# Solution ark #7.
# Variance estimation

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## Exercise 1

**(R code available)** Use the data in srs30.dat, which includes $n = 30$ units selected with simple random sampling from an artificial population of size $N = 100$.

1. Use the jackknife to estimate $V(\bar{y})$, and verify that $\hat{V}_{JK}(\bar{y}) = s^2/30$ for this data. What are the jackknife weights for jackknife replicate $j$?

**Solution:**
As SRS of size $n = 30$ from an artificial population of size $N = 100$. **a.** Use the jackknife to estimate $V(\bar{y})$ for srs30.dat. The customary jackknife estimator is given by

$$\hat{V}_{JK}(\bar{y}) = \frac{n-1}{n}\sum_{j\in s}\Big(\bar{y}_{-j} - \frac{1}{n}\sum_{j\in s}\bar{y}_{-j}\Big)^2, \quad \bar{y}_{-j} = \sum_{i\in s} w_{i(j)} y_i/N,$$

where $w_{i(j)}$ are the jackknife weights for jackknife replicate $j$:

$$w_{i(j)} = \begin{cases} 0 & \text{if} \quad i = j \\ \frac{30}{29} w_i & \text{if} \quad i \neq j \end{cases}.$$

We have $w_i = 100/30$ under SRS. Hence, $w_{i(j)} = (30/29)(100/30) = 3.44828$ for $i \neq j$. We have $\bar{y}_{-j} = \sum_{i\neq j\in s} y_i/(n-1)$, as $w_{i(j)} = N/(n-1)$ for $i \neq j$, and

$$\frac{1}{n}\sum_{j\in s}\bar{y}_{-j} = \frac{1}{n}\frac{1}{n-1}\sum_{j\in s}\sum_{i\neq j\in s} y_i = \frac{1}{n}\frac{1}{n-1}\sum_{j\in s}(n-1)y_j = \bar{y}.$$

$\bar{y}_{-j}$ can be can re-written as follows.

$$\bar{y}_{-j} = \frac{1}{n-1}\Big(\sum_{i\in s} y_i - y_j\Big) = \frac{1}{n-1}(n\bar{y} - y_j) = \bar{y} - \frac{1}{n-1}(y_j - \bar{y}).$$

Thus we have

$$\hat{V}_{JK}(\bar{y}) = \frac{n-1}{n}\frac{1}{(n-1)^2}\sum_{j\in s}(y_j - \bar{y})^2 = \frac{s_y^2}{n}.$$

For this data, we have $\bar{y} = 8.233333$, and $\bar{y}_{-1} = 8.241379$, $\bar{y}_{-2} = 8.344828, \cdots, \bar{y}_{-30} = 8.206897$.

$$\hat{V}_{JK}(\bar{y}) = \frac{30}{29}0.5509711 = 0.5326054,$$

$$s_y^2/30 = 15.97816/30 = 0.5326054\cdot$$

Using the finite population correction, we obtain

$$\left(1 - \frac{n}{N}\right)\hat{V}_{JK}(\bar{y}) = \left(1 - \frac{30}{100}\right)0.5326054 = 0.3728238\cdot$$

