STA 326 2.0 Programming and Data Analysis with

Generating Random Numbers Using Inverse Transform Method

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1. Probability distribution functions in R to generate random numbers

rbeta	beta distribution	**lnorm**	log-normal distribution
rbinom	binomial distribution	**multinom**	multinomial distribution
rcauchy	Cauchy distribution	**nbinom**	negative binomial distribution
rchisq	chi-squared distribution	**rnorm**	normal distribution
rexp	exponential distribution	**rpois**	Poisson distribution
rf	F distribution	**rt**	Student's t distribution
rgamma	gamma distribution	**runif**	uniform distribution
rgeom	geometric distribution	**weibull**	Weibull distribution
rhyper	hyper-geometric distribution		

There are other methods of generating random numbers from a particular distribution. In this lectorial we will discuss **Inverse Transform Method**.

2. Inverse transform method

Theorem 1: Probability Integral Transformation

> Let X have continuous cdf $F_X(x)$ and define the random variable Y as $Y = F_X(X)$. Then Y is uniformly distributed on (0, 1), that is, $P(Y \leq y) = y, \ 0 <$ y < 1.

Let's try to understand the theorem using an example.

Useful results to prove the theorem.

Result 1:

If F_X is strictly increasing, then F_X^{-1} is well defined by

$$F_X^{-1}(y) = x \Leftrightarrow F_X(x) = y.$$

If F_X is constant on some interval, then F_X^{-1} is not well defined by the above equation. To avoid this problem we define $F_X^{-1}(y)$ for $0 < y < 1$ by

$$F_X^{-1}(y) = inf\{x : F_X(x) \geq y\}.$$

Result 2:

If F_X is **strictly** increasing, then it is true that

$$F_X^{-1}(F_X(x)) = x.$$

Proof of Theorem 1:

For $Y = F_X(X)$ we have, for $0 < y < 1$, We can use Theorem 1 to generate random numbers from a particular distribution.

3. Steps in deriving random numbers using integral transformation method

1. Derive the cumulative distribution function of $f_X(x)$
2. Derive the inverse function $F_X^{-1}(u)$.
3. Write a function to generate random numbers.
   - Generate u from $Uniform(0,1)$.
   - compute $x = F_X^{-1}(u)$.

Example 1

Write a function to generate n random numbers from the distribution with density $f_X(x) = 3x^2,\ 0 <$ x < 1.

Step 1: Find the cumulative distribution function of $f_X(x)$,

$$F_X(x) = x^3 \text{ for } 0 < x < 1$$

Step 2: Next we need to compute $F_X^{-1}(u)$,

$$F_X^{-1}(u) = u^{\frac{1}{3}}.$$

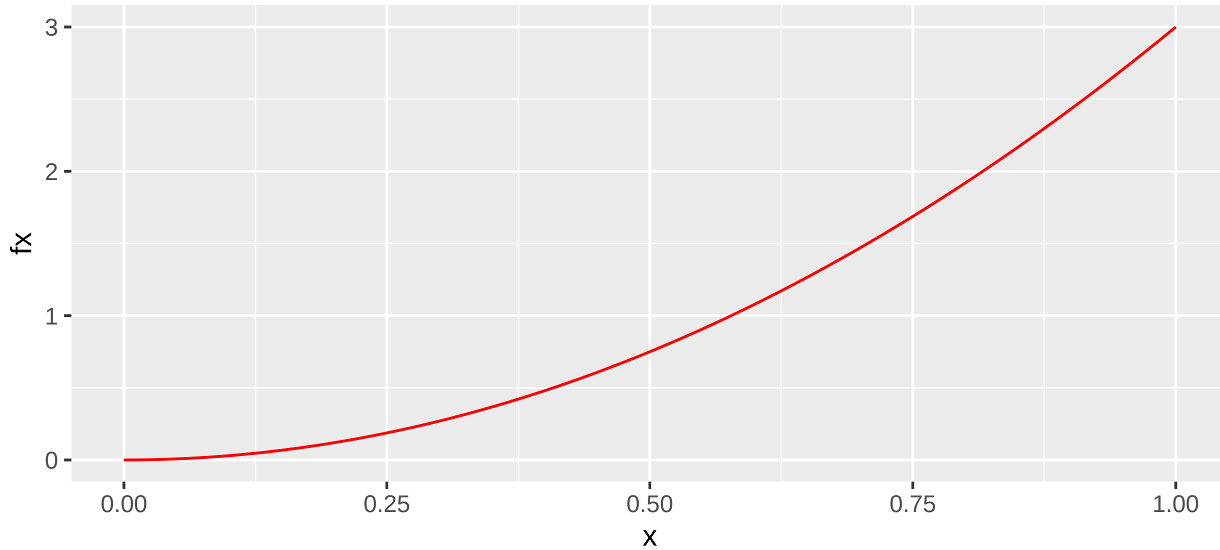
Step 3: R function

```
generate_it <- function(n){
  # Generate random numbers
  u <- runif(n)
  xgen <- u^(1/3)
  xgen

}

set.seed(2020)
generate_it(10)
```

```
## [1] 0.8648611 0.7332437 0.8520145 0.7812795 0.5143788 0.4069300 0.5054766
## [8] 0.7325562 0.1372012 0.8527963
```

