Sympy_DerivMatrix

October 1, 2020

 $\begin{bmatrix} \mathbf{y_{11}} \left(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33} \right) & \mathbf{y_{12}} \left(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33} \right) \\ \mathbf{y_{21}} \left(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33} \right) & \mathbf{y_{22}} \left(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33} \right) \\ \mathbf{y_{31}} \left(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33} \right) & \mathbf{y_{32}} \left(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33} \right) \\ \end{bmatrix}$

y = f(X)

0.0.1 Derivative of Matrix With Respect a Matrix

Let
$$X=\{x_{ij}\}$$
 be a matrix of order $m imes n$ and let

be a scalar function of
$$X$$
, so $y \in \mathbb{R}$ and $f: \mathbb{R}^{m imes n} o \mathbb{R}$,

Also let the matrix
$$Y = \{y_{ij}(X)\}$$
 be of size $p imes q$.

Then we can define the derivative of Y with respect to X as the following matrix of order $mp \times nq$:

$$\frac{\partial Y}{\partial X} = \begin{pmatrix} \frac{\partial Y}{\partial x_{11}} & \frac{\partial Y}{\partial x_{12}} & \dots & \frac{\partial Y}{\partial x_{1n}} \\ \frac{\partial Y}{\partial x_{21}} & \frac{\partial Y}{\partial x_{22}} & \dots & \frac{\partial Y}{\partial x_{23}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial Y}{\partial x_{m1}} & \frac{\partial Y}{\partial x_{m2}} & \dots & \frac{\partial Y}{\partial x_{mn}} \end{pmatrix} = \left\{ \frac{\partial y_{ij}}{\partial x_{lk}} \right\}$$

[3]: # GOT IT this is the definition of gradient matrix (matrix of □ →partial derivatives or dY/dX)
D = derive_by_array(Y, X); D

