DRAFT: Documentation of `ModelSolver` [*] A `Python`-class for analyzing dynamic algebraic models

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Abstract

This article documents the `Python` class `ModelSolver`. The class lets the user define a model in terms of equations and endogenous variables. It contains methods to solve the model subject to data in a `Pandas DataFrame`, as well as analyzing the model.

1 Background

1.1 The model

Suppose we have a model, [1]

$$
\begin{aligned}
L_1(\mathbf{x}_t, \mathbf{z}_t) &= H_1(\mathbf{x}_t, \mathbf{z}_t), \\
L_2(\mathbf{x}_t, \mathbf{z}_t) &= H_2(\mathbf{x}_t, \mathbf{z}_t), \\
&\vdots \\
L_n(\mathbf{x}_t, \mathbf{z}_t) &= H_n(\mathbf{x}_t, \mathbf{z}_t),
\end{aligned}
$$

where $\mathbf{x}_t = (x_{1,t}, x_{2,t}, \dots, x_{n,t})$ is a vector of endogenous variables, and $\mathbf{z}_t$ is a vector of exogenous variables and lags. The model can be re-written as

$$
\mathbf{F}(\mathbf{x}_t, \mathbf{z}_t) = \begin{pmatrix} L_1(\mathbf{x}_t, \mathbf{z}_t) - H_1(\mathbf{x}_t, \mathbf{z}_t) \\ L_2(\mathbf{x}_t, \mathbf{z}_t) - H_2(\mathbf{x}_t, \mathbf{z}_t) \\ \vdots \\ L_n(\mathbf{x}_t, \mathbf{z}_t) - H_n(\mathbf{x}_t, \mathbf{z}_t) \end{pmatrix},
$$

and the solution to the model is given by

$$
\mathbf{F}(\mathbf{x}_t, \mathbf{z}_t) = \mathbf{0}.
$$

On the face of it, this is a problem which the *Newton-Raphson*-algorithm handles well. The issue, however, is that n might be quite large. In the Norwegian

[*] https://github.com/statisticsnorway/model-solver.git

[1] L_i and H_i are just the left and right hand side of equation i.

Figure 1: Bipartite graph (BiGraph of model)

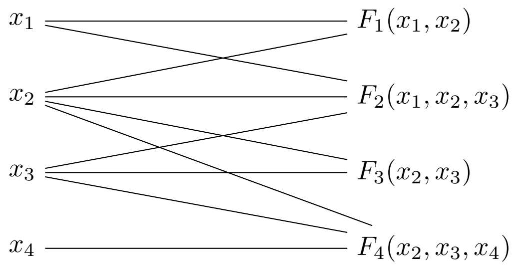
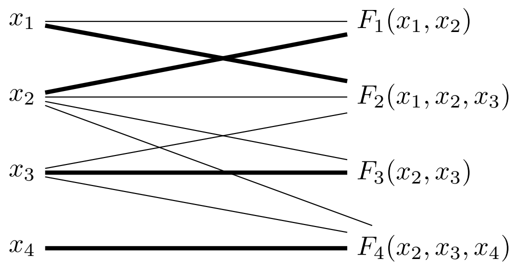


Figure 2: Maximum Bipartite Match (MBM) of BiGraph



national accounts, for instance, n is more than 15,500. Therefore, it is useful to analyze the system of equations before solving it, in order to break it down into minimal simultaneous blocks that can be solved in a particular sequence.

1.2 Block analysis

In order to analyze and divide the model into blocks, we use results from *graph theory*.

First, we construct a *bipartite graph* (BiGraph) that connects endogenous variables with equations. This is illustrated in Figure 1 for an arbitrary model with 4 equations (we omit time subscripts and exogenous variables and lags for notational convenience).

Next, we apply *maximum bipartite matching* (MBM), which assigns each endogenous variable to *one and only one* equation. The MBM is arbitrary, and Figure 2 shows one possible solution.

We use Figure 2 to determine what endogenous variables impact what *other* endogenous variables. This is illustrated in Figure 3.

Next, we use Figure 3 to construct a *directed graph* (DiGraph) that shows how the endogenous variables impact each other. The DiGraph is shown in Figure 4.

