ch1_phase2

September 28, 2020

[1]:
```
from sympy import Matrix, Symbol, derive_by_array, Lambda, Function,␣
↪MatrixSymbol, Derivative, diff, symbols
from sympy import var
from sympy.abc import x, i, j, a, b


from sympy.interactive import init_printing

init_printing(pretty_print=True, wrap_line=True, num_columns=60)
```

[2]:
```
def myvar(letter: str, i: int, j: int) -> Symbol:
    letter_ij = Symbol('{}_{}{}'.format(letter, i+1, j+1), is_commutative=True)
    return letter_ij


n,m,p = 3,3,2

X = Matrix(n, m, lambda i,j : myvar('x', i, j)); X
```

[2]:
$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

[3]:
```
W = Matrix(m, p, lambda i,j : myvar('w', i, j)); W
```

[3]:
$$\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix}$$

[4]:
```
A = MatrixSymbol('X',3,3); Matrix(A)
B = MatrixSymbol('W',3,2)
```

[5]:
```

```

[5]:
```

```

[5]:
```
v = lambda a,b: a*b
```

```
vL = Lambda((a,b), a*b)

n = Function('v') #, Lambda((a,b), a*b))

vN = lambda mat1, mat2: Matrix(mat1.shape[0], mat2.shape[1], lambda i, j:␣
 ↪Symbol("n_{}{}".format(i+1, j+1))); vN

Nelem = vN(X, W); Nelem
```

[5]:
$$\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{bmatrix}$$

[6]:
```
n(X,W)
```

[6]:
$$v\left(\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix}\right)$$

[7]:
```
n(A,B)
```

[7]:
$$v(X, W)$$

[8]:
```
n(X,W).replace(n, v) # replace works when v = python lambda
```

[8]:
$$\begin{bmatrix} w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix}$$

[9]:
```
n(X,W).subs({n: vL}) # subs works when v = sympy lambda
```

[9]:
$$\begin{bmatrix} w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix}$$

[10]:
```
n(X,W).replace(n, vL)
```

[10]:
$$\begin{bmatrix} w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix}$$

[11]:
```
n(X,W).subs({n: v})# subs() doesn't work when v is python lambda
```

[11]:
$$v\left(\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix}\right)$$

[12]:
```
Matrix(n(A,B).subs({n: vL}))
```

[12]:

$$\begin{bmatrix} W_{0,0}X_{0,0} + W_{1,0}X_{0,1} + W_{2,0}X_{0,2} & W_{0,1}X_{0,0} + W_{1,1}X_{0,1} + W_{2,1}X_{0,2} \\ W_{0,0}X_{1,0} + W_{1,0}X_{1,1} + W_{2,0}X_{1,2} & W_{0,1}X_{1,0} + W_{1,1}X_{1,1} + W_{2,1}X_{1,2} \\ W_{0,0}X_{2,0} + W_{1,0}X_{2,1} + W_{2,0}X_{2,2} & W_{0,1}X_{2,0} + W_{1,1}X_{2,1} + W_{2,1}X_{2,2} \end{bmatrix}$$

[13]:
```
#N = v(X, W); N
N = n(A,B); N
```

[13]:
$$v(X, W)$$

[14]:
```
N.replace(n, v)
```

[14]:
$$XW$$

[15]:
```
N.replace(n, v).subs({A: X, B:W}) # replacing ariable values after doing␣
↪function doesn't make the function apply directly on the values (matrices),␣
↪need to replace values before the function is replaced, so that the function␣
↪can act on them while they are given/alive.
```

[15]:
$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix}$$

[16]:
```
N.subs({n: vL, A:X, B:W})
```

[16]:
$$\begin{bmatrix} w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix}$$

[17]:
```
Nspec = N.subs({A:X, B:W}).replace(n, v); Nspec
```

[17]:
$$\begin{bmatrix} w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix}$$

[18]:
```

```

[18]:
```
N.diff(N)
```

[18]:
$$1$$

[19]:
```
N.diff(X)
```

[19]:
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

[20]:
```

```

[20]:
```

```

[20]:
```
# way 2 of declaring S (better way)
sigma = Function('sigma')

sigmaApply = Function("sigma_apply") #lambda matrix:  matrix.applyfunc(sigma)

sigmaApply_ = lambda matrix: matrix.applyfunc(sigma)

sigmaApply(A)
```

