The Method of Monte Carlo

STA 326 2.0 Programming and Data Analysis with R

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2.1 Estimate Area

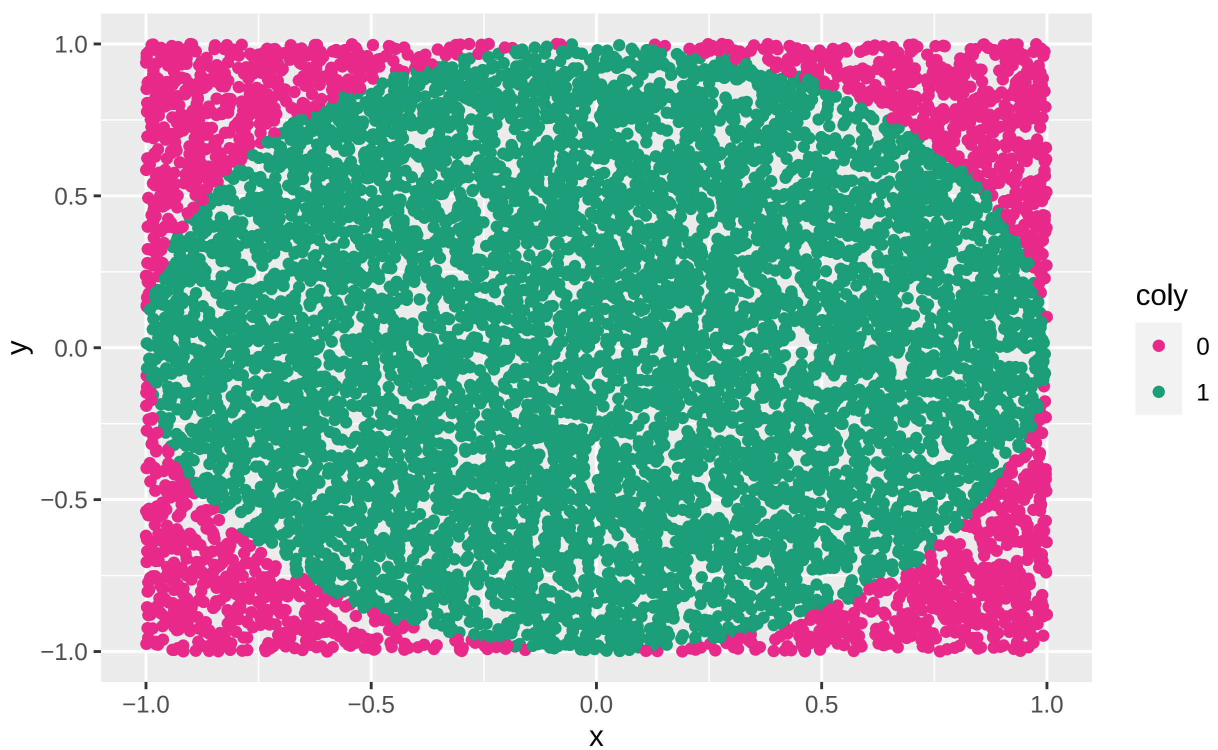
Example 2.1.1: Approximating Pi = 3.14616

