

## ch1\_phase2

September 30, 2020

```
[1]: from sympy import Matrix, Symbol, derive_by_array, Lambda, Function, MatrixSymbol, Derivative, diff, symbols
    from sympy import var
    from sympy.abc import x, i, j, a, b

    from sympy.interactive import init_printing

    init_printing(pretty_print=True, wrap_line=True, num_columns=60)
```

```
[2]: def myvar(letter: str, i: int, j: int) -> Symbol:
    letter_ij = Symbol('{}_{}'.format(letter, i+1, j+1), is_commutative=True)
    return letter_ij

n,m,p = 3,3,2

X = Matrix(n, m, lambda i,j : myvar('x', i, j)); X
```

```
[2]: 
$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

```

```
[3]: W = Matrix(m, p, lambda i,j : myvar('w', i, j)); W
```

```
[3]: 
$$\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix}$$

```

```
[4]: A = MatrixSymbol('X',3,3); Matrix(A)
    B = MatrixSymbol('W',3,2)
```

```
[5]:
```

```
[5]:
```

```
[5]: v = lambda a,b: a*b

    vL = Lambda((a,b), a*b)
```

```
n = Function('v') #, Lambda((a,b), a*b))

vN = lambda mat1, mat2: Matrix(mat1.shape[0], mat2.shape[1], lambda i, j: Symbol("n_{}".format(i+1, j+1))); vN

Nelem = vN(X, W); Nelem
```

[5]: 
$$\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{bmatrix}$$

[6]: `n(X,W)`

[6]: 
$$v \left( \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix} \right)$$

[7]: `n(A,B)`

[7]:  $v(X, W)$

[8]: `n(X,W).replace(n, v) # replace works when v = python lambda`

[8]: 
$$\begin{bmatrix} w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix}$$

[9]: `n(X,W).subs({n: vL}) # subs works when v = sympy lambda`

[9]: 
$$\begin{bmatrix} w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix}$$

[10]: `n(X,W).replace(n, vL)`

[10]: 
$$\begin{bmatrix} w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix}$$

[11]: `n(X,W).subs({n: v})# subs() doesn't work when v is python lambda`

[11]: 
$$v \left( \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix} \right)$$

[12]: `Matrix(n(A,B).subs({n: vL}))`

[12]: 
$$\begin{bmatrix} W_{0,0}X_{0,0} + W_{1,0}X_{0,1} + W_{2,0}X_{0,2} & W_{0,1}X_{0,0} + W_{1,1}X_{0,1} + W_{2,1}X_{0,2} \\ W_{0,0}X_{1,0} + W_{1,0}X_{1,1} + W_{2,0}X_{1,2} & W_{0,1}X_{1,0} + W_{1,1}X_{1,1} + W_{2,1}X_{1,2} \\ W_{0,0}X_{2,0} + W_{1,0}X_{2,1} + W_{2,0}X_{2,2} & W_{0,1}X_{2,0} + W_{1,1}X_{2,1} + W_{2,1}X_{2,2} \end{bmatrix}$$

```
[13]: #N = v(X, W); N
      N = n(A,B); N
```

```
[13]: v(X,W)
```

```
[14]: N.replace(n, v)
```

```
[14]: XW
```

```
[15]: N.replace(n, v).subs({A: X, B:W}) # replacing ariable values after doing function doesn't make the function apply directly on the values (matrices), need to
      ↪ replace values before the function is replaced, so that the function can act on them while they are given/alive.
```

```
[15]: 
$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix}$$

```

```
[16]: N.subs({n: vL, A:X, B:W})
```

```
[16]: 
$$\begin{bmatrix} w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix}$$

```

```
[17]: Nspec = N.subs({A:X, B:W}).replace(n, v); Nspec
```

```
[17]: 
$$\begin{bmatrix} w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix}$$

```

```
[18]:
```

```
[18]: N.diff(N)
```

```
[18]: 1
```

```
[19]: N.diff(X)
```

```
[19]: 
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

```

```
[20]:
```

```
[20]:
```

```
[20]: # way 2 of declaring S (better way)
      sigma = Function('sigma')

      sigmaApply = Function("sigma_apply") #lambda matrix: matrix.applyfunc(sigma)

      sigmaApply_ = lambda matrix: matrix.applyfunc(sigma)
```

```
sigmaApply(A)
```

