Yonsei University, Department of Applied Statistics

손해보험통계 필기노트

Contents

1 Review

9/4 (월)

Discrete rv:

we say X is a discrete rv if X can take a countable number of values. we denote by p its pmf

$$p(x) = \Pr(X = x)$$

Its *cdf* and the survival ft. are

$$P(x) = \Pr(X \leq x) = \sum_{y \leq x} p(y)$$

and

$$\overline{P}(x) = 1 - P(x) = \Pr(X > x) = \sum_{y > x} p(y)$$

The probability generating ft. (pgf)

$$P_X(z) = E\left(z^X\right) = \sum_{all\ x} z^x p(x)$$

Cont. rv:

For a cont. rv X, we denote by f its density ft. Its *cdf* and the survival ft. are

$$F(x) = \int_{-\infty}^{x} f(y)dy$$

and

$$\overline{F}(x) = 1 - F(x) = \int_{x}^{\infty} f(y)dy$$

For various reasons, another quantity is important:

$$h(x) = \frac{f(x)}{\overline{F}(x)} = \frac{-\frac{d}{dx}\overline{F}(x)}{\overline{F}(x)} = -\frac{d}{dx}\log \overline{F}(x)$$

This hazard rate (failure rate or force of mortality) is particularly interesting to assess the thickness of the right-hand tail of the loss dist'n

Finally, the mgf of X is

$$M_X(t) = E\left[e^{tX}\right] = \int e^{tx} f(x)dx$$

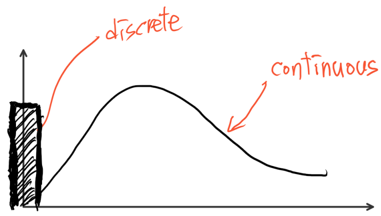
where

$$\frac{d^n}{dt^n} M_X(t)|_{t=0} = E(X^n)$$

9/6 (수)

Mixed rv

· Combination of discrete & continuous rv's



· Suppose that on intervals of the form (x_i, x_{i+1}) for $i = 1, \cdots, m \quad (m < \infty)$, there exists a density(cont. part) $f(x)$ for X.

Also, suppose that X has discrete mass points at the boundaries of these intervals (i.e., $x_1, x_2, \cdots, x_m, x_{m+1}$)

$$\Pr(X = x_i) = p(x_i), \quad i = 1, 2, \cdots, m+1$$

(Mixed rv의 분포를 그림으로 표현하려면 c.d.f로 표현해주는 것이 편함. discrete part 에서는 점프가 일어남)

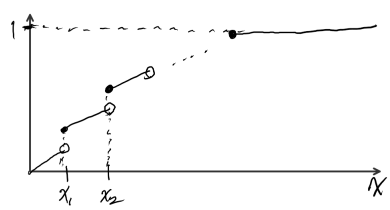
Then X is a mixed rv and we have

$$\sum_{i=1}^{m+1} p(x_i) + \sum_{i=1}^{m} \int_{x_i}^{x_{i+1}} f(x)dx = 1$$

· CDF

$$F(x) = \Pr(X \leq x) = \sum_{i=1}^{m+1} p(x_i) \cdot \mathbb{1}(x_i \leq x) + \sum_{i=1}^{m} \int_{x_i}^{x_{i+1}} f(y) \cdot \mathbb{1}(y \leq x) dy$$

(여기서, $\mathbb{1}(\cdot)$는 Indicator ft.)



· n^{th} moment

$$E(X^n) = \sum_{i=1}^{m+1} (x_i)^n p(x_i) + \sum_{i=1}^{m} \int_{x_i}^{x_{i+1}} x^n f(x)dx$$

· MGF & PGF

$$M_X(t) = E\left[e^{tX}\right] = \sum_{i=1}^{m+1} e^{tx_i} p(x_i) + \sum_{i=1}^{m} \int_{x_i}^{x_{i+1}} e^{tx} f(x)dx$$

$$P(z) = E\left[z^X\right] = \sum_{i=1}^{m+1} z^{x_i} p(x_i) + \sum_{i=1}^{m} \int_{x_i}^{x_{i+1}} z^x f(x)dx$$

Example. For a mixed rv X with a single mass point at 0 (i.e., $m = 1$, $x_1 = 0$, $x_2 = \infty$, $p(0) = p$, and $p(\infty) = 0$), it is immediate that

$$p + \int_0^\infty f(x)dx = 1$$

which implies that

$$\int_0^\infty f(x)dx = 1 - p$$

CDF of X:

$$F(x) = p + \int_0^x f(y)dy, \quad x > 0$$

Also

$$E(X^n) = \cancel{p} + \int_0^\infty x^n f(x)dx$$

$$M_X(t) = p + \int_0^\infty e^{tx} f(x)dx$$

2 Introduction

· Random events in insurance business

- Early death (mortgage, family to support)
- Destruction of property by storm, fire or other natural/ man-made hazards
- Car accidents (car damage, personal injury)
- Short & long term disability

· Collective risk model: modelization at the macro-level (portfolio-wise).

To develop a collective risk model we

(1) determine a dist^n for the total # of claims for the entire portfolio.

Then

(2) determine a dist^n for the loss amt($\equiv$ amount) per claim.

Finally

(3) combine both components to obtain a model for the total amt of claims over a given time period.

9/11 (월)

In this course we focus on the collective risk model:

$$S = \sum_{i=1}^{N} X_i$$

(여기서, N은 'frequency of loss'를 의미하며 discrete random variable이고, X_i는 'severity of loss'를 의미하며 random variable이다. S는 'aggregate loss' 를 의미한다.)

Terminologies

· Severity dist^n: Dist^n of the loss amt paid by insurer for a given loss/claim.

· Frequency dist^n: Dist^n of the # of losses/claims paid by insurer per unit time interval.

· We also refer to the total amt of all losses over a given time period as newmacroname the aggregate loss and to the total of all amts paid by the aggregate payment.

NOTE: There is a distinction to be made between the aggregate loss and the aggregate payment due to the presence of claim adjustments.
(deductible, limit, coinsurance) in an insurance policy.

3 Basic distributional quantities

The level of exposure to risk/loss X is often summarized by one number, or at least a small set of numbers, called risk measures.

[examples of risk measures]

- Mean: measure of central tendancy
- Variance: measure of dispersion
- Higher moments $E(X^k)$, $\quad k = 2, 3, \cdots$
- $E(X) + k\sqrt{Var(X)}$ for some constant k.

In what follows we consider two additional risk measures of the risk X. These two are relatively new and popular. These are also used to assess the tail-thickness of the dist^n of X.

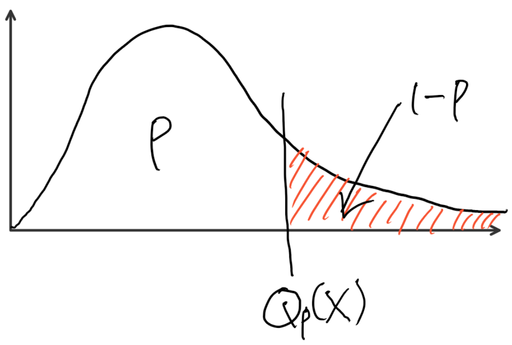
3.1 Value-at-Risk (VaR)

The Value-at-Risk of X at the $100p\%$ level, denoted $VaR_p(X)$ or $Q_p(X)$, is given by

$$VaR_p(X) = Q_p(X) = \inf\{x \in \mathbb{R} : \Pr(X > x) \le 1 - p\}$$

For a cont rv X, it is the solution to

$$\Pr(X < Q_p(X)) = 1 - p$$



· $VaR_p(X)$ of $Q_p(X)$ is the $100p^{\text{th}}$ percentile of the dist^n of X

· One would expect the risk X to exceed the threshold $Q_p(X)$ in at most $100(1-p)\%$ cases.

· Shortcomings:

- fail to communicate how bad the risk X can be in those $100(1-p)\%$ cases.
- fail to satisfy some desirable properties of a risk measure.

Example. $X \sim$ Pareto

$$F_X(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha, \quad x > 0$$

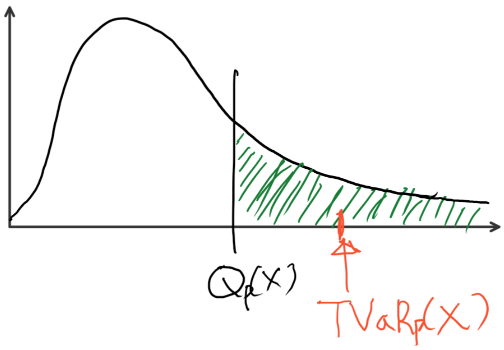
To obtain VaR,

$$F_X(Q_p(X)) = p \iff 1 - \left(\frac{\theta}{Q_p(X)+\theta}\right)^\alpha = p$$
$$\iff Q_p(X) = \theta\left[(1-p)^{-1/\alpha} - 1\right]$$

3.2 Tail-VaR ($TVaR$)

The $TVaR$ of X at the $100p\%$ level, denoted $TVaR_p(X)$, is the expected loss given that the loss exceeds $VaR_p(X)$. i.e.

$$TVaR_p(X) = E\left[X \mid X > Q_p(X)\right]$$



· More conservative than VaR:

$$TVaR_p(X) > VaR_p(X)$$

· Better indicator of the tail loss than VaR: Average loss of the risk X given that the loss exceeds VaR.

· Provides an answer to how bad the risk X can be in those $100(1-p)\%$ cases.

· a.k.a **C**onditional **T**ail **E**xpectation (CTE) or **T**ail **C**onditional **E**xpectation (TCE) or **E**xpected **S**hortfall (ES) or **C**onditional **V**aR ($CVaR$)

By definition,

$$\begin{aligned} TVaR_p(X) &= E\left[X \mid X > Q_p(X)\right] \\ &= E\left[(X - Q_p(X)) \mid X > Q_p(X)\right] + Q_p(X) \\ &= e_X\left(Q_p(X)\right) + Q_p(X), \end{aligned}$$

where, $e_X(d) = E\left[X - d \mid X > d\right]$ is the mean excess loss or

the mean residual lifetime (MRL)

Note that for a cont. X

$$\begin{aligned} e_X\left(Q_p(X)\right) &= E\left[X - Q_p(X) \mid X > Q_p(X)\right] \\ &= \frac{E\left[(X - Q_p(X)) \cdot \mathbb{1}\left\{X > Q_p(X)\right\}\right]}{\Pr\left(X > Q_p(X)\right)} \\ &= \frac{E\left[\max\left(X - Q_p(X),\ 0\right)\right]}{1-p} \\ &= \frac{E\left[\left(X - Q_p(X)\right)_+\right]}{1-p} \end{aligned}$$

where,

$$(k)_+ \equiv \max\left(k,\ 0\right)$$

Example. $X \sim$ Pareto,

$$\begin{aligned} e_X(d) &= \int_d^\infty (x-d) \cdot f_{X|X>d}(x)dx \\ &= \frac{\int_d^\infty (x-d) f(x)dx}{1-F(d)} \qquad (1) \\ &= \int_d^\infty \frac{1-F_X(x)}{1-F(d)} dx \qquad (2) \\ &= \int_d^\infty \frac{\left(\dfrac{\theta}{x+\theta}\right)^\alpha}{\left(\dfrac{\theta}{d+\theta}\right)^\alpha} dx = (d+\theta)^\alpha \frac{(d+\theta)^{-\alpha+1}}{\alpha-1} \\ &= \frac{d+\theta}{\alpha-1}, \quad \alpha > 1 \end{aligned}$$

As a result,

$$\begin{aligned} TVaR(X) &= e_X\left(Q_p(X)\right) + Q_p(X) \\ &= \frac{Q_p(X)+\theta}{\alpha-1} + Q_p(X) = \frac{\alpha Q_p(X)+\theta}{\alpha-1} \\ &= \frac{\theta\alpha\left[(1-p)^{-1/\alpha} - 1\right] + \theta}{\alpha-1} = \cdots = \frac{\theta\alpha(1-p)^{-1/\alpha}}{\alpha-1} - \theta \end{aligned}$$

식 (1)은

$$P\left[X = x \mid X > d\right] = \frac{\Pr\left(X = x \ \&\ X > d\right)}{\Pr\left(X > d\right)} = \begin{cases} \dfrac{\Pr\left(X = x\right)}{\Pr\left(X > d\right)}, & X > d \\ 0, & X < d \end{cases}$$

식 (2)는

$$\begin{aligned}
\int_d^\infty 1-F(x)dx &= \int_d^\infty \overbrace{\int_x^\infty f(y)dy}^{1-F(x)} dx \\
&= \int_d^\infty \int_d^y f(y)dxdy \quad (\because \text{Variable transformation}) \\
&= \int_d^\infty f(y)\underbrace{\left(\int_d^y 1\,dx\right)}_{=y-d}dy = \int_d^\infty f(y)(y-d)dy
\end{aligned}$$

9/13 (수)

4 Severity dist^n

4.1 Introduction

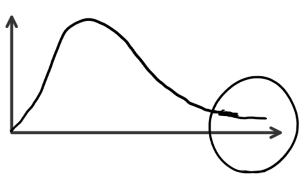
· When a loss occurs, the amount of loss is **NOT** necessarily the amount paid by insurer, due to some common forms of adjustments(= modifications) (eg. deductible, policy limit, and coinsurance)

· Thus there is a distinction to be made between the actual loss prior to the application of any adjustment (called the ground-up loss) and the amt paid by insurer. Throughout this course we denote the ground-up loss by X.

· Due to the adjustments, data of ground-up losses and the data of claim payments shall be treated differently for estimation purposes. Indeed, all losses are reported under ground-up loss basis while policy adjustments may cause some losses to not be reported (or partly reported) on the payment basis.

· Our objective is to find a reasonable model for the ground-up loss X, which has some desirable properties:

- X is distributed on $\mathbb{R}^+$ (non-negative)
- positively skewed



4.2 Parametric approach

· Treat historical losses as a sample from the underlying (unknown) dist^n of X.

· Based on the shape of the empirical dist^n, choose a continuous parametric dist^n as a candidate to model X. (MSE, MME, $\cdots$)

Def. [**Parametric dist^n**]

Parametric dist^n is defined as the set of dist^n fts, each member of which is determined by specifying one or more values called parameters.