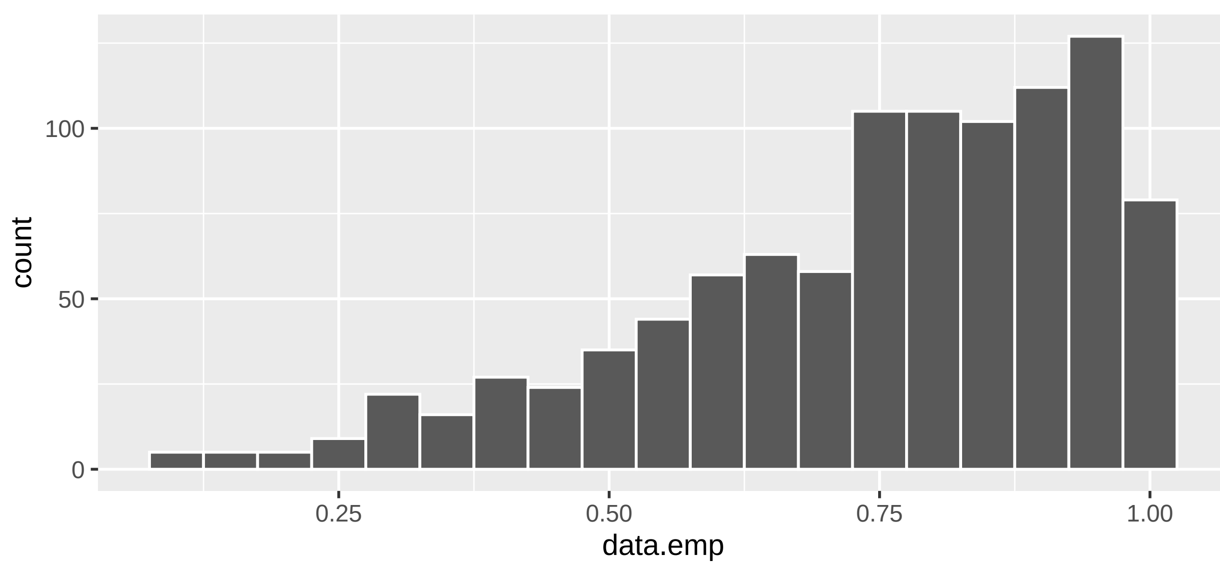
Visualisation of theoretical distribution

```r
library(tidyverse)
# Theoretical distribution values
theoretical.df <- tibble(x = seq(0, 1, 0.01), fx = 3*x^2)
ggplot(theoretical.df, aes(x = x, y = fx)) +
  geom_line(col = "red")
```



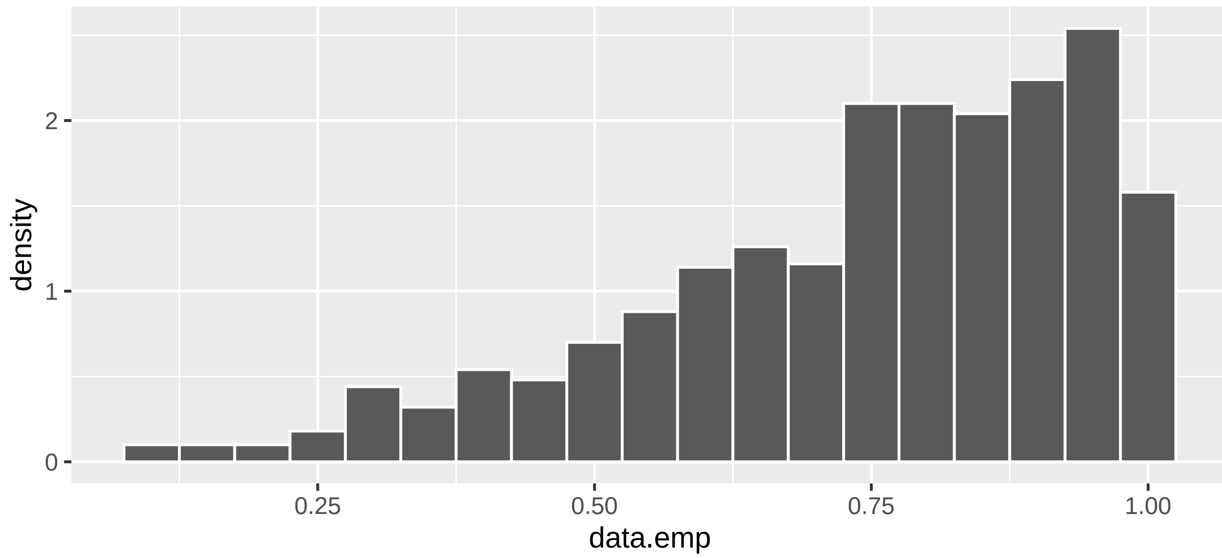
Visualize empirical distribution - counts

```r
empirical.df <- data.frame(data.emp = generate_it(1000))
# Plot empirical distribution - counts
ggplot(empirical.df, aes(x = data.emp))+
  geom_histogram(col = "white", binwidth = 0.05)
```