[20]:

```
sigmaApply(A).subs({A: X})
```

[21]:

```
sigmaApply_(A)
```

[22]:

```
sigmaApply(A).subs({A: X}).replace(sigmaApply, sigmaApply_) # NOTE: subs of functions doesn't work, replace actually evaluates the replaced function!
```

[23]:

```
S = sigmaApply(N); S
```

[24]:

Derivative(S, S)

[25]:

```
Derivative(S, S).doit()
```

[26]:

```
Derivative(S, n(A,B)).doit()
```

[27]:

```
#lambd = Function("lambda")
#Lagain = lambd(sigmaApply(n(A))); Lagain

# diff(Lagain, A) # never execute
#
```

```
S.replace(A,X).replace(B,W)
```

[29]:

$$\sigma_{apply} \left( v \left( \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix} \right) \right)$$

```
[30]: S.replace(n, v)
```

```
[30]:  $\sigma_{apply}(XW)$ 
```

```
[31]: S.subs({A:X, B:W}).replace(n, v)
```

```
[31]:  $\sigma_{apply} \left( \begin{bmatrix} w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix} \right)$ 
```

```
[32]: Sspec = S.subs({A:X, B:W}).replace(n, v).replace(sigmaApply, sigmaApply_)
Sspec
```

```
[32]:  $\begin{bmatrix} \sigma(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}) & \sigma(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13}) \\ \sigma(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}) & \sigma(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23}) \\ \sigma(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}) & \sigma(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33}) \end{bmatrix}$ 
```

```
[33]: S.replace(n, vN) #.replace(sigmaApply, sigmaApply_)
```

```
[33]:  $\sigma_{apply} \left( \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{bmatrix} \right)$ 
```

```
[34]: Selem = S.replace(n, vN).replace(sigmaApply, sigmaApply_); Selem
```

```
[34]:  $\begin{bmatrix} \sigma(n_{11}) & \sigma(n_{12}) \\ \sigma(n_{21}) & \sigma(n_{22}) \\ \sigma(n_{31}) & \sigma(n_{32}) \end{bmatrix}$ 
```

```
[35]: import itertools
```

```
elemToSpecD = dict(itertools.chain(*[[ (Nelem[i, j], Nspec[i, j]) for j in range(2)] for i in range(3)]))
```

```
elemToSpec = list(elemToSpecD.items())
```

```
Matrix(elemToSpec)
```

```
[35]:  $\begin{bmatrix} n_{11} & w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13} \\ n_{12} & w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13} \\ n_{21} & w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23} \\ n_{22} & w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23} \\ n_{31} & w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33} \\ n_{32} & w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33} \end{bmatrix}$ 
```

```
[36]: elemToSpecFuncD = dict(itertools.chain(*[[ (Nelem[i, j], Function("n_{}".format(i + 1, j + 1))(Nspec[i, j])) for j in range(2)] for i in range(3)]))
```

```
elemToSpecFunc = list(elemToSpecFuncD.items())
```

```
Matrix(elemToSpecFunc)
```

[36]:

$$\begin{bmatrix} n_{11} & \mathbf{n}_{11} (w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}) \\ n_{12} & \mathbf{n}_{12} (w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13}) \\ n_{21} & \mathbf{n}_{21} (w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}) \\ n_{22} & \mathbf{n}_{22} (w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23}) \\ n_{31} & \mathbf{n}_{31} (w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}) \\ n_{32} & \mathbf{n}_{32} (w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33}) \end{bmatrix}$$

[37]: elemToSpecFuncArgsD = dict(itertools.chain(\*[[Nelem[i, j], Function("n\_{}".format(i + 1, j + 1))(\*X,\*W)) for j in range(2)] for i in range(3)]))

```
elemToSpecFuncArgs = list(elemToSpecFuncArgsD.items())
```

```
Matrix(elemToSpecFuncArgs)
```

[37]:

$$\begin{bmatrix} n_{11} & \mathbf{n}_{11} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}) \\ n_{12} & \mathbf{n}_{12} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}) \\ n_{21} & \mathbf{n}_{21} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}) \\ n_{22} & \mathbf{n}_{22} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}) \\ n_{31} & \mathbf{n}_{31} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}) \\ n_{32} & \mathbf{n}_{32} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}) \end{bmatrix}$$