$\left[\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\left[egin{array}{cccccccccccccccccccccccccccccccccccc$	$x_3, x_{31}, x_{32}, x_{33}$
$\frac{\partial^{-1}}{\partial x_{11}}$ $\mathbf{y_{21}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33})$ $\frac{\partial^{-1}}{\partial x_{11}}$ $\mathbf{y_{22}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33})$	$oxed{x_{33}} \left[\begin{array}{cccccccccccccccccccccccccccccccccccc$	$_3, x_{31}, x_{32}, x_{33})$
$ \begin{vmatrix} \frac{\partial}{\partial x_{11}} \mathbf{y}_{31}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) & \frac{\partial}{\partial x_{11}} \mathbf{y}_{32}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{vmatrix} $		
$\begin{bmatrix} \frac{\partial}{\partial x_{21}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) & \frac{\partial}{\partial x_{21}} \mathbf{y_{12}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix}$	A F O	≒ i
$\begin{bmatrix} \frac{\partial}{\partial x_{21}} \mathbf{y}_{21}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) & \frac{\partial}{\partial x_{21}} \mathbf{y}_{22}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix}$	$\frac{1}{2}$	11
$ \frac{\partial z_{21}}{\partial x_{21}} \mathbf{y}_{31} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \\ \frac{\partial z_{21}}{\partial x_{21}} \mathbf{y}_{32} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) $	$\Gamma = \Gamma =$	
$\begin{bmatrix} \frac{\partial x_{21}}{\partial x_{31}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) & \frac{\partial x_{21}}{\partial x_{31}} \mathbf{y_{12}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \\ \end{bmatrix}$		- 1
$ \begin{vmatrix} \frac{\partial}{\partial x_{31}} & \mathbf{y}_{11} & (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \\ \frac{\partial}{\partial x_{31}} & \mathbf{y}_{21} & (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \\ \frac{\partial}{\partial x_{31}} & \mathbf{y}_{22} & (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{vmatrix} $	4 702	11
$\left[\begin{array}{ccc} \frac{\partial}{\partial x_{31}} \mathbf{y_{31}} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) & \frac{\partial}{\partial x_{31}} \mathbf{y_{32}} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{array}\right]$	$ \left[\frac{\partial}{\partial x_{32}} \mathbf{y_{31}} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \left[\frac{\partial}{\partial x_{32}} \mathbf{y_{32}} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \left[\frac{\partial}{\partial x_{33}} \mathbf{y_{31}} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] $	[3, x31, x32, x33)
D.subs({Y[0,0]: X[0,0]**2 + X[1,0]}).doit()		
$\frac{\partial}{\partial x_{11}} \mathbf{y}_{12} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, x_{34}, x_$	$\left[\begin{array}{cccccccccccccccccccccccccccccccccccc$	x_{21}, x_{22}, x_{22}
$\left\{egin{array}{c} rac{\partial z_{11}}{\partial x_{11}} \mathbf{y}_{21}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) & rac{\partial z_{11}}{\partial x_{11}} \mathbf{y}_{22}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{array}\right.$		11
1 011		
$\begin{bmatrix} \frac{\partial}{\partial x_{11}} \mathbf{y}_{31}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) & \frac{\partial}{\partial x_{11}} \mathbf{y}_{32}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \\ \frac{\partial}{\partial x_{11}} \mathbf{y}_{32}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix}$	in the second	. = 1
$\frac{\partial}{\partial x_{21}} \mathbf{y}_{12} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{23}, x_{34}, x_$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$x_3, x_{31}, x_{32}, x_{33}$
$\frac{\partial}{\partial x_{21}} \mathbf{y_{21}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33})$ $\frac{\partial}{\partial x_{21}} \mathbf{y_{22}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33})$	$oxed{x_{33}} ig igg rac{\partial}{\partial x_{22}} oldsymbol{y_{21}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) rac{\partial}{\partial x_{22}} oldsymbol{y_{22}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) igg rac{\partial}{\partial x_{23}} oldsymbol{y_{21}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) rac{\partial}{\partial x_{23}} oldsymbol{y_{22}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) egin{array}{c} \partial x_{11}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) egin{array}{c} \partial x_{11}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) egin{array}{c} \partial x_{11}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) egin{array}{c} \partial x_{11}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) egin{array}{c} \partial x_{11}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) egin{array}{c} \partial x_{11}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) egin{array}{c} \partial x_{11}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) egin{array}{c} \partial x_{11}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) egin{array}{c} \partial x_{11}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) egin{array}{c} \partial x_{11}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) egin{array}{c} \partial x_{11}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) egin{array}{c} \partial x_{11}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) egin{array}{c} \partial x_{11}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) egin{array}{c} \partial x_{11}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) egin{array}{c} \partial x_{11}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) egin{array}{c} \partial x_{11}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x$	$_3, x_{31}, x_{32}, x_{33})$
$ \frac{\partial^{2}}{\partial x_{21}} \mathbf{y}_{31}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \frac{\partial^{2}}{\partial x_{21}} \mathbf{y}_{32}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) $	$oxed{x_{33}} \left[egin{array}{c} \frac{\partial}{\partial x_{22}} \mathbf{y_{31}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) & \frac{\partial}{\partial x_{22}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) & \frac{\partial}{\partial x_{23}} \mathbf{y_{31}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) & \frac{\partial}{\partial x_{23}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) & \frac{\partial}{\partial x_{23}} \mathbf{y_{32}}(x_{21}, x_{22}, x_{23}, x_{21}, x_{22}, x_{23}, x_{22}, x_{23}, x_{23}$	$_3, x_{31}, x_{32}, x_{33})$
$0 \qquad \frac{\partial}{\partial x_{21}} \mathbf{y_{12}} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, x_{34}, x_{34}$. = 1
$\begin{vmatrix} \frac{\partial}{\partial x_{31}} \mathbf{y}_{21}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) & \frac{\partial x_{31}}{\partial x_{31}} \mathbf{y}_{22}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{vmatrix}$. 1 1
$\begin{bmatrix} \frac{\partial x_{31}}{\partial x_{31}} \mathbf{y}_{31}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) & \frac{\partial x_{31}}{\partial x_{31}} \mathbf{y}_{32}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \\ \frac{\partial x_{31}}{\partial x_{32}} \mathbf{y}_{32}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix}$, 1 1
$\frac{1}{2}$	$ \begin{bmatrix} \partial x_{32} & \mathcal{F}_{31} & (-11)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-12)^{-1} & (-1$), 31, 32, 33/]]

