

ch1_phase2

September 28, 2020

```
[1]: from sympy import Matrix, Symbol, derive_by_array, Lambda, Function,   
      ↪MatrixSymbol, Derivative, diff, symbols   
      from sympy import var   
      from sympy.abc import x, i, j, a, b   
  
      from sympy.interactive import init_printing   
  
      init_printing(pretty_print=True, wrap_line=True, num_columns=60)
```

```
[2]: def myvar(letter: str, i: int, j: int) -> Symbol:   
      letter_ij = Symbol('{}_{}_{}'.format(letter, i+1, j+1), is_commutative=True)   
      return letter_ij   
  
      n,m,p = 3,3,2   
  
      X = Matrix(n, m, lambda i,j : myvar('x', i, j)); X
```

```
[2]: 
$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

```

```
[3]: W = Matrix(m, p, lambda i,j : myvar('w', i, j)); W
```

```
[3]: 
$$\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix}$$

```

```
[4]: A = MatrixSymbol('X',3,3); Matrix(A)   
      B = MatrixSymbol('W',3,2)
```

```
[5]:
```

```
[5]:
```

```
[5]: v = lambda a,b: a*b
```

```

vL = Lambda((a,b), a*b)

n = Function('v') #, Lambda((a,b), a*b))

vN = lambda mat1, mat2: Matrix(mat1.shape[0], mat2.shape[1], lambda i, j:
↳Symbol("n_{}".format(i+1, j+1))); vN

Nelem = vN(X, W); Nelem

```

[5]:
$$\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{bmatrix}$$

[6]: `n(X,W)`

[6]:
$$v \left(\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix} \right)$$

[7]: `n(A,B)`

[7]:
$$v(X, W)$$

[8]: `n(X,W).replace(n, v) # replace works when v = python lambda`

[8]:
$$\begin{bmatrix} w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix}$$

[9]: `n(X,W).subs({n: vL}) # subs works when v = sympy lambda`

[9]:
$$\begin{bmatrix} w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix}$$

[10]: `n(X,W).replace(n, vL)`

[10]:
$$\begin{bmatrix} w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix}$$

[11]: `n(X,W).subs({n: v}) # subs() doesn't work when v is python lambda`

[11]:
$$v \left(\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix} \right)$$

[12]: `Matrix(n(A,B).subs({n: vL}))`

[12]:

$$\begin{bmatrix} W_{0,0}X_{0,0} + W_{1,0}X_{0,1} + W_{2,0}X_{0,2} & W_{0,1}X_{0,0} + W_{1,1}X_{0,1} + W_{2,1}X_{0,2} \\ W_{0,0}X_{1,0} + W_{1,0}X_{1,1} + W_{2,0}X_{1,2} & W_{0,1}X_{1,0} + W_{1,1}X_{1,1} + W_{2,1}X_{1,2} \\ W_{0,0}X_{2,0} + W_{1,0}X_{2,1} + W_{2,0}X_{2,2} & W_{0,1}X_{2,0} + W_{1,1}X_{2,1} + W_{2,1}X_{2,2} \end{bmatrix}$$

```
[13]: #N = v(X, W); N
      N = n(A,B); N
```

```
[13]: v(X, W)
```

```
[14]: N.replace(n, v)
```

```
[14]: XW
```

```
[15]: N.replace(n, v).subs({A: X, B:W}) # replacing ariable values after doing
      ↪function doesn't make the function apply directly on the values (matrices),
      ↪need to replace values before the function is replaced, so that the function
      ↪can act on them while they are given/alive.
```

```
[15]: 
$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix}$$

```

```
[16]: N.subs({n: vL, A:X, B:W})
```

```
[16]: 
$$\begin{bmatrix} w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix}$$

```

```
[17]: Nspec = N.subs({A:X, B:W}).replace(n, v); Nspec
```

```
[17]: 
$$\begin{bmatrix} w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix}$$

```

```
[18]:
```

```
[18]: N.diff(N)
```

```
[18]: 1
```

```
[19]: N.diff(X)
```

```
[19]: 
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