[20]:
$$\sigma_{apply}(X)$$

[21]:
```
sigmaApply(A).subs({A: X})
```

[21]:
$$\sigma_{apply}\left(\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}\right)$$

[22]:
```
sigmaApply_(A)
```

[22]:
$$\left(d \mapsto \sigma(d)\right)_{\circ}(X)$$

[23]:
```
sigmaApply(A).subs({A: X}).replace(sigmaApply, sigmaApply_) # NOTE: subs of␣
↪functions doesn't work, replace actually evaluates the replaced function!
```

[23]:
$$\begin{bmatrix} \sigma(x_{11}) & \sigma(x_{12}) & \sigma(x_{13}) \\ \sigma(x_{21}) & \sigma(x_{22}) & \sigma(x_{23}) \\ \sigma(x_{31}) & \sigma(x_{32}) & \sigma(x_{33}) \end{bmatrix}$$

[24]:
```
S = sigmaApply(N); S
```

[24]:
$$\sigma_{apply}(v(X, W))$$

[25]:
```
Derivative(S, S)
```

[25]:
$$\frac{\partial}{\partial \sigma_{apply}(v(X, W))} \sigma_{apply}(v(X, W))$$

[26]:
```
Derivative(S, S).doit()
```

[26]:
$$1$$

[27]:
```
Derivative(S, n(A,B)).doit()
```

[27]:
$$\frac{\partial}{\partial v(X, W)} \sigma_{apply}(v(X, W))$$

[28]:
```
#lambd = Function("lambda")
#Lagain = lambd(sigmaApply(n(A))); Lagain
```

```
# diff(Lagain, A) # never execute
#
```

[29]:
```
S.replace(A,X).replace(B,W)
```

[29]:
$$\sigma_{apply}\left(v\left(\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix}\right)\right)$$

[30]:
```
S.replace(n, v)
```

[30]:
$$\sigma_{apply}(XW)$$

[31]:
```
S.subs({A:X, B:W}).replace(n, v)
```

[31]:
$$\sigma_{apply}\left(\begin{bmatrix} w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix}\right)$$

[32]:
```
Sspec = S.subs({A:X, B:W}).replace(n, v).replace(sigmaApply, sigmaApply_)
Sspec
```

[32]:
$$\begin{bmatrix} \sigma(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}) & \sigma(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13}) \\ \sigma(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}) & \sigma(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23}) \\ \sigma(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}) & \sigma(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33}) \end{bmatrix}$$

[33]:
```
S.replace(n, vN) #.replace(sigmaApply, sigmaApply_)
```

[33]:
$$\sigma_{apply}\left(\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{bmatrix}\right)$$

[34]:
```
Selem = S.replace(n, vN).replace(sigmaApply, sigmaApply_); Selem
```

[34]:
$$\begin{bmatrix} \sigma(n_{11}) & \sigma(n_{12}) \\ \sigma(n_{21}) & \sigma(n_{22}) \\ \sigma(n_{31}) & \sigma(n_{32}) \end{bmatrix}$$

[35]:
```
import itertools

elemToSpecD = dict(itertools.chain(*[[(Nelem[i, j], Nspec[i, j]) for j in␣
 ↪range(2)] for i in range(3)]))

elemToSpec = list(elemToSpecD.items())

Matrix(elemToSpec)
```

[35]:

$$\begin{bmatrix} n_{11} & w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} \\ n_{12} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ n_{21} & w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} \\ n_{22} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ n_{31} & w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} \\ n_{32} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix}$$

[36]:
```
elemToSpecFuncD = dict(itertools.chain(*[[(Nelem[i, j], Function("n_{}{}".
 ↪format(i + 1, j + 1))(Nspec[i, j])) for j in range(2)] for i in range(3)]))

elemToSpecFunc = list(elemToSpecFuncD.items())

Matrix(elemToSpecFunc)
```

[36]:
$$\begin{bmatrix} n_{11} & \mathrm{n}_{11}\left(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}\right) \\ n_{12} & \mathrm{n}_{12}\left(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13}\right) \\ n_{21} & \mathrm{n}_{21}\left(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}\right) \\ n_{22} & \mathrm{n}_{22}\left(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23}\right) \\ n_{31} & \mathrm{n}_{31}\left(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}\right) \\ n_{32} & \mathrm{n}_{32}\left(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33}\right) \end{bmatrix}$$