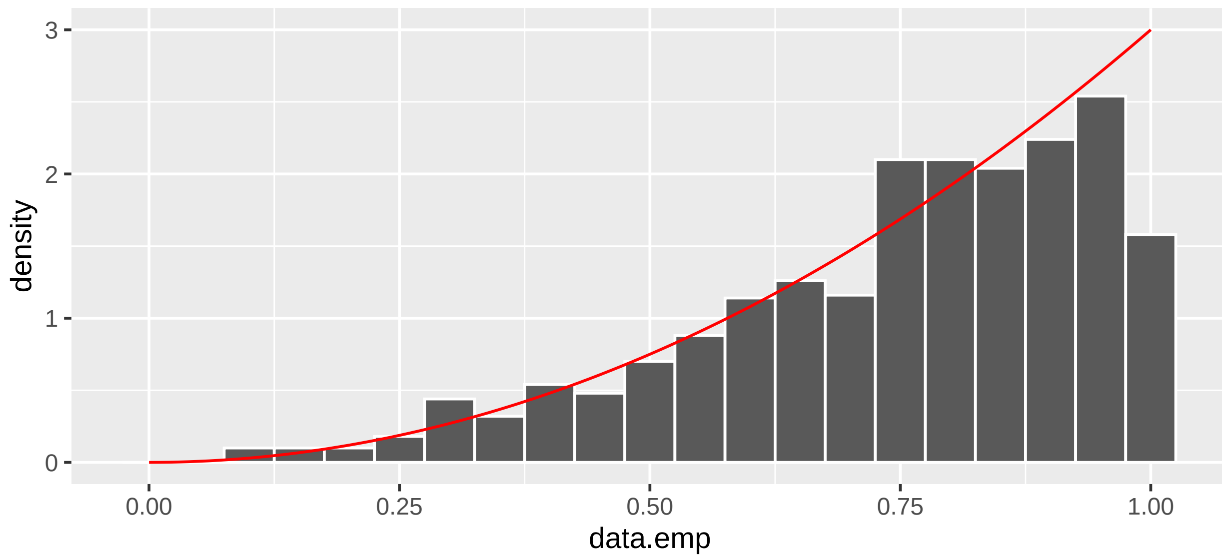
Visualize empirical distribution - density

```r
ggplot(empirical.df, aes(x = data.emp, y=..density..)) +
  geom_histogram(col = "white", binwidth = 0.05)
```



Visualize theoretical distribution and empirical distribution together

```r
ggplot(empirical.df, aes(x = data.emp, y=..density..)) +
  geom_histogram(col = "white", binwidth = 0.05) +
  geom_line(data = theoretical.df, aes(x = x, y = fx), color = 'red')
```



Function to generate random numbers and visualize theoretical and empirical distributions

```
generate_it_dist <- function(n){
  # Generate random numbers
  u <- runif(n)
  xgen <- u^(1/3)
  xgen

  # values for empirical distribution
  empirical.df <- data.frame(xgen=xgen)

  # values for the theoretical distribution
  theoretical.df <- tibble(x = seq(0, 1, 0.01),
  fx = 3*x^2)

  # arrange values and plot into a list
  list(
    xgen,
  ggplot2::ggplot(empirical.df, aes(x=xgen, y=..density..)) +
    geom_histogram(col="white", binwidth = 0.01) +
  geom_line(data = theoretical.df, aes(x = x, y = fx), color = 'red') )
}
```