```

```
[20]:
```

```
[20]:
```

```
[20]: # way 2 of declaring S (better way)
sigma = Function('sigma')

sigmaApply = Function("sigma_apply") #lambda matrix: matrix.applyfunc(sigma)

sigmaApply_ = lambda matrix: matrix.applyfunc(sigma)

sigmaApply(A)
```

[20]: $\sigma_{\text{apply}}(X)$

```
[21]: sigmaApply(A).subs({A: X})
```

[21]: $\sigma_{\text{apply}} \left(\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \right)$

```
[22]: sigmaApply_(A)
```

[22]: $(d \mapsto \sigma(d))_{\circ}(X)$

```
[23]: sigmaApply(A).subs({A: X}).replace(sigmaApply, sigmaApply_) # NOTE: subs of  $\sqcup$ 
      ↪ functions doesn't work, replace actually evaluates the replaced function!
```

[23]: $\begin{bmatrix} \sigma(x_{11}) & \sigma(x_{12}) & \sigma(x_{13}) \\ \sigma(x_{21}) & \sigma(x_{22}) & \sigma(x_{23}) \\ \sigma(x_{31}) & \sigma(x_{32}) & \sigma(x_{33}) \end{bmatrix}$

```
[24]: S = sigmaApply(N); S
```

[24]: $\sigma_{\text{apply}}(v(X, W))$

```
[25]: Derivative(S, S)
```

[25]: $\frac{\partial}{\partial \sigma_{\text{apply}}(v(X, W))} \sigma_{\text{apply}}(v(X, W))$

```
[26]: Derivative(S, S).doit()
```

[26]: 1

```
[27]: Derivative(S, n(A,B)).doit()
```

[27]: $\frac{\partial}{\partial v(X, W)} \sigma_{\text{apply}}(v(X, W))$

```
[28]: #lambd = Function("lambda")
      #Lagain = lambd(sigmaApply(n(A))); Lagain
```

```
# diff(Lagain, A) # never execute
#
```

```
[29]: S.replace(A,X).replace(B,W)
```

```
[29]:
```

$$\sigma_{\text{apply}} \left(v \left(\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix} \right) \right)$$

```
[30]: S.replace(n, v)
```

```
[30]:
```

$$\sigma_{\text{apply}}(XW)$$

```
[31]: S.subs({A:X, B:W}).replace(n, v)
```

```
[31]:
```

$$\sigma_{\text{apply}} \left(\begin{bmatrix} w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix} \right)$$

```
[32]: Sspec = S.subs({A:X, B:W}).replace(n, v).replace(sigmaApply, sigmaApply_)
Sspec
```

```
[32]:
```

$$\begin{bmatrix} \sigma(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}) & \sigma(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13}) \\ \sigma(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}) & \sigma(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23}) \\ \sigma(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}) & \sigma(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33}) \end{bmatrix}$$

```
[33]: S.replace(n, vN) #.replace(sigmaApply, sigmaApply_)
```

```
[33]:
```

$$\sigma_{\text{apply}} \left(\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{bmatrix} \right)$$

```
[34]: Selem = S.replace(n, vN).replace(sigmaApply, sigmaApply_); Selem
```

```
[34]:
```

$$\begin{bmatrix} \sigma(n_{11}) & \sigma(n_{12}) \\ \sigma(n_{21}) & \sigma(n_{22}) \\ \sigma(n_{31}) & \sigma(n_{32}) \end{bmatrix}$$

```
[35]: import itertools

elemToSpecD = dict(itertools.chain(*[(Nelem[i, j], Nspec[i, j]) for j in
↪range(2)] for i in range(3)))

elemToSpec = list(elemToSpecD.items())

Matrix(elemToSpec)
```

```
[35]:
```

