Bayesian Classifier

Eun Yi Kim

Artificial Inteligence
& Computer Vision
Laboratory

I N D E X

Artificial Inteligence
& Computer Vision
L a b o r a t o r y

Bayesian Decision Theory

- Design classifiers to make decisions subject to minimizing an expected ”risk”.
  - The simplest risk is the classification error.
  - When misclassification errors are not equally important, the risk can include the cost associated with different misclassification errors.

Terminology

- State of nature ω *(class label):*
  - e.g., ω_1 for sea bass, ω_2 for salmon

- Probabilities $P(\omega_1)$ and $P(\omega_2)$ *(priors):*
  - e.g., prior knowledge of how likely is to get a sea bass or a salmon

- Probability density function $p(x)$ *(evidence):*
  - e.g., how frequently we will measure a pattern with feature value x (e.g., x corresponds to lightness)

Terminology

- Conditional probability density $p(x/\omega_j)$ (*likelihood*) :
  - e.g., how frequently we will measure a pattern with feature value x given that the pattern belongs to class ω_j

e.g., lightness distributions between salmon/sea-bass populations

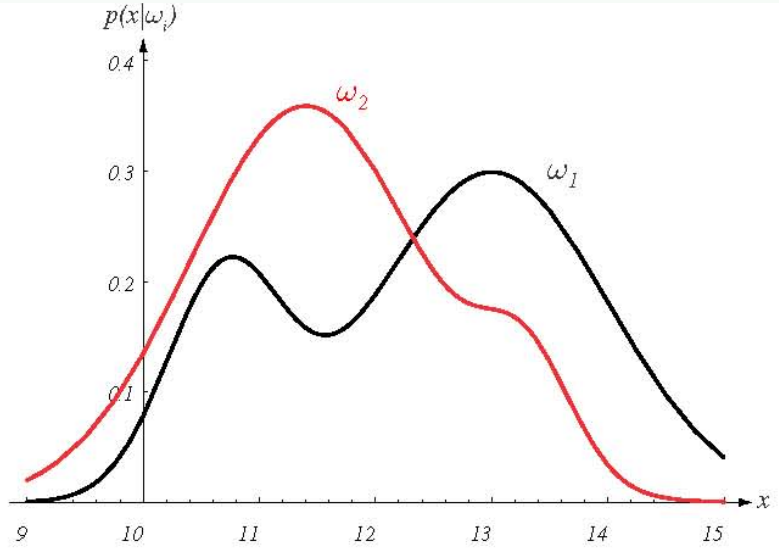


FIGURE 2.1. Hypothetical class-conditional probability density functions show the probability density of measuring a particular feature value x given the pattern is in category ω_j. If x represents the lightness of a fish, the two curves might describe the difference in lightness of populations of two types of fish. Density functions are normalized, and thus the area under each curve is 1.0. From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons,

Terminology

- Conditional probability $P(\omega_j/x)$ *(posterior)* :
  - e.g., the probability that the fish belongs to class ω_j given feature x.

Decision Rule using Prior Probability Only

Decide ω_1 if $P(\omega_1) > P(\omega_2)$; otherwise decide ω_2

$$P(error) = \begin{cases} P(\omega_1) & if\ we\ decide\ \omega_2 \\ P(\omega_2) & if\ we\ decide\ \omega_1 \end{cases}$$

or $P(error) = \min[P(\omega_1), P(\omega_2)]$

- Favours the most likely class.
- This rule will be making the same decision all times.
  - i.e., optimum if no other information is available

Decision Rule using Conditional Probability

- Using Bayes' rule:

$$P(\omega_j / x) = \frac{p(x/\omega_j)P(\omega_j)}{p(x)} = \frac{likelihood \times prior}{evidence}$$

where $p(x) = \sum_{j=1}^{2} p(x/\omega_j)P(\omega_j)$ (i.e., scale factor – sum of probs = 1)

Decide ω_1 if $P(\omega_1/x) > P(\omega_2/x)$; otherwise decide ω_2

or

Decide ω_1 if $p(x/\omega_1)P(\omega_1) > p(x/\omega_2)P(\omega_2)$; otherwise decide ω_2

or

Decide ω_1 if $p(x/\omega_1)/p(x/\omega_2) > P(\omega_2)/P(\omega_1)$; otherwise decide ω_2

likelihood ratio threshold

Decision Rule using Conditional Probability

$$P(\omega_1) = \frac{2}{3} \qquad P(\omega_2) = \frac{1}{3}$$

$p(x/\omega_j)$

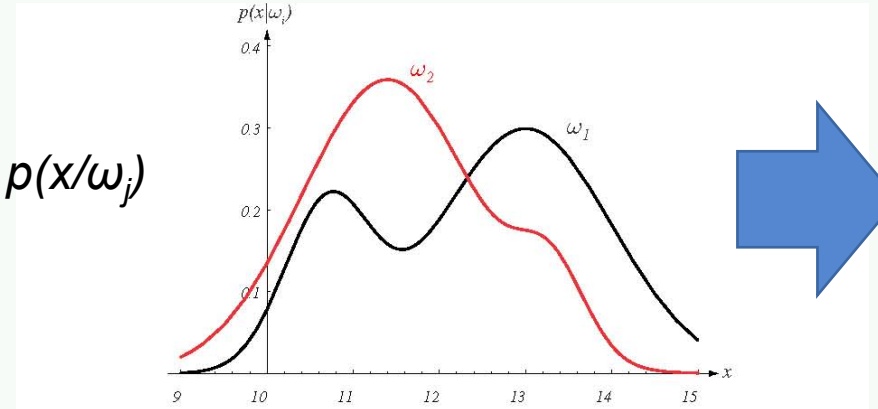


FIGURE 2.1. Hypothetical class-conditional probability density functions show the probability density of measuring a particular feature value x given the pattern is in category ω_i. If x represents the lightness of a fish, the two curves might describe the difference in lightness of populations of two types of fish. Density functions are normalized, and thus the area under each curve is 1.0. From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons,

$P(\omega_j/x)$

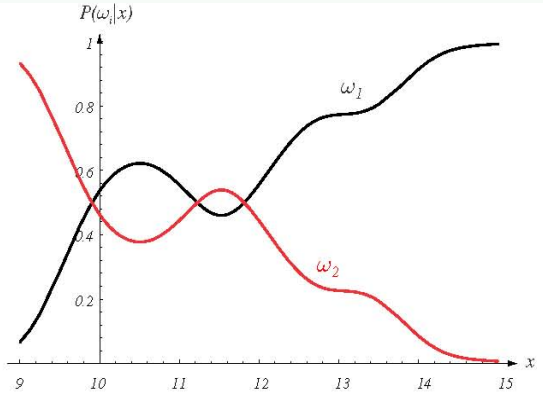


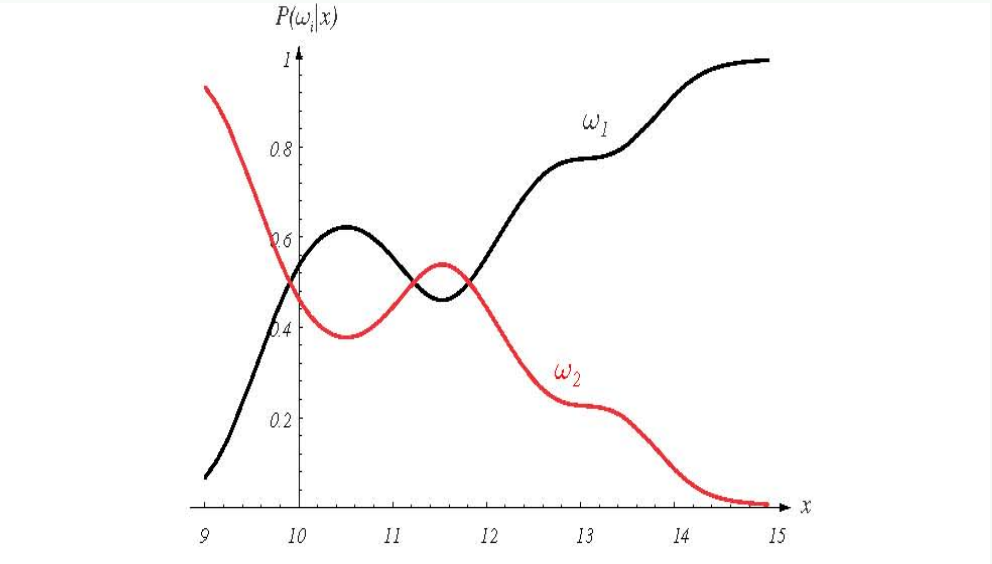
FIGURE 2.2. Posterior probabilities for the particular priors $P(\omega_1) = 2/3$ and $P(\omega_2) = 1/3$ for the class-conditional probability densities shown in Fig. 2.1. Thus in this case, given that a pattern is measured to have feature value $x = 14$, the probability it is in category ω_2 is roughly 0.08, and that it is in ω_1 is 0.92. At every x, the posteriors sum to 1.0. From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons, Inc.

Probability of Error

- The probability of error is defined as:

$$P(error/x) = \begin{cases} P(\omega_1/x) & if\ we\ decide\ \omega_2 \\ P(\omega_2/x) & if\ we\ decide\ \omega_1 \end{cases}$$

or $P(error/x) = min[P(\omega_1/x), P(\omega_2/x)]$



- What is the average probability error?

$$P(error) = \int_{-\infty}^{\infty} P(error, x)dx = \int_{-\infty}^{\infty} P(error/x)\,p(x)dx$$

- The Bayes rule is optimum, that is, it minimizes the average probability error!

Where do Probabilities come from?

- There are two competitive answers:

  (1) Relative frequency (**objective**) approach.
  - Compute probabilities from experiments.

  (2) Bayesian (**subjective**) approach.
  - Compute probabilities from models.

Example: Objective approach

- Classify cars whether they are more or less than $50K:
  - Classes: C_1 if price > $50K, C_2 if price <= $50K
  - Feature: x, the height of a car

- Use the Bayes' rule to compute the posterior probabilities:

$$P(C_i / x) = \frac{p(x / C_i)P(C_i)}{p(x)}$$

- We need to estimate $p(x/C_1)$, $p(x/C_2)$, $P(C_1)$, $P(C_2)$

Example: Objective approach

- Collect data
  - Ask drivers how much their car was and measure height.