[5]: Y.diff(X) ## GOT IT

 $\begin{bmatrix} 5 \end{bmatrix} : \begin{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{11}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) & \frac{\partial}{\partial x_{11}} \mathbf{y_{12}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{12}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{12}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{12}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{12}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{12}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{12}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{12}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{12}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{12}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{12}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{12}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{12}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{12}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{12}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{12}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{12}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{12}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{12}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \end{bmatrix} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_{1$ $\left[\frac{\partial^{11}}{\partial x_{11}}\mathbf{y_{21}}(x_{11},x_{12},x_{13},x_{21},x_{22},x_{23},x_{31},x_{32},x_{33}) - \frac{\partial^{12}}{\partial x_{11}}\mathbf{y_{22}}(x_{11},x_{12},x_{13},x_{21},x_{22},x_{23},x_{31},x_{32},x_{33})\right] - \left[\frac{\partial^{12}}{\partial x_{12}}\mathbf{y_{21}}(x_{11},x_{12},x_{13},x_{21},x_{22},x_{23},x_{31},x_{32},x_{33}) - \frac{\partial^{12}}{\partial x_{12}}\mathbf{y_{22}}(x_{11},x_{12},x_{13},x_{21},x_{22},x_{23},x_{31},x_{32},x_{33})\right] - \left[\frac{\partial^{12}}{\partial x_{12}}\mathbf{y_{21}}(x_{11},x_{12},x_{13},x_{21},x_{22},x_{23},x_{31},x_{32},x_{33}) - \frac{\partial^{12}}{\partial x_{12}}\mathbf{y_{22}}(x_{11},x_{12},x_{13},x_{21},x_{22},x_{23},x_{31},x_{32},x_{33})\right]$ $\left[\frac{\partial^{c}}{\partial x_{13}}\mathbf{y_{21}}(x_{11},x_{12},x_{13},x_{21},x_{22},x_{23},x_{31},x_{32},x_{33})\right]\left[\frac{\partial^{c}}{\partial x_{13}}\mathbf{y_{22}}(x_{11},x_{12},x_{13},x_{21},x_{22},x_{23},x_{31},x_{32},x_{33})\right]\right]$ $\frac{\partial^{1}}{\partial x_{11}} \mathbf{y_{31}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial^{1}}{\partial x_{11}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial^{1}}{\partial x_{12}} \mathbf{y_{31}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial^{1}}{\partial x_{12}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial^{1}}{\partial x_{12}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial^{1}}{\partial x_{12}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial^{1}}{\partial x_{12}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial^{1}}{\partial x_{12}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial^{1}}{\partial x_{12}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial^{1}}{\partial x_{12}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial^{1}}{\partial x_{12}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial^{1}}{\partial x_{12}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial^{1}}{\partial x_{12}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial^{1}}{\partial x_{12}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial^{1}}{\partial x_{12}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial^{1}}{\partial x_{12}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial^{1}}{\partial x_{12}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial^{1}}{\partial x_{12}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial^{1}}{\partial x_{12}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x$ $\left[\frac{\partial}{\partial x_{12}}\mathbf{y_{31}}(x_{11},x_{12},x_{13},x_{21},x_{22},x_{23},x_{31},x_{32},x_{33})\right]$ $\frac{\partial}{\partial x_2} \mathbf{y}_{11}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33})$ $\frac{\partial}{\partial x_{22}}$ **y**₁₂ $(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33})$ $\left[\frac{\partial^{2}}{\partial x_{23}}\mathbf{y_{21}}(x_{11},x_{12},x_{13},x_{21},x_{22},x_{23},x_{31},x_{32},x_{33}) - \frac{\partial^{2}}{\partial x_{23}}\mathbf{y_{22}}(x_{11},x_{12},x_{13},x_{21},x_{22},x_{23},x_{31},x_{32},x_{33})\right]$ $\frac{\delta^2}{\partial x_{21}} \mathbf{y_{31}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\delta^2}{\partial x_{22}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\delta^2}{\partial x_{22}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\delta^2}{\partial x_{22}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\delta^2}{\partial x_{21}} \mathbf{y_{32}}(x_{21}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\delta^2}{\partial x_{22}} \mathbf{y_{32}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\delta^2}{\partial x_{22}} \mathbf{y_{32}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\delta^2}{\partial x_{22}} \mathbf{y_{32}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\delta^2}{\partial x_{22}} \mathbf{y_{32}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\delta^2}{\partial