$$\begin{bmatrix} n_{11} & w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} \\ n_{12} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ n_{21} & w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} \\ n_{22} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ n_{31} & w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} \\ n_{32} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix}$$

```
[36]: elemToSpecFuncD = dict(itertools.chain(*[(Nelem[i, j], Function("n_{}_{}".
    ↪format(i + 1, j + 1))(Nspec[i, j])) for j in range(2)] for i in range(3))))

elemToSpecFunc = list(elemToSpecFuncD.items())

Matrix(elemToSpecFunc)
```

```
[36]:
```

$$\begin{bmatrix} n_{11} & n_{11} (w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}) \\ n_{12} & n_{12} (w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13}) \\ n_{21} & n_{21} (w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}) \\ n_{22} & n_{22} (w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23}) \\ n_{31} & n_{31} (w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}) \\ n_{32} & n_{32} (w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33}) \end{bmatrix}$$

```
[37]: elemToSpecFuncArgsD = dict(itertools.chain(*[(Nelem[i, j], Function("n_{}_{}".
    ↪format(i + 1, j + 1))(*X,*W)) for j in range(2)] for i in range(3))))

elemToSpecFuncArgs = list(elemToSpecFuncArgsD.items())

Matrix(elemToSpecFuncArgs)
```

```
[37]:
```

$$\begin{bmatrix} n_{11} & n_{11} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}) \\ n_{12} & n_{12} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}) \\ n_{21} & n_{21} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}) \\ n_{22} & n_{22} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}) \\ n_{31} & n_{31} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}) \\ n_{32} & n_{32} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}) \end{bmatrix}$$

```
[38]: Selem
```

```
[38]:
```

$$\begin{bmatrix} \sigma(n_{11}) & \sigma(n_{12}) \\ \sigma(n_{21}) & \sigma(n_{22}) \\ \sigma(n_{31}) & \sigma(n_{32}) \end{bmatrix}$$

```
[39]: Selem.subs(elemToSpecD)
```

```
[39]:
```

$$\begin{bmatrix} \sigma(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}) & \sigma(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13}) \\ \sigma(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}) & \sigma(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23}) \\ \sigma(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}) & \sigma(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33}) \end{bmatrix}$$

```
[40]: Selem[0,1].diff(Nelem[0,1])
```

```
[40]:
```

$$\frac{d}{dn_{12}}\sigma(n_{12})$$

```
[41]: Selem[0,1].diff(Nelem[0,1]).subs({Nelem[0,1] : Nspec[0,1]})
      #Selem[0,1].diff(Nelem[0,1]).subs(dict([Nelem[0,1] : Nspec[0,1]]))
```

```
[41]:
```

$$\left. \frac{d}{dn_{12}}\sigma(n_{12}) \right|_{n_{12}=w_{12}x_{11}+w_{22}x_{12}+w_{32}x_{13}}$$

```
[42]: Selem[0,1].diff(Nelem[0,1]).subs({Nelem[0,1] : Nspec[0,1]}).subs({Nspec[0,1] : 23})
```

```
[42]:
```

$$\left. \frac{d}{dn_{12}}\sigma(n_{12}) \right|_{n_{12}=23}$$

```
[43]: Selem[0,1].diff(Nelem[0,1]).subs({Nelem[0,1] : Nspec[0,1]}).replace(sigma, lambda x: 8*x**3)
```

```
[43]:
```

$$\left. \frac{d}{dn_{12}}8n_{12}^3 \right|_{n_{12}=w_{12}x_{11}+w_{22}x_{12}+w_{32}x_{13}}$$

```
[44]: Selem[0,1].diff(Nelem[0,1]).replace(sigma, lambda x: 8*x**3)
```

```
[44]:
```

$$\frac{d}{dn_{12}}8n_{12}^3$$

```
[45]: Selem[0,1].diff(Nelem[0,1]).replace(sigma, lambda x: 8*x**3).doit()
```

```
[45]:
```

$$24n_{12}^2$$

```
[46]: # ### GOT IT: can replace now with expression and do derivative with respect to
      # that expression.
      Selem[0,1].diff(Nelem[0,1]).subs({Nelem[0,1] : Nspec[0,1]}).replace(sigma, lambda x: 8*x**3).doit()
```

```
[46]:
```