- Determine prior probabilities $P(C_1)$, $P(C_2)$
  - e.g., 1209 samples: #C_1=221 #C_2=988

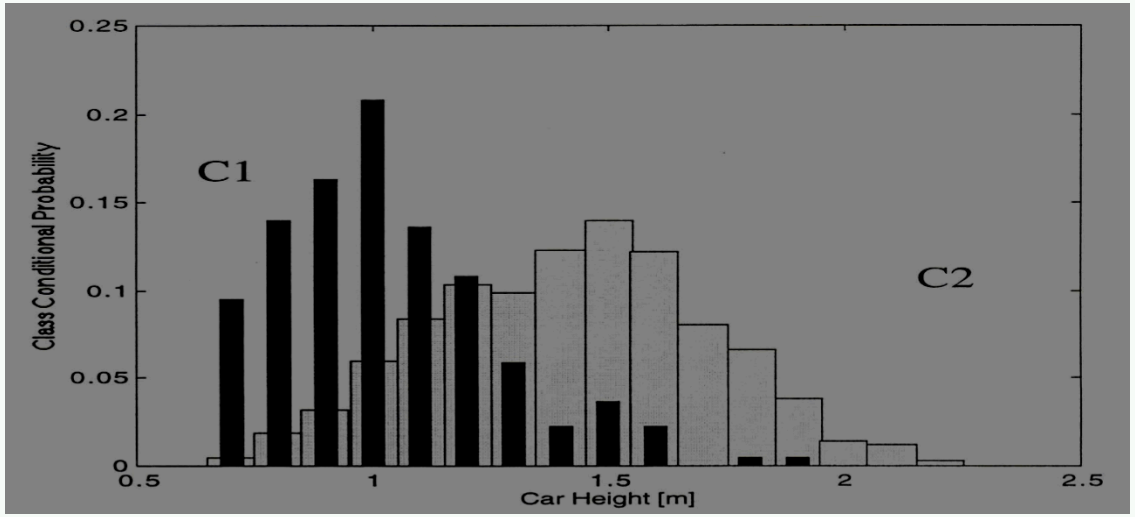
$$P(C_1) = \frac{221}{1209} = 0.183$$

$$P(C_2) = \frac{988}{1209} = 0.817$$

Example: Objective approach

- Determine class conditional probabilities (likelihood)
  - Discretize car height into bins and compute normalized histogram.

$$p(x / C_i)$$



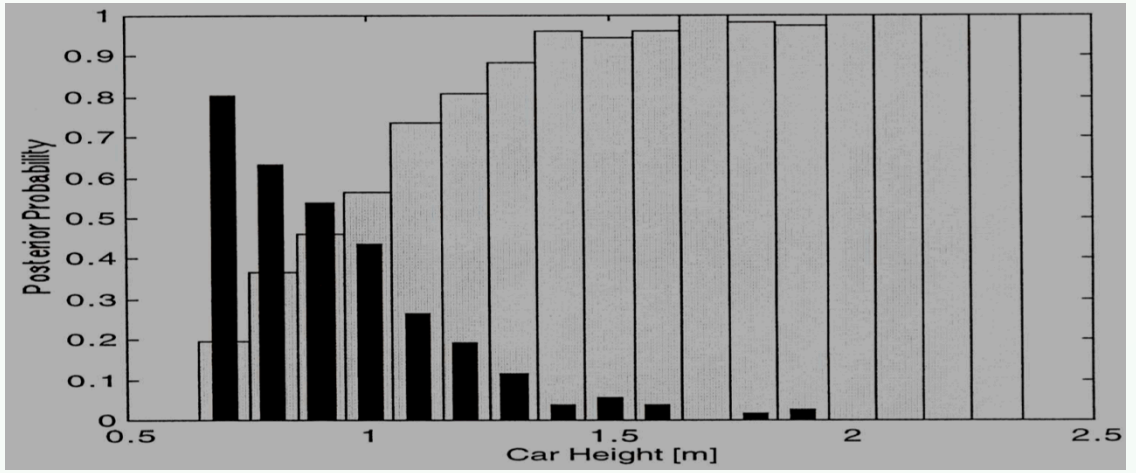
Example: Objective approach

- Calculate the posterior probability for each bin, e.g.:

$$P(C_1 / x = 1.0) = \frac{p(x = 1.0 / C_1) P(C_1)}{p(x = 1.0 / C_1) P(C_1) + p(x = 1.0 / C_2) P(C_2)} =$$

$$= \frac{0.2081 * 0.183}{0.2081 * 0.183 + 0.0597 * 0.817} = 0.438$$

$P(C_i / x)$



A More General Theory

- Use more than one features.
- Allow more than two categories.
- Allow actions other than classifying the input to one of the possible ca tegories (e.g., rejection).
- Employ a more general error function (i.e., conditional “risk”) by assoc iating a “cost” (based on a “loss” function) with different errors.

Terminology

- Features form a vector
- A set of *c* categories $\omega_1, \omega_2, \ldots, \omega_c$
- A finite set of *l* actions $\alpha_1, \alpha_2, \ldots, \alpha_l$ (typically $l \geq c$)
- A *loss* function $\lambda(\alpha_i / \omega_j)$
  - the cost associated with taking action α_i when the correct classification category is ω_j
- Conditional risk $R(\alpha_i / \mathbf{x})$ – expected loss of taking action α_i given **x**
- Classification is now performed using the $R(\alpha_i / \mathbf{x})$ instead of $P(\omega_i / \mathbf{x})$

Conditional Risk

- Suppose we take action α_i when **x** is observed.
- The **conditional risk** (or **expected loss**) with taking action α_i is defined as:

$$R(a_i / \mathbf{x}) = \sum_{j=1}^{c} \lambda(a_i / \omega_j) P(\omega_j / \mathbf{x})$$

where $$P(\omega_j / \mathbf{x}) = \frac{p(\mathbf{x} / \omega_j) P(\omega_j)}{p(\mathbf{x})}$$

Overall Risk

- The overall risk is defined as:

$$R = \int R(a(\mathbf{x}) / \mathbf{x}) p(\mathbf{x}) d\mathbf{x}$$

, where $\alpha(\mathbf{x})$ is a general decision rule that determines which action $\alpha_1, \alpha_2, \ldots, \alpha_l$ to take for every **x**.

- Use the Bayes rule to minimize R.

Overall Risk

- The *Bayes rule* minimizes *R* by:
  (i) Computing $R(\alpha_i/\mathbf{x})$ for every α_i given an **x**
  (ii) Choosing the action α_i with the minimum conditional risk $R(\alpha_i/\mathbf{x})$

- The resulting minimum R^* is called *Bayes risk* and is the best performance that can be achieved:

$$R^* = \min R$$

Example: Two-category classification

- Define
  - α_1: decide ω_1
  - α_2: decide ω_2
  - $\lambda_{ij} = \lambda(\alpha_i / \omega_j)$

- The conditional risks are:

$$R(a_i / \mathbf{x}) = \sum_{j=1}^{c} \lambda(a_i / \omega_j) P(\omega_j / \mathbf{x})$$

$$R(a_1/\mathbf{x}) = \lambda_{11} P(\omega_1/\mathbf{x}) + \lambda_{12} P(\omega_2/\mathbf{x})$$
$$R(a_2/\mathbf{x}) = \lambda_{21} P(\omega_1/\mathbf{x}) + \lambda_{22} P(\omega_2/\mathbf{x})$$

Example: Two-category classification

- Minimum risk decision rule:

$$\textbf{Decide } \omega_1 \text{ if } R(a_1/\mathbf{x}) < R(a_2/\mathbf{x}); \text{ otherwise decide } \omega_2$$

or, using

$$R(a_1/\mathbf{x}) = \lambda_{11}P(\omega_1/\mathbf{x}) + \lambda_{12}P(\omega_2/\mathbf{x})$$
$$R(a_2/\mathbf{x}) = \lambda_{21}P(\omega_1/\mathbf{x}) + \lambda_{22}P(\omega_2/\mathbf{x})$$

$$\textbf{Decide } \omega_1 \text{ if } (\lambda_{21} - \lambda_{11})P(\omega_1/\mathbf{x}) > (\lambda_{12} - \lambda_{22})P(\omega_2/\mathbf{x}); \text{ otherwise decide } \omega_2$$

or, using

$$P(\omega_j/\mathbf{x}) = \frac{p(\mathbf{x}/\omega_j)P(\omega_j)}{p(\mathbf{x})}$$

$$\textbf{Decide } \omega_1 \text{ if } \frac{p(\mathbf{x}/\omega_1)}{p(\mathbf{x}/\omega_2)} > \frac{(\lambda_{12} - \lambda_{22})}{(\lambda_{21} - \lambda_{11})}\frac{P(\omega_2)}{P(\omega_1)}; \text{ otherwise decide } \omega_2$$

likelihood ratio threshold

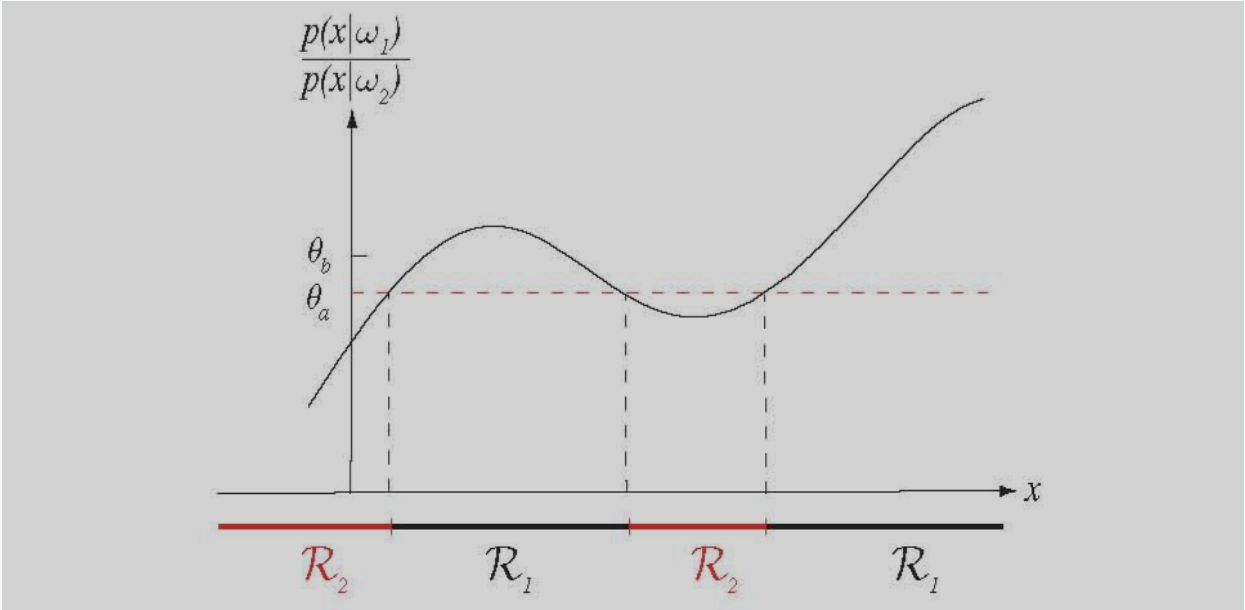
Example: Two-category classification

- Assuming general loss:

$$\textbf{Decide } \omega_1 \text{ if } \frac{p(\mathbf{x}/\omega_1)}{p(\mathbf{x}/\omega_2)} > \frac{(\lambda_{12}-\lambda_{22})}{(\lambda_{21}-\lambda_{11})}\frac{P(\omega_2)}{P(\omega_1)}; \text{ otherwise decide } \omega_2$$

