DRAFT: Documentation of `ModelSolver` A `Python`-class for analyzing dynamic algebraic models

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Abstract

This article documents the `Python` class `ModelSolver`. The class lets the user define a model in terms of equations and endogenous variables. It contains methods to solve and analyze the model.

1 Introduction

Suppose we have a model,

$$\begin{aligned}
L_1(\mathbf{x}_t, \mathbf{z}_t) &= H_1(\mathbf{x}_t, \mathbf{z}_t),\\
L_2(\mathbf{x}_t, \mathbf{z}_t) &= H_2(\mathbf{x}_t, \mathbf{z}_t),\\
&\vdots\\
L_n(\mathbf{x}_t, \mathbf{z}_t) &= H_n(\mathbf{x}_t, \mathbf{z}_t),
\end{aligned}$$

where $\mathbf{x}_t = (x_{1,t}, x_{2,t}, \dots, x_{n,t})$ is a vector of endogenous variables, and $\mathbf{z}_t$ is a vector of exogenous variables and lags. The model can be re-written as

$$\mathbf{F}(\mathbf{x}_t, \mathbf{z}_t) = \begin{pmatrix} L_1(\mathbf{x}_t, \mathbf{z}_t) - H_1(\mathbf{x}_t, \mathbf{z}_t) \\ L_2(\mathbf{x}_t, \mathbf{z}_t) - H_2(\mathbf{x}_t, \mathbf{z}_t) \\ \vdots \\ L_n(\mathbf{x}_t, \mathbf{z}_t) - H_n(\mathbf{x}_t, \mathbf{z}_t) \end{pmatrix},$$

and the solution to the model is given by

$$\mathbf{F}(\mathbf{x}_t, \mathbf{z}_t) = \mathbf{0}.$$

On the face of it, this is a problem which the *Newton-Raphson*-algorithm handles well. The issue, however, is that n might be quite large. In the Norwegian national accounts, n is somewhere between 15,000 and 16,000. Therefore, it is useful to analyze the system of equations before solving it in order to break it down into minimal simultaneous blocks that can be solved in a particular sequence.

Figure 1: Bipartite graph (BiGraph of model)

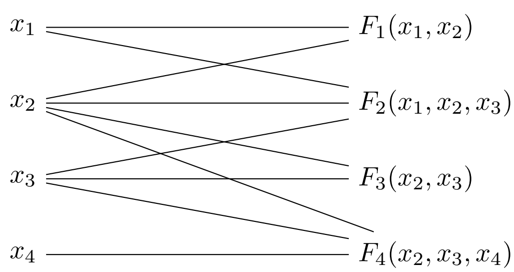
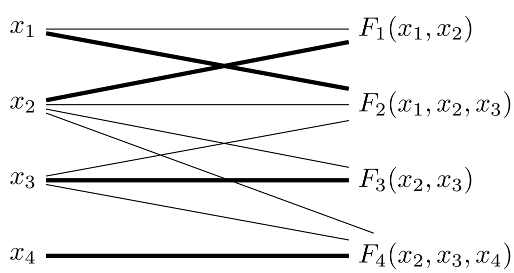


Figure 2: Maximum Bipartite Match (MBM) of BiGraph



2 Block analysis

In order to analyze and divide the model into blocks, we use results from *graph theory*. First, we construct a *bipartite graph* (BiGraph) that connects endogenous variables with equations. This is illustrated in Figure 1 for an arbitrary model with 4 equations (we omit time subscripts for notational convenience).

Next, we apply *maximum bipartite matching* (MBM), which connects each endogenous variable with one and only one equation. The MBM is arbitrary, and Figure 2 shows one possible solution.

We use Figure 2 to show what endogenous variables inpact what *other* endogenous variables. This is illustrated in Figure 3.

We use Figure 3 to construct a *directed graph* (DiGraph) that shows how endogenous variables impact each other. Figure 4 shows the DiGraph for the model.