$$\frac{A_{circle}}{A_{square}} = \frac{\pi r^2}{4r^2}$$

Equation of the unit circle center around 0: $x^2 + y^2 = r^2$

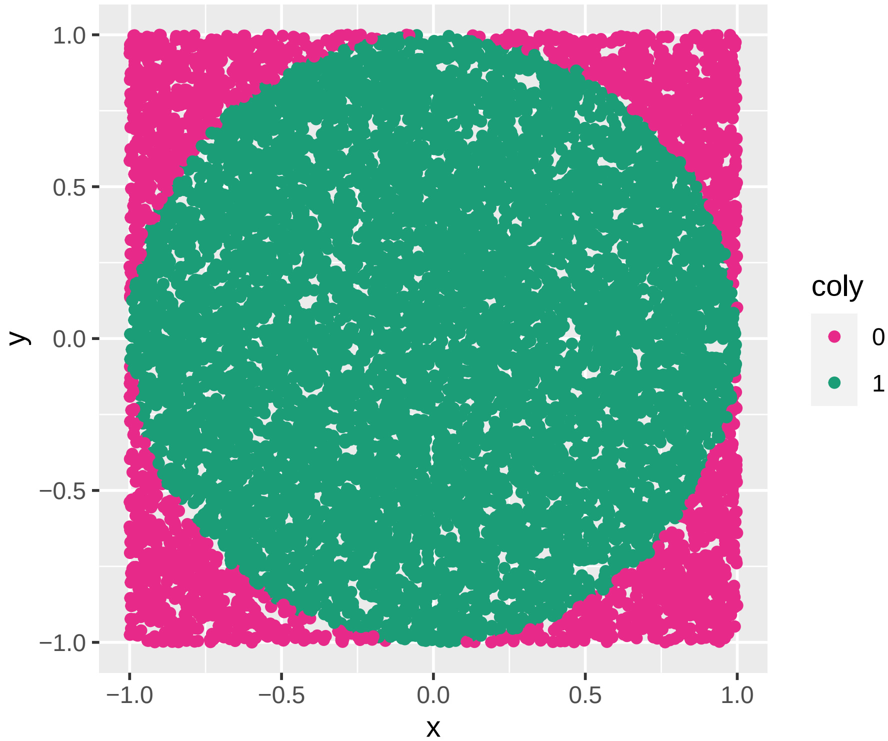
```
library(tidyverse)
x <- runif(10000, -1, 1)
y <- runif(10000, -1, 1)
fx <- x^2 + y^2
coly <-  ifelse(fx <= 1, 1, 0)
coly <- as.factor(coly)
pidf <- data.frame(x=x, y=y, coly=coly)
```

```
# without coord_equal()
ggplot(pidf, aes(x=x, y=y, col=coly)) + geom_point()  +
   scale_colour_manual(values = c("#e7298a", "#1b9e77"))
```




```
# with coord_equal()
ggplot(pidf, aes(x=x, y=y, col=coly)) + geom_point() +
```

```
  scale_colour_manual(values = c("#e7298a", "#1b9e77")) +
  coord_equal()
```




```
compute_pi_mc_sim <- function(n){
  x <- runif(n, -1, 1)
  y <- runif(n, -1, 1)
  count <- 0
  for(i in 1:n){
    if(x[i]^2 + y[i]^2 < 1 | x[i]^2 + y[i]^2 == 1){
      count = count + 1

    } else {
      count
    }

  }

 pi <- (count/n) * 4
 pi

}
compute_pi_mc_sim(100)
```

```
[1] 2.92
```

```
compute_pi_mc_sim(1000)
```

```
[1] 3.248
```

```r
compute_pi_mc_sim(10000)
```

```
[1] 3.14
```

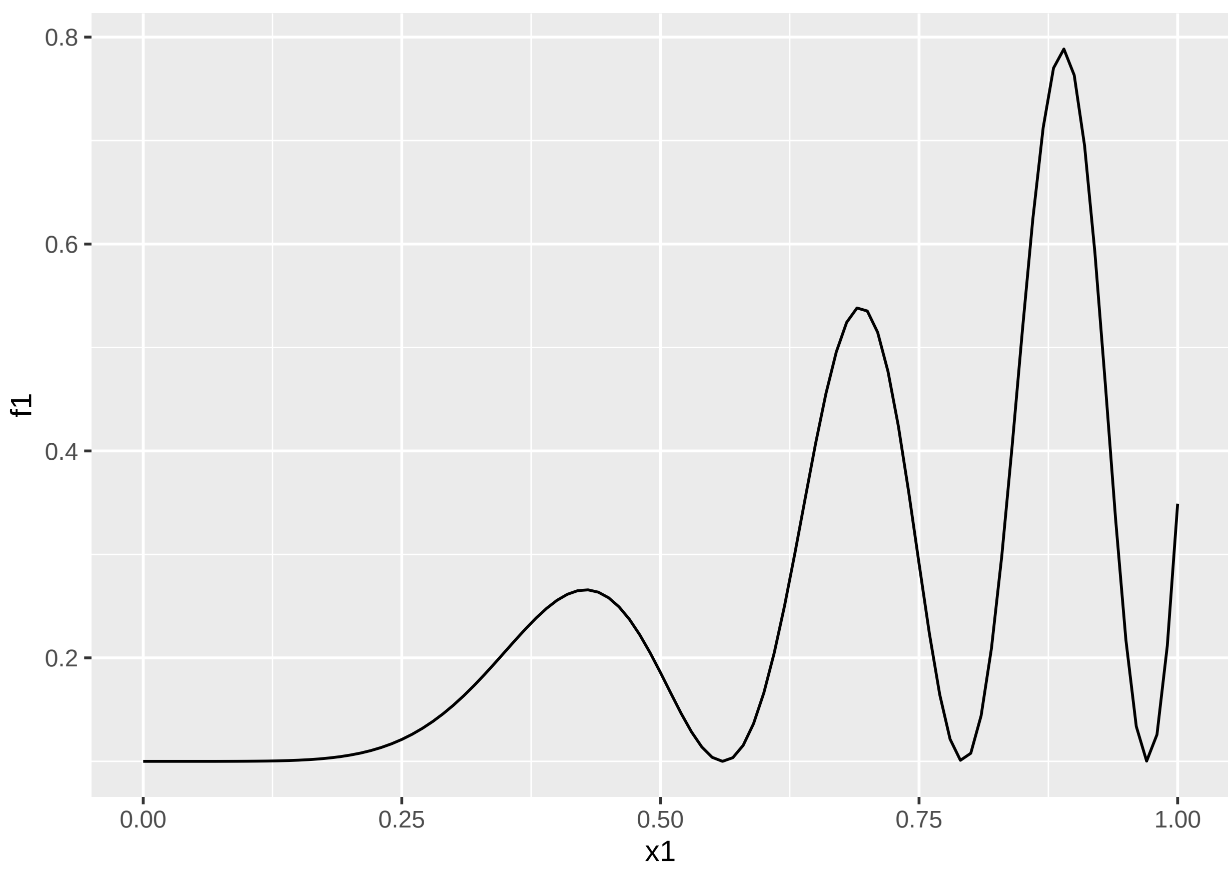
Write a function to approximate π using Monte Carlo simulations.