$$24(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13})^2$$

```
[47]: Selem[0,1].subs({Nelem[0,1] : Nspec[0,1]}).diff(X[0,1]).subs({Nelem[0,1] : Nspec[0,1]})
```

```
[47]:
```

$$w_{22} \left. \frac{d}{d\xi_1}\sigma(\xi_1) \right|_{\xi_1=w_{12}x_{11}+w_{22}x_{12}+w_{32}x_{13}}$$

```
[48]: Selem
```

```
[48]:
```

$$\begin{bmatrix} \sigma(n_{11}) & \sigma(n_{12}) \\ \sigma(n_{21}) & \sigma(n_{22}) \\ \sigma(n_{31}) & \sigma(n_{32}) \end{bmatrix}$$

```
[49]: nt = Nelem.subs(elemToSpecFunc); nt
```

[49]:
$$\begin{bmatrix} n_{11}(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}) & n_{12}(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13}) \\ n_{21}(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}) & n_{22}(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23}) \\ n_{31}(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}) & n_{32}(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33}) \end{bmatrix}$$

[50]: `st = Selem.subs(elemToSpecFunc); st`

[50]:
$$\begin{bmatrix} \sigma(n_{11}(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})) & \sigma(n_{12}(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13})) \\ \sigma(n_{21}(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23})) & \sigma(n_{22}(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23})) \\ \sigma(n_{31}(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33})) & \sigma(n_{32}(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33})) \end{bmatrix}$$

[51]: `init_printing(num_columns=10, wrap_line=True, pretty_print=True)#`

`st.diff(nt)`

[51]:
$$\begin{bmatrix} \frac{\partial}{\partial n_{11}(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})} \sigma(n_{11}(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})) & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{\partial}{\partial n_{21}(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23})} \sigma(n_{21}(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23})) & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{\partial}{\partial n_{31}(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33})} \sigma(n_{31}(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33})) & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{\partial}{\partial n_{12}(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13})} \sigma(n_{12}(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13})) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{\partial}{\partial n_{22}(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23})} \sigma(n_{22}(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23})) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{\partial}{\partial n_{32}(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33})} \sigma(n_{32}(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33})) \end{bmatrix}$$

[52]: `st[0,0].diff(st[0,0].args[0])`

[52]:
$$\frac{\partial}{\partial n_{11}(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})} \sigma(n_{11}(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}))$$

[53]: `st[0,0].diff(X[0,0])`

[53]:
$$w_{11} \frac{\partial}{\partial n_{11}(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})} \sigma(n_{11}(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})) \frac{d}{d\xi_1} n_{11}(\xi_1) \Big|_{\xi_1=w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}}$$

[54]: `st[0,0].diff(st[1,0].args[0])`

[54]:
$$0$$

[55]: `Selem.diff(Selem)`

[55]:

$$\begin{bmatrix} \left[\begin{array}{cc} \frac{d}{dn_{11}}\sigma(n_{11}) & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right] & \left[\begin{array}{cc} 0 & \frac{d}{dn_{12}}\sigma(n_{12}) \\ 0 & 0 \\ 0 & 0 \end{array} \right] \\ \left[\begin{array}{cc} \frac{d}{dn_{21}}\sigma(n_{21}) & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right] & \left[\begin{array}{cc} 0 & \frac{d}{dn_{22}}\sigma(n_{22}) \\ 0 & 0 \\ 0 & 0 \end{array} \right] \\ \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \frac{d}{dn_{31}}\sigma(n_{31}) & 0 \end{array} \right] & \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & \frac{d}{dn_{32}}\sigma(n_{32}) \end{array} \right] \end{bmatrix}$$

```
[56]: Selem.diff(Selem).subs(elemToSpecFunc)
```