[38]: Selem

[38]:

$$\begin{bmatrix} \sigma(n_{11}) & \sigma(n_{12}) \\ \sigma(n_{21}) & \sigma(n_{22}) \\ \sigma(n_{31}) & \sigma(n_{32}) \end{bmatrix}$$

[39]: Selem.subs(elemToSpecD)

[39]:

$$\begin{bmatrix} \sigma(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}) & \sigma(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13}) \\ \sigma(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}) & \sigma(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23}) \\ \sigma(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}) & \sigma(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33}) \end{bmatrix}$$

[40]: Selem[0,1].diff(Nelem[0,1])

[40]:

$$\frac{d}{dn_{12}}\sigma(n_{12})$$

[41]: Selem[0,1].diff(Nelem[0,1]).subs({Nelem[0,1] : Nspec[0,1]})  
#Selem[0,1].diff(Nelem[0,1]).subs(dict([Nelem[0,1] : Nspec[0,1]]))

[41]:

$$\left. \frac{d}{dn_{12}}\sigma(n_{12}) \right|_{n_{12}=w_{12}x_{11}+w_{22}x_{12}+w_{32}x_{13}}$$

[42]: Selem[0,1].diff(Nelem[0,1]).subs({Nelem[0,1] : Nspec[0,1]}).subs({Nspec[0,1] : 23})

[42]:

$$\left. \frac{d}{dn_{12}} \sigma(n_{12}) \right|_{n_{12}=23}$$

```
[43]: Selem[0,1].diff(Nelem[0,1]).subs({Nelem[0,1] : Nspec[0,1]}).replace(sigma, lambda x: 8*x**3)
```

$$\left. \frac{d}{dn_{12}} 8n_{12}^3 \right|_{n_{12}=w_{12}x_{11}+w_{22}x_{12}+w_{32}x_{13}}$$

```
[44]: Selem[0,1].diff(Nelem[0,1]).replace(sigma, lambda x: 8*x**3)
```

$$\frac{d}{dn_{12}} 8n_{12}^3$$

```
[45]: Selem[0,1].diff(Nelem[0,1]).replace(sigma, lambda x: 8*x**3).doit()
```

$$24n_{12}^2$$

```
[46]: # ### GOT IT: can replace now with expression and do derivative with respect to that expression.
Selem[0,1].diff(Nelem[0,1]).subs({Nelem[0,1] : Nspec[0,1]}).replace(sigma, lambda x: 8*x**3).doit()
```

$$24(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13})^2$$

```
[47]: Selem[0,1].subs({Nelem[0,1] : Nspec[0,1]}).diff(X[0,1])#.subs({Nelem[0,1] : Nspec[0,1]})
```

$$w_{22} \left. \frac{d}{d\xi_1} \sigma(\xi_1) \right|_{\xi_1=w_{12}x_{11}+w_{22}x_{12}+w_{32}x_{13}}$$

```
[48]: Selem
```

$$\begin{bmatrix} \sigma(n_{11}) & \sigma(n_{12}) \\ \sigma(n_{21}) & \sigma(n_{22}) \\ \sigma(n_{31}) & \sigma(n_{32}) \end{bmatrix}$$

```
[49]: nt = Nelem.subs(elemToSpecFunc); nt
```

$$\begin{bmatrix} \mathbf{n}_{11}(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}) & \mathbf{n}_{12}(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13}) \\ \mathbf{n}_{21}(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}) & \mathbf{n}_{22}(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23}) \\ \mathbf{n}_{31}(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}) & \mathbf{n}_{32}(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33}) \end{bmatrix}$$

```
[50]: st = Selem.subs(elemToSpecFunc); st
```

$$\begin{bmatrix} \sigma(\mathbf{n}_{11}(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})) & \sigma(\mathbf{n}_{12}(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13})) \\ \sigma(\mathbf{n}_{21}(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23})) & \sigma(\mathbf{n}_{22}(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23})) \\ \sigma(\mathbf{n}_{31}(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33})) & \sigma(\mathbf{n}_{32}(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33})) \end{bmatrix}$$

```
[51]: init_printing(num_columns=10, wrap_line=True, pretty_print=True)#
```

```
st.diff(nt)
```