[37]:
```
elemToSpecFuncArgsD = dict(itertools.chain(*[[(Nelem[i, j], Function("n_{}{}".
 ↪format(i + 1, j + 1))(*X,*W)) for j in range(2)] for i in range(3)]))

elemToSpecFuncArgs = list(elemToSpecFuncArgsD.items())

Matrix(elemToSpecFuncArgs)
```

[37]:
$$\begin{bmatrix} n_{11} & \mathrm{n}_{11}\left(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}\right) \\ n_{12} & \mathrm{n}_{12}\left(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}\right) \\ n_{21} & \mathrm{n}_{21}\left(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}\right) \\ n_{22} & \mathrm{n}_{22}\left(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}\right) \\ n_{31} & \mathrm{n}_{31}\left(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}\right) \\ n_{32} & \mathrm{n}_{32}\left(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}\right) \end{bmatrix}$$

[38]:
```
Selem
```

[38]:
$$\begin{bmatrix} \sigma(n_{11}) & \sigma(n_{12}) \\ \sigma(n_{21}) & \sigma(n_{22}) \\ \sigma(n_{31}) & \sigma(n_{32}) \end{bmatrix}$$

[39]:
```
Selem.subs(elemToSpecD)
```

[39]:
$$\begin{bmatrix} \sigma(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}) & \sigma(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13}) \\ \sigma(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}) & \sigma(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23}) \\ \sigma(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}) & \sigma(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33}) \end{bmatrix}$$

[40]:
```
Selem[0,1].diff(Nelem[0,1])
```

[40]:

$$\frac{d}{dn_{12}}\sigma(n_{12})$$

[41]:
```
Selem[0,1].diff(Nelem[0,1]).subs({Nelem[0,1] : Nspec[0,1]})
#Selem[0,1].diff(Nelem[0,1]).subs(dict([{Nelem[0,1] : Nspec[0,1]}]))
```

[41]:
$$\left.\frac{d}{dn_{12}}\sigma(n_{12})\right|_{n_{12}=w_{12}x_{11}+w_{22}x_{12}+w_{32}x_{13}}$$

[42]:
```
Selem[0,1].diff(Nelem[0,1]).subs({Nelem[0,1] : Nspec[0,1]}).subs({Nspec[0,1] :
↪23})
```

[42]:
$$\left.\frac{d}{dn_{12}}\sigma(n_{12})\right|_{n_{12}=23}$$

[43]:
```
Selem[0,1].diff(Nelem[0,1]).subs({Nelem[0,1] : Nspec[0,1]}).replace(sigma,
↪lambda x: 8*x**3)
```

[43]:
$$\left.\frac{d}{dn_{12}}8n_{12}^3\right|_{n_{12}=w_{12}x_{11}+w_{22}x_{12}+w_{32}x_{13}}$$

[44]:
```
Selem[0,1].diff(Nelem[0,1]).replace(sigma, lambda x: 8*x**3)
```

[44]:
$$\frac{d}{dn_{12}}8n_{12}^3$$

[45]:
```
Selem[0,1].diff(Nelem[0,1]).replace(sigma, lambda x: 8*x**3).doit()
```

[45]:
$$24n_{12}^2$$

[46]:
```
# ### GOT IT: can replace now with expression and do derivative with respect to
↪that expression.
Selem[0,1].diff(Nelem[0,1]).subs({Nelem[0,1] : Nspec[0,1]}).replace(sigma,
↪lambda x: 8*x**3).doit()
```

[46]:
$$24\left(w_{12}x_{11}+w_{22}x_{12}+w_{32}x_{13}\right)^2$$

[47]:
```
Selem[0,1].subs({Nelem[0,1] : Nspec[0,1]}).diff(X[0,1])#.subs({Nelem[0,1] :
↪Nspec[0,1]})
```

[47]:
$$w_{22}\left.\frac{d}{d\xi_1}\sigma(\xi_1)\right|_{\xi_1=w_{12}x_{11}+w_{22}x_{12}+w_{32}x_{13}}$$

[48]:
```
Selem
```

[48]:
$$\begin{bmatrix}\sigma(n_{11}) & \sigma(n_{12})\\ \sigma(n_{21}) & \sigma(n_{22})\\ \sigma(n_{31}) & \sigma(n_{32})\end{bmatrix}$$