- Assuming zero-one loss:

Decide ω_1 if $p(x/\omega_1)/p(x/\omega_2) > P(\omega_2)/P(\omega_1)$ otherwise **decide** ω_2



(decision regions)

$$\theta_a = P(\omega_2)/P(\omega_1)$$

$$\theta_b = \frac{P(\omega_2)(\lambda_{12}-\lambda_{22})}{P(\omega_1)(\lambda_{21}-\lambda_{11})}$$

Special case: Zero-One Loss Function

- Assign the same loss to all errors:

$$\lambda(a_i/\omega_j) = \begin{cases} 0 & i = j \\ 1 & i \neq j \end{cases}$$

- What is the conditional risk in this case?

$$R(a_i/\mathbf{x}) = \sum_{j=1}^{c} \lambda(a_i/\omega_j) P(\omega_j/\mathbf{x}) = \sum_{i \neq j} P(\omega_j/\mathbf{x}) = 1 - P(\omega_i/\mathbf{x})$$

Special case: Zero-One Loss Function

- The decision rule becomes:

Decide ω_1 if $R(a_1/\mathbf{x}) < R(a_2/\mathbf{x})$; otherwise decide ω_2

or **Decide** ω_1 if $1 - P(\omega_1/\mathbf{x}) < 1 - P(\omega_2/\mathbf{x})$; otherwise decide ω_2

or **Decide** ω_1 if $P(\omega_1/\mathbf{x}) > P(\omega_2/\mathbf{x})$; otherwise decide ω_2

- The overall risk becomes the average probability error!

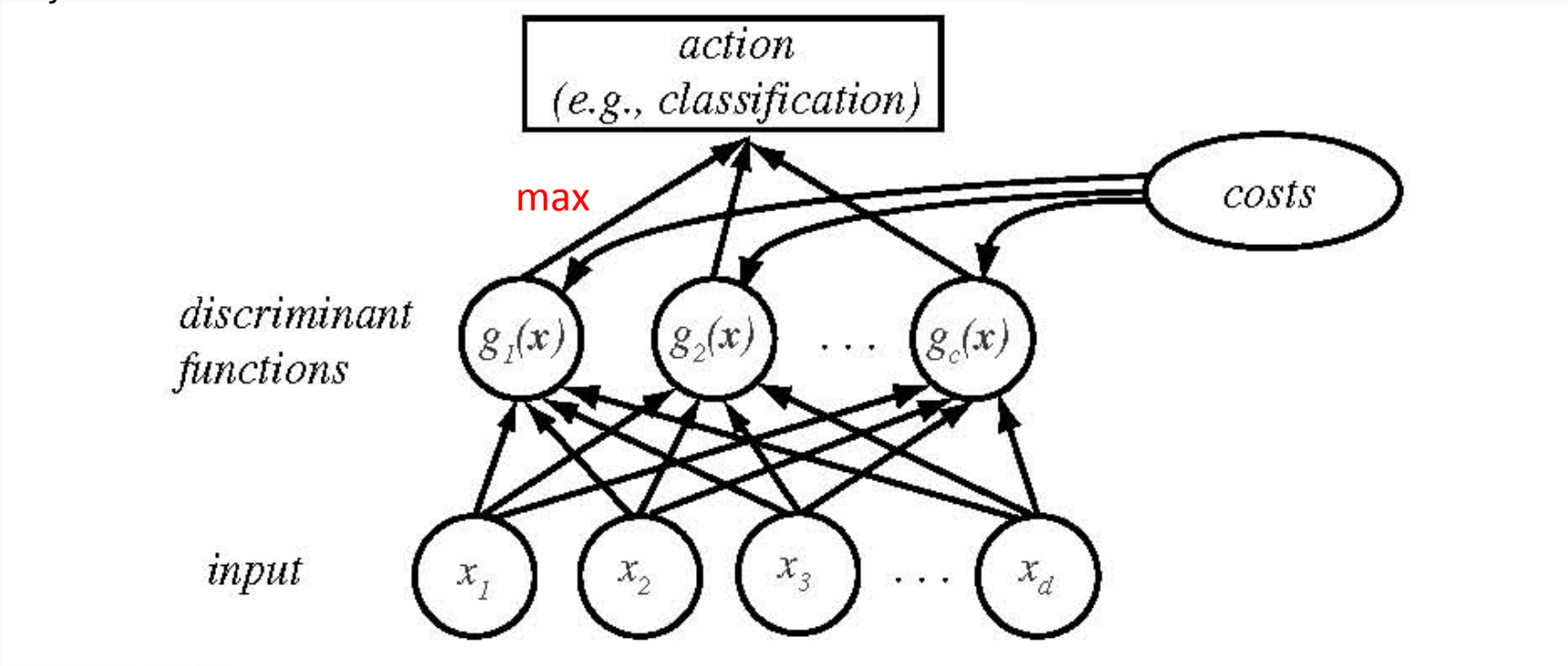
Discriminant Functions

- Represent a classifier through discriminant functions

$$g_i(x),\ i = 1, \ldots, c$$

- A feature vector **x** is assigned to class ω_i if:

$$g_i(\mathbf{x}) > g_j(\mathbf{x}) \quad \text{for all} \quad j \neq i$$



Discriminants for Bayes Classifier

- Assuming a general loss function:

$$g_i(\mathbf{x}) = -R(\alpha_i / \mathbf{x})$$

- Assuming the zero-one loss function:

$$g_i(\mathbf{x}) = P(\omega_i / \mathbf{x})$$

Discriminants for Bayes Classifier

- Replacing $g_i(\mathbf{x})$ with $f(g_i(\mathbf{x}))$, where $f()$ is monotonically increasing, does not change the classification results.

$g_i(x)=P(\omega_i/x)$

$$g_i(\mathbf{x}) = \frac{p(\mathbf{x}/\omega_i)P(\omega_i)}{p(\mathbf{x})}$$

$$g_i(\mathbf{x}) = p(\mathbf{x}/\omega_i)P(\omega_i)$$

$$g_i(\mathbf{x}) = \ln p(\mathbf{x}/\omega_i) + \ln P(\omega_i)$$

we'll use this extensively!

Case of two categories

- More common to use a single discriminant function (*dichotomizer*) instead of two:

$$g(\mathbf{x}) = g_1(\mathbf{x}) - g_2(\mathbf{x})$$

Decide ω_1 if $g(\mathbf{x}) > 0$; otherwise decide ω_2

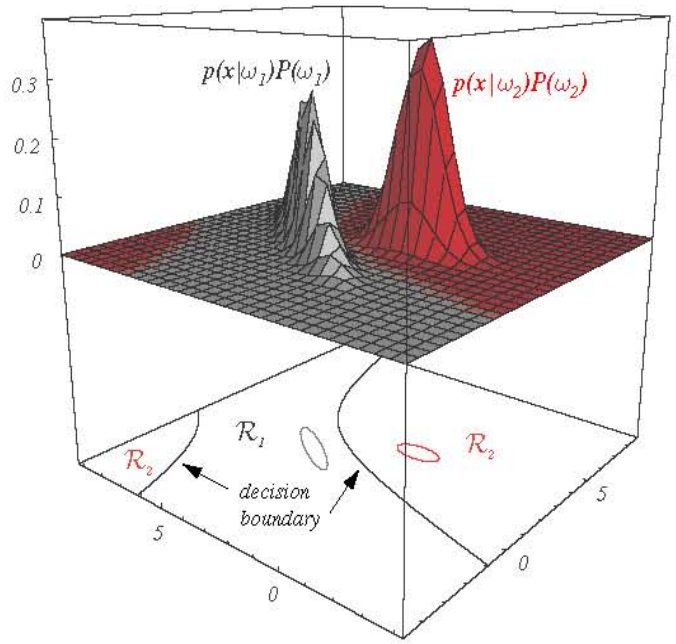
Examples:

$$g(\mathbf{x}) = P(\omega_1 / \mathbf{x}) - P(\omega_2 / \mathbf{x})$$

$$g(\mathbf{x}) = \ln \frac{p(\mathbf{x} / \omega_1)}{p(\mathbf{x} / \omega_2)} + \ln \frac{P(\omega_1)}{P(\omega_2)}$$

Decision Regions and Boundaries

- Discriminants divide the feature space in *decision regions* R_1, $R_{2, \ldots,} R_c$, separated by *decision boundaries*.