[51]:

$$\left[ \begin{array}{cc} \left[ \begin{array}{cc} \frac{\partial}{\partial \mathbf{n}_{11} (w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})} \sigma(\mathbf{n}_{11} (w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})) & 0 \\ 0 & 0 \\ 0 & 0 \end{array} & \left[ \begin{array}{cc} 0 & \frac{\partial}{\partial \mathbf{n}_{12} (w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13})} \sigma(\mathbf{n}_{12} (w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13})) \\ 0 & 0 \\ 0 & 0 \end{array} \end{array} \right. \\ \left. \begin{array}{cc} \left[ \begin{array}{cc} \frac{\partial}{\partial \mathbf{n}_{21} (w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23})} \sigma(\mathbf{n}_{21} (w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23})) & 0 \\ 0 & 0 \\ 0 & 0 \end{array} & \left[ \begin{array}{cc} 0 & \frac{\partial}{\partial \mathbf{n}_{22} (w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23})} \sigma(\mathbf{n}_{22} (w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23})) \\ 0 & 0 \\ 0 & 0 \end{array} \end{array} \right. \right. \\ \left. \left. \begin{array}{cc} \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} & \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] \right. \right. \\ \left. \left. \left[ \begin{array}{cc} \frac{\partial}{\partial \mathbf{n}_{31} (w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33})} \sigma(\mathbf{n}_{31} (w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33})) & 0 \end{array} \right] & \left[ \begin{array}{cc} 0 & \frac{\partial}{\partial \mathbf{n}_{32} (w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33})} \sigma(\mathbf{n}_{32} (w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33})) \end{array} \right] \right] \right]$$

[52]: `st[0,0].diff(st[0,0].args[0])`

[52]:

$$\frac{\partial}{\partial \mathbf{n}_{11} (w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})} \sigma(\mathbf{n}_{11} (w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}))$$

[53]: `st[0,0].diff(X[0,0])`

[53]:

$$w_{11} \frac{\partial}{\partial \mathbf{n}_{11} (w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})} \sigma(\mathbf{n}_{11} (w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})) \frac{d}{d\xi_1} \mathbf{n}_{11} (\xi_1) \Big|_{\xi_1 = w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}}$$

[54]: `st[0,0].diff(st[1,0].args[0])`

[54]: 0

[55]: `Selem.diff(Nelem)`

[55]:

$$\left[ \begin{array}{cc} \left[ \begin{array}{cc} \frac{d}{dn_{11}} \sigma(n_{11}) & 0 \\ 0 & 0 \\ 0 & 0 \end{array} & \left[ \begin{array}{cc} 0 & \frac{d}{dn_{12}} \sigma(n_{12}) \\ 0 & 0 \\ 0 & 0 \end{array} \right] \right. \\ \left. \begin{array}{cc} \left[ \begin{array}{cc} 0 & 0 \\ \frac{d}{dn_{21}} \sigma(n_{21}) & 0 \\ 0 & 0 \end{array} & \left[ \begin{array}{cc} 0 & \frac{d}{dn_{22}} \sigma(n_{22}) \\ 0 & 0 \\ 0 & 0 \end{array} \right] \right. \right. \\ \left. \left. \begin{array}{cc} \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} & \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] \right. \right. \\ \left. \left. \left[ \begin{array}{cc} \frac{d}{dn_{31}} \sigma(n_{31}) & 0 \end{array} \right] & \left[ \begin{array}{cc} 0 & \frac{d}{dn_{32}} \sigma(n_{32}) \end{array} \right] \right] \right]$$

[56]: `Selem.diff(Nelem).subs(elemToSpecFunc)`

[56]:

$$\left[ \begin{array}{cc} \left[ \begin{array}{cc} \frac{\partial}{\partial \mathbf{n}_{11} (w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})} \sigma(\mathbf{n}_{11} (w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})) & 0 \\ 0 & 0 \\ 0 & 0 \end{array} & \left[ \begin{array}{cc} 0 & \frac{\partial}{\partial \mathbf{n}_{12} (w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13})} \sigma(\mathbf{n}_{12} (w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13})) \\ 0 & 0 \\ 0 & 0 \end{array} \right] \right. \\ \left. \begin{array}{cc} \left[ \begin{array}{cc} \frac{\partial}{\partial \mathbf{n}_{21} (w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23})} \sigma(\mathbf{n}_{21} (w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23})) & 0 \\ 0 & 0 \\ 0 & 0 \end{array} & \left[ \begin{array}{cc} 0 & \frac{\partial}{\partial \mathbf{n}_{22} (w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23})} \sigma(\mathbf{n}_{22} (w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23})) \\ 0 & 0 \\ 0 & 0 \end{array} \right] \right. \right. \\ \left. \left. \begin{array}{cc} \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} & \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] \right. \right. \\ \left. \left. \left[ \begin{array}{cc} \frac{\partial}{\partial \mathbf{n}_{31} (w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33})} \sigma(\mathbf{n}_{31} (w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33})) & 0 \end{array} \right] & \left[ \begin{array}{cc} 0 & \frac{\partial}{\partial \mathbf{n}_{32} (w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33})} \sigma(\mathbf{n}_{32} (w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33})) \end{array} \right] \right] \right]$$



[57]: # CAN even replace elements after have done an operation on them!!! replacing n\_21 \* 2 with the number 4.  
Sspec.subs({Nspec[0, 0]: 3}).replace(sigma, lambda x: 2 \* x).replace(Nspec[2, 1] \* 2, 4)

[57]: 
$$\begin{bmatrix} 6 & 2w_{12}x_{11} + 2w_{22}x_{12} + 2w_{32}x_{13} \\ 2w_{11}x_{21} + 2w_{21}x_{22} + 2w_{31}x_{23} & 2w_{12}x_{21} + 2w_{22}x_{22} + 2w_{32}x_{23} \\ 2w_{11}x_{31} + 2w_{21}x_{32} + 2w_{31}x_{33} & 4 \end{bmatrix}$$

[58]: lambda = Function("lambda")  
lambda\_ = lambda matrix : sum(matrix)  
  
vN(X, W)

[58]: 
$$\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{bmatrix}$$

[59]: vN(A, B)

[59]: 
$$\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{bmatrix}$$

[60]: L = lambda(S); L

[60]:  $\lambda(\sigma_{apply}(v(X, W)))$

[61]: Nelem

[61]: 
$$\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{bmatrix}$$

[62]: L.replace(n, vN)

[62]: 
$$\lambda\left(\sigma_{apply}\left(\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{bmatrix}\right)\right)$$

[63]: L.replace(n, vN).replace(sigmaApply, sigmaApply\_)

[63]: 
$$\lambda\left(\begin{bmatrix} \sigma(n_{11}) & \sigma(n_{12}) \\ \sigma(n_{21}) & \sigma(n_{22}) \\ \sigma(n_{31}) & \sigma(n_{32}) \end{bmatrix}\right)$$

[64]: L.replace(n, v)

[64]:  $\lambda(\sigma_{apply}(XW))$

[65]: L.replace(n, v).replace(sigmaApply, sigmaApply\_)

[65]:

$$\lambda((d \mapsto \sigma(d))_{\circ}(XW))$$

[66]:

```
L.subs({A:X, B:W}).replace(n, vL).replace(sigmaApply, sigmaApply_)
```

[66]:

$$\lambda\left(\begin{bmatrix}\sigma(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}) & \sigma(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13}) \\ \sigma(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}) & \sigma(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23}) \\ \sigma(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}) & \sigma(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33})\end{bmatrix}\right)$$

[67]:

```
L.replace(n, vN)
```

[67]:

$$\lambda\left(\sigma_{apply}\left(\begin{bmatrix}n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32}\end{bmatrix}\right)\right)$$

[68]:

```
L.replace(n, vN).subs({A:X, B:W}).replace(sigmaApply, sigmaApply_).replace(lambd, lambd_)
```

[68]:

$$\sigma(n_{11}) + \sigma(n_{12}) + \sigma(n_{21}) + \sigma(n_{22}) + \sigma(n_{31}) + \sigma(n_{32})$$

[69]:

```
from sympy import symbols, Derivative

x, y, r, t = symbols('x y r t') # r (radius), t (angle theta)
f, g, h = symbols('f g h', cls=Function)
h = g(f(x))

Derivative(h, f(x)).doit()
```