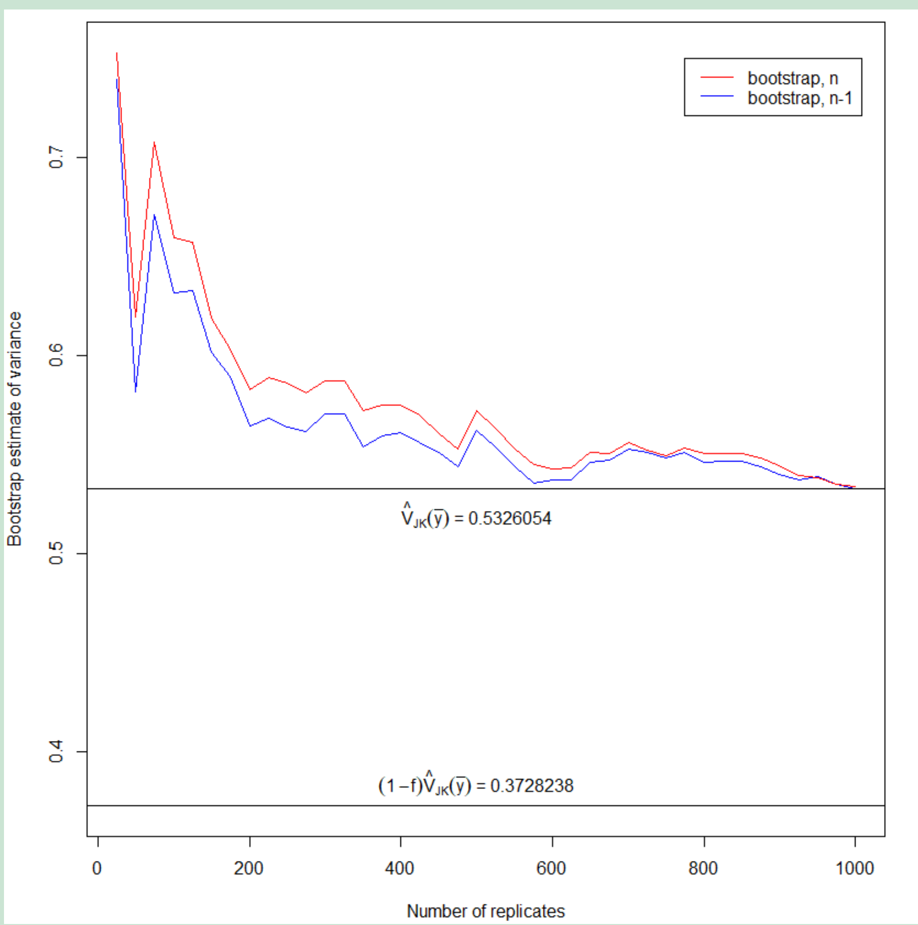
2. Find also the bootstrap estimate of $V(\bar{y})$.

**Solution:** An estimate of the population mean $\bar{y}$ at the $b$th bootstrap replicate is found by, respectively with the naive bootstrap and the rescaling bootstrap with $m = n - 1$

$$\bar{y}^*_{r,MC} = \frac{\sum_{i\in s} \tau_{ri} y_i}{n}, \quad \bar{y}^*_r = \frac{\sum_{i\in s} w^*_{ri} y_i}{\sum_{i\in s} w^*_{ri}}, \quad w^*_{ri} = \tau^*_{ri}\left(\frac{n}{n-1}\right)\left(\frac{N}{n}\right)\cdot$$

Here, $\bar{y}^*_{r,MC} = \bar{y}^*_r$, as the rescaling bootstrap with $m = n - 1$ reduces to the naive bootstrap with replicates of size of $n - 1$.

$$\hat{V}_{boot}(\bar{y}) = \frac{1}{R-1}\sum_{r=1}^{R}\left(\bar{y}^*_r - \frac{1}{R}\sum_{r=1}^{R}\bar{y}^*_r\right)^2\cdot$$



# Exercise 2

**(R code available)** The file agsrs.dat contains data from an SRS of 300 of the 3 078 counties. Let $y_i$ be total acreage of farms in county $i$ in 1992 and $x_i$ be total acreage of farms in county $i$ in 1987. Use the jackknife and the bootstrap to estimate the variance of the ratio estimator $\hat{B}_r = \bar{y}/\bar{x}$. How do they compare with the linearization estimator?

**Solution:**

An SRS of $n = 300$ of the $N = 3\,078$ counties. $y_i$: total acreage of farms in county $i$ in 1992; $x_i$: total acreage of farms in county $i$ in 1987. Estimate $V(\hat{B})$ with $\hat{B} = \bar{y}/\bar{x}$ using the resampling methods. We have

$$\hat{B} = 297\,897/301\,953.7 = 0.9865652\cdot$$

For the jackknife, we have

$$\hat{B}_{-j} = \frac{\bar{y}_{-j}}{\bar{x}_{-j}}, \quad \bar{y}_{-j} = \frac{1}{n-1}\sum_{i \neq j \in s} y_i,\ \bar{x}_{-j} = \frac{1}{n-1}\sum_{i \neq j \in s} x_i,$$

$$\frac{1}{n}\sum_{j \in s} \hat{B}_{-j} = 0.9865651\cdot$$

The jackknife variance estimator provides

$$\hat{V}_{JK}(\hat{B}) = \frac{299}{300}\sum_{j \in s}\left(\hat{B}_{-j} - 0.9865651\right)^2 = 3.707 \times 10^{-05},$$

$$\left(1 - \frac{300}{3\,078}\right)\hat{V}_{JK}(\hat{B}) = 3.346 \times 10^{-05}\cdot$$