· Examples of non-negative, skewed parametric dist^{n}s: Exponential, Gamma, Weibull, Lognormal, Pareto, $\cdots$

· In what follows, some well-known techniques are proposed to create new parametric dist^{n}s from existing ones.

4.2.1 Multiplication by a constant

Let X be a continuous rv with density f_X and cdf F_X. Define Y=cX for constant $c > 0$. Then

$$\begin{aligned} F_Y(y) &= \Pr(Y \le y) = \Pr(cX \le y) \\ &= \Pr\left(X \le \frac{y}{c}\right) = F_X\left(\frac{y}{c}\right) \\ f_Y(y) &= \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_Y\left(\frac{y}{c}\right) = \frac{1}{c}f_X\left(\frac{y}{c}\right) \end{aligned}$$

Def. [**Scale dist^n**]

We say that a parametric dist^n is a scale dist^n if, when X is from the set of dist^{n}s, so is $Y = cX$.

Example. $X \sim \mathrm{Exp}(\theta)$, 여기서 θ 는 scale parameter.

$$f_X(x) = \frac{1}{\theta}e^{-\frac{x}{\theta}}, \quad x > 0$$

Letting $Y = cX$, it follows that

$$f_Y(y) = \frac{1}{c}f_X\left(\frac{y}{c}\right) = \frac{1}{c}\frac{1}{\theta}e^{-\frac{y}{c\theta}}, \quad y > 0$$

, which implies that $Y \sim \text{Exp}(c\theta)$. Thus the exponential dist^n is a scale dist^n.

Def.

For rv's with non-negative support, a scale parameter is a parameter for a scale dist^n that meets 2 conditions:

1. When a scale dist^n is multiplied by a constant, the parameter is also multiplied by the same constant.
2. All the other parameters remain unchanged.

Example. If $X \sim \text{Exp}(\theta)$, then $Y \sim \text{Exp}(c\theta) \Longrightarrow \theta$ is a scale parameter.

Example. If X is a exponential rv with density

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0$$

then $Y = cX$ has density

$$f_Y(y) = \frac{\lambda}{c} e^{-\frac{\lambda}{c} x}, \quad y > 0$$

$\Longrightarrow$ clearly, λ is **NOT** a scale parameter. (1번 정의에 위배됨. '상수의 곱'이여야되는데 '상수의 나누기'이기 때문에.)

9.18 (월)

4.2.2 Raising to a power

Let X be a continuous rv with density $f(x)$ and cdf $F(x)$ with $F(0) = 0$. Define $Y = X^{\frac{1}{\tau}}$

a) For $\tau > 0$ (called transformed)Y

$$\begin{aligned} F_Y(y) &= \Pr\left(X^{1/\tau} \le y\right) = \Pr\left(X \le y^\tau\right) = F_X(y^\tau) \\ f_Y(y) &= \frac{d}{dy}F_Y(y) = \tau y^{\tau-1} f_X(y^\tau) \end{aligned}$$

b) For $\tau < 0$ (called inverse transformed; when $\tau = -1$, simply called "inverse")

$$\begin{aligned} F_Y(y) &= \Pr\left(X^{1/\tau} \le y\right) = P\left(X > y^\tau\right) \\ &= 1 - F_X(y^\tau) \\ f_Y(y) &= \frac{d}{dy}F_Y(y) = \underbrace{-\tau}_{>0} y^{\tau-1} f_X(y^\tau) \end{aligned}$$

Example. $X \sim \text{Exp}(\theta)$ and let $Y = X^{1/\tau}$, $\tau > 0$. Then

$$\begin{aligned} F_Y(y) &= F_X(y^\tau) = 1 - \exp\left(-\frac{y^\tau}{\theta}\right) \\ &= 1 - e^{-\left(\frac{y}{\alpha}\right)^\tau} \end{aligned}$$

where $\alpha = \theta^{1/\tau} \implies Y \sim \text{Weibull}(\alpha, \tau)$

4.2.3 Exponentiation

Let X be a continuous rv with density $f_X(x)$ and cdf $F_X(x)$. Define $Y = e^X$. Then

$$\begin{aligned} F_Y(y) &= \Pr\left(e^X \le y\right) = \Pr\left(X \le \log y\right), \quad \log \equiv \ln \\ &= F_X(\log y) \\ f_Y(y) &= \frac{1}{y} f_X(\log y) \end{aligned}$$

Example. $X \sim N(\mu, \sigma^2)$ and $Y = e^X$.

$$
\begin{aligned}
F_Y(y) &= \Phi\left(\frac{\log y - \mu}{\sigma}\right), \quad \Phi: \text{ cdf of } Z \sim N(0,1), \quad \phi: \text{ density of } Z \\
f_y(Y) &= \frac{d}{dy} F_Y(y) \\
&= \phi\left(\frac{\log y - \mu}{\sigma}\right) \cdot \frac{1}{y\sigma} \\
&= \frac{1}{y\sigma} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{(\log y - \mu)^2}{2\sigma^2}}
\end{aligned}
$$

4.2.4 Mixture of dist^{n}s

· useful technique to create new dist^{n}s

· Define X conditional on a second (indep) rv, say Θ (called the mixing rv), Θ can be either discrete or continuous.

· The resulting unconditional dist^n of X is a mixed dist^n

Discrete n-point mixture

Let Θ be a discrete rv on the integers $\{1, 2, \cdots, n\}$ with

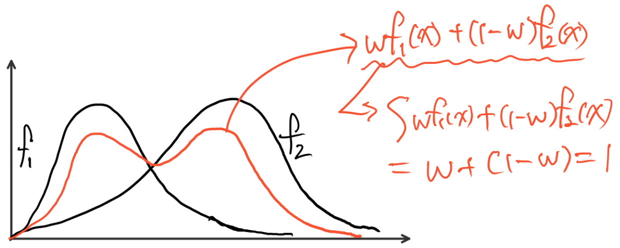
$$\Pr(\Theta = i) = a_i, \quad i = 1, 2, \cdots, n$$

where $a_i \geq 0$ and $a_1 + \cdots + a_n = 1$. Also let Y_i be a rv distributed as $X|\Theta = i$, ie,

$$F_{Y_i}(x) = \Pr(X \leq x \mid \Theta = i), \quad x \geq 0$$

the unconditional rv X is said to have a mixed dist^n with cdf

$$
\begin{aligned}
F_X(X) &= \sum_{i=1}^{n} \Pr(\Theta = i) \Pr(X \leq x \mid \Theta = i) = \sum_{i=1}^{n} a_i F_{Y_i}(x) \\
f_X(x) &= \sum a_i f_{Y_i}(x)
\end{aligned}
$$



Example. $Y_i \sim \text{Exp}(1/i), \quad i = 1, 2, 3$ Define X to be an equal mixture of these 3 dist^{n}s. Then

$$\begin{aligned}
f_X(x) &= \frac{1}{3} \times e^{-x} + \frac{1}{3} \times \left(\frac{1}{2} e^{-x/2}\right) + \frac{1}{3} \times \left(\frac{1}{3} e^{-x/3}\right) \\
\overline{F}_X(x) &= \frac{1}{3} \left\{ e^{-x} + e^{-x/2} + e^{-x/3} \right\} \\
E(X) &= \frac{1}{3} \left\{ E(Y_1) + E(Y_2) + E(Y_3) \right\}
\end{aligned}$$

Continuous mixture

Let Θ be a continuous rv with density $\pi(\theta)$ and let $X|\Theta = \theta$ have (conditional) cdf

$$F_{X|\Theta}(x|\theta) = \Pr\left(X \le x \mid \Theta = \theta\right)$$

The (unconditional) dist^n of X has cdf

$$F_X(x) = \Pr\left(X \le x\right) = \int_{\text{all } \Theta} F_{X|\Theta}(x|\theta) \pi(\theta) d\theta$$

Example. $X|\Lambda = \lambda \sim \text{Exp}(\lambda)$ and let Λ be a gamma rv with mean α/θ and variance α/θ^2. The unconditional density of X is

$$\begin{aligned}
f(x) &= \int_0^\infty \underbrace{\lambda e^{-\lambda x}}_{f(x|\lambda)} \times \underbrace{\frac{\theta^\alpha \lambda^{\alpha-1} e^{-\theta\lambda}}{\Gamma(\alpha)}}_{\pi(\lambda)} d\lambda \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^\alpha e^{-\lambda(x+\theta)} d\lambda \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{(x+\theta)^{\alpha+1}} \underbrace{\int_0^\infty \frac{(x+\theta)^{\alpha+1} \lambda^\alpha e^{-\lambda(x+\theta)}}{\Gamma(\alpha+1)} d\lambda}_{1} \\
&= \alpha \frac{\theta^\alpha}{(x+\theta)^{\alpha+1}} \implies X \sim \text{Pareto}(\alpha, \theta)
\end{aligned}$$

(exponential, gamma, normal) dist^n 는 기억할것!

Erlang mixture class

Among the large class of mixed dist^{n}s, the class of mixed Erlangs stands out for its flexibility and tractability in the loss modelling context.

An Erlang dist^n is a special case of a gamma dist^n for which the shape parameter α

is a positive integer.

$$f_X(x) = \frac{\left(\frac{1}{\theta}\right)^\alpha x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)}, \quad x > 0, \quad \alpha \in \mathbb{N}^+$$

An Erlang dist^n admits a closed-form expression for its cdf:

$$\begin{aligned}
F_X(x|j,\theta) &= \int_0^x f_X(y)dy = \int_0^x \frac{\left(\frac{1}{\theta}\right)^j y^{j-1} e^{-\frac{y}{\theta}}}{(j-1)!} dy, \quad \text{where } j \in \mathbb{N}^+ \\
&= \frac{\left(\frac{1}{\theta}\right)^j}{(j-1)!} \int_0^x y^{j-1} e^{-\frac{y}{\theta}} dy \\
&= \frac{(1/\theta)^j}{(j-1)!} \left\{ (-\theta) y^{j-1} e^{-y/\theta} |_{y=0}^x + \theta(j-1) \int_0^x y^{j-2} e^{-y/\theta} dy \right\} \\
&= -\frac{\left(\frac{x}{\theta}\right)^{j-1} e^{-\frac{x}{\theta}}}{(j-1)!} + \underbrace{\frac{\left(\frac{1}{\theta}\right)^{j-1}}{\Gamma(j-1)} \int_0^x y^{j-2} e^{-\frac{y}{\theta}} dy}_{F(x \,|\, j-1, \theta)} \\
&= \cdots \\
&= 1 - \sum_{k=0}^{j-1} \frac{\left(\frac{x}{\theta}\right)^k e^{-\frac{x}{\theta}}}{k!}
\end{aligned}$$

A mixture of Erlang dist^{n}s involves the density

$$f(x) = \sum_{j=1}^\infty a_j \frac{\left(\frac{1}{\theta}\right)^j x^{j-1} e^{-\frac{x}{\theta}}}{(j-1)!}, \quad x > 0$$

with $\{a_j\}$ is a discrete probability measure. Its cdf is

$$\begin{aligned}
F(x) &= \int_0^x f(y)dy = \sum_{j=1}^\infty a_j \int_0^x \frac{\left(\frac{1}{\theta}\right)^j y^{j-1} e^{-\frac{y}{\theta}}}{(j-1)!} dy \\
&= \sum_{j=1}^\infty a_j \left(1 - \sum_{k=0}^{j-1} \frac{\left(\frac{x}{\theta}\right)^k e^{-\frac{x}{\theta}}}{k!} \right) \\
&= 1 - \sum_{k=0}^\infty \frac{\left(\frac{x}{\theta}\right)^k e^{-\frac{x}{\theta}}}{k!} \sum_{j=k+1}^\infty a_j \\
&= 1 - \sum_{k=0}^\infty \overline{A}_k \frac{\left(\frac{x}{\theta}\right)^k e^{-\frac{x}{\theta}}}{k!}, \quad \text{where } \overline{A}_k = \sum_{j=k+1}^\infty a_j
\end{aligned}$$

$\overline{A}_k \longrightarrow$ survival ft indicated

Its Laplace Transform is

$$\widetilde{f}(s) = E\left[e^{-sX}\right] = \int_0^\infty e^{-sx} \left\{ \sum_{j=1}^{\infty} a_j \frac{\left(\frac{1}{\theta}\right)^j x^{j-1} e^{-\frac{x}{\theta}}}{(j-1)!} \right\} dx$$

$$= \sum_{j=1}^{\infty} a_j \left(\frac{1}{1+\theta s}\right)^j \overset{\text{def}}{=} A\left(\frac{1}{1+\theta s}\right), \quad \text{where } A(z) = \sum_{j=1}^{\infty} a_j z^j$$

is the pgf associated to the probability measure $\{a_j\}$. It turns out that many dist^{n}s have a Laplace Transform of the form.

9/20 (수)

(Review)

Erlang mixture

$$f(x) = \sum_{j=1}^{\infty} a_j \frac{\left(\frac{1}{\theta}\right)^j x^{j-1} e^{-x/\theta}}{(j-1)!}, \quad \text{where } a_j : \text{weight}, \ \sum a_j = 1$$

Laplace Transform

$$\widetilde{f}_X(s) = E\left(e^{-sX}\right) = \sum_{j=1}^{\infty} a_j \left(\frac{1}{1+\theta s}\right)^j \equiv A\left(\frac{1}{1+\theta s}\right)$$

$A(z) = \sum a_j z^j$

<u>NOTE</u>

Let $\mathbb{C} = \left\{ A(z) : \ A(z) = \sum_{j=1}^{\infty} a_j z^j, \quad \sum_{j=1}^{\infty} a_j = 1 \ \& \ a_j \geq 0 \right\}$.
Then the following properties hold.