[69]:

$$\frac{d}{df(x)}g(f(x))$$

[70]:

```
# Never do this gives recursion ERROR (max depth exceeded)
# h = g(f(A))
# Derivative(h, A).doit()
```

[71]:

[71]:

```
from sympy.abc import a, b

Llower = lambd(sigmaApply(n(a, b)))
Llower
```

[71]:

$$\lambda(\sigma_{apply}(v(a, b)))$$

[72]:

```
Derivative(Llower, a).doit()
```

[72]:

$$\frac{\partial}{\partial \sigma_{apply}(v(a, b))}\lambda(\sigma_{apply}(v(a, b)))\frac{\partial}{\partial v(a, b)}\sigma_{apply}(v(a, b))\frac{\partial}{\partial a}v(a, b)$$

[73]:

[73]: # ### WAY 1: of substituting to differentiate with respect to expression:  
n\_ij = Function('n\_ij')  
n\_ij(A,B) # (N[0,0]); n\_ij

[73]:  $n_{ij}(X,W)$

[74]: n\_ij(A,B).args

[74]: (X, W)

[75]: # sigma(n\_ij).diff(n\_ij).replace(n\_ij, N[0,0]) # ERROR cannot deriv wi.r.t to the expression w11\*x11 + ...  
  
sigma(n\_ij(A,B)).diff(n\_ij(A,B))

[75]:  $\frac{\partial}{\partial n_{ij}(X,W)} \sigma(n_{ij}(X,W))$

[76]: sigma(n\_ij(\*X,\*W)).diff(X[0,0])

[76]:  $\frac{\partial}{\partial x_{11}} n_{ij}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}) \frac{\partial}{\partial n_{ij}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32})} \sigma(n_{ij}(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}))$

[77]: nab\_ij = n\_ij(A,B)  
sigma(nab\_ij).diff(nab\_ij)#.subs({nab\_ij : Nspec[0, 0]})

[77]:  $\frac{\partial}{\partial n_{ij}(X,W)} \sigma(n_{ij}(X,W))$

[78]: sigma(nab\_ij).diff(nab\_ij).subs({nab\_ij : Nspec[2, 1]})

[78]:  $\left. \frac{d}{d\xi} \sigma(\xi) \right|_{\xi=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}}$

[79]: sigma(nab\_ij).diff(nab\_ij).subs({nab\_ij : Nspec[2,1]}).subs({X[2,1]:77777})

[79]:  $\left. \frac{d}{d\xi} \sigma(\xi) \right|_{\xi=w_{12}x_{31}+77777w_{22}+w_{32}x_{33}}$

[80]: sigma(nab\_ij).diff(nab\_ij).subs({nab\_ij : 23}) # ERROR if using replace() since it says can't calc derivs w.r.t to the x\_11\*w\_11 + ...

[80]:  $\left. \frac{d}{d\xi} \sigma(\xi) \right|_{\xi=23}$

[81]: sigma(nab\_ij).diff(nab\_ij).subs({nab\_ij : Nspec[2,1]}).doit()

[81]:  $\left. \frac{d}{d\xi} \sigma(\xi) \right|_{\xi=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}}$

[82]: sigma(nab\_ij).subs({nab\_ij : Nspec[2,1]}).diff(X[2,1])

[82]:

$$\sigma(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33})$$

[83]: # Substituting the value of the function n\_ij first, and THEN differentiating with respect to something in that substitution. (X\_21)  
sigma(nab\_ij).subs({nab\_ij : Nspec[2,1]}).diff(X[2,1])

[83]:  $w_{22} \frac{d}{d\xi_1} \sigma(\xi_1) \Big|_{\xi_1=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}}$

[84]: Selem[2,1].subs({Nelem[2,1] : Nspec[2,1]}).diff(X[2,1])

[84]:  $w_{22} \frac{d}{d\xi_1} \sigma(\xi_1) \Big|_{\xi_1=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}}$

[85]: # ### WAY 2:  
n\_11 = Function('n\_11')(Nspec[0, 0]); n\_11

[85]:  $\mathbf{n}_{11}(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})$

[86]: sigma(n\_11)

[86]:  $\sigma(\mathbf{n}_{11}(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}))$