[49]:
```
nt = Nelem.subs(elemToSpecFunc); nt
```

[49]:
$$\begin{bmatrix} \mathrm{n}_{11}\left(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}\right) & \mathrm{n}_{12}\left(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13}\right) \\ \mathrm{n}_{21}\left(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}\right) & \mathrm{n}_{22}\left(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23}\right) \\ \mathrm{n}_{31}\left(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}\right) & \mathrm{n}_{32}\left(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33}\right) \end{bmatrix}$$

[50]:
```
st = Selem.subs(elemToSpecFunc); st
```

[50]:
$$\begin{bmatrix} \sigma(\mathrm{n}_{11}\left(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}\right)) & \sigma(\mathrm{n}_{12}\left(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13}\right)) \\ \sigma(\mathrm{n}_{21}\left(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}\right)) & \sigma(\mathrm{n}_{22}\left(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23}\right)) \\ \sigma(\mathrm{n}_{31}\left(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}\right)) & \sigma(\mathrm{n}_{32}\left(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33}\right)) \end{bmatrix}$$

[51]:
```
init_printing(num_columns=10, wrap_line=True, pretty_print=True)#




st.diff(nt)
```

[51]:
$$\left[\begin{bmatrix} \frac{\partial}{\partial \mathrm{n}_{11}\left(w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}\right)}\sigma(\mathrm{n}_{11}\left(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}\right)) & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{\partial}{\partial \mathrm{n}_{12}\left(w_{12}x_{11}+w_{22}x_{12}+w_{32}x_{13}\right)}\sigma(\mathrm{n}_{12}\left(w_{12}x_{11}\right. \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right.$$
$$\left.\begin{bmatrix} 0 & 0 \\ \frac{\partial}{\partial \mathrm{n}_{21}\left(w_{11}x_{21}+w_{21}x_{22}+w_{31}x_{23}\right)}\sigma(\mathrm{n}_{21}\left(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}\right)) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \frac{\partial}{\partial \mathrm{n}_{22}\left(w_{12}x_{21}+w_{22}x_{22}+w_{32}x_{23}\right)}\sigma(\mathrm{n}_{22}\left(w_{12}x_{21}\right. \\ 0 & 0 \end{bmatrix}\right.$$
$$\left.\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\partial}{\partial \mathrm{n}_{31}\left(w_{11}x_{31}+w_{21}x_{32}+w_{31}x_{33}\right)}\sigma(\mathrm{n}_{31}\left(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}\right)) & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{\partial}{\partial \mathrm{n}_{32}\left(w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}\right)}\sigma(\mathrm{n}_{32}\left(w_{12}x_{31}\right. \end{bmatrix}\right.$$

[52]:
```
st[0,0].diff(st[0,0].args[0])
```

[52]:
$$\frac{\partial}{\partial \mathrm{n}_{11}\left(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}\right)}\sigma(\mathrm{n}_{11}\left(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}\right))$$

[53]:
```
st[0,0].diff(X[0,0])
```

[53]:
$$w_{11}\frac{\partial}{\partial \mathrm{n}_{11}\left(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}\right)}\sigma(\mathrm{n}_{11}\left(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}\right))\ \frac{d}{d\xi_1}\mathrm{n}_{11}\left(\xi_1\right)\Big|_{\xi_1 = w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}}$$

[54]:
```
st[0,0].diff(st[1,0].args[0])
```

[54]:
$$0$$

[55]:
```
Selem.diff(Nelem)
```

[55]:

$$\begin{bmatrix}\begin{bmatrix}\frac{d}{dn_{11}}\sigma(n_{11}) & 0\\ 0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & \frac{d}{dn_{12}}\sigma(n_{12})\\ 0 & 0\\ 0 & 0\end{bmatrix}\\ \begin{bmatrix}0 & 0\\ \frac{d}{dn_{21}}\sigma(n_{21}) & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & \frac{d}{dn_{22}}\sigma(n_{22})\\ 0 & 0\end{bmatrix}\\ \begin{bmatrix}0 & 0\\ 0 & 0\\ \frac{d}{dn_{31}}\sigma(n_{31}) & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & \frac{d}{dn_{32}}\sigma(n_{32})\end{bmatrix}\end{bmatrix}$$