1. For $\forall A_1(z)$ & $A_2(z) \ \in \mathbb{C}$, we have

$$\omega A_1(z) + (1-\omega) A_2(z) \in \mathbb{C}, \quad 0 \leq \omega \leq 1$$

Proof. Let $A_1(z) = \sum_{j=1}^{\infty} a_j z^j$ and $A_2(z) = \sum_{j=1}^{\infty} b_j z^j$. Then

$$\begin{aligned} \omega A_1(z) + (1-\omega) A_2(z) &= \sum_{j=1}^{\infty} \left[\omega a_j + (1-\omega) b_j\right] z^j \\ &= \sum_{j=1}^{\infty} c_j z^j \\ \text{where } \sum_{j=1}^{\infty} c_j &= 1, \quad c_j \geq 0 \end{aligned}$$

□

2. For $A_1(z)$ and $A_2(z) \in \mathbb{C}$, we have

$$A_1(z) \times A_2(z) \in \mathbb{C}$$

Proof.

$$\left(\sum_{j=1}^{\infty} a_j z^j\right)\left(\sum_{i=1}^{\infty} b_i z^i\right) = z^2(a_1b_1) + z^3(a_1b_2 + a_2b_1) + z^4(a_1b_3 + a_2b_2 + a_3b_1)$$
$$+ \cdots + z^k\left(\sum_{j=1}^{k-1} a_j b_{k-j}\right) + \cdots$$
$$= \sum_{j=1}^{\infty} c_j z^j \ \in \mathbb{C}$$
$$\text{where } c_k = \sum_{j=1}^{k-1} a_j b_{k-j}, \quad k = 2, 3, \cdots, \quad c_1 = 0$$

$$\underbrace{\overbrace{(a_1 + a_2 + \cdots)}^{1}\overbrace{(b_1 + b_2 + \cdots)}^{1}}_{\text{Expand}} = 1 \iff \sum_{j=1}^{\infty} c_j = 1$$

□

3. If we define

$$\mathbb{C}_0 = \left\{ B(z) : \ \sum_{j=0}^{\infty} b_j z^j, \quad \sum_{j=0}^{\infty} b_j = 1 \right\}$$

Then. for $\forall A(z) \in \mathbb{C}$ and $\forall B(z) \in \mathbb{C}_0$, we have $A(z)B(z) \in \mathbb{C}$

Example. Let $X = Y_1 + Y_2$ where Y_i is an exponential rv with mean θ_i $(i = 1, 2)$. The rv's Y_i are assumed indep. WLOG, we assume that $\theta_1 \geq \theta_2$. Show that X is a member of Erlang Mixture class.

Solution. Laplace Transform of X is

$$\widetilde{f}_X(s) = E\left[e^{-sX}\right] = E\left[e^{-s(Y_1+Y_2)}\right] \overset{\text{indep.}}{=} \frac{1}{1+\theta_1 s} \times \frac{1}{1+\theta_2 s}$$

Given That, for any θ_n $(0 < \theta_n \leq \theta_i)$,

$$\frac{1}{1+\theta_1 s} = \frac{1}{1+\theta_n s} \times \frac{\theta_n/\theta_1}{1 - \left(1 - \frac{\theta_n}{\theta_1}\right)\frac{1}{1+\theta_n s}}$$

(좌변에서 우변으로의 식 변환 idea를 생각해내는 것은 어렵지만 일단 받아들이자.)

$$\frac{1}{1+\theta_1 s} = \frac{1}{1+\theta_n s} \times \frac{\theta_n/\theta_1}{1 - \left(1 - \frac{\theta_n}{\theta_1}\right)\frac{1}{1+\theta_n s}}$$
$$\overset{\text{def}}{=} A_{1,n}\left(\frac{1}{1+\theta_n s}\right)$$

where

$$\begin{aligned} A_{1,n}(z) &= z \times \frac{\theta_n/\theta_1}{1 - \left(1 - \frac{\theta_n}{\theta_1}\right) z} \\ &= \underbrace{\underbrace{z}_{\mathbb{C}} \underbrace{\frac{\theta_n}{\theta_1} \left(1 + \left(1 - \frac{\theta_n}{\theta_1}\right) z + \left(1 - \frac{\theta_n}{\theta_1}\right)^2 z^2 + \cdots\right)}_{\mathbb{C}_0}}_{\mathbb{C}} \end{aligned}$$

Thus

$$\widetilde{f}_X(s) = \frac{1}{1+\theta_1 s} \times \frac{1}{1+\theta_2 s} = A_{1,2}\left(\frac{1}{1+\theta_2 s}\right) \times \frac{1}{1+\theta_2 s}$$

If we set $z = \dfrac{1}{1+\theta_2 s}$,

$$\widetilde{f}_X(s) = \underbrace{A_{1,2}(z)}_{\mathbb{C}} \times \underbrace{z}_{\mathbb{C}} \in \mathbb{C}$$

Q.E.D

Example. X has density

$$f_X(x) = \sum_{i=1}^{n} p_i \frac{\left(\frac{1}{\theta_i}\right)^{k_i} x^{k_i - 1} e^{-x/\theta_i}}{(k_i - 1)!}, \quad x > 0$$

(countable shape & scale mixture of Erlangs)

9/25 (월)

Example.

$$f(x) = \sum_{i=1}^{n} p_i \frac{\left(\frac{1}{\theta_i}\right)^{k_i} x^{k_i-1} e^{-x/\theta_i}}{(k_i-1)!}, \quad \longleftarrow \text{ Different scale para.}$$

$\{k_i\}_{i=1}^{n}$ are positive integers, $\sum\limits_{i=1}^{n} p_i = 1$ with $p_i \geq 0$

Solution. WLOG(Without Loss Of Generality), let us assume that $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$. The Laplace Transform of X:

$$\widetilde{f}(s) = E\left[e^{-sX}\right] = \sum_{i=1}^{n} p_i \left(\frac{1}{1+\theta_i s}\right)^{k_i}$$

using $A_{1,n}(z)$ notation, it follows that

$$\widetilde{f}(s) = \sum_{i=1}^{n} p_i \left[A_{i,n}\left(\frac{1}{1+\theta_n s}\right)\right]^{k_i}, \quad \left(A_{i,n}\left(\frac{1}{1+\theta_n s}\right) \text{ is member of } \mathbb{C}\right)$$

or, equivalently,

$$\widetilde{f}(s) = A\left(\frac{1}{1+\theta_n s}\right),$$

where $A(z) = \sum\limits_{i=1}^{n} p_i \left(A_{i,n}(z)\right)^{k_i} \in \mathbb{C}$

Q.E.D

so, $f(x)$ can be written as

$$f(x) = \sum_{j=1}^{\infty} w_j \frac{\left(\frac{1}{\theta_n}\right)^{n} x^{j-1} e^{-x/\theta_n}}{(j-1)!} \quad \Longleftarrow \text{ Erlang Mixture}$$

for some w_j $(j = 1, 2, \cdots)$

4.3 Some important criteria

4.3.1 Role of parameters

· Any mathematical model is a simplified description of the reality

· Mathematical models have different levels of complexity

· Generally speaking, its level of complexity is measured by the number of parameters in the model.

$\Longrightarrow$ more parameters, more complex the model is.

· Pros & Cons of simple vs complex models

· Pros of simple models

- easier to estimate parameters.
- more stable over time.

· Cons of simple models

- too superficial

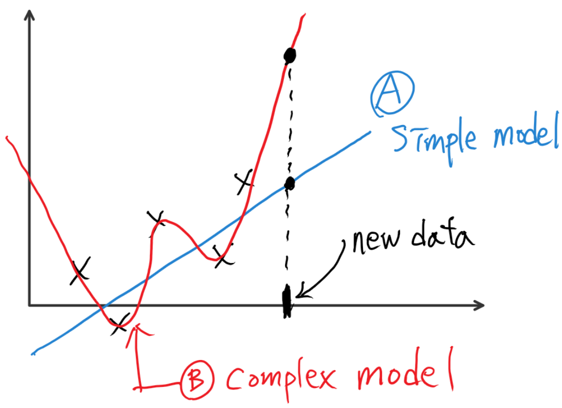
· Pros of complex models

- many parameters can match the reality better
- allow to factor in irregularities in the data

· Cons of complex models

- over-fitting
- not stable over time

· The principle of parsimony states that the simplest model that adequately reflects reality should be used.



4.3.2 Tail of the dist^n

· It is a primary concern for the insurer to appropriately quantify the thickness (heaviness) in the right-hand tail of the given dist^n

· We look at different criteria used to measure the tail.

· We assume X has a continuous dist^n with density $f_X(x) = \dfrac{d}{dx}F_X(x)$. The tail of X is an interval of the form (x, ∞) with probability

$$\Pr(X \in (x, \infty)) = \Pr(X > x) = 1 - F_X(x) = \overline{F}_X(x)$$

where $\overline{F}_X(x)$ is a survival ft.

(1) Existence of moments

The k^{th} moment of X (non-negative rv):

$$E\left(X^k\right) = \int_0^\infty x^k f_X(x)dx$$

If the density $f_X(x)$ takes on large values for large x, the integral may not converge.

· Existence of only moments up to a certain order indicates a heavy-tail dist^n (eg. Pareto)

· Existence of all the moments indicates a light-tail dist^n (eg. normal, gamma)

Note that the fact that the k^{th} moment of X does not exist implies that the mgf of X does not exist as well.

$$M_X(t) = E\left[e^{tX}\right] = 1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} E\left[X^k\right]$$

therefore, one can look at the existence of the mgf to infer on the thickness in the tail of X.

(2) Limiting ratio

The survival ft can also be used to assess the tail thickness. For this criterion, we use

$$\lim_{x\to\infty} \frac{\overline{F}_X(x)}{\overline{F}_Y(x)} \qquad \Longleftarrow \text{ ratio of two survival fts}$$

Suppose that the limit is c (clearly non-negative), Then

· If $c = 0$, $\overline{F}_X(x)$ goes to 0 faster than $\overline{F}_Y(x) \Longrightarrow Y$ has a heavier tail than X.

· If $0 < c < \infty$, $\overline{F}_X(x)$ is similar to $\overline{F}_Y(x)$ for large value of $x \Longrightarrow X$ and Y have similar tails.

·If $c = +\infty$, $\overline{F}_X(x)$ goes to 0 at a much slower pace than $\overline{F}_Y(x) \Longrightarrow Y$ has a lighter tail than X.

From the L'Hospital's rule, it is equivalent to compare the ratio of their densities:

$$\lim_{x\to\infty} \frac{\overline{F}_X(x)}{\overline{F}_Y(x)} = \lim_{x\to\infty} \frac{f_X(x)}{f_Y(x)} = c$$

Example. Pareto tail vs. Gamma tail.

$$X \;\sim\; \text{Pareto}(\alpha, \theta) \qquad \text{and} \qquad Y \;\sim\; Gam(\tau, \lambda)$$

It follows that

$$\begin{aligned}
\lim_{x\to\infty} \frac{f_{\text{Pareto}}(x)}{f_{\text{gamma}}(x)} &= \lim_{x\to\infty} \frac{\alpha \dfrac{\theta^\alpha}{(x+\theta)^{\alpha+1}}}{\dfrac{x^{\tau-1}e^{-x/\lambda}}{\lambda^\tau \Gamma(\tau)}} \\
&= \underbrace{\alpha\theta^\alpha \lambda^\tau \Gamma(\tau)}_{\text{Const.}} \times \lim_{x\to\infty} \frac{e^{x/\lambda}}{x^{\tau-1}(x+\theta)^{\alpha+1}} \\
&= +\infty
\end{aligned}$$

$\therefore$ Pareto dist^n has a heavier tail than the gamma dist^n

(3) Hazard rate (Failure rate)

The hazard rate ft $h_X(x)$ is defined as

$$h_X(x) = \frac{f_X(x)}{\overline{F}_X(x)}, \quad x > 0$$

in relation with the concept of tail, the following probabilistic interpretation is relevant:

$$\begin{aligned}
h_X(x)dx = \frac{f_X(x)}{\overline{F}_X(x)} &\approx \frac{\Pr\left(X \in (x, x+dx)\right)}{\overline{F}_X(x)} \\
&= \Pr\left(X \in (x, x+dx) \mid X > x\right) \\
&= 1 - \Pr\left(X > x + dx \mid X > x\right)
\end{aligned}$$

9/27 (수)

(3) hazard rate

$$\begin{aligned} h_X(x)dx &\fallingdotseq \Pr\left(X \in (x, x+dx) \mid X > x\right) \\ &= 1 - \Pr\left(X > x + dx \mid X > x\right) \end{aligned}$$

· If $h(x)$ is a decreasing ft for large x, then it is less and less likely that $X \in (x,\ x+dx)$ (where X is a random time of death) given that $X > x$ (thus more and more likely that $X > x + dx \mid X > x$) $\implies X$ has a heavy tail.

($X > x$: Condition that X is alive at time x,
$X \in (x,\ x + dx)$: X dies in $(x,\ x + dx)$)

· If $h(x)$ is an increasing ft for large x, then it is more and more likely that $X \in (x,\ x+dx)$ given that $X > x$ (thus less and less likely that $X > x+dx \mid X > x$) $\implies X$ has a light tail.

· Terminology

- $F_X(x)$ is said to be decreasing failure rate (DFR) if $h_X(x)$ is a non-increasing ft of x
- $F_X(x)$ is said to be increasing failure rate (IFR) if $h_X(x)$ is a non-decreasing ft of x

NOTE) Exponential dist^n is the only dist^n to be both IFR and DFR

Example. $X \sim$ Pareto(α, θ)

$$\begin{aligned} h(x) = \frac{f(x)}{\overline{F}(x)} &\overset{\text{해보기}}{=} \frac{\alpha}{x+\theta}, \quad x > 0 \\ &\implies \text{DFR} \end{aligned}$$

(4) Residual lifetime and its mean

Define residual lifetime rv

$$T_x = X - x \mid X > x, \qquad x \geq 0$$

The survival ft of T_x :

$$\begin{aligned}\overline{F}_{T_x}(t) &= \Pr(T_x > t)) = \Pr(X - x > t \mid X > x) \\ &= \Pr(X > x + t \mid X > x) = \frac{\overline{F}_X(x+t)}{\overline{F}_X(x)}, \quad t \geq 0\end{aligned}$$

Density of T_x :

$$f_{T_x}(t) = \frac{f_X(x+t)}{\overline{F}_X(X)}$$

we refer to the mean of T_x as the mean excess loss or mean residual lifetime (MRL)

$$\begin{aligned}e_X(x) = E(T_x) &= \int_0^\infty t f_{T_x}(t)dt \\ &= \int_0^\infty t \cdot \frac{f_X(x+t)}{\overline{F}_X(x)}dt = \int_x^\infty (t-x)\frac{f_X(t)}{\overline{F}_X(x)}dt\end{aligned}$$

using integration by parts,

$$e_X(x) = \int_x^\infty \frac{\overline{F}_X(t)}{\overline{F}_X(x)}dt = \int_0^\infty \frac{\overline{F}_X(x+t)}{\overline{F}_X(x)}dt$$

Def.