The ratio estimate at the $r$th bootstrap replicate is found by

$$\hat{B}^*_r = \frac{\bar{y}^*_r}{\bar{x}^*_r}, \quad \bar{y}^*_r = \frac{\sum_{i \in s} \tau_{ri} y_i}{m},\ \bar{x}^*_r = \frac{\sum_{i \in s} \tau_{ri} x_i}{m}, \quad m = n, n-1\cdot$$

$$\hat{V}_{boot}(\hat{B}) = \frac{1}{R-1}\sum_{r=1}^{R}\left(\hat{B}^*_r - \frac{1}{R}\sum_{r=1}^{R}\hat{B}^*_r\right)^2\cdot$$

The variance estimator with the linearization method,

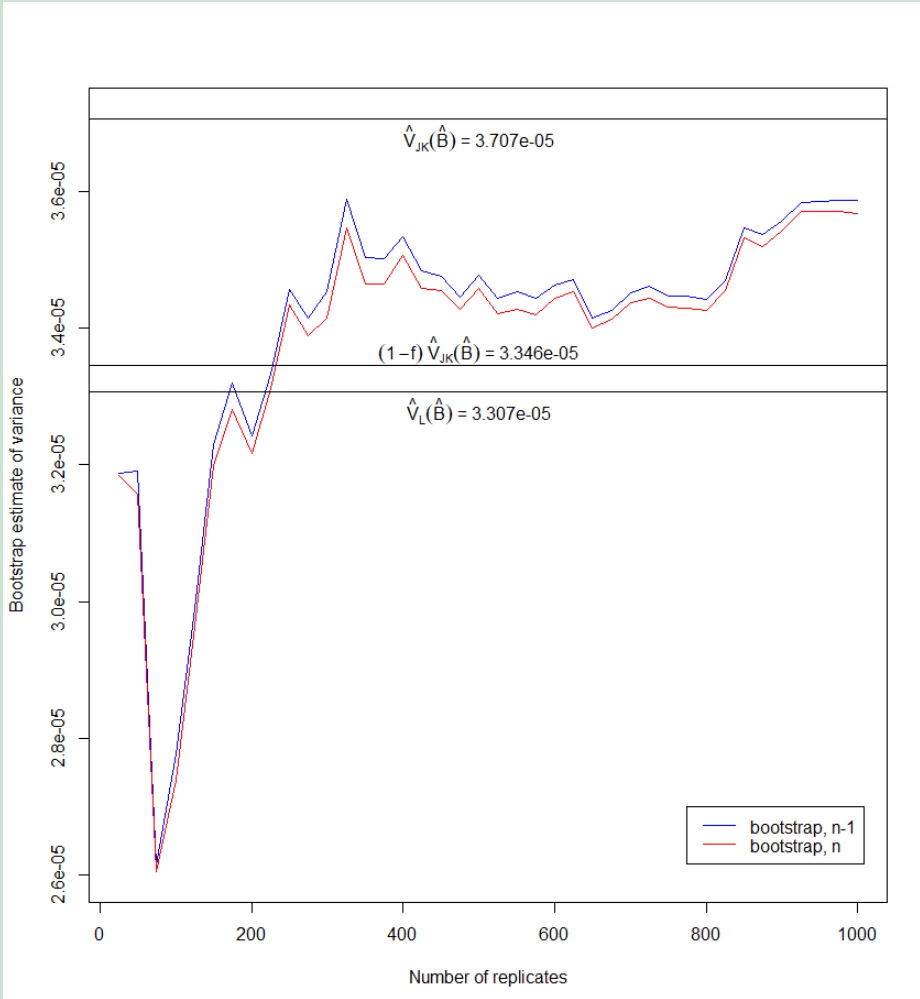
$$\hat{V}_L(\hat{B}) = (1-f)\frac{1}{\bar{x}^2}\frac{s_e^2}{n},$$

where

$$s_e^2 = \frac{1}{n-1}\sum_{i \in s}(e_i - \bar{e})^2, \quad e_i = y_i - \hat{B}x_i,$$

provides

$$\hat{V}_L(\hat{B}) = \left(1 - \frac{300}{3\,078}\right)\frac{1}{(301\,953.7)^2}\frac{1\,002\,179\,462}{300} = 3.307 \times 10^{-05}\cdot$$




# Exercise 3

**(R code available)** Foresters want to estimate the average age of trees in a stand. Determining age is cumbersome, because one needs to count the tree rings on a core taken from the tree. In general, though, the older the tree, the larger the diameter, and diameter is easy to measure. The foresters measure the diameter of all 1 132 trees and find that the population mean equals 10.3. They then randomly select 20 trees for age measurement.