[56]:
```
Selem.diff(Nelem).subs(elemToSpecFunc)
```

[56]:
$$\begin{bmatrix}\begin{bmatrix}\frac{\partial}{\partial \mathrm{n}_{11}\,(w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13})}\sigma(\mathrm{n}_{11}\,(w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13})) & 0\\ 0 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & \frac{\partial}{\partial \mathrm{n}_{12}\,(w_{12}x_{11}+w_{22}x_{12}+w_{32}x_{13})}\sigma(\mathrm{n}_{12}\,(w_{12}x_{11}\\ 0 & 0\\ 0 & 0\end{bmatrix}\\ \begin{bmatrix}0 & 0\\ \frac{\partial}{\partial \mathrm{n}_{21}\,(w_{11}x_{21}+w_{21}x_{22}+w_{31}x_{23})}\sigma(\mathrm{n}_{21}\,(w_{11}x_{21}+w_{21}x_{22}+w_{31}x_{23})) & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & \frac{\partial}{\partial \mathrm{n}_{22}\,(w_{12}x_{21}+w_{22}x_{22}+w_{32}x_{23})}\sigma(\mathrm{n}_{22}\,(w_{12}x_{21}\\ 0 & 0\end{bmatrix}\\ \begin{bmatrix}0 & 0\\ 0 & 0\\ \frac{\partial}{\partial \mathrm{n}_{31}\,(w_{11}x_{31}+w_{21}x_{32}+w_{31}x_{33})}\sigma(\mathrm{n}_{31}\,(w_{11}x_{31}+w_{21}x_{32}+w_{31}x_{33})) & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & \frac{\partial}{\partial \mathrm{n}_{32}\,(w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33})}\sigma(\mathrm{n}_{32}\,(w_{12}x_{31}\end{bmatrix}$$

[57]:
```
# CAN even replace elements after have done an operation on them!!! replacing␣
↪n_21 * 2 with the number 4.
Sspec.subs({Nspec[0, 0]: 3}).replace(sigma, lambda x: 2 * x).replace(Nspec[2,␣
↪1] * 2, 4)
```

[57]:
$$\begin{bmatrix}6 & 2w_{12}x_{11}+2w_{22}x_{12}+2w_{32}x_{13}\\ 2w_{11}x_{21}+2w_{21}x_{22}+2w_{31}x_{23} & 2w_{12}x_{21}+2w_{22}x_{22}+2w_{32}x_{23}\\ 2w_{11}x_{31}+2w_{21}x_{32}+2w_{31}x_{33} & 4\end{bmatrix}$$

[58]:
```
lambd = Function("lambda")
lambd_ = lambda matrix : sum(matrix)


vN(X, W)
```

[58]:
$$\begin{bmatrix}n_{11} & n_{12}\\ n_{21} & n_{22}\\ n_{31} & n_{32}\end{bmatrix}$$

[59]:
```
vN(A, B)
```

[59]:
$$\begin{bmatrix}n_{11} & n_{12}\\ n_{21} & n_{22}\\ n_{31} & n_{32}\end{bmatrix}$$

[60]:
```
L = lambd(S); L
```

[60]:
$$\lambda(\sigma_{apply}(v(X, W)))$$

[61]:
```
Nelem
```

[61]:
$$\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{bmatrix}$$

[62]:
```
L.replace(n, vN)
```

[62]:
$$\lambda\left(\sigma_{apply}\left(\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{bmatrix}\right)\right)$$

[63]:
```
L.replace(n, vN).replace(sigmaApply, sigmaApply_)
```

[63]:
$$\lambda\left(\begin{bmatrix} \sigma(n_{11}) & \sigma(n_{12}) \\ \sigma(n_{21}) & \sigma(n_{22}) \\ \sigma(n_{31}) & \sigma(n_{32}) \end{bmatrix}\right)$$

[64]:
```
L.replace(n, v)
```

[64]:
$$\lambda(\sigma_{apply}(XW))$$

[65]:
```
L.replace(n, v).replace(sigmaApply, sigmaApply_)
```

[65]:
$$\lambda((d \mapsto \sigma(d))_{\circ}(XW))$$

[66]:
```
L.subs({A:X, B:W}).replace(n, vL).replace(sigmaApply, sigmaApply_)
```

[66]:
$$\lambda\left(\begin{bmatrix} \sigma(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}) & \sigma(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13}) \\ \sigma(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}) & \sigma(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23}) \\ \sigma(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}) & \sigma(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33}) \end{bmatrix}\right)$$