- $F_X(x)$ is said to be increasing mean residual lifetime (IMRL) if $e_X(x)$ is a non-decreasing ft in $x \Longrightarrow$ heavy tail
- $F_X(x)$ is said to be DMRL if $e_X(x)$ is a non-increasing ft in $x \Longrightarrow$ light tail

your age (x) MRL : $e_x(x)$

$x=25 \longrightarrow 60 = e_x(25)$

$x=45 \longrightarrow 40 = e_x(40)$

$x=60 \longrightarrow 25 = e_x(25)$

Humans $\rightarrow$ DMRL

Softwares $\rightarrow$ IMRL

· There exists a connection between the concept of IMRL/DMRL and the concept of DFR/IFR

Note that

$$\overline{F}_X(x) = e^{-\int_0^x h_X(y)dy}, \quad \Longleftarrow \text{ DIY}$$

which implies that

$$\begin{aligned}\frac{\overline{F}_X(x+t)}{\overline{F}_X(x)} = \frac{e^{-\int_0^{x+t} h(y)dy}}{e^{-\int_0^x h(y)dy}} &= e^{-\int_x^{x+t} h(y)dy} \\ &= e^{-\int_0^t h(x+y)dy}\end{aligned}$$

Assume that $F_X(x)$ is DFR. Then for $x_1 \le x_2$ and $y \ge 0$, we have

$$h_X(x_1+y) \ge h_X(x_2+y) \Longleftarrow \text{ DFR}$$

$$\begin{aligned}\int_0^t h_X(x_1+y)dy &\ge \int_0^t h_X(x_2+y)dy \\ -\int_0^t h_X(x_1+y)dy &\le -\int_0^t h_X(x_2+y)dy \\ e^{-\int_0^t h_X(x_1+y)dy} &\le e^{-\int_0^t h_X(x_2+y)dy} \\ \frac{\overline{F}_X(x_1+t)}{\overline{F}_X(x_1)} &\le \frac{\overline{F}_X(x_2+t)}{\overline{F}_X(x_2)}, \quad x_1 \le x_2\end{aligned}$$

This immediately implies that

$$e_X(x_1) = \int_0^\infty \frac{\overline{F}_X(x_1+t)}{\overline{F}_X(x_1)}dt \le \int_0^\infty \frac{\overline{F}_X(x_2+t)}{\overline{F}_X(x_2)}dt = e_X(x_2)$$

for $\forall\ x_1 \le x_2$

$$\Longrightarrow \begin{cases} X \text{ is DFR} \overset{\not\Leftarrow}{\Longrightarrow} X \text{ is IMRL} \\ \\ X \text{ is IFR} \overset{\not\Leftarrow}{\Longrightarrow} X \text{ is DMRL} \quad \Longleftarrow \text{ DIY} \end{cases}$$

10/11 (수)

Example. $X \sim \text{Weibull}(\tau,\ \theta)$

$$
\begin{aligned}
f(x) &= \frac{\tau\left(\frac{x}{\theta}\right)^{\tau-1} e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{\theta}, \quad x > 0 \\
F(x) &= 1 - e^{-(x/\theta)^{\tau}} \\
h(x) &= \tau \frac{x^{\tau-1}}{\theta^{\tau}}
\end{aligned}
$$

$$
\Rightarrow \begin{cases} \text{if } \tau > 1, & h(x) \text{ is increasing } \Rightarrow X \text{ is IFR} \\ & \qquad\qquad\qquad\qquad \Rightarrow X \text{ is DMRL} \\ \text{if } 0 < \tau < 1, & h(x) \text{ is decreasing } \Rightarrow X \text{ is DFR } \Rightarrow X \text{ is IMRL} \end{cases}
$$

4.4 Policy adjustments

Insurance policies contain adjustments (modifications) to determine the amt to be paid for a given ground-up loss. common policy adjustments are listed here:

1. Policy limit u ($u > 0$)

· for any given loss, the maximum amt paid by the insurer is u

· so the insurer pays the minimum between the ground-up loss X and u

· if there are no other adjustments on the policy, the insurer pays

$$
\min(X,\ u) = \begin{cases} X, & X < u \\ u, & X \geq u \end{cases}
$$

2. Ordinary deductible d ($d \geq 0$)

· for any given loss, the first d dollars falls on the insured

· If there are no other adjustments on the policy, the insurer pays

$$\max(X-d,\ 0) = (X-d)_+ = \begin{cases} 0, & X < d \\ X-d, & X \geq d \end{cases}$$

3. Coinsurance factor α ($0 < \alpha < 1$)

· Insurer pays a proportion α of the loss amt; the remaining $(1-\alpha)$ falls on the insured.

· if there are no other adjustments on the policy, the insurer pays αX

4. Franchise deductible d ($d \geq 0$)

· differs from the ordinary deductible in that when the loss exceeds d, the deductible is waived and the full loss is paid by the insurer

· If there are no other adjustments, the insurer pays

$$X \cdot \underbrace{\mathbb{1}\{X \geq d\}}_{\text{Indicator ft.}} = \begin{cases} 0, & X < d \\ X, & X \geq d \end{cases}$$

10/16 (월)

It is possible that more than one policy adjustment is included in a policy. In this case we assume that policy adjustments are applied in the following order, unless otherwise specified

1. policy limit
2. policy deductible
3. coinsurance

For example, if there is an ordinary deductible d and a policy limit u, then the maximum payment by the insurer would be $u - d$ (**NOT** u). In general the insurer pays

$$\begin{cases} 0, & X \le d \\ X - d, & d < X \le u \\ u - d, & X > u \end{cases} \implies \underbrace{\max\left\{ \overbrace{\min(X,\ u)}^{\text{policy limit}} - d,\ 0 \right\}}_{\text{ordinary deductible}}$$

If the policy also has a coinsurance factor α, the payment by the insurer becomes

$$\alpha \cdot \max\{\, \min(X,\ u) - d,\ 0\} = \begin{cases} 0, & X \le d \\ \alpha(X - d), & d < X \le u \\ \alpha(u - d), & X > u \end{cases}$$

Note that the application of deductibles and limits are examples of **truncation** and **censoring**

Def.

· A truncated loss is a loss that is **NOT** observed. For a policy with a deductible d, any loss below the deductible are truncated (**NOT** reported to insurer)

· A censored loss is a loss that is observed to occur, but whose value is **NOT** known. For a policy with a policy limit u, any losses above the limit are censored in that those losses are observed but the exact loss amounts are unknown.

when considering the amt paid by the insurer, it is typical to consider (and dstinguish between) 2 types of reporting methods for an amt paid.

1. Loss basis: Y_L = amount paid per loss
   - there is an entry for each loss (even if the amt paid by the insurer 0)
   - includes the 0 amounts on some small losses
   - for a policy with a limit u, deductible d, and a coinsurance α,

   $$Y_L = \alpha \cdot \max\{\ \min(X,\ u) - d,\ 0\ \}$$

   - In the presence of a deductible d, the dist^n of Y_L has a prob. mass at 0 of $F_X(d) = \Pr(X \le d)$
2. Payment basis Y_p = amt paid per payment
   - only the non-zero payments of the insurer are recorded
   - NOT all losses have entries
   - the dist^n of Y_p does NOT have a prob. mass at 0

$$\begin{array}{cccc} \text{eg.} & \{0,0,1,2\} & \text{vs} & \{1,2\} \\ \text{Average} & \dfrac{3}{4} & \text{vs} & \dfrac{3}{2} \end{array}$$

The two reporting methods are linked as follows . The Y_p is defined as the amt paid on a loss given that this loss generates a non-zero payment. Thus

$$Y_p = Y_L \mid Y_L > 0$$

Letting $F_{Y_L}(y) = \Pr(Y_L \le y)$, the cdf of Y_p is

$$\begin{aligned} F_{Y_p}(y) &= \Pr(Y_p \le y) = \Pr(Y_L \le y \mid Y_L > 0) \\ &= \frac{\Pr(0 < Y_L \le y)}{\Pr(Y_L > 0)} = \frac{\Pr(Y_L \le y) - \Pr(Y_L \le 0)}{1 - \Pr(Y_L \le 0)} \\ &= \frac{F_{Y_L}(y) - F_{Y_L}(0)}{1 - F_{Y_L}(0)} \quad (\ \Pr(Y_L \le 0) = \Pr(Y_L = 0)\ ) \end{aligned}$$

Assuming $F_{Y_L}(y)$ is differentiable for $y > 0$, the density of Y_p is

$$f_{Y_p}(y) = \frac{d}{dy} F_{Y_p}(y) = \frac{f_{Y_L}(y)}{1 - F_{Y_L}(0)}$$

In conclusion, it is sufficient to know Y_L to determine the cdf and density of Y_p.

Remark: If $\underbrace{F_{Y_L}(0) = 0}_{\text{no deductible case.}}$ then

$$F_{Y_p}(y) = F_{Y_L}(y),$$

which implies that there is no difference between Y_L and Y_p. However, if there is a policy deductible, we know that

$$F_{Y_L}(0) = \Pr(X \leq d) \neq 0$$

and there is a meaningful difference between Y_L and Y_p.

In what follows, we consider the effect of some policy adjustments on the dist^n of Y_L and Y_p. We assume that the ground-up loss X is a continuous rv with density f_X and cdf F_X.

(아래 **4.4.1** ~ **4.4.4**는 추가 자료 (*Summary_polMod.pdf*) 와 함께 진행)

4.4.1 Analysis of Y_L under ordinary deductible

$$Y_L = \begin{cases} 0, & X < d \\ X - d, & d < X < u \\ u - d, & X > u \end{cases}$$

The goal is to present Y_L in terms of the ground-up loss X

(1)
$$\begin{array}{ccc} X & & Y_L \\ \hline 0.01 \times d & \searrow & \\ 0.02 \times d & \longrightarrow & 0 \\ \vdots & & \\ 0.99 \times d & \nearrow & \end{array}$$

there is a prob. mass (or spike) at $Y_L = 0$ because infinitely many different X values are mapped to 0

(2) The prob. mass size at $Y_L = 0$:

$$\Pr(Y_L = 0) = \Pr(X < d) = F_X(d)$$

(3) At $y = 0$, the cdf value of Y_L is

$$F_{Y_L}(0) = \Pr(Y_L = 0) = F_X(d)$$

(4) No prob. mass when $d < X < u \iff 0 < Y_L < u - d$

(5) It is clear that in this range

$$\underbrace{\Pr(Y_L = y)}_{\text{Heuristic}} = \Pr(X - d = y) = \Pr(X = y + d) = f_X(y + d)$$

and therefore

$$F_{Y_L}(y) = F_X(y + d)$$

(6)

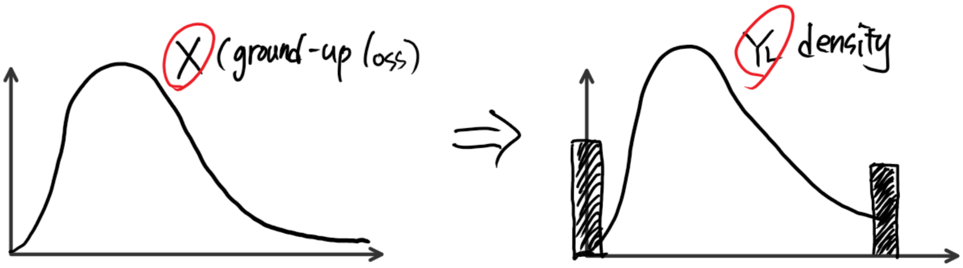
$$\begin{array}{ccc} X & & Y_L \\ \hline u & \searrow & \\ 1.1 \times u & \longrightarrow & u - d \\ 1.2 \times u & \nearrow & \\ \vdots & & \end{array}$$

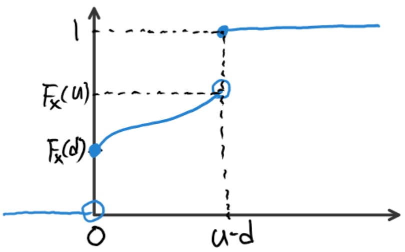
there is a prob. mass at $Y_L = u - d$ with a size of $\Pr(Y_L = u - d) = \Pr(X > u) = \underbrace{1 - F_X(u)}_{(7)}$

(8) The maximum value of Y_L can take is $u - d$ For $Y_L = u - d$, we simply set $F_{Y_L}(u - d) = 1$

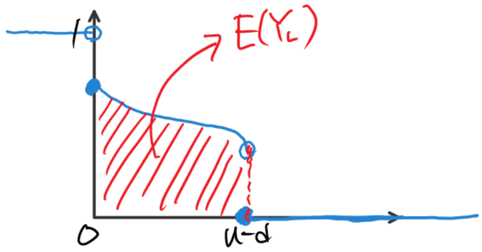
$\Longrightarrow$ From the table and using the right continuity of cdf, we finally obtain

$$F_{Y_L}(y) = \begin{cases} F_X(y + d), & 0 \le y < u - d \\ 1, & y \ge u - d \end{cases}$$





< cdf of Y_L >



< Survival ft. of Y_L >

$$
\begin{aligned}
E(Y_L) &= \int_0^\infty 1 - F_{Y_L}(y)dy = \int_0^{u-d} 1 - F_{Y_L}(y)dy \\
&\underbrace{=}_{(5)} \int_0^{u-d} 1 - F_X(y+d)dy \underbrace{=}_{t=y+d} \int_d^u 1 - F_X(t)dt
\end{aligned}
$$

10/25 (수)

(Recall)

$$Y_P = Y_L \mid Y_L > 0$$

4.4.2 Analysis of Y_P (ordinary deductible)

$$Y_P = \begin{cases} \text{undefined}, & X < d \\ X - d, & d < X < u \\ u - d, & X > u \end{cases}$$

(there is no equal sign yet.)

[Handout]

(1) when $X < d$, Y_P is **NOT** defined, so its pdf & cdf also undefined.