| Tree No. | Diameter, $x$ | Age, $y$ | Tree No. | Diameter, $x$ | Age, $y$ |
|---|---|---|---|---|---|
| 1 | 12.0 | 125 | 11 | 5.7 | 61 |
| 2 | 11.4 | 119 | 12 | 8.0 | 80 |
| 3 | 7.9 | 83 | 13 | 10.3 | 114 |
| 4 | 9.0 | 85 | 14 | 12.0 | 147 |
| 5 | 10.5 | 99 | 15 | 9.2 | 122 |
| 6 | 7.9 | 117 | 16 | 8.5 | 106 |
| 7 | 7.3 | 69 | 17 | 7.0 | 82 |
| 8 | 10.2 | 133 | 18 | 10.7 | 88 |
| 9 | 11.7 | 154 | 19 | 9.3 | 97 |
| 10 | 11.3 | 168 | 20 | 8.2 | 99 |

Using the jackknife and the bootstrap, estimate the standard error for the regression estimate of the population age of trees in a stand. How do the jackknife and bootstrap compare with the standard error calculated using linearization methods?

**Solution:**

Survey of trees, SRS. $N = 1\,132$ and $n = 20$. $y_i$: age of the $i$th tree: $x_i$: the diameter of the $i$th tree. The population average: $\bar{x} = 10.3$. Estimate $V(\bar{y}_{reg})$ using the resampling methods.

We have

$$\hat{B}_1 = \frac{\sum_{i\in s}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i\in s}(x_i - \bar{x})^2} = 12.24966,$$

$$\hat{B}_0 = \bar{y} - \hat{B}_1\bar{x} = 107.4 - 12.24966(9.405) = -7.808087,$$

$$\bar{y}_{reg} = \hat{B}_0 + \hat{B}_1\bar{X} = -7.808087 + 12.24966(10.3) = 118.3634\cdot$$

For the jackknife, we have

$$\bar{y}_{reg,(j)} = \hat{B}_{0(j)} + \hat{B}_{1(j)}\bar{X},$$

$$\hat{B}_{0(j)} = \bar{y}_{-j} - \hat{B}_{1(j)}\bar{x}_{-j}, \quad \hat{B}_{1(j)} = \frac{\sum_{i\neq j\in s}(x_i - \bar{x}_{-j})(y_i - \bar{y}_{-j})}{\sum_{i\neq j\in s}(x_i - \bar{x}_{-j})^2}\cdot$$

$$\frac{1}{n}\sum_{j\in s}\bar{y}_{reg(j)} = 118.3579\cdot$$

$$\widehat{SE}_{JK}(\bar{y}_{reg}) = \sqrt{\frac{19}{20}\sum_{j\in s}(\bar{y}_{reg(j)} - 118.3579)^2} = 5.384611\cdot$$

$$\sqrt{1 - \frac{20}{1\,103}}\widehat{SE}_{JK}(\bar{y}_{reg}) = 0.9911267(5.384611) = 5.336832\cdot$$

The regression estimate of the mean at the $r$th bootstrap replicate is given by

$$\bar{y}^*_{reg(r)} = \hat{B}^*_{0(r)} - \hat{B}^*_{1(r)}\bar{x},$$

$$\hat{B}^*_{0(r)} = \bar{y}_r - \hat{B}^*_{1(r)}\bar{x}_r,$$

$$\hat{B}^*_{1(r)} = \frac{\sum_{i\in s}\tau_{ri}(x_i - \bar{x}_r)(y_i - \bar{y}_r)}{\sum_{i\in s}\tau_{ri}(x_i - \bar{x}_r)^2}\cdot$$

$$\widehat{SE}_{boot}(\bar{y}_{reg}) = \sqrt{\frac{1}{R-1}\sum_{r=1}^{R}\left(\bar{y}^*_{reg(r)} - \frac{1}{R}\sum_{r=1}^{B}\bar{y}^*_{reg(r)}\right)^2}\cdot$$