x_{22}} \mathbf{y_{32}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\delta^2}{\partial x_{22}} \mathbf{y_{32}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\delta^2}{\partial x_{22}} \mathbf{y_{32}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\delta^2}{\partial x_{22}} \mathbf{y_{32}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\delta^2}{\partial x_{22}} \mathbf{y_{32}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\delta^2}{\partial x_{22}} \mathbf{y_{32}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\delta^2}{\partial x_{22}} \mathbf{y_{32}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\delta^2}{\partial x_{22}} \mathbf{y_{32}}(x_{21}, x_{22}, x_{23}, x$ $\left| \frac{\partial}{\partial x_{23}} \mathbf{y_{31}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right| = \frac{\partial}{\partial x_{23}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right|$ $\left\lceil \frac{\partial}{\partial x_{21}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{21}} \mathbf{y_{12}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right\rceil - \left\lceil \frac{\partial}{\partial x_{22}} \mathbf{y_{11}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{22}} \mathbf{y_{12}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right\rceil - \left\lceil \frac{\partial}{\partial x_{22}} \mathbf{y_{11}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{22}} \mathbf{y_{12}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right\rceil - \left\lceil \frac{\partial}{\partial x_{22}} \mathbf{y_{11}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{22}} \mathbf{y_{12}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right\rceil - \left\lceil \frac{\partial}{\partial x_{22}} \mathbf{y_{11}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{22}} \mathbf{y_{12}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right\rceil - \left\lceil \frac{\partial}{\partial x_{22}} \mathbf{y_{11}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{22}} \mathbf{y_{12}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{22}} \mathbf{y_{12}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right\rceil - \left\lceil \frac{\partial}{\partial x_{22}} \mathbf{y_{12}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{22}} \mathbf{y_{12}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right\rceil - \left\lceil \frac{\partial}{\partial x_{22}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{22}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right\rceil - \left\lceil \frac{\partial}{\partial x_{22}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right\rceil - \left\lceil \frac{\partial}{\partial x_{22}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{22}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right\rceil - \left\lceil \frac{\partial}{\partial x_{22}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}) \right\rceil - \left\lceil \frac{\partial}{\partial x_{22}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{23},$ $\frac{1}{\partial x_{31}} \mathbf{y_{21}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{22}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{21}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{21}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) - \frac{\partial}{\partial x_{23}} \mathbf{y_{22}}(x_{21}, x_{22}, x_{23}, x_{2$ $\left[\left[\frac{\partial}{\partial x_{31}} \mathbf{y_{31}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \quad \left[\frac{\partial}{\partial x_{32}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \quad \left[\frac{\partial}{\partial x_{33}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \quad \left[\frac{\partial}{\partial x_{33}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \quad \left[\frac{\partial}{\partial x_{33}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \quad \left[\frac{\partial}{\partial x_{33}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \quad \left[\frac{\partial}{\partial x_{33}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \quad \left[\frac{\partial}{\partial x_{33}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \quad \left[\frac{\partial}{\partial x_{33}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \quad \left[\frac{\partial}{\partial x_{33}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \quad \left[\frac{\partial}{\partial x_{33}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \quad \left[\frac{\partial}{\partial x_{33}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \quad \left[\frac{\partial}{\partial x_{33}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \quad \left[\frac{\partial}{\partial x_{33}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \quad \left[\frac{\partial}{\partial x_{33}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \quad \left[\frac{\partial}{\partial x_{33}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \quad \left[\frac{\partial}{\partial x_{33}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \quad \left[\frac{\partial}{\partial x_{33}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) \right] \quad \left[\frac{\partial}{\partial x_{33}} \mathbf{y_{32}}(x_{11}, x_{12}, x_{13}, x_{22}, x_{23}, x_$