(2) For y, $0 < y < u - d$, we have

$$\begin{aligned}\Pr(Y_P = y) &= \Pr(Y_L = y \mid Y_L > 0) = \frac{\Pr(Y_L = y \ \& \ Y_L > 0)}{1 - \Pr(Y_L = 0)}, \quad (Y_L = y \ \& \ Y_L > 0 \Leftrightarrow Y_L = y) \\ &= \frac{\Pr(X - d = y)}{1 - F_X(d)} = \frac{\Pr(X = y + d)}{1 - F_X(d)} = \frac{f_X(y+d)}{1 - F_X(d)}\end{aligned}$$

(3) cdf of Y_P:

$$\begin{aligned}\Pr(Y_P \le y) &= \int_0^y \mathrm{pdf}_{Y_P}(t)dt = \int_0^y \frac{f_X(d+t)}{1 - F_X(d)}dt \\ &= \left[\frac{F_X(d+t)}{1 - F_X(d)}\right]_0^y = \frac{F_X(d+y) - F_X(d)}{1 - F_X(d)}\end{aligned}$$

(4) There is a prob. at $Y_P = u - d$ with size

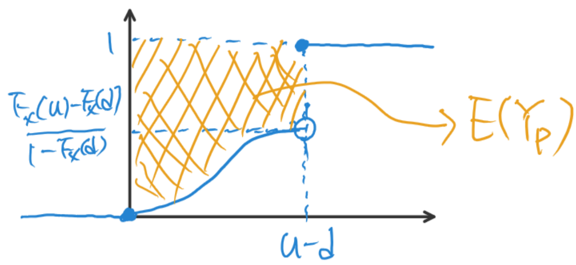
$$\begin{aligned}\Pr(Y_P = u - d) &= \Pr(Y_L = u - d \mid Y_L > 0) \\ &= \frac{\Pr(Y_L = u - d \ \& \ Y_L > 0)}{1 - \Pr(Y_L = 0)} = \frac{\Pr(X > u)}{1 - F_X(d)} = \frac{1 - F_X(u)}{1 - F_X(d)}\end{aligned}$$

(5) The maximum value of Y_P is $u - d \Leftarrow$ Table / Right-continuitiy of cdf

$$\Rightarrow F_{Y_P}(u - d) = 1$$

$\therefore$ The cdf of Y_P

$$F_{Y_P}(y) = \begin{cases} \dfrac{F_X(y+d) - F_X(d)}{1 - F_X(d)}, & 0 \le y < u - d \\ 1, & y \ge u - d \end{cases}$$



<cdf of Y_P>

$$E(Y_P) = \int_0^\infty 1 - F_{Y_P}(y)dy = \int_0^{u-d} 1 - \frac{F_X(d+y) - F_X(d)}{1 - F_X(d)} dy + \cancelto{0}{\left(\int_{u-d}^\infty 0 dy \right)}$$

$$= \frac{1}{1 - F_X(d)} \int_0^{u-d} 1 - F_X(y+d)dy$$

$$\underset{t=y+d}{=} \frac{1}{1 - F_X(d)} \int_d^u 1 - F_X(t)dt$$

or,

$$E(Y_P) = E(Y_L \mid Y_L > 0) = \frac{E(Y_L)}{\Pr(Y_L > 0)} \overset{\text{Sec 4.4.1}}{=} \frac{\int_d^u 1 - F_X(t)dt}{1 - F_X(d)}$$

4.4.3 Analysis of Y_L (franchise deductible)

$$Y_L = \begin{cases} 0, & X < d \\ X. & d < X < u \\ u, & X > u \end{cases}$$

(1) Same as Y_L under ordinary deductible

(2) For $d < X < u$, $\Pr(Y_L = y) = \Pr(X = y) \Rightarrow f_{Y_L}(y) = f_X(y)$

(3) There is a prob. mass at $Y_L = u$ with size

$$\Pr(Y_L = u) = \Pr(X > u) = 1 - F_X(u)$$

(4) The maximum Y_L can take is $u \Rightarrow F_{Y_L}(u) = 1$

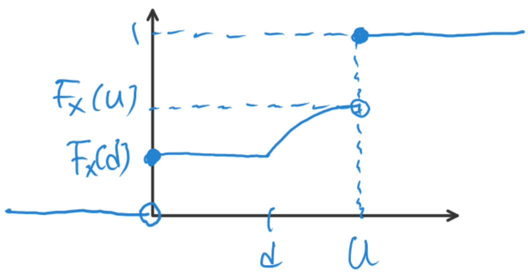
$\therefore$ cdf of Y_L

$$F_{Y_L}(y) = \begin{cases} F_X(d), & \cancel{y=0}, \quad 0 \le y < d \\ F_X(y), & d \le y < u \\ 1, & y \ge u \end{cases}$$

However, what about $0 < y < d$?

Ans) since it is impossible for Y_L to be in (0, d), the prob in this range must be zero.
$\Rightarrow$ Revise the first line of cdf above to

$$F_X(d), \quad 0 \le y < d$$



<cdf of Y_L (franchise deductible)>

$$E(Y_L) = \int_0^\infty 1 - F_{Y_L}(y)dy = \int_0^d 1 - F_X(d)dy + \int_d^u 1 - F_X(y)dy + \int_u^\infty 0dy$$
$$= d\,[1 - F_X(d)] + \int_d^u 1 - F_X(y)dy$$

10/30 (월)

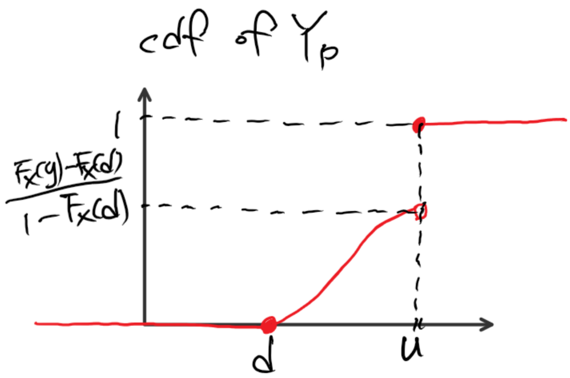
4.4.4 Analysis of Y_P under franchise deductible

$$Y_P = \begin{cases} \text{undefined}, & X < d \\ X, & d < x < u \\ u, & X > u \end{cases}$$

It is straightfoward to derive each line in the Table.

$\Rightarrow$ cdf of Y_P:

$$F_{Y_P}(y) = \begin{cases} 0, & y < d \\ \dfrac{F_X(y) - F_X(d)}{1 - F_X(d)}, & d \le y < u \\ 1, & y \ge u \end{cases}$$



$$\begin{aligned} E(Y_P) &= \int_0^\infty 1 - F_{Y_P}(y)dy \\ &= \int_0^d (1-0)dy + \int_d^u 1 - \frac{F_X(y) - F_X(d)}{1 - F_X(d)}dy + \cancelto{0}{\int_0^\infty (1-1)dy} \\ &= d + \frac{1}{1 - F_X(d)} \int_d^u 1 - F_X(y)dy \end{aligned}$$

4.4.5 Coinsurance factor α $(0 < \alpha < 1)$

With coinsurance the amt paid by the insurer is

$$Y_L = \alpha Y_L^*$$

$$\text{and} \;\; Y_P = \alpha Y_P^*$$

where Y_L^* and Y_P^* denote the amt paid (per loss & per payment, respectively) by the insurer on an identical insurance contract without coinsurance (covered in §4.4.1 ~ §4.4.4). We can therefore apply the results of creating dist^n - Multiplication by a constant.

It follows that

· cdf:

$$F_{Y_L}(y) = F_{Y_L^*}(\frac{y}{\alpha}) \quad \text{and} \quad F_{Y_P}(y) = F_{Y_P^*}(\frac{y}{\alpha})$$

· mean:

$$E(Y_L) = \alpha E(Y_L^*) \quad \text{and} \quad E(Y_P) = \alpha E(Y_P^*)$$

By introducing one or more policy adjustments, it is interesting to determine the proportion of loss (on average) reduction. A common measure for this is the loss elimination ratio (LER):

$$\text{LER} = 1 - \frac{E(Y_L)}{E(X)} \quad \longleftarrow \text{ 암기!}$$

Here $\dfrac{E(Y_L)}{E(X)}$ stands for the % of loss retained by the insurer.

For instance, a policy that contains only an ordinary deductible d,

$$\begin{aligned} \text{LER} &= 1 - \frac{E\left[\max(X-d,0)\right]}{E(X)} \\ &= 1 - \frac{E(X) - E\left[\min(X,d)\right]}{E(X)} \qquad (3) \\ &= \frac{E\left[\min\left(X,d\right)\right]}{E(X)} \qquad (4) \end{aligned}$$

(3) 은 다음으로부터 얻어진다.

$$\begin{aligned} \max\left(X-d,0\right) &= X + \max\left(-d,-X\right) \\ &= X - \min\left(d,X\right) \end{aligned}$$

(4) 의 분자는 다음과 같다.

$$\begin{aligned} \int_0^d xf(x)dx + \int_d^\infty df(x)dx &= \int_0^d xf(x)dx + d\left[1-F_X(d)\right] \\ &= \int_0^d 1 - F_X(x)dx \end{aligned}$$

마지막 등식은 $\int_0^d xf(x)dx = [xF_X(x)]_0^d - \int_0^d F_X(x)dx$ 로 부터 얻어진다.

Example. X is an equal mixture of an exponential with mean 250 and an exponential with mean 550

(a) If an ordinary deductible d is applied, find the cdf of the per payment variable Y_P.

Solution. Note that

$$f_X(x) = \underbrace{\frac{1}{2}}_{\text{weights}} \times \frac{1}{250}e^{-\frac{x}{250}} + \underbrace{\frac{1}{2}}_{\text{weights}} \times \frac{1}{550}e^{-\frac{x}{550}}$$

$$F_X(x) = 1 - \frac{1}{2}\left\{e^{-\frac{x}{250}} + e^{-\frac{x}{550}}\right\}$$

It follows that

$$\begin{aligned}
F_{Y_P}(y) &= \frac{F_X(y+d) - F_X(d)}{1 - F_X(d)} \\
&= \frac{\frac{1}{2}\left\{e^{-\frac{d}{250}} + e^{-\frac{d}{550}}\right\} - \frac{1}{2}\left\{e^{-\frac{y+d}{250}} + e^{-\frac{y+d}{550}}\right\}}{\frac{1}{2}\left\{e^{-\frac{d}{250}} + e^{-\frac{d}{550}}\right\}} \\
&= 1 - \frac{e^{-\frac{y+d}{250}} + e^{-\frac{y+d}{550}}}{e^{-\frac{d}{250}} + e^{-\frac{d}{550}}} \\
&= 1 - \left\{\underbrace{p(d)}_{\text{new weight}} \times e^{-\frac{y}{250}} + \underbrace{(1-p(d))}_{\text{new weight}} e^{-\frac{y}{550}}\right\}
\end{aligned}$$

where

$$p(d) = \frac{e^{-\frac{d}{250}}}{e^{-\frac{d}{250}} + e^{-\frac{d}{550}}}$$

$\therefore$ Y_P is a different mixture of the same exponentials $\Rightarrow$

$$f_{Y_P}(y) = p(d) \times \frac{1}{250}e^{-\frac{y}{250}} + (1-p(d)) \times \frac{1}{550}e^{-\frac{d}{550}}, \quad \text{for } y \geq 0$$

(b) calculate the mean of var

Solution.

$$E(Y_P) = p(d) \cdot 250 + (1-p(d)) \cdot 550$$

(c) Determine the expression for the LER

$$
\begin{aligned}
LER &= 1 - \frac{E(Y_L)}{E(X)} \\
&= 1 - \frac{E(Y_P)(1 - F_{Y_L}(0))}{E(X)} \\
&= 1 - \frac{E(Y_P)(1 - F_X(d))}{E(X)}
\end{aligned} \tag{5}
$$

(5)는

$$
\begin{aligned}
Y_P &= Y_L \mid Y_L > 0 \\
E(Y_P) &= \frac{E(Y_L \& Y_L > 0)}{P(Y_L > 0)} = \frac{E(Y_L)}{P(Y_L > 0)}
\end{aligned}
$$

$\Rightarrow$ plug

$$
\begin{aligned}
E(X) &= \frac{1}{2} \times 250 + \frac{1}{2} \times 550 = 400 \\
E(Y_P) &= p(d) \times 250 + (1 - p(d)) \times 550 \quad \leftarrow \text{from (b)} \\
1 - F_X(d) &= \frac{1}{2}\left\{e^{-\frac{d}{250}} + e^{-\frac{d}{550}}\right\}
\end{aligned}
$$

(d) Suppose that the policy has a policy limit u, an ordinary deductible d, and a coinsurance α. Identify the cdf of the per loss variable Y_L.

Solution. Define

$$
Y_L^* = \max\{\min(X, u) - d, 0\} = \begin{cases} 0, & X \le d \\ X - d, & d < X \le u \\ u - d, & X > u \end{cases}
$$

We know that

$$
\begin{aligned}
F_{Y_L^*}(y) &= \begin{cases} F_X(y + d), & 0 \le y < u - d \\ 1, & y \ge u - d \end{cases} \\
&= \begin{cases} 1 - \frac{1}{2}\left\{e^{-\frac{y+d}{250}} + e^{-\frac{y+d}{550}}\right\}, & 0 \le y < u - d \\ 1, & y \ge u - d \end{cases}
\end{aligned}
$$

Now define $Y_L = \alpha Y_L^*$. Then

$$
\begin{aligned}
F_{Y_L}(y) &= F_{Y_L^*}(\frac{y}{\alpha}) \\
&= \begin{cases} 1 - \frac{1}{2}\left\{ e^{-\frac{(y/\alpha)+d}{250}} + e^{-\frac{(y/\alpha)+d}{550}} \right\}, & 0 \le \frac{y}{\alpha} < u - d \\ 1, & \frac{y}{\alpha} \ge u - d \end{cases} \\
&= \begin{cases} 1 - \frac{1}{2}\left\{ e^{-\frac{(y/\alpha)+d}{250}} + e^{-\frac{(y/\alpha)+d}{550}} \right\}, & 0 \le y < \alpha(u - d) \\ 1, & y \ge \alpha(u - d) \end{cases}
\end{aligned}
$$

5 Frequency - Models for the number of claims / payments

5.1 Introduction

The aggregate claim amt under the collective risk model

$$
S = \begin{cases} \sum_{i=1}^{N} X_i, & N > 0 \\ 0, & N = 0 \end{cases}, \quad N\text{은 Random frequency, } X_i\text{는 Random severity}
$$

Here S is a sum of a random number N of claim or payment amts. Our objective in this chapter is to study the random variable that generates claims / payments i.e, N.