Decision boundary is defined by:

$$g_1(x)=g_2(x)$$

Discriminant Function for Multivariate Gaussian Density

$$N(\mu,\Sigma) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} exp[-\frac{1}{2}(\mathbf{x}-\mu)^t\Sigma^{-1}(\mathbf{x}-\mu)]$$

- Consider the following discriminant function:

$$g_i(\mathbf{x}) = \ln p(\mathbf{x}/\omega_i) + \ln P(\omega_i)$$

- If $p(\mathbf{x}/\omega_i) \sim N(\mu_i, \Sigma_i)$, then

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\mu_i)^t\Sigma_i^{-1}(\mathbf{x}-\mu_i) - \frac{d}{2}ln2\pi - \frac{1}{2}\ln|\Sigma_i| + \ln P(\omega_i)$$

Multivariate Gaussian Density: Case I

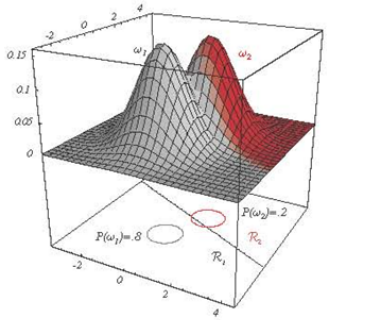
$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\mu_i)^t\Sigma_i^{-1}(\mathbf{x}-\mu_i) - \frac{d}{2}ln2\pi - \frac{1}{2}\ln|\Sigma_i| + \ln P(\omega_i)$$

- $\Sigma_i=\sigma^2 I$ (diagonal matrix)
  - This is true when features are uncorrelated (or statistically independent) with same variance
  - The clusters have spherical shape and same size (centered at μ_i)

- If we disregard $\frac{d}{2}ln2\pi$ and $\frac{1}{2}\ln|\Sigma_i|$ (constants):

$$g_i(\mathbf{x}) = -\frac{\|\mathbf{x}-\mu_i\|^2}{2\sigma^2} + \ln P(\omega_i)$$

where $\|\mathbf{x}-\mu_i\|^2 = (\mathbf{x}-\mu_i)^t(\mathbf{x}-\mu_i)$



Multivariate Gaussian Density: Case I

- Expanding the above expression:

$$g_i(\mathbf{x}) = -\frac{1}{2\sigma^2}\left[\mathbf{x}^t\mathbf{x} - 2\mu_i^t\mathbf{x} + \mu_i^t\mu_i\right] + \ln P(\omega_i)$$

- Disregarding $\mathbf{x}^t\mathbf{x}$ (constant), we get a linear discriminant:

$$\boxed{g_i(\mathbf{x}) = \mathbf{w}_i^t\mathbf{x} + w_{i0}}$$

where $\mathbf{w}_i = \frac{1}{\sigma^2}\mu_i$, and $w_{i0} = -\frac{1}{2\sigma^2}\mu_i^t\mu_i + \ln P(\omega_i)$

Multivariate Gaussian Density: Case I

- Decision boundary is determined by hyperplanes; setting $g_i(\mathbf{x}) = g_j(\mathbf{x})$:

$$\boxed{\mathbf{w}^t(\mathbf{x} - \mathbf{x_0}) = 0}$$

where $\mathbf{w} = \mu_i - \mu_j$, and $\mathbf{x}_0 = \frac{1}{2}(\mu_i + \mu_j) - \frac{\sigma^2}{\|\mu_i - \mu_j\|^2} \ln \frac{P(\omega_i)}{P(\omega_j)} (\mu_i - \mu_j)$

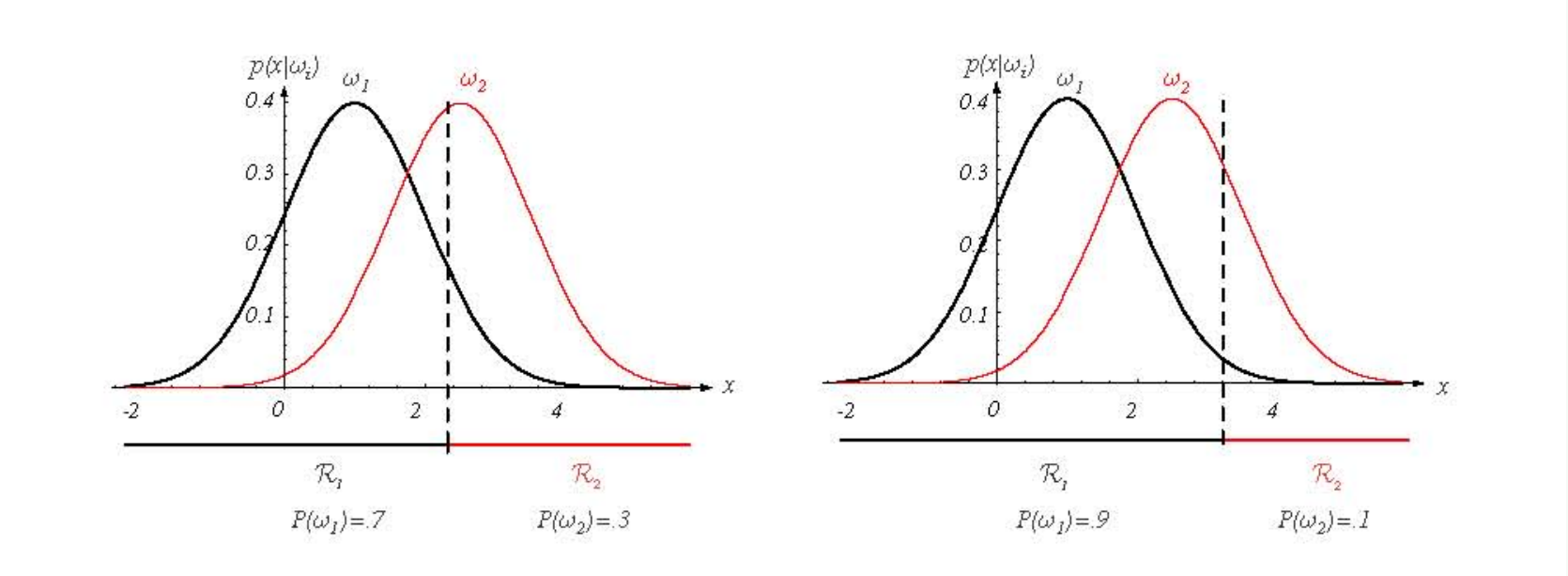
Multivariate Gaussian Density: Case I

- Properties of decision boundary:
  - It passes through $\mathbf{x_0}$
  - It is orthogonal to the line linking the means.
  - What happens when $P(\omega_i) = P(\omega_j)$?
  - If $P(\omega_i) \neq P(\omega_j)$, then $\mathbf{x_0}$ shifts away from the most likely category.
  - If σ is very small, the position of the boundary is insensitive to $P(\omega_i)$ and $P(\omega_j)$

$$\mathbf{w}^t(\mathbf{x} - \mathbf{x_0}) = 0$$

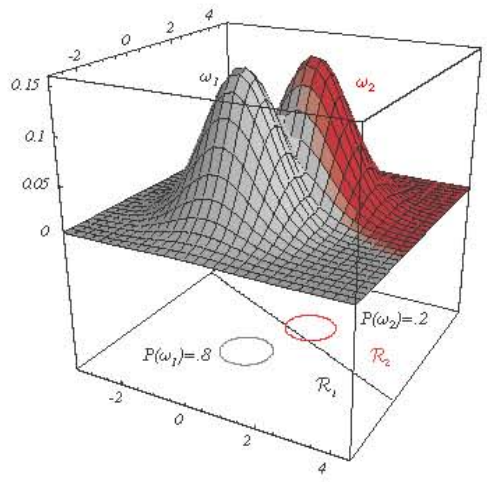
$$\text{where } \mathbf{w} = \mu_i - \mu_j, \text{ and } \mathbf{x}_0 = \frac{1}{2}(\mu_i + \mu_j) - \frac{\sigma^2}{\|\mu_i - \mu_j\|^2} \ln \frac{P(\omega_i)}{P(\omega_j)}(\mu_i - \mu_j)$$

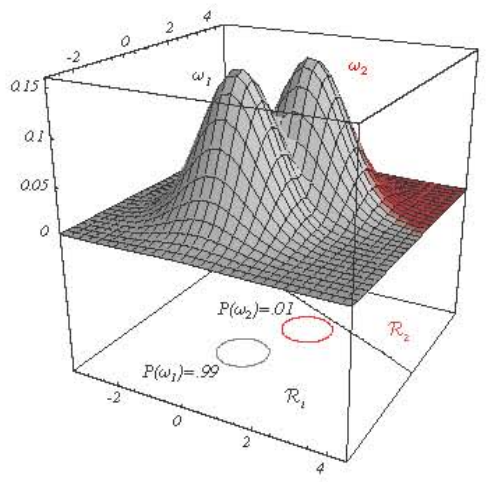
Multivariate Gaussian Density: Case I



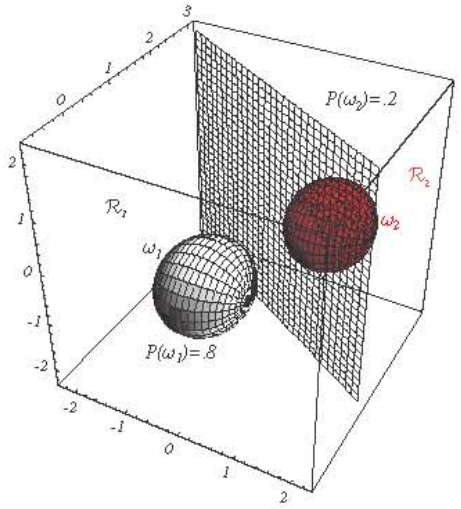
If $P(\omega_i) \neq P(\omega_j)$, then $\mathbf{x_0}$ shifts away from the most likely category.

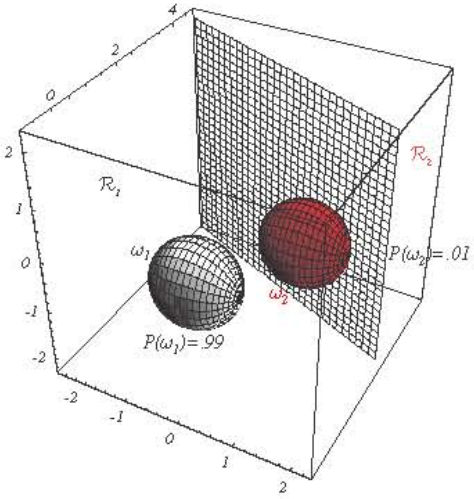
Multivariate Gaussian Density: Case I





If $P(\omega_i) \neq P(\omega_j)$, then $\mathbf{x_0}$ shifts away from the most likely category.





Multivariate Gaussian Density: Case I

- When $P(\omega_i)$ are equal, then the discriminant becomes:

$$g_i(\mathbf{x}) = -\frac{\|\mathbf{x}-\mu_i\|^2}{2\sigma^2} + \ln P(\omega_i) \quad \Rightarrow \quad g_i(\mathbf{x}) = -\|\mathbf{x}-\mu_i\|^2$$

- This is the Euclidean distance (or minimum distance classifier)

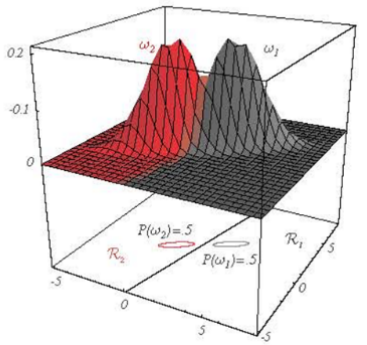
- Recall the assumptions for Case I:
  - Diagonal covariance matrix (true for uncorrelated or statistically independent features.
  - All features have the same variance.