[87]: assert Nspec[0,0] == n\_11.args[0]  
sigma(n\_11).subs({n\_11 : n\_11.args[0]})

[87]:  $\sigma(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})$

[88]: sigma(n\_11).diff(n\_11) #.replace(n\_ij, n\_ij.args[0])

[88]:  $\frac{\partial}{\partial \mathbf{n}_{11}(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})} \sigma(\mathbf{n}_{11}(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}))$

[89]: sigma(n\_11).diff(n\_11).subs({n\_11 : n\_11.args[0]}).subs({X[0,0]:77777})

[89]:  $\frac{d}{d\xi} \sigma(\xi) \Big|_{\xi=77777w_{11}+w_{21}x_{12}+w_{31}x_{13}}$

[90]: sigma(n\_11).diff(n\_11).subs({n\_11 : n\_11.args[0]}).replace(n\_11.args[0], 23) # same as subs in this case

[90]:  $\frac{d}{d\xi} \sigma(\xi) \Big|_{\xi=23}$

[91]: sigma(n\_11).diff(X[0,0])

[91]:  $w_{11} \frac{\partial}{\partial \mathbf{n}_{11}(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})} \sigma(\mathbf{n}_{11}(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13})) \frac{d}{d\xi_1} \mathbf{n}_{11}(\xi_1) \Big|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}}$

[92]: id = Lambda(x, x)

sigma(n\_11).diff(X[0,0]).subs({n\_11 : id})

[92]:  $w_{11} \frac{d}{d\xi_1} \mathbf{n}_{11}(\xi_1) \Big|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}} \frac{d}{d\xi} \sigma(\xi) \Big|_{\xi=(x \mapsto x)}$

[93]: # NOTE: so I don't think WAY 2 is correct because here it doesn't simplify the derivative d n11 / d eps11, since this should equal 1 because now n11 = eps11.  $\square$   
  $\hookrightarrow$ Correct one is below (repeated from above)  
sigma(n\_11).diff(X[0,0]).subs({n\_11 : Nspec[0,0]})

[93]:  $w_{11} \frac{d}{d\xi_1} \mathbf{n}_{11}(\xi_1) \Big|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}} \frac{d}{d\xi} \sigma(\xi) \Big|_{\xi=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}}$

[94]: # CORRECT WAY 1  
sigma(n\_11).subs({n\_11 : Nspec[0,0]}).diff(X[0,0])

[94]:  $w_{11} \frac{d}{d\xi_1} \sigma(\xi_1) \Big|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}}$

[95]: # CORRECT WAY 2  
  
sigma(nab\_ij).subs({nab\_ij : Nspec[0,0]}).diff(X[0,0])

[95]:  $w_{11} \frac{d}{d\xi_1} \sigma(\xi_1) \Big|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}}$

[96]: # CORRECT WAY 3  
Selem[2,1].subs({Nelem[2,1] : Nspec[2,1]}).diff(X[2,1])

[96]:  $w_{22} \frac{d}{d\xi_1} \sigma(\xi_1) \Big|_{\xi_1=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}}$

[97]: sigma(n\_11) # WAY 1: sigma argument is already hardcoded

[97]:  $\sigma(\mathbf{n}_{11}(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}))$

[98]: sigma(nab\_ij) # Way 2: sigma argument is function of matrixsymbol (better than 1)

[98]:  $\sigma(\mathbf{n}_{ij}(X, W))$

[99]: Selem[2,1] # WAY 3: sigma argument is just symbol and we replace it as function with argument hardcoded only later. (better than 2)

[99]:  $\sigma(n_{32})$

[100]: L

[100]:  $\lambda(\sigma_{apply}(v(X, W)))$

[101]: assert Selem == S.replace(n, vN).replace(sigmaApply, sigmaApply\_)  
  
Selem

[101]: 
$$\begin{bmatrix} \sigma(n_{11}) & \sigma(n_{12}) \\ \sigma(n_{21}) & \sigma(n_{22}) \\ \sigma(n_{31}) & \sigma(n_{32}) \end{bmatrix}$$

[102]: `L.replace(n, vN).replace(sigmaApply, sigmaApply_)`

[102]: 
$$\lambda \left( \begin{bmatrix} \sigma(n_{11}) & \sigma(n_{12}) \\ \sigma(n_{21}) & \sigma(n_{22}) \\ \sigma(n_{31}) & \sigma(n_{32}) \end{bmatrix} \right)$$