Finally, find the *strong components* of the DiGraph, as shown in Figure 5. The strong components are nodes that are connected such that every node can be reached from every other. A *condencation* of the DiGraph is a (new) DiGraph with each node being the strong components of the latter. The condensation can be illustrated as in Figure 6. Each node of the condensation corresponds to a block of the model, and the arrows decide the sequence the blocks are solved. In this example, x_1 and x_2 must be solved first in one simultaneous block. Next,

Figure 3: Graph of what endogenous variables impact what *other* endogenous variables

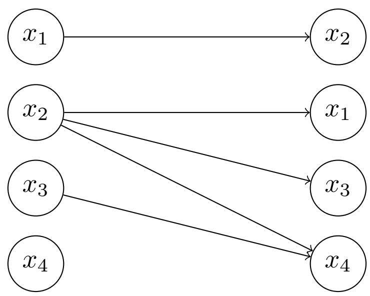


Figure 4: Directed graph (DiGraph) of what endogenous variables impact what *other* endogenous variables

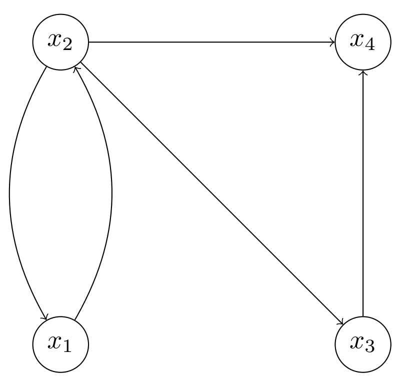


Figure 5: Condensation of DiGraph

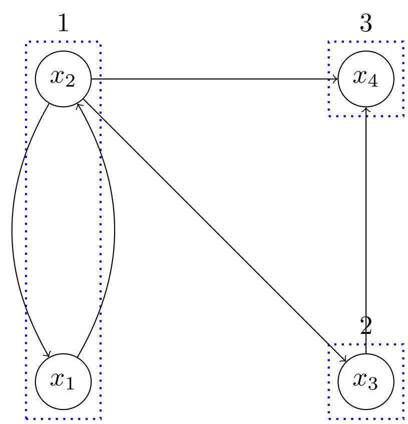
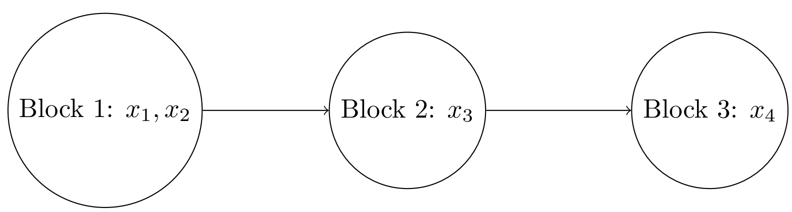


Figure 6: Condensed DiGraph



x_3 is solved (taking x_1 and x_2 as given by the solution to block 1). Finally, x_4 is solved (taking x_1, x_2 and x_3 as given by the solutions to block 1 and 2).

3 Simulation code

With the blocks of the model given by the condensation of the DiGraph of the model, we can generate *simulation code*. That is *either*

- a function that takes exogenous input and returns the endogenous value (if the block is a definition as described below), or
- a symbolic *objective function* and *Jacobian matrix* to be sent to a Newton-Raphson algorithm.

A block is a said to be a *definition* if and only if it consists of one equation *and* if the endogenous variable is alone on the left hand side of the equation (and not on the right hand side).

4 Solution

The $k+1$st iteration of the Newton-Raphson algorithm is given by

$$\mathbf{x}_t^{(k+1)} = \mathbf{x}_t^{(k)} - \mathbf{J}_{\mathbf{F}}^{-1}(\mathbf{x}_t^{(k)}, \mathbf{z}_t)\mathbf{F}(\mathbf{x}_t^{(k)}, \mathbf{z}_t).$$

We stop if $\max\left(\left|\mathbf{x}_t^{(k+1)} - \mathbf{x}_t^{(k)}\right|\right) \leq \varepsilon$, where ε is a tolerance level.