Multivariate Gaussian Density: Case II

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\mu_\mathbf{i})^t\Sigma_i^{-1}(\mathbf{x}-\mu_\mathbf{i}) - \frac{d}{2}ln2\pi - \frac{1}{2}\ln|\Sigma_i| + \ln P(\omega_i)$$

- $\Sigma_i = \Sigma$ (not necessarily diagonal)

- The clusters have hyperellipsoidal shape and same size (centered at μ).



- If we disregard $\frac{d}{2}ln2\pi$ and $\frac{1}{2}\ln|\Sigma_i|$ (constants):

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\mu_\mathbf{i})^t\Sigma^{-1}(\mathbf{x}-\mu_\mathbf{i}) + \ln P(\omega_i)$$

- Expanding the above expression and disregarding the quadratic term:

$$\boxed{\begin{array}{c} g_i(\mathbf{x}) = \mathbf{w}_i^t\mathbf{x} + w_{i0} \\ (\textit{linear discriminant}) \end{array}}$$

where $\mathbf{w}_i = \Sigma^{-1}\mu_i$, and $w_{i0} = -\frac{1}{2}\mu_i^t\Sigma^{-1}\mu_i + \ln P(\omega_i)$

Multivariate Gaussian Density: Case II

- Decision boundary is determined by hyperplanes; setting $g_i(\mathbf{x}) = g_j(\mathbf{x})$:

$$\mathbf{w}^t(\mathbf{x} - \mathbf{x_0}) = 0$$

where $\mathbf{w} = \Sigma^{-1}(\mu_i - \mu_j)$ and $\mathbf{x}_0 = \frac{1}{2}(\mu_i + \mu_j) - \frac{ln[P(\omega_i)/P(\omega_j)]}{(\mu_i - \mu_j)^t\Sigma^{-1}(\mu_i - \mu_j)}(\mu_i - \mu_j)$

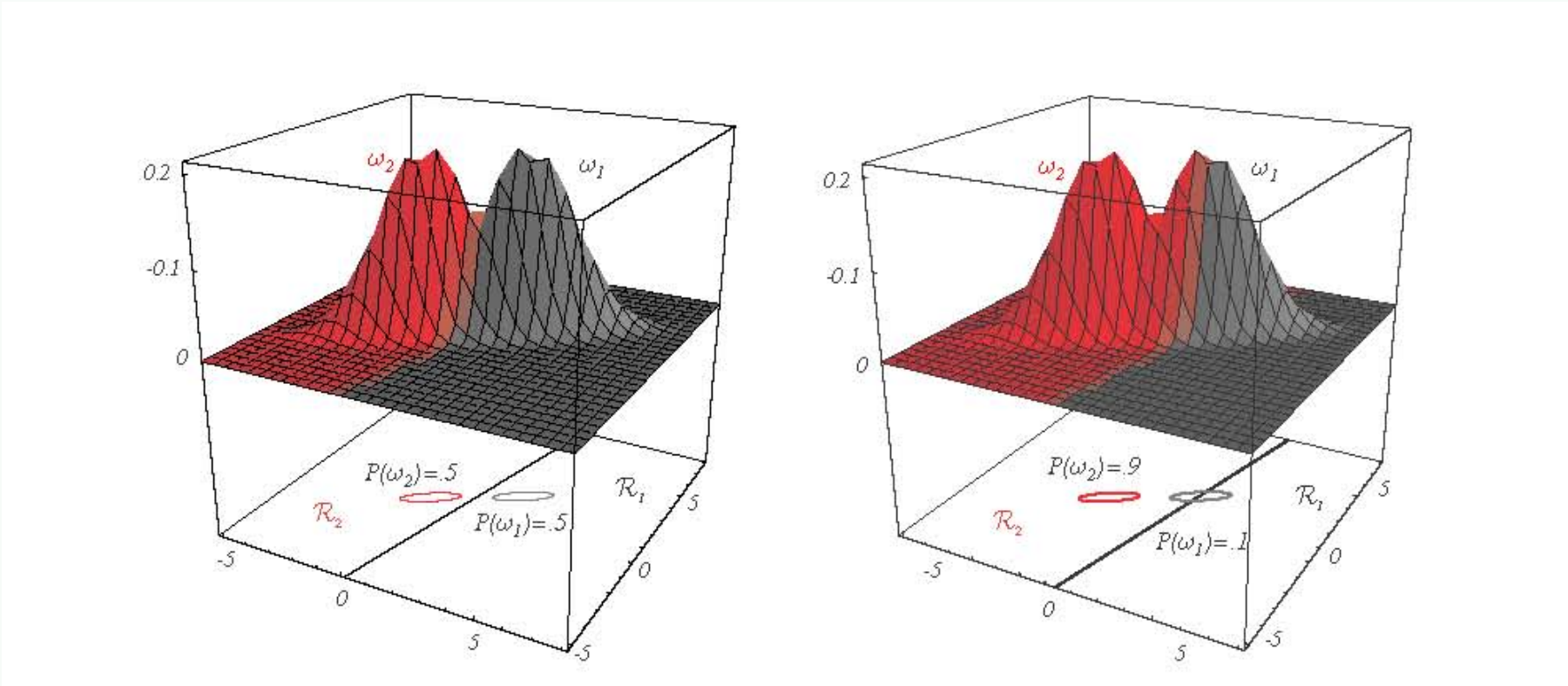
Multivariate Gaussian Density: Case II

- Properties of hyperplane (decision boundary):
  - It passes through $\mathbf{x_0}$
  - It is not orthogonal to the line linking the means.
  - What happens when $P(\omega_i) = P(\omega_j)$?
  - If $P(\omega_i) \neq P(\omega_j)$, then $\mathbf{x_0}$ shifts away from the most likely category.

$$\mathbf{w}^t(\mathbf{x} - \mathbf{x_0}) = 0$$

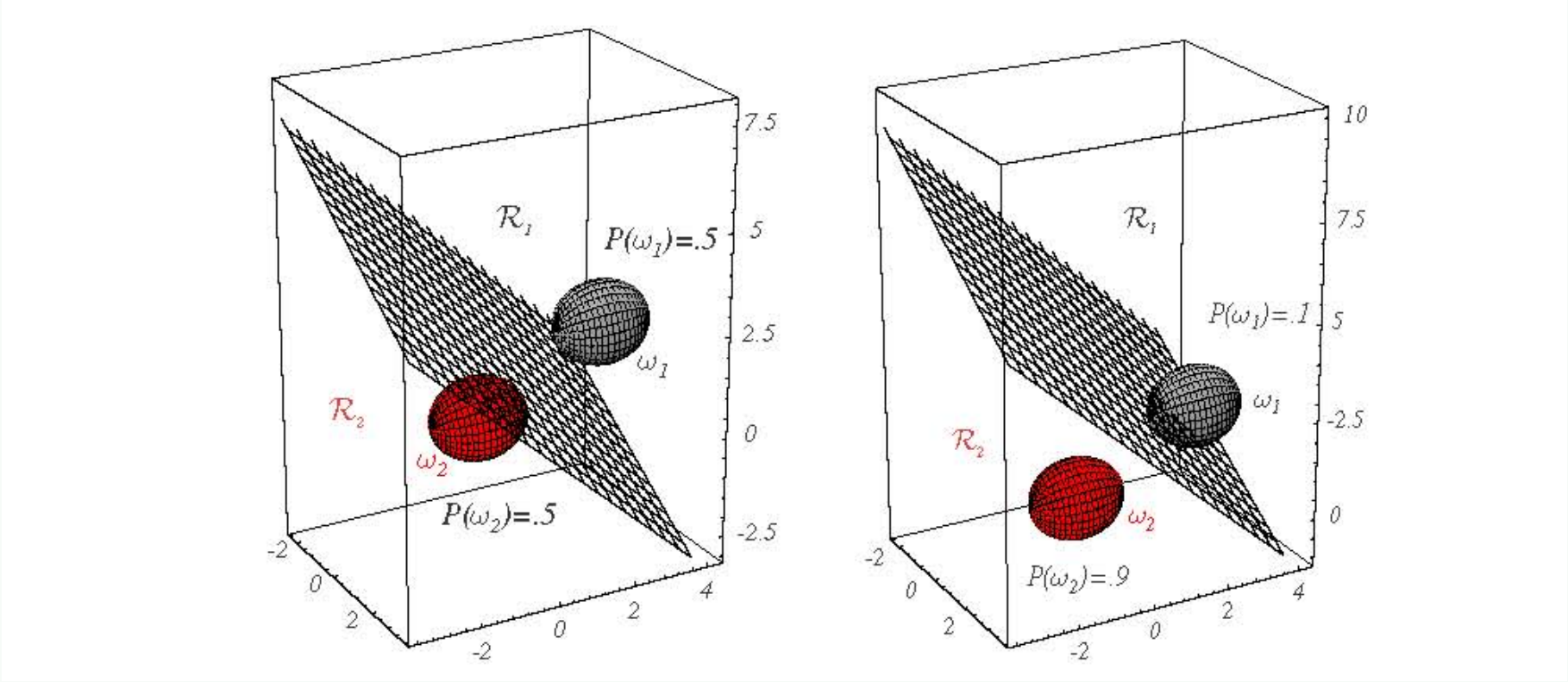
$$\text{where } \mathbf{w} = \Sigma^{-1}(\mu_i - \mu_j) \text{ and } \mathbf{x}_0 = \frac{1}{2}(\mu_i + \mu_j) - \frac{ln[P(\omega_i)/P(\omega_j)]}{(\mu_i - \mu_j)^t \Sigma^{-1}(\mu_i - \mu_j)}(\mu_i - \mu_j)$$

Multivariate Gaussian Density: Case II



If $P(\omega_i) \neq P(\omega_j)$, then $\mathbf{x_0}$ shifts away from the most likely category.

Multivariate Gaussian Density: Case II



If $P(\omega_i) \neq P(\omega_j)$, then $\mathbf{x_0}$ shifts away from the most likely category.

