T)^{-1} = (A^{-1})^T$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

The norm of a vector

The **norm** of a vector **x** is written $||\mathbf{x}||$.
The norm represents the euclidean length of a vector.

$$\begin{aligned} ||\mathbf{x}|| &= \sqrt{\sum_{i=0}^{n-1} x_i^2} \\ &= \sqrt{x_0^2 + x_1^2 + \ldots + x_{n-1}^2} \end{aligned}$$



Eigenvectors

For a square matrix A, the eigenvector is defined as

$$A\mathbf{u}_i = \lambda_i \mathbf{u}_i$$

Where $\mathbf{u}_i$ is an **eigenvector** and λ_i is its corresponding **eigenvalue**.

In general, eigenvalues are complex numbers, but if A is symmetric, they are real.

Eigenvectors describe how a matrix transforms a vector, and can be used to define a basis space, namely the eigenspace.

Who cares? The eigenvectors of a covariance matrix have some very interesting properties.