The # of claims / payments in a given period of time is modeled as a discrete and integer-valued rv N with pmf:

$$
p_k = \Pr(N = k), \quad k = 0, 1, 2, \cdots
$$

Let $f_N(t)$ be the associated pgf:

$$
P_N(t) = E(t^N) = \sum_{k=0}^{\infty} t^k p_k
$$

which is properly defined whenever the sum converges.

Properties of pgf

1.

$$P_N(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \{k! \cdot p_k\}$$

$$\Rightarrow \frac{d^n}{dt^n} P_N(t)|_{t=0} \stackrel{\text{def}}{=} P_N^{(n)}(0) = n! p_n$$

Hence $p_n = \dfrac{P_N^{(n)}(0)}{n!}$.

2. Also, one can identify the factorial moment of N via its pgf

$$P_N(t) = \sum_{k=0}^{\infty} t^k p_k$$

It follows that

$$P_N^{(1)}(t) = \frac{d}{dt} P_N(t) = \sum_{k=0}^{\infty} k t^{k-1} p_k = \sum_{k=1}^{\infty} k t^{k-1} p_k,$$

which implies that

$$P_N^{(1)}(1) = \sum_{k=1}^{\infty} k p_k = E(N)$$

similarly,

$$P_N^{(2)}(t) = \frac{d^2}{dt^2} P_N(t) = \sum_{k=1}^{\infty} k(k-1) t^{k-2} p_k = \sum_{k=2}^{\infty} k(k-1) t^{k-2} p_k,$$

which yields

$$P_N^{(2)}(1) = E\left[N(N-1)\right]$$

More generally

$$P_N^{(n)}(1) = E\left[N(N-1)\cdots(N-n+1)\right]$$

3. pgf uniquely determines a dist^n , just like mgf.

11/1 (수)

5.2 Some possible candidates

5.2.1 Selection of basic dist^n

5.2.1.1 poisson dist^n

The pmf of a poisson rv N with parameter $\lambda > 0$ is

$$p_k = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, \cdots$$

A Poisson rv N has a mean equal to its variance

$$E(N) = Var(N) = \lambda$$

The pgf of N is

$$\begin{aligned} P_N(t) = E(t^N) &= \sum_{k=0}^{\infty} t^k \left\{ \frac{e^{-\lambda}\lambda^k}{k!} \right\} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \\ &= e^{-\lambda} e^{\lambda t} \underbrace{\sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!}}_{=1} \\ &= e^{-\lambda(1-t)} \end{aligned}$$

<u>Properties</u>

· If N_1, N_2, $\cdots$, N_m are indep. Poisson rv's with parameters λ_1, λ_2, $\cdots$, λ_m, respectively, then

$$N = N_1 + \cdots + N_m$$

has a Poisson rv with mean $\lambda_1 + \cdots + \lambda_m$

Solution.

$$\begin{aligned} P_N(t) &= E\left[t^N\right] \\ &= E\left[t^{N_1+\cdots+N_m}\right] \\ &= E\left[t^{N_1}\right]\cdots E\left[t^{N_m}\right] \qquad \because \text{indep.} \\ &= e^{\lambda_1(t-1)} \times \cdots \times e^{\lambda_m(t-1)} \\ &= e^{\lambda(t-1)}, \qquad \lambda = \lambda_1 + \cdots + \lambda_m \end{aligned}$$

which is the pgf of a Poisson rv with mean λ.

· Suppose that the overall # of events N in a certain time period is Pois(λ). Suppose that each event is one of k distinct types of events (independently of each others) and that given an event occurs, the event is of type i with prob p_i ($p_1 + \cdots + p_k = 1$). Then, for each fixed i, the # of events of type i, say N_i, has a Poisson dist^n with parameter λp_i. Furthermore, $N_1, \cdots, N_k$ are independent.

$$N \begin{cases} \text{type Ⅰ } 20\% & \longleftarrow \ N_1 \\ \text{type Ⅱ } 30\% & \longleftarrow \ N_2 \\ \text{type Ⅲ } 50\% & \longleftarrow \ N_3 \end{cases}$$

Solution. WLOG, assume $k = 2$.

From the above asumption, we have

$$\begin{aligned} P(N_1 = n_1,\ N_2 = n_2 \mid N = n_1 + n_2) &= \binom{n_1+n_2}{n_1} (p_1)^{n_1} (p_2)^{n_2}, \quad (p_1 + p_2 = 1) \\ &= (n_1+n_2)! \cdot \frac{(p_1)^{n_1}}{n_1!} \cdot \frac{(p_2)^{n_2}}{n_2!} \end{aligned}$$

and

$$P(N_1 = n_1,\ N_2 = n_2 \mid N = \underbrace{n}_{\text{eg) } n_1+n_2+1}) = 0, \qquad \text{if } n \neq n_1 + n_2$$

$$P(N_1 = n_1,\ N_2 = n_2 \mid N = n) = \begin{cases} (n_1+n_2)! \cdot \dfrac{(p_1)^{n_1}}{n_1!} \cdot \dfrac{(p_2)^{n_2}}{n_2!}, & n = n_1 + n_2 \\ 0, & n \neq n_1 + n_2 \end{cases}$$

It follows that

$$\begin{aligned} p(N_1 = n_1,\ N_2 = n_2) &= \sum_n P(N_1 = n_1,\ N_2 = n_2 \mid N = n)\,\mathrm{Pr}\,(N = n) \\ &\underset{\text{only } n = n_1 + n_2}{=} (n_1+n_2)! \cdot \frac{(p_1)^{n_1}}{n_1!} \cdot \frac{(p_2)^{n_2}}{n_2!} \times \frac{e^{-\lambda}\lambda^{n_1+n_2}}{(n_1+n_2)!} \\ &= \frac{(\lambda p_1)^{n_1}}{n_1!} \cdot \frac{(\lambda p_2)^{n_2}}{n_2!} e^{-\lambda} \\ &= \frac{(\lambda p_1)^{n_1}\, e^{-\lambda p_1}}{n_1!} \times \frac{(\lambda p_2)^{n_2}\, e^{-\lambda p_2}}{n_2!} \end{aligned}$$

one deduces that

$$\begin{aligned} \mathrm{Pr}\,(N_1 = n_1) &= \sum_{n_2} P(N_1 = n_1,\ N_2 = n_2) \\ &= \frac{(\lambda p_1)^{n_1}\, e^{-\lambda p_1}}{n_1!} \sum_{n_2=0}^{\infty} \frac{(\lambda p_2)^{n_2}\, e^{-\lambda p_2}}{n_2!} \\ &= \frac{(\lambda p_1)^{n_1}\, e^{-\lambda p_1}}{n_1!} \quad \longleftarrow \ \mathrm{Pois}(\lambda p_1) \end{aligned}$$

Similary,

$$\mathrm{Pr}\,(N_2 = n_2) = \frac{(\lambda p_2)^{n_2}\, e^{-\lambda p_2}}{n_2!} \quad \longleftarrow \ \mathrm{Pois}\,(\lambda p_2)$$

$\Rightarrow$ clearly, $P\,(N_1 = n_1,\ N_2 = n_2) = P\,(N_1 = n_1)\,P\,(N_2 = n_2)$ one concludes that N_1& N_2 are indep.

11/6 (월)

5.2.1.2 Binomial dist^n

Given that a binomial rv has a finite support, the use of a binomial dist^n for the # of claims / payments N implies that there is a maximum # of claims / payments that can occur.

For a binomial rv N with parameters $m \in \mathbb{N}^+$ and $q \in (0,1)$, the pmf is given by

$$p_k = \binom{m}{k} q^k (1-q)^{m-k}, \quad k = 0, 1, \cdots, m$$

The mean of N is greater than its variance

$$E(N) = mq > Var(N) = mq(1-q)$$

The pgf of N:

$$\begin{aligned} P_N(t) &= \sum_{k=0}^{m} t^k \left\{ \binom{m}{k} q^k (1-q)^{m-k} \right\} \\ &= \sum_{k=0}^{m} \binom{m}{k} (tq)^k (1-q)^{m-k} \\ &= \underbrace{\sum_{k=0}^{m} \left\{ \binom{m}{k} \left(\frac{tq}{1-q+tq} \right)^k \left(\frac{1-q}{1-q+tq} \right)^{m-k} \right\}}_{=1} (1-q+tq)^m \\ &= (1-q+tq)^m \end{aligned}$$

5.2.1.3 Negative Binomial (NB)

For N a neg. bin. rv with parameter $r > 0$ and $\beta > 0$, the pmf is

$$p_k = \binom{k+r-1}{k} \left(\frac{1}{1+\beta} \right)^r \left(\frac{\beta}{1+\beta} \right)^k, \qquad k = 0, 1, 2, \cdots$$

where $\binom{x}{k} = \dfrac{\Gamma(x+1)}{k!\Gamma(x-k+1)}$ for a non-negative integer k and any real x.

For an integer r, an NB rv can be viewed as the # of failures until reaching the r^{th} success

The mean of N is less than its variance

$$E(N) = r\beta \; < \; Var(N) = r\beta(1+\beta)$$

(NOTE)

$$X \sim Gam(\alpha, \beta) \Rightarrow E(X) = \alpha\beta, \quad Var(X) = \alpha\beta^2$$

NB is the discrete counterpart of gamma dist^n

The pgf of N

$$\begin{aligned}
P_N(t) &= \sum_{k=0}^{\infty} t^k \left\{ \binom{k+r-1}{k} \left(\frac{1}{1+\beta}\right)^r \left(\frac{\beta}{1+\beta}\right)^k \right\} \\
&= \sum_{k=0}^{\infty} \binom{k+r-1}{k} \left(\frac{1}{1+\beta}\right)^r \left(\frac{t\beta}{1+\beta}\right)^k \\
&= \frac{\left(\frac{1}{1+\beta}\right)^r}{\left(1-\frac{t\beta}{1+\beta}\right)^r} \underbrace{\sum_{k=0}^{\infty} \binom{k+r-1}{k} \left(1-\frac{t\beta}{1+\beta}\right)^r \left(\frac{t\beta}{1+\beta}\right)^k}_{=1} \\
&= \frac{\left(\frac{1}{1+\beta}\right)^r}{\left(1-\frac{t\beta}{1+\beta}\right)^r} = (1-(t-1)\beta)^{-r}
\end{aligned}$$

Remark. when $r = 1$, the resulting dist^n is called the geometric dist^n which has the memoryless property. (geometric dist^n can be viewd as the discrete counterpart of Exponential dist^n)

Example. The NB is another example of a mixture. If $N \mid \Lambda = \lambda \sim \text{Pois}(\lambda)$ and $\Lambda \sim \text{Gam}(\alpha, \theta)$. then the unconditional dist^n of N is NB(α, θ)

Solution.

$$P_N(t) = E\left[t^N\right] = E_\Lambda\left[E\left[t^N \mid \Lambda\right]\right] = E_\Lambda\left[e^{\Lambda(t-1)}\right]$$

using the mgf of Gam(α, θ), it follows that

$$P_N(t) = E\left[e^{\Lambda(t-1)}\right] = \left(\frac{1}{1-\theta(t-1)}\right)^\alpha = (1-\theta(t-1))^{-\alpha}$$

which is the pgf of a NB rv with $r = \alpha$ and $\beta = \theta$.

5.2.2 The (a, b, 0) class of dist^{n}s

The Poisson, Bin, and NB, dist^{n}s all belong to a class of discrete dist^n called the (a, b, 0) class. Indeed, we say that discrete non-negative rv N with pmf p_k $(k = 0, 1, \cdots)$ is a member of the (a, b, 0) class if there exists two constants a and b such that, for all $k = 1, 2, \cdots$

$$p_k = \left(a + \frac{b}{k}\right) p_{k-1},$$

or

$$\frac{p_k}{p_{k-1}} = a + \frac{b}{k}, \qquad k = 1, 2, \cdots$$

Example. Poisson

$$\frac{p_k}{p_{k-1}} = \frac{\frac{e^{-\lambda}\lambda^k}{k!}}{\frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}} = \frac{\lambda}{k} \quad \Rightarrow \quad \alpha = 0, \quad b = \lambda$$

Example. Bin(m, q)

$$\begin{aligned} \frac{p_k}{p_{k-1}} &= \frac{\binom{m}{k} q^k (1-q)^{m-k}}{\binom{m}{k-1} q^{k-1} (1-q)^{m-(k-1)}} = \cdots \\ &= -\frac{q}{1-q} + \frac{m+1}{k} \cdot \frac{q}{1-q} \ \Rightarrow \ a = \frac{-q}{1-q}, \quad b = \frac{(m+1)q}{1-q} \end{aligned}$$

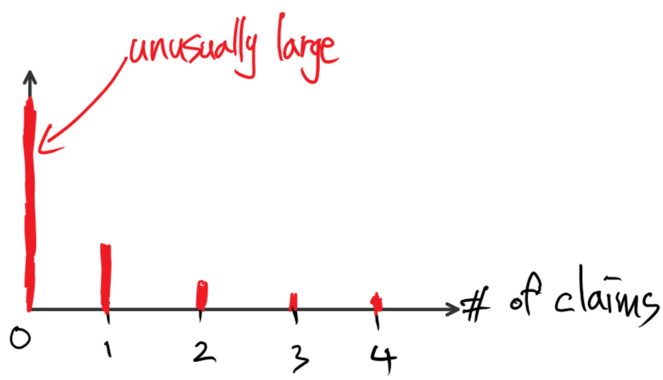
Example. NB(r, β) (DIY)

$$\Rightarrow \ a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta}$$

It can be shown that these three dist^{n}s are the only possible dist^{n}s in the (a, b, 0) class.