Multivariate Gaussian Density: Case II

- When $P(\omega_i)$ are equal, then the discriminant becomes:

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\mu_\mathbf{i})^t\Sigma^{-1}(\mathbf{x}-\mu_\mathbf{i}) + \ln P(\omega_i)$$

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\mu_\mathbf{i})^t\Sigma^{-1}(\mathbf{x}-\mu_\mathbf{i})$$

- This is known as the Mahalanobis distance (i.e., Mahalanobis distance classifier)

Multivariate Gaussian Density: Case III

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\mu_i)^t\Sigma_i^{-1}(\mathbf{x}-\mu_i) - \frac{d}{2}ln2\pi - \frac{1}{2}\ln|\Sigma_i| + \ln P(\omega_i)$$

- Σ_i= arbitrary (each class has its own covariance matrix)

- The clusters have different shapes and sizes (centered at μ).

- If we disregard $\frac{d}{2}ln2\pi$ (constant):

$$g_i(x) = x^tW_ix + w_i^tx + w_{i0}$$

(*quadratic discriminant*)

where $\mathbf{W}_i = -\frac{1}{2}\Sigma_i^{-1}$, $\mathbf{w}_i = \Sigma_i^{-1}\mu_i$, and $w_{i0} = -\frac{1}{2}\mu_i^t\Sigma_i^{-1}\mu_i - \frac{1}{2}ln|\Sigma_i| + \ln P(\omega_i)$

- Decision boundary is determined by hyperquadrics; setting $g_i(\mathbf{x}) = g_j(\mathbf{x})$

e.g., hyperplanes, pairs of hyperplanes, hyperspheres, hyperellipsoids, hyperparaboloids etc.

Multivariate Gaussian Density: Case III

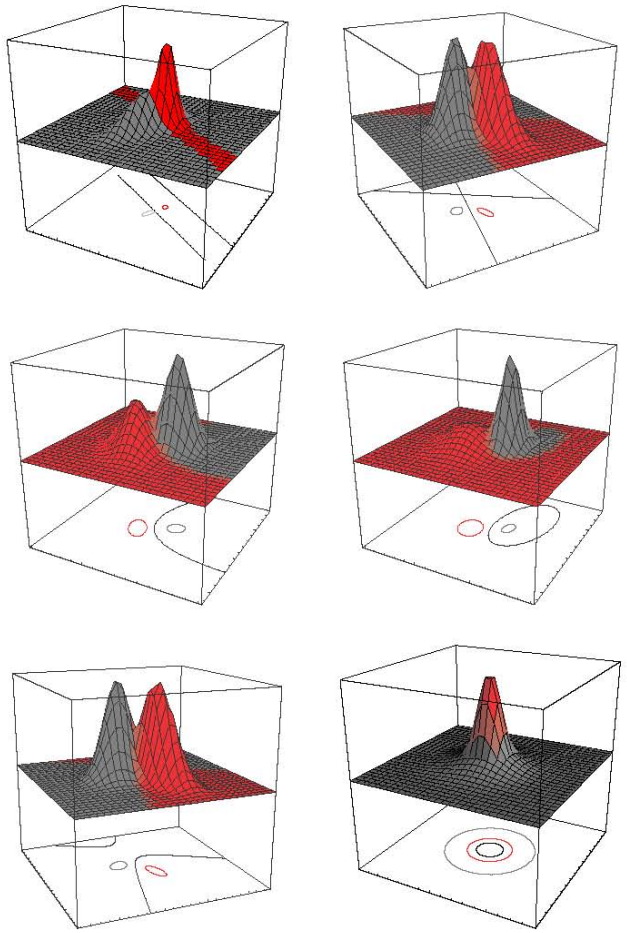
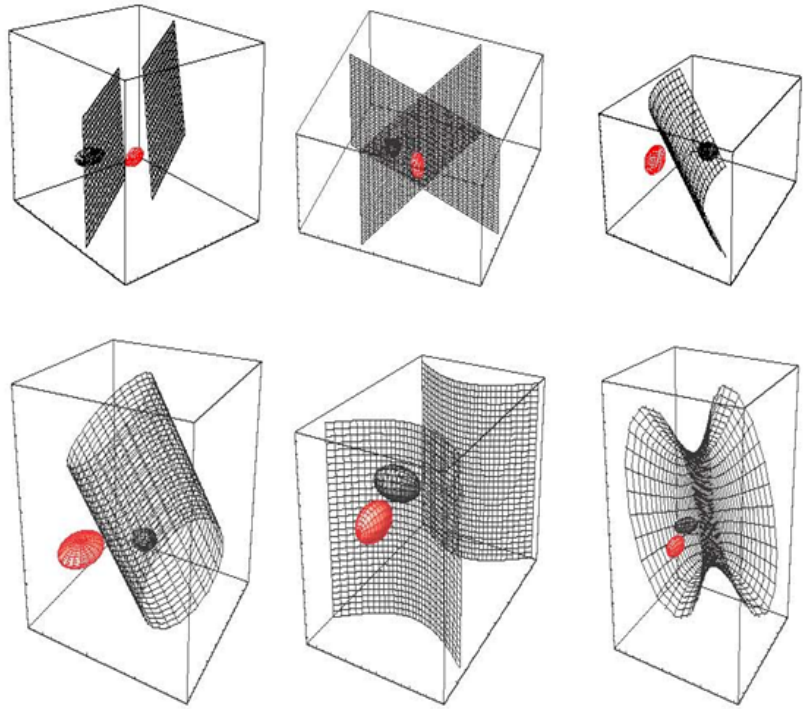


FIGURE 2.14. Arbitrary Gaussian distributions lead to Bayes decision boundaries that are general hyperquadrics. Conversely, given any hyperquadric, one can find two Gaussian distributions whose Bayes decision boundary is that hyperquadric. These variances are indicated by the contours of constant probability density. From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons, Inc.



non-linear decision boundaries

Multivariate Gaussian Density: Case III

Example

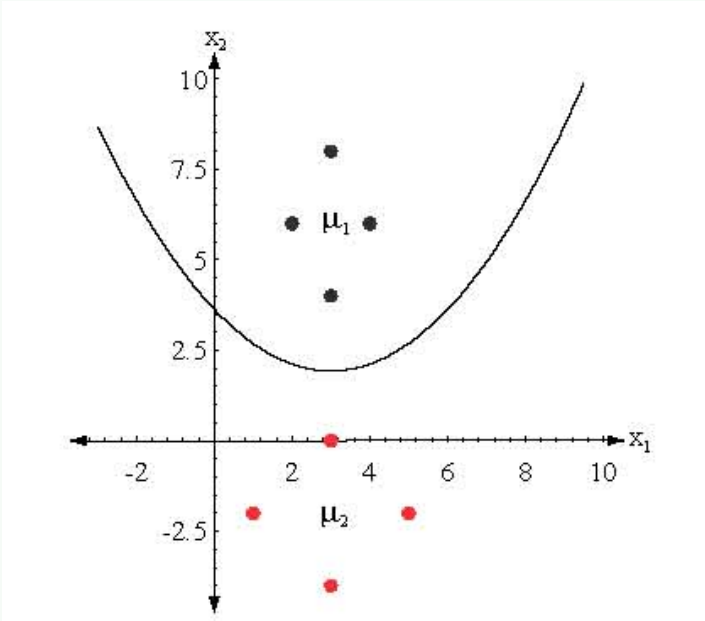
$$\boldsymbol{\mu}_1 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}; \quad \boldsymbol{\Sigma}_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } \boldsymbol{\mu}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}; \quad \boldsymbol{\Sigma}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Decision boundary:

$$x_2 = 3.514 - 1.125x_1 + 0.1875x_1^2.$$

$P(\omega_1)=P(\omega_2)$

boundary does not pass through midpoint of μ_1,μ_2



Error Bounds

- Exact error calculations could be difficult – easier to estimate error bounds.

$$P(error) = \int P(error, \mathbf{x})\, d\mathbf{x} = \int P(error/\mathbf{x}) p(\mathbf{x})\, d\mathbf{x}$$

$$P(error/\mathbf{x}) = \begin{cases} P(\omega_1/\mathbf{x}) & \text{if we decide } \omega_2 \\ P(\omega_2/\mathbf{x}) & \text{if we decide } \omega_1 \end{cases}$$

or $\min[P(\omega_1/x), P(\omega_2/x)]$

$$P(\omega_j / x) = \frac{p(x/\omega_j)P(\omega_j)}{p(x)}$$

- Using the inequality:

$$min[a, b] \le a^{\beta} b^{1-\beta}, \qquad a, b \ge 0, 0 \le \beta \le 1$$

P(error) $= \int min[p(\mathbf{x}/\omega_1)P(\omega_1),\ p(\mathbf{x}/\omega_2)P(\omega_2)]d\mathbf{x} \le$

$$P^{\beta}(\omega_1)P^{1-\beta}(\omega_2) \int p^{\beta}(\mathbf{x}/\omega_1)\ p^{1-\beta}(\mathbf{x}/\omega_2)d\mathbf{x} = e^{-\kappa(\beta)}$$

Error Bounds

- If the class conditional distributions are Gaussian, then

$$\int p^{\beta}(\mathbf{x}/\omega_1)\, p^{1-\beta}(\mathbf{x}/\omega_2)d\mathbf{x} = e^{-\kappa(\beta)}$$

where:

$$k(\beta) = \frac{\beta(1-\beta)}{2}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)^t \left[(1-\beta)\boldsymbol{\Sigma}_1 + \beta\boldsymbol{\Sigma}_2\right]^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) + \frac{1}{2}\ln\frac{\left|(1-\beta)\boldsymbol{\Sigma}_1 + \beta\boldsymbol{\Sigma}_2\right|}{|\boldsymbol{\Sigma}_1|^{1-\beta}|\boldsymbol{\Sigma}_2|^{\beta}}.$$

Warning: certain prints of our textbook have a typo!

determinant

Error Bounds

- The *Chernoff* bound is obtained by minimizing $e^{-\kappa(\beta)}$
  - This is a 1-D optimization problem, regardless to the dimensionality of the class conditional densities.