$$\begin{bmatrix} \left[\begin{array}{cc} \frac{\partial}{\partial n_{11}(w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13})}\sigma(n_{11}(w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13})) & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right] & \left[\begin{array}{cc} 0 & \frac{\partial}{\partial n_{12}(w_{12}x_{11}+w_{22}x_{12}+w_{32}x_{13})}\sigma(n_{12}(w_{12}x_{11}+w_{22}x_{12}+w_{32}x_{13})) \\ 0 & 0 \\ 0 & 0 \end{array} \right] \\ \left[\begin{array}{cc} \frac{\partial}{\partial n_{21}(w_{11}x_{21}+w_{21}x_{22}+w_{31}x_{23})}\sigma(n_{21}(w_{11}x_{21}+w_{21}x_{22}+w_{31}x_{23})) & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right] & \left[\begin{array}{cc} 0 & \frac{\partial}{\partial n_{22}(w_{12}x_{21}+w_{22}x_{22}+w_{32}x_{23})}\sigma(n_{22}(w_{12}x_{21}+w_{22}x_{22}+w_{32}x_{23})) \\ 0 & 0 \\ 0 & 0 \end{array} \right] \\ \left[\begin{array}{cc} \frac{\partial}{\partial n_{31}(w_{11}x_{31}+w_{21}x_{32}+w_{31}x_{33})}\sigma(n_{31}(w_{11}x_{31}+w_{21}x_{32}+w_{31}x_{33})) & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right] & \left[\begin{array}{cc} 0 & \frac{\partial}{\partial n_{32}(w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33})}\sigma(n_{32}(w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33})) \\ 0 & 0 \\ 0 & 0 \end{array} \right] \end{bmatrix}$$

```
[57]: # CAN even replace elements after have done an operation on them!!! replacing
      ↪ n_21 * 2 with the number 4.
      Sspec.subs({Nspec[0, 0]: 3}).replace(sigma, lambda x: 2 * x).replace(Nspec[2,
      ↪ 1] * 2, 4)
```

$$\begin{bmatrix} 6 & 2w_{12}x_{11} + 2w_{22}x_{12} + 2w_{32}x_{13} \\ 2w_{11}x_{21} + 2w_{21}x_{22} + 2w_{31}x_{23} & 2w_{12}x_{21} + 2w_{22}x_{22} + 2w_{32}x_{23} \\ 2w_{11}x_{31} + 2w_{21}x_{32} + 2w_{31}x_{33} & 4 \end{bmatrix}$$

```
[58]: lambda = Function("lambda")
      lambda_ = lambda matrix : sum(matrix)

      vN(X, W)
```

$$\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{bmatrix}$$

```
[59]: vN(A, B)
```

$$\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{bmatrix}$$

```
[60]: L = lambda(S); L
```

[60]: $\lambda(\sigma_{\text{apply}}(v(X, W)))$

[61]: Nelem

[61]:
$$\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{bmatrix}$$

[62]: L.replace(n, vN)

[62]:
$$\lambda\left(\sigma_{\text{apply}}\left(\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{bmatrix}\right)\right)$$

[63]: L.replace(n, vN).replace(sigmaApply, sigmaApply_)

[63]:
$$\lambda\left(\begin{bmatrix} \sigma(n_{11}) & \sigma(n_{12}) \\ \sigma(n_{21}) & \sigma(n_{22}) \\ \sigma(n_{31}) & \sigma(n_{32}) \end{bmatrix}\right)$$

[64]: L.replace(n, v)

[64]: $\lambda(\sigma_{\text{apply}}(XW))$

[65]: L.replace(n, v).replace(sigmaApply, sigmaApply_)

[65]: $\lambda((d \mapsto \sigma(d))_{\circ}(XW))$

[66]: L.subs({A:X, B:W}).replace(n, vL).replace(sigmaApply, sigmaApply_)

[66]:
$$\lambda\left(\begin{bmatrix} \sigma(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}) & \sigma(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13}) \\ \sigma(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}) & \sigma(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23}) \\ \sigma(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}) & \sigma(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33}) \end{bmatrix}\right)$$

[67]: L.replace(n, vN)