[67]:
```
L.replace(n, vN)
```

[67]:
$$\lambda\left(\sigma_{apply}\left(\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{bmatrix}\right)\right)$$

[68]:
```
L.replace(n, vN).subs({A:X, B:W}).replace(sigmaApply, sigmaApply_).
↪replace(lambd, lambd_)
```

[68]:
$$\sigma(n_{11}) + \sigma(n_{12}) + \sigma(n_{21}) + \sigma(n_{22}) + \sigma(n_{31}) + \sigma(n_{32})$$

[69]:
```
from sympy import symbols, Derivative

x, y, r, t = symbols('x y r t') # r (radius), t (angle theta)
f, g, h = symbols('f g h', cls=Function)
```

```
h = g(f(x))

Derivative(h, f(x)).doit()
```

[69]:
$$\frac{d}{df(x)} g(f(x))$$

[70]:
```
# Never do this gives recursion ERROR (max depth exceeded)
# h = g(f(A))
# Derivative(h, A).doit()
```

[71]:

[71]:
```
from sympy.abc import a, b

Llower = lambd(sigmaApply(n(a, b)))
Llower
```

[71]:
$$\lambda(\sigma_{apply}(v(a,b)))$$

[72]:
```
Derivative(Llower, a).doit()
```

[72]:
$$\frac{\partial}{\partial \sigma_{apply}(v(a,b))} \lambda(\sigma_{apply}(v(a,b))) \frac{\partial}{\partial v(a,b)} \sigma_{apply}(v(a,b)) \frac{\partial}{\partial a} v(a,b)$$

[73]:

[73]:
```
# ### WAY 1: of substituting to differentiate with respect to expression:
n_ij = Function('n_ij')
n_ij(A,B) # (N[0,0]); n_ij
```

[73]:
$$\mathrm{n_{ij}}\left(X, W\right)$$

[74]:
```
n_ij(A,B).args
```

[74]:
$$\left(X,\ W\right)$$

[75]:
```
# sigma(n_ij).diff(n_ij).replace(n_ij, N[0,0]) # ERROR cannot deriv wi.r.t to␣
↪the expression w11*x11 + ...

sigma(n_ij(A,B)).diff(n_ij(A,B))
```

[75]:
$$\frac{\partial}{\partial\, \mathrm{n_{ij}}\left(X, W\right)} \sigma(\mathrm{n_{ij}}\left(X, W\right))$$

[76]:
```
sigma(n_ij(*X,*W)).diff(X[0,0])
```

[76]:
$$\frac{\partial}{\partial x_{11}} \mathrm{n_{ij}}\left(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}\right) \frac{\partial}{\partial\, \mathrm{n_{ij}}\left(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32},\right.}$$

[77]:
```
nab_ij = n_ij(A,B)
sigma(nab_ij).diff(nab_ij)#.subs({nab_ij : Nspec[0, 0]})
```

[77]:
$$\frac{\partial}{\partial \mathrm{n_{ij}}\left(X, W\right)} \sigma(\mathrm{n_{ij}}\left(X, W\right))$$

[78]:
```
sigma(nab_ij).diff(nab_ij).subs({nab_ij : Nspec[2, 1]})
```

[78]:
$$\left.\frac{d}{d\xi}\sigma(\xi)\right|_{\xi=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}}$$

[79]:
```
sigma(nab_ij).diff(nab_ij).subs({nab_ij : Nspec[2,1]}).subs({X[2,1]:77777})
```

[79]:
$$\left.\frac{d}{d\xi}\sigma(\xi)\right|_{\xi=w_{12}x_{31}+77777w_{22}+w_{32}x_{33}}$$

[80]:
```
sigma(nab_ij).diff(nab_ij).subs({nab_ij : 23}) # ERROR if using replace() since␣
↪it says can't calc derivs w.r.t to the x_11*w_11 + ...
```

[80]:
$$\left.\frac{d}{d\xi}\sigma(\xi)\right|_{\xi=23}$$

[81]:
```
sigma(nab_ij).diff(nab_ij).subs({nab_ij : Nspec[2,1]}).doit()
```

[81]:
$$\left.\frac{d}{d\xi}\sigma(\xi)\right|_{\xi=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}}$$