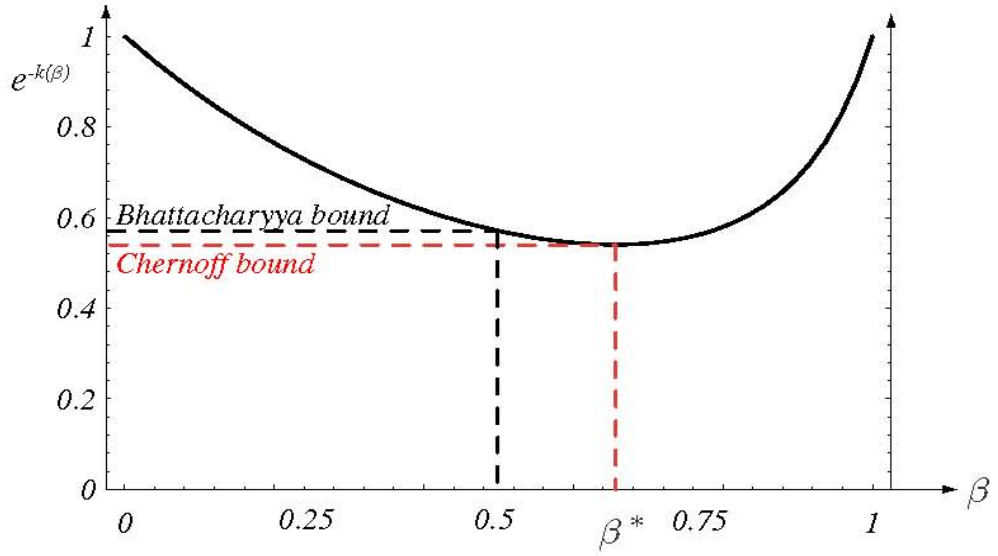


FIGURE 2.18. The Chernoff error bound is never looser than the Bhattacharyya bound. For this example, the Chernoff bound happens to be at $\beta^* = 0.66$, and is slightly tighter than the Bhattacharyya bound ($\beta = 0.5$). From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons, Inc.

Error Bounds

- The *Bhattacharyya* bound is obtained by setting β=0.5
  - Easier to compute than Chernoff error but looser.

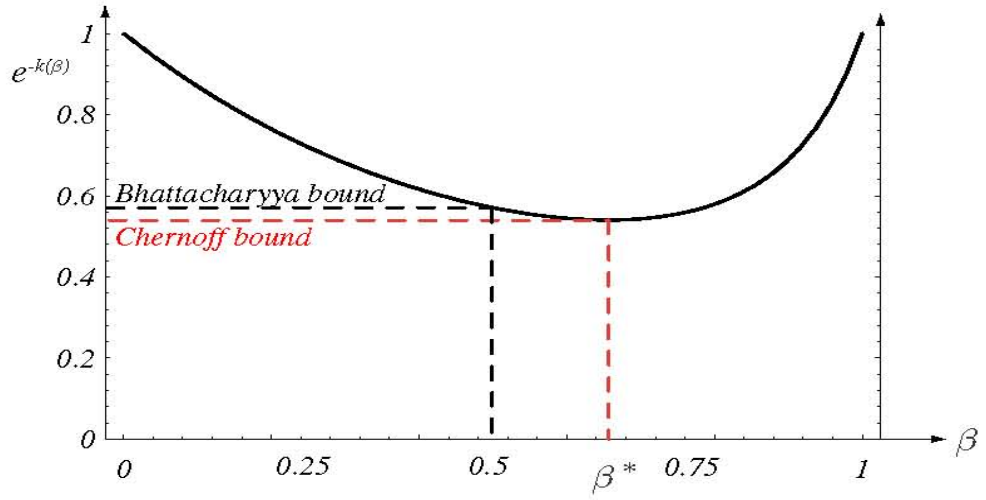


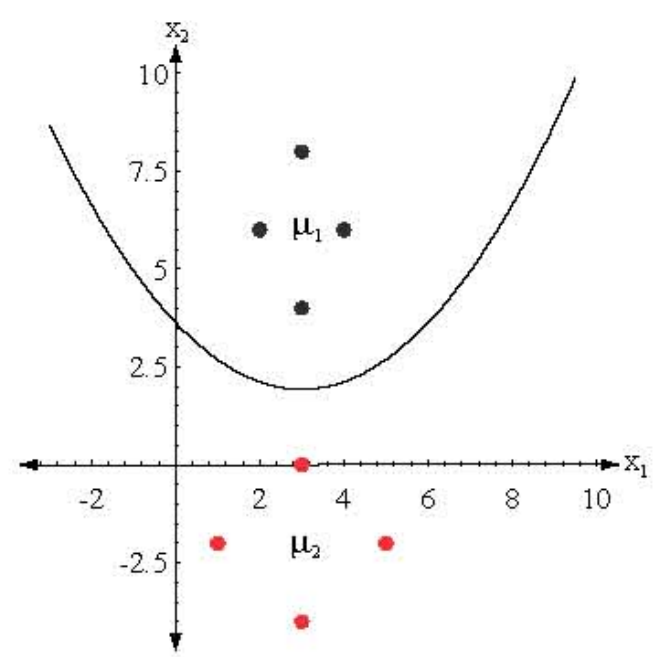
FIGURE 2.18. The Chernoff error bound is never looser than the Bhattacharyya bound. For this example, the Chernoff bound happens to be at $\beta^* = 0.66$, and is slightly tighter than the Bhattacharyya bound ($\beta = 0.5$). From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons, Inc.

- **Note**: the Chernoff and Bhattacharyya bounds will not be good error bounds if the densities are not Gaussian.

Multivariate Gaussian Density: Case III

Example

$$k(\beta) = \frac{\beta(1-\beta)}{2}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)^t\left[(1-\beta)\boldsymbol{\Sigma}_1+\beta\boldsymbol{\Sigma}_2\right]^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2) + \frac{1}{2}\ln\frac{\left|(1-\beta)\boldsymbol{\Sigma}_1+\beta\boldsymbol{\Sigma}_2\right|}{|\boldsymbol{\Sigma}_1|^{1-\beta}|\boldsymbol{\Sigma}_2|^{\beta}}.$$



$$\boldsymbol{\mu}_1 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}; \quad \boldsymbol{\Sigma}_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\boldsymbol{\mu}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}; \quad \boldsymbol{\Sigma}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

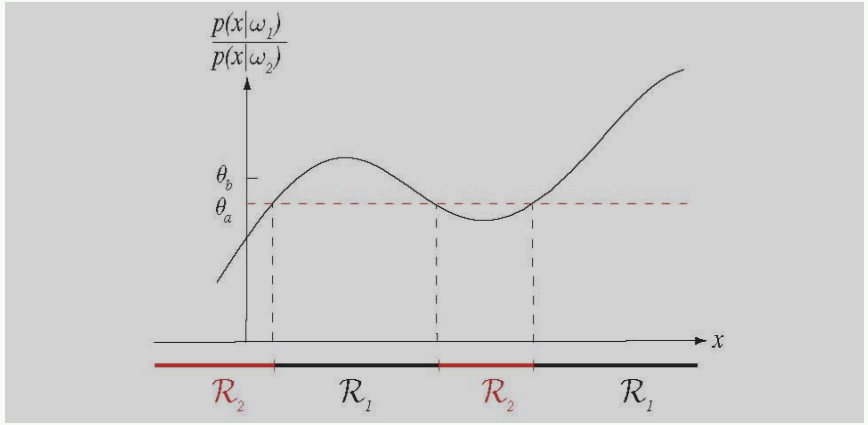
Bhattacharyya error:

k(0.5)=4.06

$$P(error) \leq 0.0087$$

Receiver Operating Characteristic (ROC) Curve

- Every classifier typically employs some kind of a threshold.



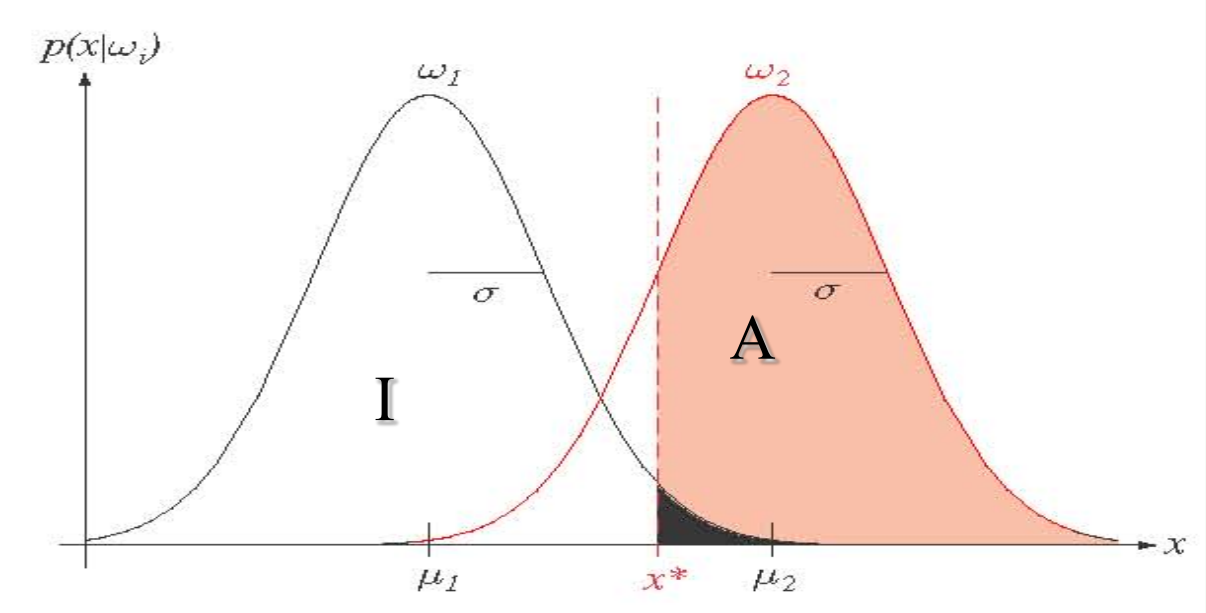
$$\theta_a = P(\omega_2)/P(\omega_1)$$

$$\theta_b = \frac{P(\omega_2)(\lambda_{12}-\lambda_{22})}{P(\omega_1)(\lambda_{21}-\lambda_{11})}$$

- Changing the threshold can affect the performance of the classifier.
- ROC curves allow us to evaluate/compare the performance of a classifier using different thresholds.