[6]: Yval = Y.subs({Y[0,0]: X[0,0]**2 + X[0,1]*X[1,0] - X[1,1], Y[0,1]: X[1,1]**3 + 4* X[0,1] + X[0,0] - X[1,0], Y[1,0]: X[1,0] * X[0,0] + 3*X[0,1] * X[1,1],

```
Y[1,1]: X[1,1] + X[1,0] + X[0,1] + X[0,0],
Y[2,0]: 2*X[0,0]**2 * X[0,1] * 3*X[1,0] + 4*X[1,1],
Y[2,1]: 3*X[0,1] - 5*X[1,1] * X[0,0] - X[1,0]**2})
[6]: \begin{bmatrix} x_{11}^2 + x_{12}x_{21} - x_{22} & x_{11} + 4x_{12} - x_{21} + x_{22}^3 \\ x_{11}x_{21} + 3x_{12}x_{22} & x_{11} + x_{12} + x_{21} + x_{22} \end{bmatrix}
                                                                                        \begin{bmatrix} 6x_{11}^2x_{12}x_{21} + 4x_{22} & -5x_{11}x_{22} + 3x_{12} - x_{21}^2 \end{bmatrix}
       [7]: DYval = D.subs({Y[0,0]: X[0,0]**2 + X[0,1]*X[1,0] - X[1,1], Y[0,1]: X[1,1]**3 + 4* X[0,1] + X[0,0] - X[1,0], Y[1,0]: X[1,0] * X[0,0] + 3*X[0,1] * X[1,1], Y[1,1]: X[1,1] + X[1,0] + X[0,1] + X[0,0], Y[2,0]: 2*X[0,0]**2 * X[0,1] * 3*X[1,0] + 4*X[1,1], Y[2,1]: 3*X[0,1] - 5*X[1,1] * X[0,0] - X[1,0]**2})

DYval
              \begin{bmatrix} \frac{\partial}{\partial x_{11}} \left(x_{11}^2 + x_{12}x_{21} - x_{22}\right) & \frac{\partial}{\partial x_{11}} \left(x_{11} + 4x_{12} - x_{21} + x_{22}^3\right) \\ \frac{\partial}{\partial x_{11}} \left(x_{11}x_{21} + 3x_{12}x_{22}\right) & \frac{\partial}{\partial x_{11}} \left(x_{11} + 4x_{12} - x_{21} + x_{22}\right) \\ \frac{\partial}{\partial x_{12}} \left(x_{11}x_{21} + 3x_{12}x_{22}\right) & \frac{\partial}{\partial x_{12}} \left(x_{11} + x_{12} + x_{21} + x_{22}\right) \\ \frac{\partial}{\partial x_{12}} \left(x_{11}x_{21} + 3x_{12}x_{22}\right) & \frac{\partial}{\partial x_{12}} \left(x_{11} + x_{12} + x_{21} + x_{22}\right) \\ \frac{\partial}{\partial x_{12}} \left(x_{11} + x_{12} + x_{21} + x_{22}\right) & \frac{\partial}{\partial x_{12}} \left(x_{11} + x_{12} + x_{21} + x_{22}\right) \\ \frac{\partial}{\partial x_{13}} \left(x_{11}x_{21} + 3x_{12}x_{22}\right) & \frac{\partial}{\partial x_{13}} \left(x_{11} + x_{12} + x_{21} + x_{22}\right) \\ \frac{\partial}{\partial x_{13}} \left(x_{11}x_{21} + 3x_{12}x_{22}\right) & \frac{\partial}{\partial x_{13}} \left(x_{11} + x_{12} + x_{21} + x_{22}\right) \\ \frac{\partial}{\partial x_{13}} \left(x_{11}x_{21} + 3x_{12}x_{22}\right) & \frac{\partial}{\partial x_{13}} \left(x_{11} + x_{12} + x_{21} + x_{22}\right) \\ \frac{\partial}{\partial x_{13}} \left(x_{11}x_{21} + 3x_{12}x_{22}\right) & \frac{\partial}{\partial x_{13}} \left(x_{11} + x_{12} + x_{21} + x_{22}\right) \\ \frac{\partial}{\partial x_{13}} \left(x_{11}x_{21} + 3x_{12}x_{22}\right) & \frac{\partial}{\partial x_{13}} \left(x_{11} + x_{12} + x_{21} + x_{22}\right) \\ \frac{\partial}{\partial x_{13}} \left(x_{11}x_{21} + 3x_{12}x_{22}\right) & \frac{\partial}{\partial x_{13}} \left(x_{11} + x_{12} + x_{21} + x_{22}\right) \\ \frac{\partial}{\partial x_{13}} \left(x_{11}x_{21} + 3x_{12}x_{22}\right) & \frac{\partial}{\partial x_{13}} \left(x_{11} + x_{12} + x_{21} + x_{22}\right) \\ \frac{\partial}{\partial x_{13}} \left(x_{11}x_{21} + 3x_{12}x_{22}\right) & \frac{\partial}{\partial x_{13}} \left(x_{11}x_{21} + x_{22}x_{21} + x_{22}\right) \\ \frac{\partial}{\partial x_{13}} \left(x_{11