Finally, we find the *strong components* of the DiGraph, as shown in Figure 5. A strong component is a set of nodes that are connected such that every node can be reached from every other (traversing the arrows). A *condensation* of the DiGraph is a (new) DiGraph with each node being a strong component (of the former). The condensation can be illustrated as in Figure 6. Each node

Figure 3: Graph of what endogenous variables impact what *other* endogenous variables

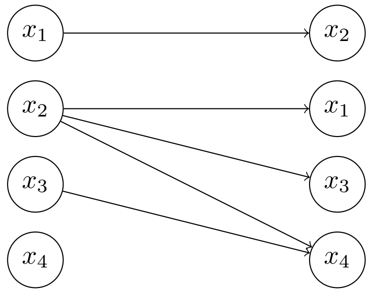


Figure 4: Directed graph (DiGraph) of what endogenous variables impact what *other* endogenous variables

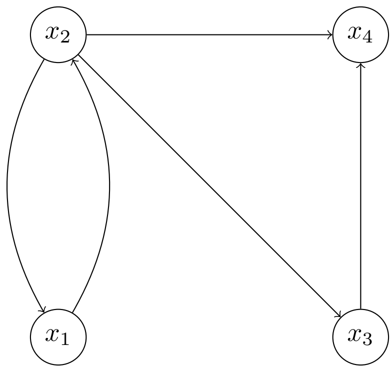


Figure 5: Condensation of DiGraph

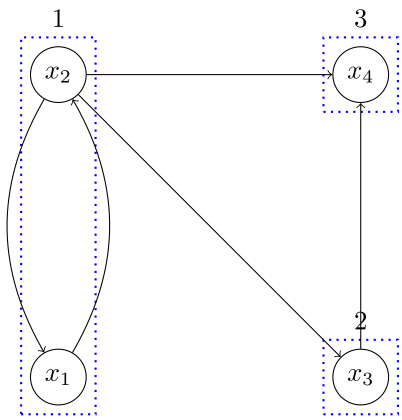
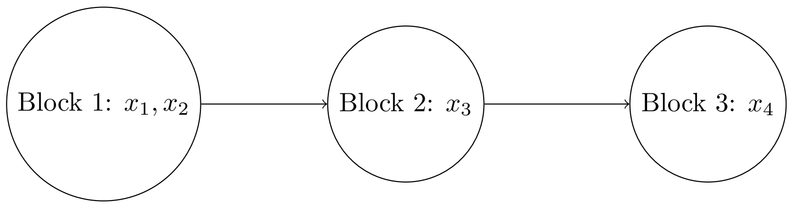


Figure 6: Condensed DiGraph



of the condensation corresponds to a block of the model, and the arrows decide the sequence in which the blocks are to be solved.

In this example, x_1 and x_2 must be solved first in one simultaneous block. Next, x_3 is solved (taking x_1 and x_2 as given by the solution to block 1). Finally, x_4 is solved (taking x_1, x_2 and x_3 as given by the solutions to block 1 and 2).

1.3 Simulation code

With the blocks of the model as given by the condensation of the DiGraph as previously discussed, we can generate *simulation code* for each block. That is *either*

- a (definition) function that takes exogenous input and returns the endogenous value (if the block is a definition, as described below), *or*

- a symbolic *objective function* and *Jacobian matrix* that takes exogenous input and (initial) values for the endogenous variables, which are to be sent to a Newton-Raphson algorithm.

A block is a said to be a *definition if and only if* it 1) consists of one equation and 2) the only thing on the left hand side of the equation is the endogenous variable of that equation (and that endogenous variable does *not* show up on the right hand side of that equation too).

1.4 Solution

If the block is a definition as discussed above, the solution is given by the definition function.

If the block to be solved is not a definition, it is sent to a Newton-Raphson algorithm, which in turn returns the solution. The $k+1$st iteration of the Newton-Raphson algorithm is given by

$$\mathbf{x}_t^{(k+1)} = \mathbf{x}_t^{(k)} - \mathbf{J}_{\mathbf{F}}^{-1}(\mathbf{x}_t^{(k)}, \mathbf{z}_t)\mathbf{F}(\mathbf{x}_t^{(k)}, \mathbf{z}_t).$$

We stop if $\max\left(\left|\mathbf{x}_t^{(k+1)} - \mathbf{x}_t^{(k)}\right|\right) \leq \varepsilon$, where ε is a tolerance level.

2 Examples of use

The `ModelSolver` object is instantiated by

```
model = ModelSolver(equations, endogenous)
```

where `equations` and `endogenous` are lists containing equations and endogenous variables as strings in lists, e.g.,

```
equations = [
'x1 = a1',
'x2 = a2',
'0.2*x1+0.7*x2 = 0.1*ca+0.8*cb+0.3*i1',
'0.8*x1+0.3*x2 = 0.9*ca+0.2*cb+0.1*i2',
'k1 = k1(-1)+i1',
'k2 = k2(-1)+i2'
]
endogenous = ['x1', 'x2', 'ca', 'cb', 'k1', 'k2']
```

More to TBA.