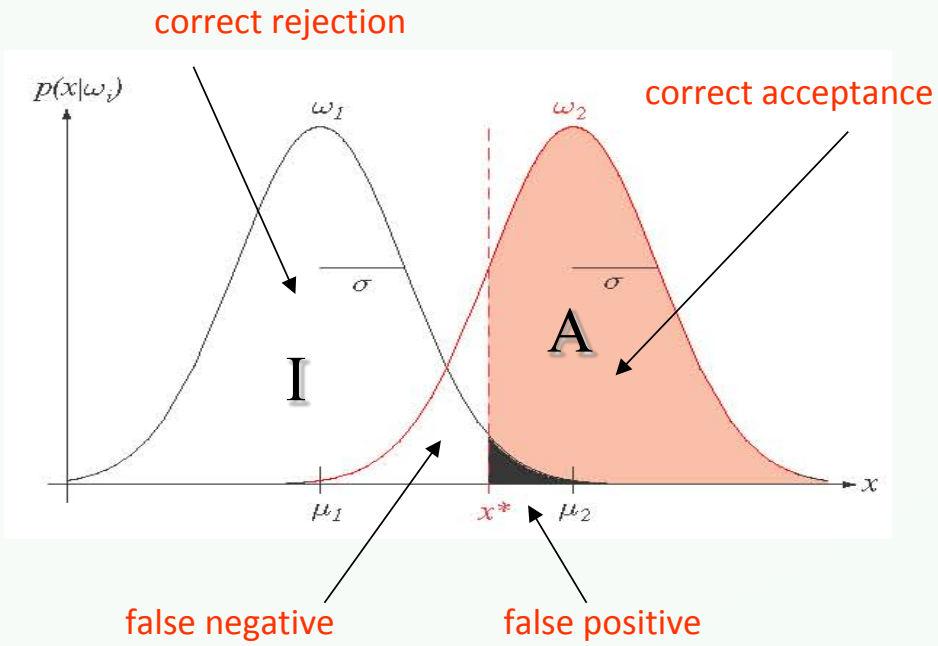
Example: Person Authentication

- Authenticate a person using biometrics (e.g., fingerprints).
- There are two possible distributions (i.e., classes):
  - *Authentic* (A) and *Impostor* (I)



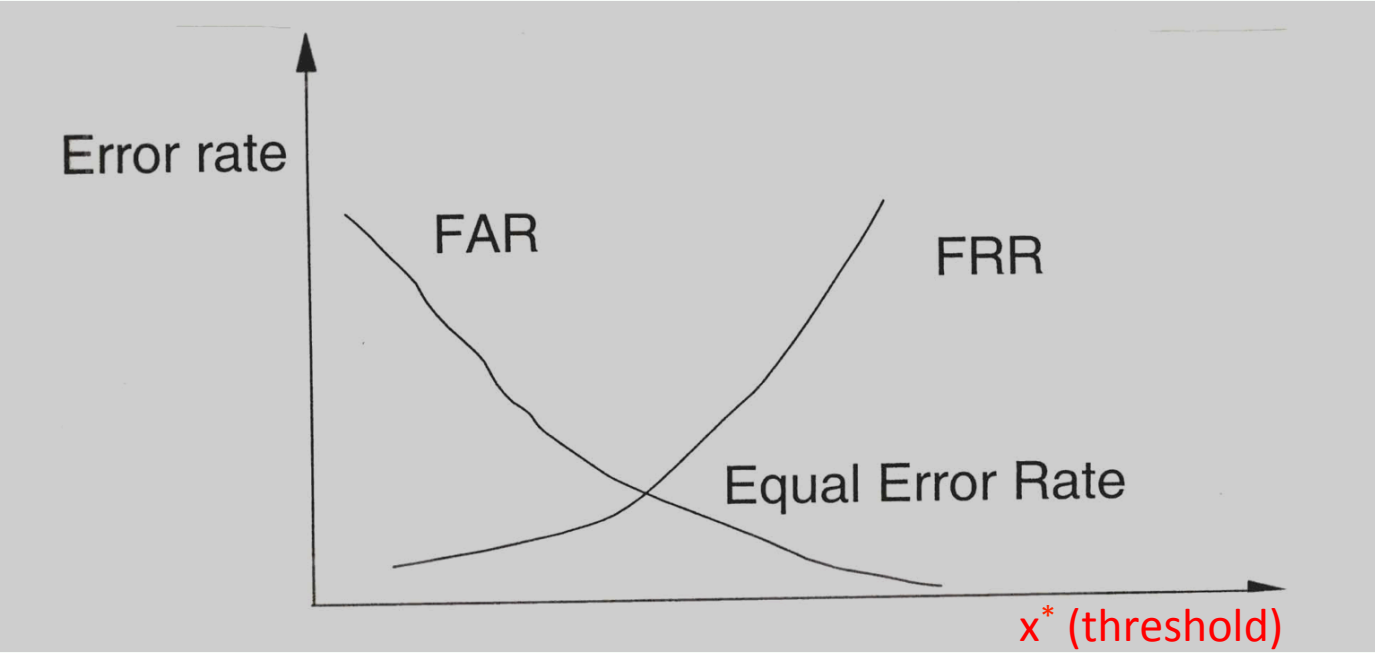
Example: Person Authentication

- Possible decisions:
  - (1) **correct acceptance** (true positive):
    - X belongs to A, and we decide A
  - (2) **incorrect acceptance** (false positive):
    - X belongs to I, and we decide A
  - (3) **correct rejection** (true negative):
    - X belongs to I, and we decide I
  - (4) **incorrect rejection** (false negative):
    - X belongs to A, and we decide I



Error vs Threshold

ROC Curve

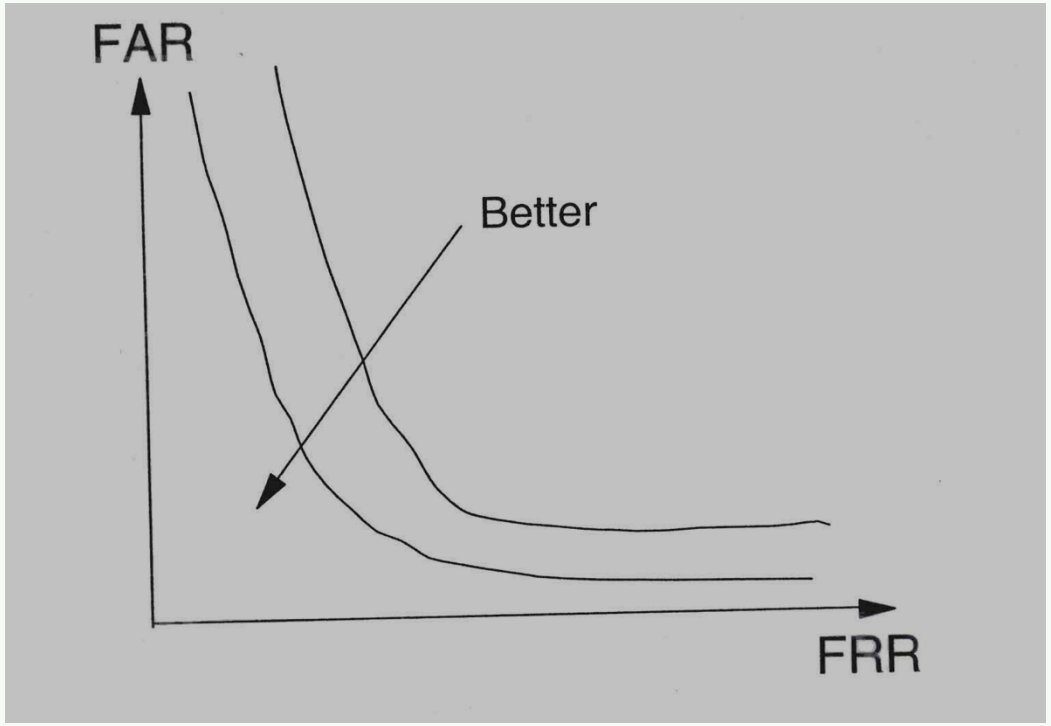


FAR: False Accept Rate (False Positive)
FRR: False Reject Rate (False Negative)

False Negatives vs False Positives

More common to plot
FRR vs FAR

ROC Curve

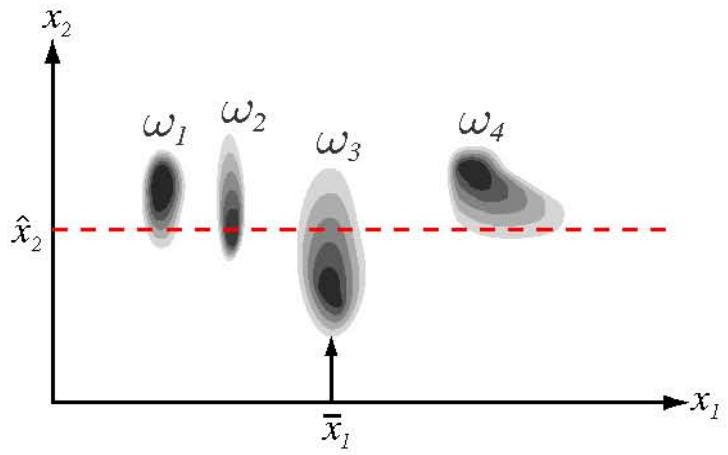


FAR: False Accept Rate (False Positive)
FRR: False Reject Rate (False Negative)

Missing Features

- Suppose **x**=(x_1,x_2) is a test vector where x_1 is missing and $x_2 = \hat{x}_2$
  - how would we classify it?

Example:



- If we set x_1 equal to the average value, we will classify **x** as ω_3
- But $p(\hat{x}_2 / \omega_2)$ is larger; should we classify **x** as ω_2 ?

Marginalize Posterior Probability

- Suppose $\mathbf{x}=[\mathbf{x}_g, \mathbf{x}_b]$ ($\mathbf{x}_g$: good features, $\mathbf{x}_b$: bad features)
- Derive the Bayes rule using the good features:

$$P(\omega_i/\mathbf{x}_g) \overset{p}{=} \frac{P(\omega_i, \mathbf{x}_g)}{p(\mathbf{x}_g)} \overset{p}{=} \frac{\int P(\omega_i, \mathbf{x}_g, \mathbf{x}_b)d\mathbf{x}_b}{p(\mathbf{x}_g)} =$$

$$\frac{\int P(\omega_i/\mathbf{x}_g, \mathbf{x}_b)p(\mathbf{x}_g, \mathbf{x}_b)d\mathbf{x}_b}{p(\mathbf{x}_g)} = \frac{\int P(\omega_i/\mathbf{x}_g, \mathbf{x}_b)p(\mathbf{x})d\mathbf{x}_b}{\int p(\mathbf{x})d\mathbf{x}_b}$$

marginalize over "bad" features.

Decide ω_1 if $P(\omega_1/\mathbf{x}_g) > P(\omega_2/\mathbf{x}_g)$; otherwise decide ω_2

Quiz

- When:
- What: Bayesian Decision Theory (case studies will not be inclu
  ded in the quiz)