5.2.3 The (a, b, 1) class of dist^{n}s

At times, the (a, b, 0) class of dist^{n}s do not adequately discribe the characteristics of some insurance data. In particular, it is frequent to encounter situations where the fit provided by any of the (a, b, 0) dist^n is poor for the prob associated to No claim / payment (i.e. $\Pr(N = 0)$)



Idea: modify the (a, b, 0) class by

· maintaining the recursive relationship

$$\frac{p_k}{p_{k-1}} = a + \frac{b}{k}, \qquad k = 2, 3, 4, \cdots$$

· choose the prob of your own choice for

$$\Pr(N = 0) = p_0$$

We distinguish two cases:

1. If the new prob for $N = 0$ is zero (i.e. $\Pr(N = 0) = 0$), the resulting dist^n is referred to as a zero-truncated (ZT) dist^n
2. If the new prob for $N = 0$ is different than p_0 suggested by the (a, b, 0) class, then the resulting dist^n is referred to as a zero-modified (ZM) dist^n

Consider that $N \in (a, b, 0)$ with $p_k = \Pr(N = k)$ for $k = 0, 1, \cdots$ and pgf

$$P_N(z) = \sum_{k=0}^{\infty} z^k p_k$$

1. The prob of its corresponding ZT dist^n are denoted p_k^T, so that $p_0^T = 0$. For the (a, b, 1) recursion to be valid, we set

   $$p_k^T = \beta p_k \qquad \text{for some } \beta,$$

   given that
   $$\frac{p_k^T}{p_{k-1}^T} = \frac{\beta p_k}{\beta p_{k-1}} = \frac{p_k}{p_{k-1}} = a + \frac{b}{k}, \quad k = 2, 3, 4, \cdots$$

   From $\sum_{k=1}^{\infty} p_k^T = 1$, we have

   $$\sum_{k=1}^{\infty} p_k^T = \sum_{k=1}^{\infty} \beta p_k = \beta(1 - p_0) = 1$$

   $$\therefore\ \beta = \frac{1}{1 - p_0} > 1 \qquad \text{and} \qquad p_k^T = \frac{p_k}{1 - p_0}, \quad k = 1, 2, \cdots$$

   The pgf of the ZT dist^n:

$$\begin{aligned} P_N^T(z) &= \sum_{k=0}^{\infty} z^k p_k^T \underset{\text{by def.}}{\overset{p_0^T=0}{=}} \sum_{k=1}^{\infty} z^k \frac{p_k}{1-p_k} \\ &= \frac{\sum_{k=0}^{\infty} z^k p_k - z^0 p_0}{1-p_0} = \frac{P_N(z) - p_0}{1-p_0} \end{aligned}$$

11/8 (수)

Original $(a, b, 0)$: $p_0, p_1, p_2, \cdots$

ZT: $\underbrace{p_0^T}_{=0}, p_1^T, p_2^T, \cdots$

ZM: $\underbrace{p_0^M}_{=\text{your choice}}, p_1^M, p_2^M, \cdots$

$$P_n^T(z) = \frac{P_N(z) - p_0}{1 - p_0}$$

2. The prob. of the corresponding ZM dist^n are denoted by p_k^M, $k = 0, 1, 2, \cdots$ Once p_0^M has been chosen (arbitrary), The other probabilities are determined such that that $(a, b, 1)$ recursion holds. i.e,

$$\frac{p_k^M}{p_{k-1}^M} = a + \frac{b}{k}, \quad k = 2, 3, \cdots$$

This is true if

$$p_k^M = \beta p_k, \quad k = 1, 2, \cdots \tag{6}$$

Given that

$$\sum_{k=0}^{\infty} p_k^M = 1$$

it follows that

$$\begin{aligned}
\sum_{k=0}^{\infty} p_k^M &= p_0^M + \sum_{k=1}^{\infty} p_k^M \\
&= p_0^M + \sum_{k=1}^{\infty} \beta p_k \qquad \because (6) \\
&= p_0^M + \beta(1 - p_0) \qquad \because \sum_{k=1}^{\infty} p_k = 1 - p_0 \\
&= 1 \qquad \Rightarrow \beta = \frac{1 - p_0^M}{1 - p_0}, \quad \begin{cases} p_0^M : \text{ your choice} \\ p_0 : \text{ original } \Pr(N = 0) \text{ of } (a, b, 0) \text{ class} \end{cases}
\end{aligned}$$

Thus

$$p_k^M = \frac{1 - p_0^M}{1 - p_0} p_k, \quad k = 1, 2, \cdots$$

the pgf of ZM dist^n:

$$
\begin{aligned}
P_N^M(z) &= \sum_{k=0}^{\infty} z^k p_k^M = p_0^M + \sum_{k=1}^{\infty} z^k p_k^M \\
&= p_0^M + \sum_{k=1}^{\infty} z^k \frac{1-p_0^M}{1-p_0} p_k \\
&= p_0^M + \frac{1-p_0^M}{1-p_0} \times \underbrace{\sum_{k=1}^{\infty} z^k p_k}_{=\sum_{k=0}^{\infty} z^k p_k - p_0 = P_N(z) - p_0} \\
&= p_0^M + \frac{1-p_0^M}{1-p_0}\left(P_N(z) - p_0\right)
\end{aligned}
$$

Somtimes we rewrite

$$
P_N^M(z) = \frac{p_0^M - p_0}{1-p_0} + \frac{1-p_0^M}{1-p_0} P_N(z)
$$

From the pgf relationship, it is easy to derive the moments for the ZM dist^n.

$$
E\left[N^M\right] = \frac{1-p_0^M}{1-p_0} E(N)
$$

or, more generally,

$$
E\left[\left(N^M\right)^k\right] = \frac{1-p_0^M}{1-p_0} E\left[N^k\right]
$$

Example. $N \sim$ Pois(2) and we consider a ZM version of N with $p_0^M = 0.3$. The pgf of N^M is

$$
P_N^M(z) = 0.3 + \frac{1-0.3}{1-e^{-2}}\left(e^{2(t-1)} - e^{-2}\right)
$$

or

$$
P_N^M(z) = \frac{0.3 - e^{-2}}{1-e^{-2}} \times (1) + \frac{1-0.3}{1-e^{-2}} \times (e^{2(t-1)}),
$$

which implies that the ZM Poisson is a mixture between a <u>degenerate dist^n at 0</u> and the original Poisson rv.

Example. (ZT NB) $N \sim NB(r,\beta)$. Its ZT version has pmf of the form

$$\begin{aligned}
p_k^T &= \frac{1}{1-p_0}p_k = \frac{\binom{k+r-1}{k}\left(\frac{1}{1+\beta}\right)^r\left(\frac{\beta}{1+\beta}\right)^k}{1-\left(\frac{1}{1+\beta}\right)^r} \\
&= \frac{\Gamma(k+r)}{k!\Gamma(r)}\cdot\frac{\left(\frac{\beta}{1+\beta}\right)^k}{(1+\beta)^r-1} = \frac{r\Gamma(k+r)}{k!\Gamma(r+1)}\times\frac{\left(\frac{\beta}{1+\beta}\right)^k}{(1+\beta)^r-1} \\
&= \frac{(-r)\Gamma(k+r)}{k!\Gamma(r+1)}\times\frac{\left(\frac{\beta}{1+\beta}\right)^k}{1-(1+\beta)^r}, \qquad k=1,2,\cdots
\end{aligned}$$

- This it also a valid pmf for $-1 < r < 0 \quad \& \quad \beta > 0$ (known as the extended truncated neg. bin)
- Note that $p_0 < 0$ for $-1 < r < 0$ under the original NB, but here we don't care about the value at 0.

11/13 (월)

In addition to the ZM (ZT) version of each of the $(a, b, 0)$ dist^{n}s, there exist other members of the $(a, b, 1)$ class of dist^n one of them is the so-called Logarithmic (and its ZM version). To derive this pmf, we consider sum

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad |x| < 1$$

Integrating both sides from 0 to q $(0 < q < 1)$,

$$\int_0^q \sum_{k=0}^{\infty} x^k dx = \int_0^q \frac{1}{1-x} dx$$
$$\sum_{k=0}^{\infty} \frac{q^{k+1}}{k+1} = -\log(1-q)$$

Rearranging this yields

$$\sum_{k=0}^{\infty} \frac{q^{k+1}}{(k+1)\left(-\log(1-q)\right)} = 1$$

or, equivalently $(k \leftarrow k+1)$

$$\sum_{k=1}^{\infty} \frac{q^k}{k\left(-\log(1-q)\right)} = 1$$

Letting $q = \dfrac{\beta}{1+\beta}, \quad (\beta > 0)$,

$$\sum_{k=1}^{\infty} \frac{\left(\frac{\beta}{1+\beta}\right)^k}{k \log(1+\beta)} = 1$$

A rv N is said to have a logarithmic dist^n with parameter $\beta > 0$ if its pmf is of the form

$$p_k = \frac{\left(\frac{\beta}{1+\beta}\right)^k}{k \log(1+\beta)}, \quad k = 1, 2, \cdots$$

5.2.4 Mixture of dist^{n}s

Suppose that $N \mid \Theta = \theta$ has conditional pmf $\Pr(N = n \mid \Theta = \theta)$ and the mixing rv Θ is either

- discrete: pmf $\Pr(\Theta = \theta_i) = a_i, \quad i = 1, \cdots$

- continuous: density $g_\Theta(\theta)$

Both the dist^n of Θ and $N \mid \Theta = \theta$ are assumed known.

Then the unconditional pmf of N is

$$\Pr(N=n) = \begin{cases} \sum_{i=1}^{m} a_i \Pr(N=n \mid \Theta=\theta), & \text{if } \Theta \text{ is disc.} \\ \int \Pr(N=n \mid \Theta=\theta)\, g(\theta) d\theta, & \text{if } \Theta \text{ is cont.} \end{cases}$$

Example. $N \mid \Theta = \theta \ \sim \ \text{Pois}(\theta)$ and Θ has density

$$g(\theta) = \frac{\lambda^2}{\lambda+1}(\theta+1)e^{-\lambda\theta}, \quad \theta > 0$$

where $\lambda > 0$, Determine the unconditional pmf of N

Solution. First it is interesting to note that

$$\begin{aligned} g(\theta) &= \frac{\lambda^2}{\lambda+1}\theta e^{-\lambda\theta} + \frac{\lambda^2}{\lambda+1}e^{-\lambda\theta} \\ &= \frac{1}{\lambda+1}\left\{\underbrace{\lambda^2\theta e^{-\lambda\theta}}_{\text{Erl}(2,1/\lambda)}\right\} + \frac{\lambda}{\lambda+1}\left\{\underbrace{\lambda e^{-\lambda\theta}}_{\substack{\text{Erl}(1,1/\lambda) \\ =\text{Exp}(1/\lambda)}}\right\} \end{aligned}$$

$\Rightarrow g(\theta)$ is a weighted average(2 - point mixture) of two Erlang dist^{n}s

Then,

$$\begin{aligned} \Pr(N=n) &= \int_0^\infty \Pr(N=n \mid \Theta=\theta)\, g(\theta) d\theta \\ &= \int_0^\infty \frac{e^{-\theta}\theta^n}{n!}\left\{\frac{1}{\lambda+1}\left(\lambda^\theta \theta e^{-\lambda\theta}\right) + \frac{\lambda}{\lambda+1}\left(\lambda e^{-\lambda\theta}\right)\right\} d\theta \\ &= \frac{\lambda^2}{\lambda+1}\cdot\frac{1}{n!}\int_0^\infty \underbrace{\theta^{n+1}e^{-(\lambda+1)\theta}}_{\text{gamma density's kernel}} d\theta + \frac{\lambda^2}{\lambda+1}\cdot\frac{1}{n!}\int_0^\infty \theta^n e^{-(\lambda+1)\theta} d\theta \\ &= \frac{\lambda^2}{\lambda+1}\cdot\frac{1}{n!}\cdot\frac{(n+1)!}{(\lambda+1)^{n+2}} + \frac{\lambda^2}{\lambda+1}\cdot\frac{1}{n!}\cdot\frac{n!}{(\lambda+1)^{n+1}} \\ &= \frac{1}{\lambda+1}p_1(n) + \frac{\lambda}{\lambda+1}p_2(n) \end{aligned}$$

where

$$p_1(n) = \frac{(n+1)\lambda^2}{(\lambda+1)^{n+2}} = \binom{n+2-1}{n}\left(\frac{\lambda}{\lambda+1}\right)^2\left(\frac{1}{\lambda+1}\right)^n \ \sim \ NB\left(2, \frac{1}{n}\right)$$

and

$$p_2(n) = \frac{\lambda}{(\lambda+1)^{n+1}} = \frac{\lambda}{\lambda+1}\left(\frac{1}{\lambda+1}\right)^n \ \sim \ NB\left(1, \frac{1}{\lambda}\right)$$

$\Rightarrow N$ is a 2-point mixture of two NB's with mixing weights $\frac{1}{\lambda+1}$ and $\frac{\lambda}{\lambda+1}$, respectively

5.2.5 Compounding random variables

A large class of frequency dist^{n}s can be created by the process of compounding any 2 discrete dist^{n}s

Idea: Suppose that N and M are two discrete non-negative interger-valued rv's. Define S as

$$S = \begin{cases} M_1 + \cdots + M_N, & N > 0 \\ 0, & N = 0 \end{cases} = \begin{cases} \sum_{i=1}^{N} M_i, & N > 0 \\ 0, & N = 0 \end{cases}$$

where each M_i has the same dist^n as M. Also, the rv's $M_1, M_2, \cdots$ and N are indep.

M_i : # of claims per each accidents.