Example 2.1.2: YOUR TURN

Estimate area under the curve withing $x \in [0, 1]$ using Monte Carlo simulation approach.

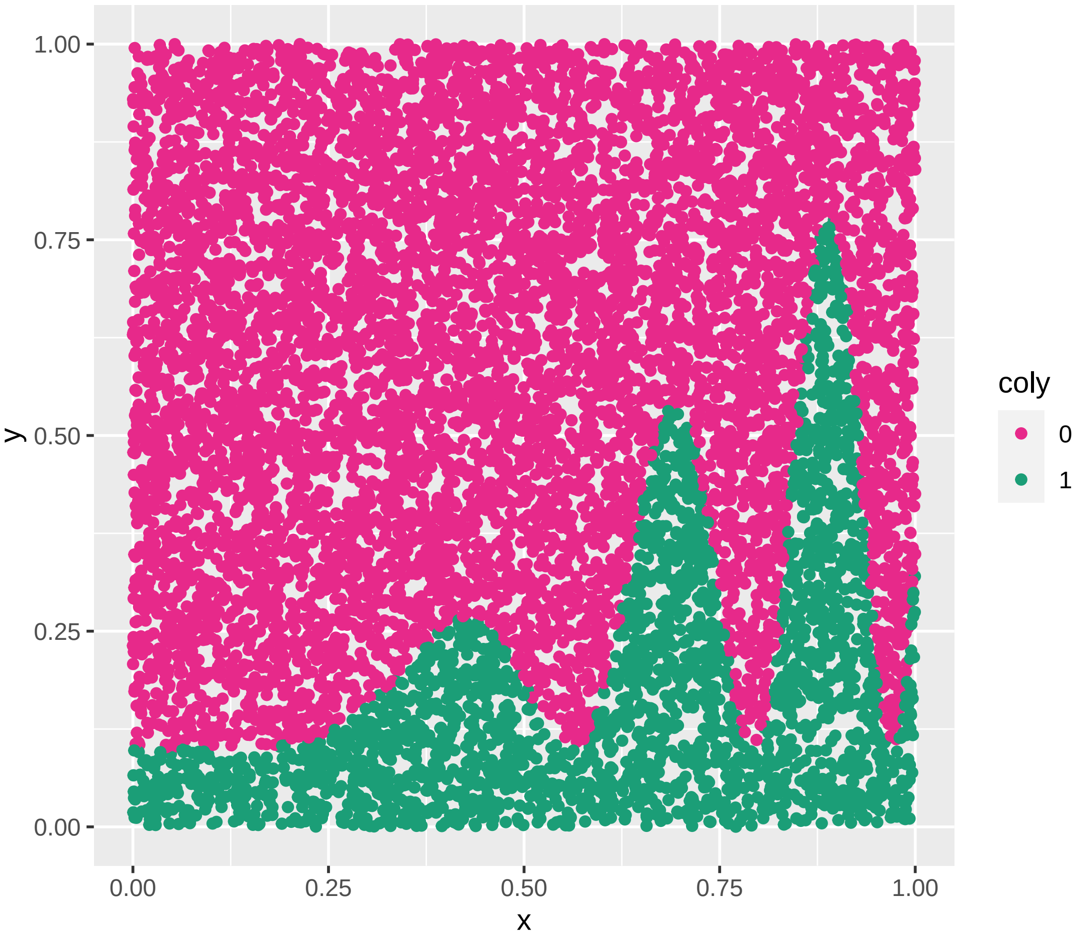
$f(x) = sin(10x)^{2sin(x)}x + 0.1$

```r
x1 <- seq(0, 1, 0.01)
f1 <- ((sin(10*x1^2))^2*sin(x1))*x1+0.1
df1 <- data.frame(x=x1, y=f1)
ggplot(df1, aes(x=x1, y=f1))+geom_line() + coord_equal()
```




```r
set.seed(2020)
x = runif(10000, min =0 , max =1 )
y = runif(10000, min =0 , max =1 )
fx <- ((sin(10*x^2))^2*sin(x))*x+0.1
```

```
coly <-  ifelse(y < fx, 1, 0)
coly <- as.factor(coly)
df2 <- data.frame(x=x, y=y, coly=coly)
ggplot(df2, aes(x=x, y=y, col=coly)) + geom_point() +   scale_colour_manual(values = c("#e7298a", "#1b9
```