[103]: `#L.replace(n, vN).replace(sigmaApply, sigmaApply_).diff(Nelem[0,0])`

[104]: `Lsum = L.replace(n, vN).replace(sigmaApply, sigmaApply_).replace(lambd, lambd_)  
Lsum`

[104]: 
$$\sigma(n_{11}) + \sigma(n_{12}) + \sigma(n_{21}) + \sigma(n_{22}) + \sigma(n_{31}) + \sigma(n_{32})$$

[105]: `Lsum.diff(Nelem)`

[105]: 
$$\begin{bmatrix} \frac{d}{dn_{11}}\sigma(n_{11}) & \frac{d}{dn_{12}}\sigma(n_{12}) \\ \frac{d}{dn_{21}}\sigma(n_{21}) & \frac{d}{dn_{22}}\sigma(n_{22}) \\ \frac{d}{dn_{31}}\sigma(n_{31}) & \frac{d}{dn_{32}}\sigma(n_{32}) \end{bmatrix}$$

[106]: `Lsum.subs(elemToSpec)#.diff(X[2,1])`

[106]: 
$$\sigma(w_{11}x_{11} + w_{21}x_{12} + w_{31}x_{13}) + \sigma(w_{11}x_{21} + w_{21}x_{22} + w_{31}x_{23}) + \sigma(w_{11}x_{31} + w_{21}x_{32} + w_{31}x_{33}) + \sigma(w_{12}x_{11} + w_{22}x_{12} + w_{32}x_{13}) + \sigma(w_{12}x_{21} + w_{22}x_{22} + w_{32}x_{23}) + \sigma(w_{12}x_{31} + w_{22}x_{32} + w_{32}x_{33})$$

[107]: `Lsum.subs(elemToSpec).diff(X)`

[107]: 
$$\begin{bmatrix} w_{11} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}} + w_{12} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{12}x_{11}+w_{22}x_{12}+w_{32}x_{13}} & w_{21} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}} + w_{22} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{12}x_{11}+w_{22}x_{12}+w_{32}x_{13}} & w_{31} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{11}x_{11}+w_{21}x_{12}+w_{31}x_{13}} + w_{32} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{12}x_{11}+w_{22}x_{12}+w_{32}x_{13}} \\ w_{11} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{11}x_{21}+w_{21}x_{22}+w_{31}x_{23}} + w_{12} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{12}x_{21}+w_{22}x_{22}+w_{32}x_{23}} & w_{21} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{11}x_{21}+w_{21}x_{22}+w_{31}x_{23}} + w_{22} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{12}x_{21}+w_{22}x_{22}+w_{32}x_{23}} & w_{31} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{11}x_{21}+w_{21}x_{22}+w_{31}x_{23}} + w_{32} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{12}x_{21}+w_{22}x_{22}+w_{32}x_{23}} \\ w_{11} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{11}x_{31}+w_{21}x_{32}+w_{31}x_{33}} + w_{12} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}} & w_{21} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{11}x_{31}+w_{21}x_{32}+w_{31}x_{33}} + w_{22} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}} & w_{31} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{11}x_{31}+w_{21}x_{32}+w_{31}x_{33}} + w_{32} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=w_{12}x_{31}+w_{22}x_{32}+w_{32}x_{33}} \end{bmatrix}$$

[108]: `specToElemD = {v : k for k, v in elemToSpecD.items()}  
  
Lsum.subs(elemToSpecD).diff(X).subs(specToElemD)`

[108]: 
$$\begin{bmatrix} w_{11} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{11}} + w_{12} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{12}} & w_{21} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{11}} + w_{22} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{12}} & w_{31} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{11}} + w_{32} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{12}} \\ w_{11} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{21}} + w_{12} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{22}} & w_{21} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{21}} + w_{22} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{22}} & w_{31} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{21}} + w_{32} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{22}} \\ w_{11} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{31}} + w_{12} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{32}} & w_{21} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{31}} + w_{22} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{32}} & w_{31} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{31}} + w_{32} \frac{d}{d\xi_1}\sigma(\xi_1) \Big|_{\xi_1=n_{32}} \end{bmatrix}$$