The linearization variance estimator provides

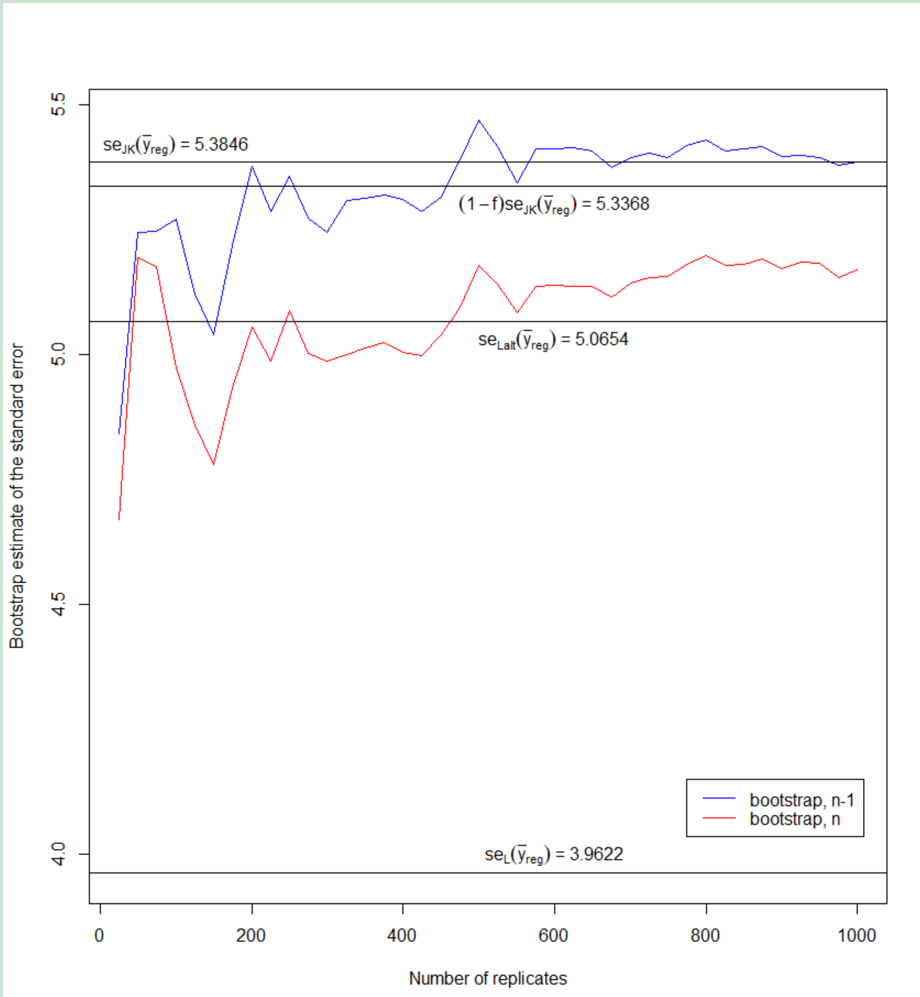
$$\widehat{SE}_L(\bar{y}_{reg}) = \sqrt{\left(1 - \frac{20}{1\,132}\right)\frac{s_e^2}{20}} = \sqrt{\left(1 - \frac{20}{1\,132}\right)\frac{319.6277}{20}} = 3.9622,$$

where $s_e^2$ is the sample variance of $e_i = y_i - \hat{B}_0 - \hat{B}_1 x_i$.

The alternative linearization estimator of the variance (Lohr 2019, p. 459), which is also the one used by the *survey* R package, provides

$$\widehat{SE}_{Lalt}(\bar{y}_{reg}) = \sqrt{\hat{V}\left(\sum_{i\in s} w_i g_i e_i\right)} = 5.065371, \quad w_i = N/n,$$

where $g_i = 1 + (\boldsymbol{t_x} - \boldsymbol{\hat{t}_x})^\top \left(\sum_{i\in s} w_i \boldsymbol{x_i}\boldsymbol{x_i}^\top\right)^{-1} \boldsymbol{x_i}$, with $\boldsymbol{t_x} = (N, X)^\top$, $\boldsymbol{\hat{x}_x} = (\hat{N}, \hat{X})^\top$, and $\boldsymbol{x_i} = (1, x_i)^\top$.




# Exercise 4

**(R code available)** The American Statistical Association (ASA) studied whether it should offer a certification designation for its members, so that statisticians meeting the qualifications could be designated as "Certified Statisticians." In 1994, the ASA surveyed its membership about this issue, with data in file certify.dat. The survey was sent to all 18 609 members; 5 001 responses were obtained. Results from the survey were reported in the October 1994 issue of Amstat News.
Assume that in 1994, the ASA membership had the following characteristics: 55% have PhD's and 38% have Master's degrees; 29% work in industry, 34% work in academia, and 11% work in government. The cross-classification between education and workplace was unavailable. There is a nonresponse. However, treat as if the respondents were selected with a probability sampling from the list of all members. Find the raking estimate of the total number of ASA members for this exercise. Estimate the total number

of members opposing certification in 1994 using and use the bootstrap to estimate its variance and construct a 95% confidence interval.