[67]:
$$\lambda\left(\sigma_{\text{apply}}\left(\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{bmatrix}\right)\right)$$

[68]: L.replace(n, vN).subs({A:X, B:W}).replace(sigmaApply, sigmaApply_).
→replace(lambd, lambd_)

[68]: $\sigma(n_{11}) + \sigma(n_{12}) + \sigma(n_{21}) + \sigma(n_{22}) + \sigma(n_{31}) + \sigma(n_{32})$

[69]: from sympy import symbols, Derivative

x, y, r, t = symbols('x y r t') # r (radius), t (angle theta)
f, g, h = symbols('f g h', cls=Function)

```
h = g(f(x))

Derivative(h, f(x)).doit()
```

[69]: $\frac{d}{df(x)}g(f(x))$

```
[70]: # Never do this gives recursion ERROR (max depth exceeded)
# h = g(f(A))
# Derivative(h, A).doit()
```

[71]:

```
[71]: from sympy.abc import a, b

Llower = lambd(sigmaApply(n(a, b)))
Llower
```

[71]: $\lambda(\sigma_{\text{apply}}(v(a, b)))$

```
[72]: Derivative(Llower, a).doit()
```

[72]: $\frac{\partial}{\partial \sigma_{\text{apply}}(v(a, b))} \lambda(\sigma_{\text{apply}}(v(a, b))) \frac{\partial}{\partial v(a, b)} \sigma_{\text{apply}}(v(a, b)) \frac{\partial}{\partial a} v(a, b)$

[73]:

```
[73]: # ### WAY 1: of substituting to differentiate with respect to expression:
n_ij = Function('n_ij')
n_ij(A,B) # (N[0,0]); n_ij
```

[73]: $n_{ij}(X, W)$

```
[74]: n_ij(A,B).args
```

[74]: (X, W)

```
[75]: # sigma(n_ij).diff(n_ij).replace(n_ij, N[0,0]) # ERROR cannot deriv wi.r.t to
↳ the expression w11*x11 + ...

sigma(n_ij(A,B)).diff(n_ij(A,B))
```

[75]: $\frac{\partial}{\partial n_{ij}(X, W)} \sigma(n_{ij}(X, W))$

```
[76]: sigma(n_ij(*X,*W)).diff(X[0,0])
```

[76]: $\frac{\partial}{\partial x_{11}} n_{ij}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}) \frac{\partial}{\partial n_{ij}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32},$

```
[77]: nab_ij = n_ij(A,B)
      sigma(nab_ij).diff(nab_ij)#.subs({nab_ij : Nspec[0, 0]})
```

[77]:
$$\frac{\partial}{\partial n_{ij}} \sigma(n_{ij}(X, W))$$

```
[78]: sigma(nab_ij).diff(nab_ij).subs({nab_ij : Nspec[2, 1]})
```

[78]:
$$\left. \frac{d}{d\xi} \sigma(\xi) \right|_{\xi=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}}$$

```
[79]: sigma(nab_ij).diff(nab_ij).subs({nab_ij : Nspec[2,1]}).subs({X[2,1]:77777})
```

[79]:
$$\left. \frac{d}{d\xi} \sigma(\xi) \right|_{\xi=w_{12}x_{31}+77777w_{22}+w_{32}x_{33}}$$

```
[80]: sigma(nab_ij).diff(nab_ij).subs({nab_ij : 23}) # ERROR if using replace() since
      ↪ it says can't calc derivs w.r.t to the x_11*w_11 + ...
```

[80]:
$$\left. \frac{d}{d\xi} \sigma(\xi) \right|_{\xi=23}$$

```
[81]: sigma(nab_ij).diff(nab_ij).subs({nab_ij : Nspec[2,1]}).doit()
```

[81]:
$$\left. \frac{d}{d\xi} \sigma(\xi) \right|_{\xi=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}}$$

```
[82]: sigma(nab_ij).subs({nab_ij : Nspec[2,1]}).diff(X[2,1])
```