                                                                          \begin{bmatrix} \frac{\partial \pi_{11}}{\partial \pi_{11}} (x_{11}x_{21} + 3x_{12}x_{22}) & \frac{\partial \pi_{11}}{\partial \pi_{11}} (x_{11} + x_{12} + x_{21} + x_{22}) \\ \frac{\partial \pi_{11}}{\partial \pi_{11}} (6x_{11}^2x_{12}x_{21} + 4x_{22}) & \frac{\partial \pi_{11}}{\partial \pi_{11}} (-5x_{11}x_{22} + 3x_{12} - x_{21}^2) \\ \frac{\partial \pi_{11}}{\partial \pi_{11}} (x_{11}x_{21} + x_{12}x_{21} - x_{22}) & \frac{\partial \pi_{11}}{\partial \pi_{11}} (-5x_{11}x_{22} + 3x_{12} - x_{21}^2) \\ \frac{\partial \pi_{12}}{\partial \pi_{22}} (x_{11}x_{21} + x_{12}x_{21} - x_{22}) & \frac{\partial \pi_{12}}{\partial \pi_{22}} (x_{11} + x_{12} + x_{21} + x_{22}) \\ \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11}x_{21} + 3x_{12}x_{22}) & \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11} + x_{12} + x_{21} + x_{22}) \\ \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11}x_{21} + 3x_{12}x_{22}) & \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11} + x_{12} + x_{21} + x_{22}) \\ \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11}x_{21} + 3x_{12}x_{22}) & \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11} + x_{12} + x_{21} + x_{22}) \\ \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11}x_{21} + 3x_{12}x_{22}) & \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11} + x_{12} + x_{21} + x_{22}) \\ \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11}x_{21} + 3x_{12}x_{22}) & \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11} + x_{12} + x_{21} + x_{22}) \\ \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11}x_{21} + 3x_{12}x_{22}) & \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11} + x_{12} + x_{21} + x_{22}) \\ \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11}x_{21} + 3x_{12}x_{22}) & \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11} + x_{12} + x_{21} + x_{22}) \\ \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11}x_{21} + 3x_{12}x_{22}) & \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11} + x_{12} + x_{21} + x_{22}) \\ \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11}x_{21} + 3x_{12}x_{22}) & \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11} + x_{12} + x_{21} + x_{22}) \\ \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11}x_{21} + 3x_{12}x_{22}) & \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11} + x_{12} + x_{21} + x_{22}) \\ \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11}x_{21} + 3x_{12}x_{22}) & \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11} + x_{12} + x_{21} + x_{22}) \\ \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11}x_{21} + x_{21} + x_{22}) & \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11} + x_{12} + x_{21} + x_{22}) \\ \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11}x_{21} + x_{21} + x_{22}) & \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11} + x_{12} + x_{21} + x_{22}) \\ \frac{\partial \pi_{22}}{\partial \pi_{22}} (x_{11}x_{21} + x_{21} + x_{22}) & \frac{\partial