[82]:
```
sigma(nab_ij).subs({nab_ij : Nspec[2,1]})#.diff(X[2,1])
```

[82]:
$$\sigma(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33})$$

[83]:
```
# Substituting the value of the function n_ij first, and THEN differentiating␣
↪with respect to something in that substitution. (X_21)
sigma(nab_ij).subs({nab_ij : Nspec[2,1]}).diff(X[2,1])
```

[83]:
$$w_{22}\left.\frac{d}{d\xi_1}\sigma(\xi_1)\right|_{\xi_1=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}}$$

[84]:
```
Selem[2,1].subs({Nelem[2,1] : Nspec[2,1]}).diff(X[2,1])
```

[84]:
$$w_{22}\left.\frac{d}{d\xi_1}\sigma(\xi_1)\right|_{\xi_1=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}}$$

[85]:
```
# ### WAY 2:
n_11 = Function('n_11')(Nspec[0, 0]); n_11
```

[85]:
$$\mathrm{n_{11}}\left(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}\right)$$

[86]:
```
sigma(n_11)
```

[86]:
$$\sigma(\mathrm{n}_{11}\left(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}\right))$$

[87]:
```
assert Nspec[0,0] == n_11.args[0]

sigma(n_11).subs({n_11 : n_11.args[0]})
```

[87]:
$$\sigma(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})$$

[88]:
```
sigma(n_11).diff(n_11) #.replace(n_ij, n_ij.args[0])
```

[88]:
$$\frac{\partial}{\partial \mathrm{n}_{11}\left(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}\right)}\sigma(\mathrm{n}_{11}\left(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}\right))$$

[89]:
```
sigma(n_11).diff(n_11).subs({n_11 : n_11.args[0]}).subs({X[0,0]:77777})
```

[89]:
$$\frac{d}{d\xi}\sigma(\xi)\Big|_{\xi=77777w_{11}+w_{21}x_{12}+w_{31}x_{13}}$$

[90]:
```
sigma(n_11).diff(n_11).subs({n_11 : n_11.args[0]}).replace(n_11.args[0], 23) #␣
↪same as subs in this case
```

[90]:
$$\frac{d}{d\xi}\sigma(\xi)\Big|_{\xi=23}$$

[91]:
```
sigma(n_11).diff(X[0,0])
```

[91]:
$$w_{11}\frac{\partial}{\partial \mathrm{n}_{11}\left(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}\right)}\sigma(\mathrm{n}_{11}\left(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}\right))\ \frac{d}{d\xi_1}\mathrm{n}_{11}\left(\xi_1\right)\Big|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}}$$

[92]:
```
id = Lambda(x, x)

sigma(n_11).diff(X[0,0]).subs({n_11 : id})
```

[92]:
$$w_{11}\ \frac{d}{d\xi_1}\mathrm{n}_{11}\left(\xi_1\right)\Big|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}}\ \frac{d}{d\xi}\sigma(\xi)\Big|_{\xi=(x\mapsto x)}$$

[93]:
```
# NOTE: so I don't think WAY 2 is correct because here it doesn't simplify the␣
↪derivative d n11 / d eps11, since this should equal 1 because now n11 =␣
↪eps11. Correct one is below (repeated from above)
sigma(n_11).diff(X[0,0]).subs({n_11 : Nspec[0,0]})
```

[93]:
$$w_{11}\ \frac{d}{d\xi_1}\mathrm{n}_{11}\left(\xi_1\right)\Big|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}}\ \frac{d}{d\xi}\sigma(\xi)\Big|_{\xi=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}}$$

[94]:
```
# CORRECT WAY 1
sigma(n_11).subs({n_11 : Nspec[0,0]}).diff(X[0,0])
```

[94]:
$$w_{11}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}}$$

[95]:
```
# CORRECT WAY 2

sigma(nab_ij).subs({nab_ij : Nspec[0,0]}).diff(X[0,0])
```

[95]:
$$w_{11}\left.\frac{d}{d\xi_1}\sigma(\xi_1)\right|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}}$$

[96]:
```
# CORRECT WAY 3
Selem[2,1].subs({Nelem[2,1] : Nspec[2,1]}).diff(X[2,1])
```

[96]:
$$w_{22}\left.\frac{d}{d\xi_1}\sigma(\xi_1)\right|_{\xi_1=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}}$$

[97]:
```
sigma(n_11) # WAY 1: sigma argument is already hardcoded
```

[97]:
$$\sigma(\mathrm{n}_{11}\left(w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}\right))$$