[82]:
$$\sigma(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33})$$

```
[83]: # Substituting the value of the function n_ij first, and THEN differentiating
      ↪ with respect to something in that substitution. (X_21)
      sigma(nab_ij).subs({nab_ij : Nspec[2,1]}).diff(X[2,1])
```

[83]:
$$w_{22} \left. \frac{d}{d\xi_1} \sigma(\xi_1) \right|_{\xi_1=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}}$$

```
[84]: Selem[2,1].subs({Nelem[2,1] : Nspec[2,1]}).diff(X[2,1])
```

[84]:
$$w_{22} \left. \frac{d}{d\xi_1} \sigma(\xi_1) \right|_{\xi_1=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}}$$

```
[85]: # ### WAY 2:
      n_11 = Function('n_11')(Nspec[0, 0]); n_11
```

[85]:
$$n_{11}(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})$$

```
[86]: sigma(n_11)
```

[86]: $\sigma(n_{11} (w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}))$

[87]: `assert Nspec[0,0] == n_11.args[0]`
`sigma(n_11).subs({n_11 : n_11.args[0]})`

[87]: $\sigma(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})$

[88]: `sigma(n_11).diff(n_11) #.replace(n_ij, n_ij.args[0])`

[88]:
$$\frac{\partial}{\partial n_{11} (w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})} \sigma(n_{11} (w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}))$$

[89]: `sigma(n_11).diff(n_11).subs({n_11 : n_11.args[0]}).subs({X[0,0]:77777})`

[89]:
$$\left. \frac{d}{d\xi} \sigma(\xi) \right|_{\xi=77777w_{11}+w_{21}x_{12}+w_{31}x_{13}}$$

[90]: `sigma(n_11).diff(n_11).subs({n_11 : n_11.args[0]}).replace(n_11.args[0], 23) #`
`→ same as subs in this case`

[90]:
$$\left. \frac{d}{d\xi} \sigma(\xi) \right|_{\xi=23}$$

[91]: `sigma(n_11).diff(X[0,0])`

[91]:
$$w_{11} \frac{\partial}{\partial n_{11} (w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})} \sigma(n_{11} (w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})) \left. \frac{d}{d\xi_1} n_{11} (\xi_1) \right|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}}$$

[92]: `id = Lambda(x, x)`
`sigma(n_11).diff(X[0,0]).subs({n_11 : id})`

[92]:
$$w_{11} \left. \frac{d}{d\xi_1} n_{11} (\xi_1) \right|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}} \left. \frac{d}{d\xi} \sigma(\xi) \right|_{\xi=(x \rightarrow x)}$$

[93]: `# NOTE: so I don't think WAY 2 is correct because here it doesn't simplify the`
`→ derivative d n11 / d eps11, since this should equal 1 because now n11 =`
`→ eps11. Correct one is below (repeated from above)`
`sigma(n_11).diff(X[0,0]).subs({n_11 : Nspec[0,0]})`

[93]:
$$w_{11} \left. \frac{d}{d\xi_1} n_{11} (\xi_1) \right|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}} \left. \frac{d}{d\xi} \sigma(\xi) \right|_{\xi=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}}$$

[94]: `# CORRECT WAY 1`
`sigma(n_11).subs({n_11 : Nspec[0,0]}).diff(X[0,0])`

[94]:
$$w_{11} \left. \frac{d}{d\xi_1} \sigma(\xi_1) \right|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}}$$

[95]: *# CORRECT WAY 2*

```
sigma(nab_ij).subs({nab_ij : Nspec[0,0]}).diff(X[0,0])
```

[95]: $w_{11} \frac{d}{d\xi_1} \sigma(\xi_1) \Big|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}}$

[96]: *# CORRECT WAY 3*

```
Selem[2,1].subs({Nelem[2,1] : Nspec[2,1]}).diff(X[2,1])
```

[96]: $w_{22} \frac{d}{d\xi_1} \sigma(\xi_1) \Big|_{\xi_1=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}}$