2.2 Monte Carlo Integration

Suppose we want to calculate the integral $\int_a^b g(x)dx$ for a continuous function g over the closed and bounded interval [a, b]. If the anti-derivative of g does not exist, then numerical integration is in order. A simple numerical technique is the method of Monte Carlo. We can write the integral as

$$\int_a^b g(x)dx = (b-a)\int_a^b \frac{1}{a-b}dx = (b-a)E[g(X)],$$

where X has the $uniform(a,b)$ distribution.

To compute the estimated value first a set of random numbers X_1, X_2, X_n of size n is, then compute $Y_i = (b-a)g(X_i)$. Then $\bar{Y}$ is a consistent estimate of $\int_a^b g(x)dx$.

Example 2.2.1: Approximating Pi using Monte Carlo Integration.

Let $g(x) = 4\sqrt{1-x^2}$ for $0 < x < 1$. Then,

$$\pi = \int_0^1 g(x)dx = E[g(X)],$$

where X has the $uniform(0,1)$ distribution.

Write an R function to obtain point estimate and 95% confidence interval for *pi* using the sample sizes, 100, 1000, 10000 , 100000 and fill the blanks in the following table.

```
pi_mi <- function(n){

  random.uniform <- runif(n)
  sample_gx_values <- 4*sqrt(1-random.uniform^2)
  # point estimate
  y_bar <- mean(sample_gx_values)
  # standard error
  se <- sqrt(var(sample_gx_values)/n)
  # 95% confidence interval
  interval_estimate_lower <- y_bar - (1.96 * se)
  interval_estimate_upper <- y_bar + (1.96 * se)
  #output
  tibble::tibble(pi=y_bar,
                 CI.lower=interval_estimate_lower,
                 CI.upper=interval_estimate_upper)

}

pi_mi(100)
```

```
# A tibble: 1 x 3
     pi CI.lower CI.upper
  <dbl>    <dbl>    <dbl>
1  2.97     2.78     3.16
```

```
pi_mi(1000)
```

```
# A tibble: 1 x 3
     pi CI.lower CI.upper
  <dbl>    <dbl>    <dbl>
1  3.15     3.10     3.21
```

```
pi_mi(10000)
```

```
# A tibble: 1 x 3
     pi CI.lower CI.upper
  <dbl>    <dbl>    <dbl>
1  3.14     3.12     3.15
```

```r
pi_mi(100000)
```

```
# A tibble: 1 x 3
     pi CI.lower CI.upper
  <dbl>    <dbl>    <dbl>
1  3.15     3.14     3.15
```

YOUR TURN:

Example 2.2.2: YOUR TURN

Write an R function to estimate $log 2$ using Monte Carlo Integration method. Obtain a point estimate, and 95% confidence interval for estimate using 10000 simulations and compare it to the true value.

Hint

$log\ 2 = \int_0^1 \frac{1}{x+1} dx.$

Example 2.2.3: YOUR TURN

2.3 Hypothesis testing based on simulations

Steps:

Let N be the number of simulations.

1. Set $k = 1$ and $I = 0$.
2. Simulate a random sample of size n from the distribution X and compute the test statistic t.
3. If test statistic $>$ critical value, increase I by 1.
4. Repeat the process until $k = N$
5. Calculate the p-value