[98]:
```
sigma(nab_ij) # Way 2: sigma argument is function of matrixsymbol (better than␣
↪1)
```

[98]:
$$\sigma(\mathrm{n}_{\mathrm{ij}}\left(X,W\right))$$

[99]:
```
Selem[2,1] # WAY 3: sigma argument is just symbol and we replace it as function␣
↪with argument hardcoded only later. (better than 2)
```

[99]:
$$\sigma(n_{32})$$

[100]:
```
L
```

[100]:
$$\lambda(\sigma_{apply}(v(X,W)))$$

[101]:
```
assert Selem == S.replace(n, vN).replace(sigmaApply, sigmaApply_)

Selem
```

[101]:
$$\begin{bmatrix}\sigma(n_{11}) & \sigma(n_{12})\\ \sigma(n_{21}) & \sigma(n_{22})\\ \sigma(n_{31}) & \sigma(n_{32})\end{bmatrix}$$

[102]:
```
L.replace(n, vN).replace(sigmaApply, sigmaApply_)
```

[102]:
$$\lambda\left(\begin{bmatrix}\sigma(n_{11}) & \sigma(n_{12})\\ \sigma(n_{21}) & \sigma(n_{22})\\ \sigma(n_{31}) & \sigma(n_{32})\end{bmatrix}\right)$$

[103]:
```
#L.replace(n, vN).replace(sigmaApply, sigmaApply_).diff(Nelem[0,0])
```

[104]:
```
Lsum = L.replace(n, vN).replace(sigmaApply, sigmaApply_).replace(lambd, lambd_)
Lsum
```

[104]:
$$\sigma(n_{11}) + \sigma(n_{12}) + \sigma(n_{21}) + \sigma(n_{22}) + \sigma(n_{31}) + \sigma(n_{32})$$

[105]:
```
Lsum.diff(Nelem)
```

[105]:
$$\begin{bmatrix} \frac{d}{dn_{11}}\sigma(n_{11}) & \frac{d}{dn_{12}}\sigma(n_{12}) \\ \frac{d}{dn_{21}}\sigma(n_{21}) & \frac{d}{dn_{22}}\sigma(n_{22}) \\ \frac{d}{dn_{31}}\sigma(n_{31}) & \frac{d}{dn_{32}}\sigma(n_{32}) \end{bmatrix}$$

[106]:
```
Lsum.subs(elemToSpec)#.diff(X[2,1])
```

[106]:
$$\sigma(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}) \; + \; \sigma(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}) \; + \; \sigma(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}) \; + \; \sigma(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13}) + \sigma(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23}) + \sigma(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33})$$

[107]:
```
Lsum.subs(elemToSpec).diff(X)
```

[107]:
$$\begin{bmatrix} w_{11}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}} + w_{12}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=w_{12}x_{11}+w_{22}x_{12}+w_{32}x_{13}} & w_{21}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}} \\ w_{11}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=w_{11}x_{21}+w_{21}x_{22}+w_{31}x_{23}} + w_{12}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=w_{12}x_{21}+w_{22}x_{22}+w_{32}x_{23}} & w_{21}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=w_{11}x_{21}+w_{21}x_{22}+w_{31}x_{23}} \\ w_{11}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=w_{11}x_{31}+w_{21}x_{32}+w_{31}x_{33}} + w_{12}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}} & w_{21}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=w_{11}x_{31}+w_{21}x_{32}+w_{31}x_{33}} \end{bmatrix}$$

[108]:
```
specToElemD = {v : k for k, v in elemToSpecD.items()}

Lsum.subs(elemToSpecD).diff(X).subs(specToElemD)
```

[108]:
$$\begin{bmatrix} w_{11}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=n_{11}} + w_{12}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=n_{12}} & w_{21}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=n_{11}} + w_{22}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=n_{12}} & w_{31}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=n_{11}} + w_{32} \\ w_{11}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=n_{21}} + w_{12}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=n_{22}} & w_{21}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=n_{21}} + w_{22}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=n_{22}} & w_{31}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=n_{21}} + w_{32} \\ w_{11}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=n_{31}} + w_{12}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=n_{32}} & w_{21}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=n_{31}} + w_{22}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=n_{32}} & w_{31}\ \frac{d}{d\xi_1}\sigma(\xi_1)\Big|_{\xi_1=n_{31}} + w_{32} \end{bmatrix}$$