**Solution:**

ASA survey of certification designation for its members. $n = N = 18\,609$ and $n = 5\,001$. Treat respondents as the sample selected from the list of members.

Sample distribution by education and workplace is given as follows:

| | Industry | Academia | Other | |
|---|---|---|---|---|
| PhD | 798 | 1 787 | 451 | 3 036 |
| non-PhD | 1 011 | 434 | 520 | 1 965 |
| | 1 809 | 2 221 | 971 | 5 001 |

Only the marginal totals are known in the population. Using the raking weights, we obtain

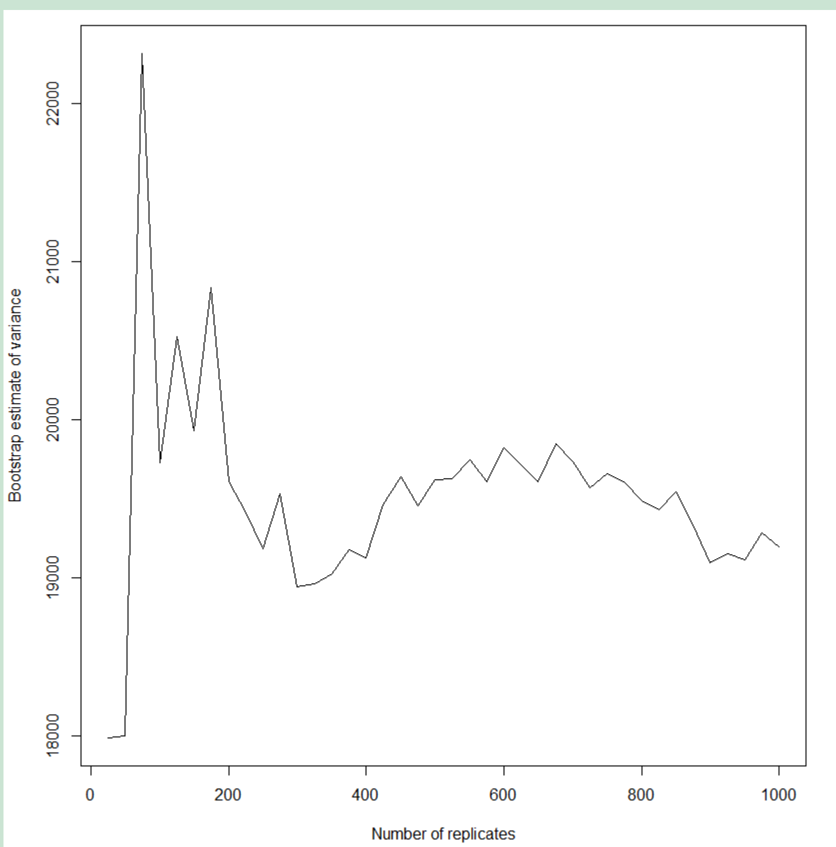
$$\hat{Y}_{rak} = \sum_{i \in s} w_{i(rak)} y_i = 7\,313.734,$$

where $y_i = 1$ if member $i$ opposes certification and zero otherwise. There is no analytical expression for the iterative raking. For each bootstrap replication, raking is applied to the subsample selected. Initial weights are set to $w^*_{0ri} = w_i n/m$, where $w_i = 18\,806/5\,001 = 3.721056$ and $m$ is the subsample size. Raking weights are then used to find the bootstrap estimates.

With $R = 1000$ and $m = n - 1$, we obtain

$$\frac{1}{1000} \sum_{r=1}^{1000} \hat{Y}^*_{rak(r)} = 7\,315.3,$$

$$\hat{V}_{boot}(\hat{Y}_{rak}) = \frac{1}{999} \sum_{r=1}^{1000} (\hat{Y}^*_{rak(r)} - 7\,315.3)^2 = 19\,196.81\cdot$$



The standard 95% CI is given by

$$7\,313.734 \pm 1.96\sqrt{19\,196.81} = [7\,042.176, 7\,134.180]$$

The 95% CI obtained from the 2.5th and the 97.5th percentiles of the distribution of the bootstrap estimates $\hat{Y}^*_{rak(r)}$ is given as follows.

$$[7\,040.938, 7\,133.355]\cdot$$

- Both methods provide similar bounds.
- The sample size is very large, making the normality assumption reasonable.