Run the following codes and check the outputs.

```
# Sample size 10
generate_it_dist(10)

# Sample size 100
n100 <- generate_it_dist(100)
n100
n100[[1]]
n100[[2]]

# Sample size 10000
n10000 <- generate_it_dist(10000)
n10000[[2]]
```

Example 2

i) Write a function to generate random numbers from the $Exponential(\lambda)$ distribution using the inverse transformation method.

```
exp_generator <- function(n, lambda){
  u <- runif(n, 0, 1)
  exp.values <- -log(1-u)/lambda
    exp.values
}
```

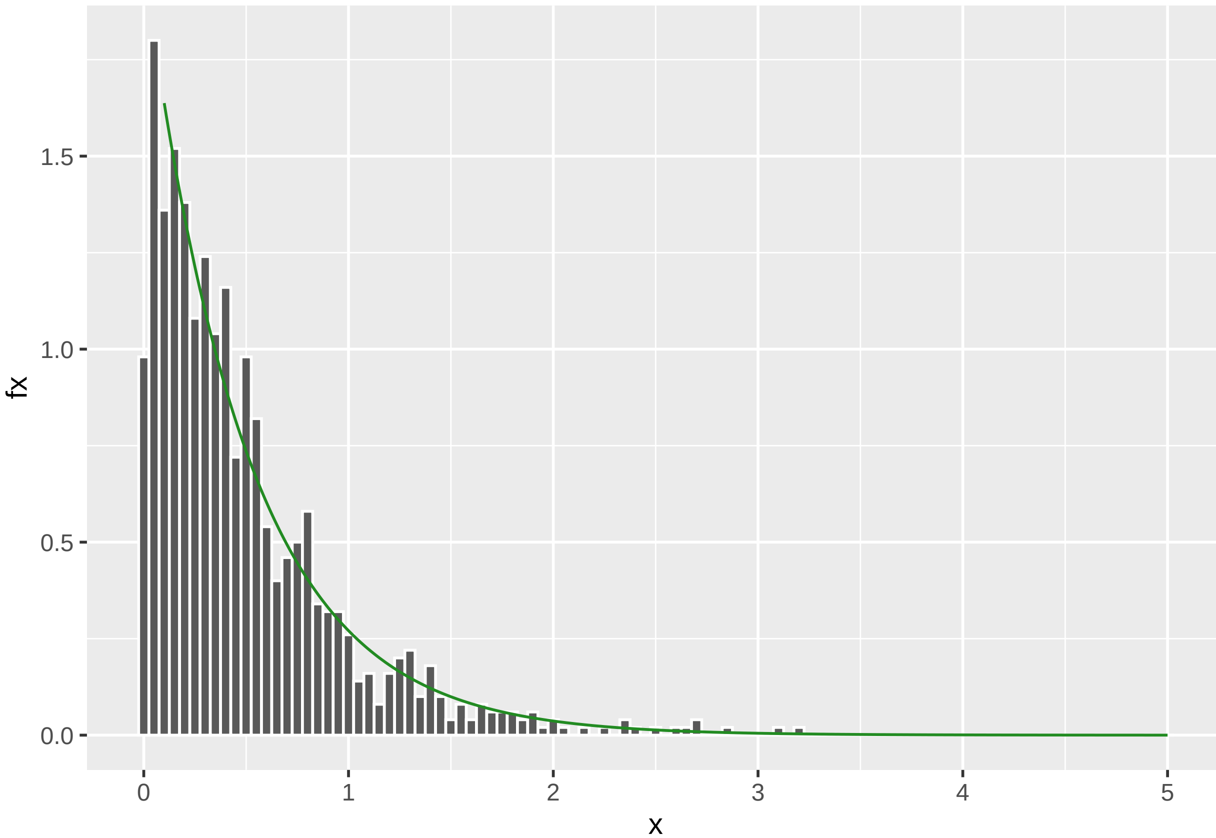
ii) Generate 1000 random numbers from the $Exponential(2)$ distribution.

```
set.seed(111)
exp2 <- exp_generator(1000, 2)
```

```
exp2
```

iii) Graph the density histogram of the sample with the $Exponential(2)$ density superimposed for comparison.

```
# values for empirical distribution
empirical.df.exp <- data.frame(xgen=exp2)
# values for the theoretical distribution
theoretical.df <- tibble(x = seq(0.1, 5, 0.01),
  fx = 2*exp(-2*x))

ggplot(empirical.df.exp, aes(x = xgen, y=..density..)) +
  geom_histogram(col = "white", binwidth = 0.05) +
  geom_line(data = theoretical.df, aes(x = x, y = fx), color = "forestgreen") +
  labs(x="x", y="fx")
```


1.5
1.0
0.5
0.0
fx
0
1
2
3
4
5
x