[97]: `sigma(n_11)` *# WAY 1: sigma argument is already hardcoded*

[97]: $\sigma(n_{11}(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}))$

[98]: `sigma(nab_ij)` *# Way 2: sigma argument is function of matrixsymbol (better than ↪ 1)*

[98]: $\sigma(n_{ij}(X, W))$

[99]: `Selem[2,1]` *# WAY 3: sigma argument is just symbol and we replace it as function ↪ with argument hardcoded only later. (better than 2)*

[99]: $\sigma(n_{32})$

[100]: `L`

[100]: $\lambda(\sigma_{apply}(v(X, W)))$

[101]: `assert Selem == S.replace(n, vN).replace(sigmaApply, sigmaApply_)`
`Selem`

[101]:
$$\begin{bmatrix} \sigma(n_{11}) & \sigma(n_{12}) \\ \sigma(n_{21}) & \sigma(n_{22}) \\ \sigma(n_{31}) & \sigma(n_{32}) \end{bmatrix}$$

[102]: `L.replace(n, vN).replace(sigmaApply, sigmaApply_)`

[102]:
$$\lambda \left(\begin{bmatrix} \sigma(n_{11}) & \sigma(n_{12}) \\ \sigma(n_{21}) & \sigma(n_{22}) \\ \sigma(n_{31}) & \sigma(n_{32}) \end{bmatrix} \right)$$

[103]: *#L.replace(n, vN).replace(sigmaApply, sigmaApply_).diff(Nelem[0,0])*

[104]: `Lsum = L.replace(n, vN).replace(sigmaApply, sigmaApply_).replace(lambd, lambd_)`
`Lsum`

[104]: $\sigma(n_{11}) + \sigma(n_{12}) + \sigma(n_{21}) + \sigma(n_{22}) + \sigma(n_{31}) + \sigma(n_{32})$

[105]: `Lsum.diff(Nelem)`

[105]:
$$\begin{bmatrix} \frac{d}{dn_{11}}\sigma(n_{11}) & \frac{d}{dn_{12}}\sigma(n_{12}) \\ \frac{d}{dn_{21}}\sigma(n_{21}) & \frac{d}{dn_{22}}\sigma(n_{22}) \\ \frac{d}{dn_{31}}\sigma(n_{31}) & \frac{d}{dn_{32}}\sigma(n_{32}) \end{bmatrix}$$

[106]: `Lsum.subs(elemToSpec)#.diff(X[2,1])`

[106]: $\sigma(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}) + \sigma(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}) + \sigma(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}) + \sigma(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13}) + \sigma(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23}) + \sigma(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33})$

[107]: `Lsum.subs(elemToSpec).diff(X)`

[107]:
$$\begin{bmatrix} w_{11} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}} & + w_{12} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{12}x_{11}+w_{22}x_{12}+w_{32}x_{13}} & w_{21} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}} \\ w_{11} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{11}x_{21}+w_{21}x_{22}+w_{31}x_{23}} & + w_{12} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{12}x_{21}+w_{22}x_{22}+w_{32}x_{23}} & w_{21} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{11}x_{21}+w_{21}x_{22}+w_{31}x_{23}} \\ w_{11} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{11}x_{31}+w_{21}x_{32}+w_{31}x_{33}} & + w_{12} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}} & w_{21} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{11}x_{31}+w_{21}x_{32}+w_{31}x_{33}} \end{bmatrix}$$

[108]: `specToElemD = {v : k for k, v in elemToSpecD.items()}`

`Lsum.subs(elemToSpecD).diff(X).subs(specToElemD)`

[108]:
$$\begin{bmatrix} w_{11} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{11}} + w_{12} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{12}} & w_{21} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{11}} + w_{22} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{12}} & w_{31} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{11}} + w_{32} \\ w_{11} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{21}} + w_{12} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{22}} & w_{21} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{21}} + w_{22} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{22}} & w_{31} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{21}} + w_{32} \\ w_{11} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{31}} + w_{12} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{32}} & w_{21} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{31}} + w_{22} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{32}} & w_{31} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{31}} + w_{32} \end{bmatrix}$$