                     [8]: DYval.doit()
                                                                                                \begin{bmatrix} 2x_{11} & 1 \\ x_{21} & 1 \\ 12x_{11}x_{12}x_{21} & -5x_{22} \end{bmatrix} & \begin{bmatrix} x_{21} & 4 \\ 3x_{22} & 1 \\ 6x_{11}^2x_{21} & 3 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} x_{12} & -1 \\ x_{11} & 1 \\ 6x_{11}^2x_{12} & -2x_{21} \end{bmatrix} & \begin{bmatrix} -1 & 3x_{22}^2 \\ 3x_{12} & 1 \\ 4 & -5x_{11} \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}
```

```
[9]: # ### GOAL: testing the A kronecker B rule for diff of Y = AXB
            from sympy import Lambda 1, m, n, q = 3, 5, 4, 2
             A = Matrix(l, m, lambda i, j: var('a', i, j))
X = Matrix(m, n, lambda i, j: var('x', i, j))
W = Matrix(n, q, lambda i, j: var('w', i, j))
             Y = X*W; Y
   \begin{vmatrix} w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} + w_{41}x_{24} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} + w_{42}x_{24} \\ w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} + w_{41}x_{34} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} + w_{42}x_{34} \end{vmatrix}
               \left| w_{11}x_{41} + w_{21}x_{42} + w_{31}x_{43} + w_{41}x_{44} \right| \quad w_{12}x_{41} + w_{22}x_{42} + w_{32}x_{43} + w_{42}x_{44}
                \left[ w_{11}x_{51} + w_{21}x_{52} + w_{31}x_{53} + w_{41}x_{54} \quad w_{12}x_{51} + w_{22}x_{52} + w_{32}x_{53} + w_{42}x_{54} \right] 
  [10]: from sympy.matrices import zeros
           E_12 = zeros(m, n)

E_12[1-1,2-1] = 1

E_12
                 \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}
               \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}
                 \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}
              \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}
  [11]: Y = X*W; Y
[11]: [w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} + w_{41}x_{14} \quad w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} + w_{42}x_{14}]
               \left| w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} + w_{41}x_{24} \right| = \left| w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} + w_{42}x_{24} \right|
               \left| w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} + w_{41}x_{34} \right| = \left| w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} + w_{42}x_{34} \right|
              \begin{bmatrix} w_{11}x_{41} + w_{21}x_{42} + w_{31}x_{43} + w_{41}x_{44} & w_{12}x_{41} + w_{22}x_{42} + w_{32}x_{43} + w_{42}x_{44} \\ w_{11}x_{51} + w_{21}x_{52} + w_{31}x_{53} + w_{41}x_{54} & w_{12}x_{51} + w_{22}x_{52} + w_{32}x_{53} + w_{42}x_{54} \end{bmatrix}
  [12]: E_12*W
```

assert Matrix(derive_by_array(Y, X[0,1]) == Y.diff(X[0,1])