Notation:

- The dist^n of N is called the primary dist^n with pmf p_k and pgf $P_N(z)$
- The dist^n of M is called the secondary dist^n with pmf f_k, pgf $P_M(z)$ and mgf $M_M(t)$

(i) pgf of S

$$P_S(t) = E\left(t^S\right) = E\left[E\left[t^S \mid N\right]\right]$$

given that

$$\begin{aligned} E\left[t^S \mid N = n\right] &= E\left[t^{M_1 + \cdots + M_n}\right] = E\left[t^{M_1}\right] \cdots E\left[t^{M_n}\right] \\ &= \left(P_M(t)\right)^n, \quad \text{for } \forall n \in \{0, 1, \cdots\} \end{aligned}$$

it follows that

$$E\left[t^S \mid N\right] = \left(P_M(t)\right)^N$$

Then $P_S(t) = E\left[\left(P_M(t)\right)^N\right] = P_N\left(P_M(t)\right)$

(ii) mgf of S

(Hint: $M_X(z) = E\left(e^{zX}\right) = E\left(\left(e^z\right)^X\right) = P_X\left(e^z\right)$ **)**

$$\begin{aligned} M_S(r) &= E\left[e^{rS}\right] \\ &= E\left[\left(e^r\right)^S\right] = P_S\left(e^r\right) \\ &\overset{(\,\mathrm{i}\,)}{=} P_N\left(P_M\left(e^r\right)\right) \overset{\text{Hint}}{=} P_N\left(M_M(r)\right) \end{aligned}$$

(iii) $E(S)$

$$E(S) = E\left[E(S \mid N)\right]$$

where

$$\begin{aligned} E(S \mid N = n) &= E\left[M_1 + \cdots + M_N \mid N = n\right] = E\left[M_1 + \cdots + M_n\right] \\ &= nE(M), \quad \text{for } \forall n \in \{0, 1, 2, \cdots\} \end{aligned}$$

It follows that

$$E\left[S \mid N\right] = NE(M)$$

which implies that

$$E(S) = E\left[NE(M)\right] = E(N)E(M)$$

(iv) $Var(S) = E\left[Var(S|N)\right] + Var\left[E(S|N)\right]$

where $E(S|N) = NE(M)$ and $Var(S|N) = NVar(M)$

Thus

$$\begin{aligned} Var(S) &= E\left[N \cdot Var(M)\right] + Var\left[NE(M)\right] \\ &= E(N)Var(M) + Var(N)\left(E(M)\right)^2 \end{aligned}$$

11/15 (수)

(v) pmf of S

Let $\{g_k\}_{k\geq 0}$ be the pmf of S, i.e. $\Pr(S=k)=g_k$

Also recall that $\{p_n\}_{n\geq 0}$ and $\{f_k\}_{k\geq 0}$ are the pmf of N and M, respectively.

(1) Calculation of g_0

$$
\begin{aligned}
g_0 &= \Pr(S=0) \\
&= \underbrace{\Pr(S=0 \mid N=0)}_{=1} \underbrace{\Pr(N=0)}_{p_0} + \sum_{n=1}^{\infty} \Pr(S=0 \mid N=n) \times \underbrace{\Pr(N=n)}_{p_n} \\
&= p_0 + \sum_{n=1}^{\infty} \Pr(M_1+\cdots+M_n=0)\, p_n \\
&= p_0 + \sum_{n=1}^{\infty} \underbrace{\Pr(M_1=\cdots=M_n=0)}_{(f_0)^n}\, p_n = P_N(f_0)
\end{aligned}
$$

(2) calculation of $\{g_k\}_{k\geq 1}$

$$
\begin{aligned}
g_k = \Pr(S=k) &= \underbrace{\Pr[S=k \mid N=0]}_{=0}\, p_0 + \sum_{n=1}^{\infty} \Pr[S=k \mid N=n]\, p_n \\
&= \sum_{n=1}^{\infty} \Pr[M_1+\cdots+M_n=k]\, p_n = \sum_{n=1}^{\infty} f_k^{*n} p_n,
\end{aligned}
$$

where

$$
f_k^{*n} = \Pr[M_1+\cdots+M_n=k]
$$

is the pmf associated to the n-fold convolution of f_k with itself (i.e. the pmf of the sum of n iid rv's with pmf $\{f_k\}_{k\geq 0}$)

Example. Suppose that $p_0=0.4$, $p_1=0.4$ and $p_2=0.2$. Also we have $f_0=0.5$, $f_1=0.3$, and $f_2=0.2$. Find the pmf of S.

Solution.

$$
\begin{aligned}
N &= 0,1,2 \quad \leftarrow \{p_n\} \\
M &= 0,1,2 \quad \leftarrow \{f_k\} \\
S &= 0 \sim 4
\end{aligned}
$$

First, we identify the 2-fold convolution of f_k

$$
\begin{aligned}
f_0^{*2} &= \Pr\left[M_1 + M_2 = 0\right] = \Pr\left[M_1 = M_2 = 0\right] = \left(f_0\right)^2 = 0.25 \\
f_1^{*2} &= \Pr\left[M_1 + M_2 = 1\right] = \Pr\left[M_1 = 1, M_2 = 0\right] + \Pr\left[M_1 = 0, M_2 = 1\right] \\
&= 2 f_0 f_1 = 2 \times 0.5 \times 0.3 = 0.3 \\
f_2^{*2} &= \Pr\left[M_1 + M_2 = 2\right] \\
&= \Pr\left[M_1 = 2, M_2 = 0\right] + \Pr\left[M_1 = M_2 = 1\right] + \Pr\left[M_1 = 0, M_2 = 2\right] \\
&= f_2 f_0 + f_1 f_1 + f_0 f_2 = 0.29 \\
f_3^{*2} &= \Pr\left[M_1 + M_2 = 3\right] = \Pr\left[M_1 = 2, M_2 = 1\right] + \Pr\left[M_1 = 1, M_2 = 2\right] \\
&= 2 f_1 f_2 = 0.12 \\
f_4^{*2} &= \Pr\left[M_1 + M_2 = 4\right] = \Pr\left[M_1 = M_2 = 2\right] = \left(f_2\right)^2 = 0.04
\end{aligned}
$$

It follows that

$$
\begin{aligned}
g_0 &= p_0 + p_1 f_0 + p_2 \underbrace{\left(f_0^{*2}\right)}_{(f_0)^2} = 0.4 + 0.4 \times 0.5 + 0.2 \times 0.25 = 0.65 \\
g_1 &= p_1 f_1 + p_2 \left(f_1^{*2}\right) = 0.4 \times 0.3 + 0.2 \times 0.3 = 0.18 \\
g_2 &= p_1 f_2 + p_2 \left(f_2^{*2}\right) = 0.4 \times 0.2 + 0.2 \times 0.29 = 0.138 \\
g_3 &= p_2 \left(f_3^{*2}\right) = 0.2 \times 0.12 = 0.024 \\
g_4 &= p_2 \left(f_4^{*2}\right) = 0.2 \times 0.04 = 0.008
\end{aligned}
$$

In general, the n-fold convolution f_k^{*n} can be computed recursively via

$$
\begin{aligned}
f_k^{*n} &= \Pr\left[M_1 + \cdots + M_n = k\right] \\
&= \sum_{j=0}^{k} \Pr\left[M_1 + \cdots + M_n = k \mid M_n = j\right] \Pr\left[M_n = j\right] \\
&= \sum_{j=0}^{k} \Pr\left[M_1 + \cdots + M_{n-1} = k - j\right] \times f_j \\
&= \sum_{j=0}^{k} f_{k-j}^{*(n-1)} \times f_j
\end{aligned}
$$

However, the evaluation of the n-fold convolution for large n is a cumbersome operation.

Fortunately, under some restrictions on the primary dist^n a recursive fomula, known as Panjer recursion, can be used to obtain $\{g_k\}$ efficiently.

11/20 (월)

5.2.5.1 Panjer recursion for the $(a, b, 0)$ class

Suppose that N is a $(a, b, 0)$ member, i.e.

$$\frac{p_n}{p_{n-1}} = a + \frac{b}{n}, \quad n = 1, 2, \cdots$$

Equivalently,

$$\begin{aligned} np_n &= (na + b)p_{n-1} \\ &= ((n-1)a + a + b)\, p_{n-1} \\ &= a(n-1)p_{n-1} + (a+b)p_{n-1} \end{aligned}$$

Recall that the pgf of S is given by

$$P_S(t) = P_N\left(P_M(t)\right),$$

which implies that

$$P'_S(t) = P'_M(t)P'_N\left(P_M(t)\right) = P'_M(t)\sum_{n=1}^{\infty} n\left(P_M(t)\right)^{n-1} \cdot p_n$$

마지막 등호는 $P_N(z) = \sum\limits_{n=0}^{\infty} z^n p_n, \quad P'_N(z) = \sum\limits_{n=1}^{\infty} nz^{n-1}p_n$ 에 의해.

Capitalizing m the Last equality, we multiply both sides of (*) by $P'_M(t)\left(P_M(t)\right)^{n-1}$:

$$P'_M(t)n\left(P_M(t)\right)^{n-1} p_n = aP'_M(t)(n-1)\left(P_M(t)\right)^{n-1} p_{n-1} + (a+b)P'_M(t)\left(P_M(t)\right)^{n-1} p_{n-1}$$

Summing the eqn. over n from 1 to ∞ gives

$$\begin{aligned} \sum_{n=1}^{\infty} P'_M(t)n\left(P_M(t)\right)^{n-1} p_n &= aP'_M(t)\sum_{n=1}^{\infty}(n-1)\left(P_M(t)\right)^{n-1} p_{n-1} \\ &\quad + (a+b)P'_M(t)\sum_{n=1}^{\infty}\left(P_M(t)\right)^{n-1} p_{n-1} \\ &= aP_M(t)\left\{P'_M(t)\sum_{n=0}^{\infty} n\left(P_M(t)\right)^{n-1} p_n\right\} \\ &\quad + (a+b)P'_M(t)\sum_{n=0}^{\infty}\left(P_M(t)\right)^{n} p_n \\ \Rightarrow \qquad P'_S(t) &= aP_M(t)P'_S(t) + (a+b)P'_M(t)P_S(t) \end{aligned}$$

여기서 $P'_S(t) = \left\{P'_M(t)\sum_{n=0}^{\infty} n\left(P_M(t)\right)^{n-1} p_n\right\}$, $P_S(t) = \sum_{n=0}^{\infty}\left(P_M(t)\right)^{n} p_n$.

By expanding both sides as a ft of t, one gets

$$\sum_{i=0}^{\infty} t^{i-1}\{ig_i\} = a\left\{\sum_{i=0}^{\infty} t^i f_i\right\}\left\{\sum_{j=0}^{\infty} t^{j-1} j g_j\right\} + (a+b)\left\{\sum_{i=0}^{\infty} t^{i-1} i f_i\right\}\left\{\sum_{j=0}^{\infty} t^j g_j\right\}, \quad |t|<1$$

As a result, the coefficient of t^{k-1} $(k=1,2,3,\cdots)$ must be the same on both sides of the equality. i.e,

$$\begin{aligned} kg_k &= a\sum_{i+j=k} jf_i g_i + (a+b)\sum_{i+j=k} if_i g_j \\ &= a\sum_{i+j=k} \underbrace{(i+j)}_{k} f_i g_j + b\sum_{i+j=k} if_i g_j \\ &= ak\sum_{i+j=k} f_i g_j + b\sum_{i+j=k} if_i g_j \\ &\underbrace{=}_{j=k-i} ak\sum_{i=0}^{k} f_i g_{k-i} + b\sum_{i=0}^{k} if_i g_{k-i} \\ &= akf_0 g_k + ak\sum_{i=1}^{k} f_i g_{k-i} + b\sum_{i=1}^{k} if_i g_{k-1} \end{aligned}$$

which implies that

$$g_k = \frac{\sum_{i=1}^{k}(ak+bi)f_i g_{k-i}}{k-akf_0} = \frac{1}{1-af_0}\sum_{i=1}^{k}\left(a+\frac{bi}{k}\right) f_i g_{k-i}, \quad k=1,2,3,\cdots$$

(Q: 왜 $i+j=k$가 되나요? A:

$$\left\{\sum_{i=0}^{\infty} t^i f_i\right\}\left\{\sum_{j=0}^{\infty} t^{j-1} j g_j\right\} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} jf_i g_j t^{i+j-1}$$

$$\sum_{i+j=k} :(i,j)$$

$$\begin{aligned} &(0,k) \\ &(1,k-1) \\ &\vdots \\ &(k,0) \end{aligned}$$

)

Theorem. *For $N \in (a, b, 0)$ class, the pmf $\{g_k\}_{k\geq 0}$ can be calculated recursively*

$$g_k = \frac{1}{1 - af_0} \sum_{i=1}^{k} \left(a + \frac{b}{k} i \right) f_i g_{k-i}, \quad k = 1, 2, \cdots$$

where the starting point is

$$g_0 = P_N(f_0)$$

Definition. A comp. rv

$$S = \begin{cases} \sum_{i=1}^{N} M_i, & N > 0 \\ 0, & N = 0 \end{cases}$$

with N a Pois / NB / Bin is called a compound Pois / NB / Bin rv.

Example. S is a comp. Pois with poisson parameter $\lambda > 0$ and the secondary dist^n is Bernoulli with mean q. Determine the pmf of S.

Solution. [Approach 1] Using the pgf.

$$P_S(t) = P_N\left(P_M(t)\right)$$

where $P_N(t) = e^{\lambda(t-1)} \leftarrow N \sim \text{Pois}\,(\lambda)$

and

$$\begin{aligned} P_M(t) = E\left(t^M\right) &= \Pr\left(M = 0\right) t^0 + \Pr\left(M = 1\right) t^1 \\ &= 1 - q + qt \end{aligned}$$

It follows that

$$P_S(t) = P_N\left(P_M(t)\right) = e^{\lambda(1-q+qt-1)} = e^{\lambda q(t-1)}$$

$\therefore$ S is a Poisson with poisson parameter λq.

[Approach 2]

For $N \sim \text{Pois}\,(\lambda)$, we know that $a = 0$ and $b = \lambda$. The starting point of Panjer recursion is

$$g_0 = P_N(f_0) = e^{\lambda(f_0-1)} = e^{\lambda(1-q-1)} = e^{-\lambda q}$$

Also, for $k = 1, 2, \cdots$,

$$g_k = \lambda \sum_{i=1}^{k} \frac{i}{k} f_i g_{k-i} = \frac{\lambda}{k} f_1 g_{k-1} = \frac{\lambda q}{k} g_{k-1}$$

$\Rightarrow g_k$ itself satisfies the $(a, b, 0)$ recursion. Given that

$$g_k = \left(a + \frac{b}{k}\right) g_{k-1}, \quad k = 1, 2, \cdots$$

for $a = 0$ and $b = \lambda q$, S must be Poisson with mean λq.

11/22 (수)

$$
\begin{aligned}
N_1 &\sim \text{Pois}(\lambda_1) \\
&\quad\quad\xrightarrow{\text{ind}} N_1 + N_2 \sim \text{Pois}(\lambda_1 + \lambda_2) \\
N_2 &\sim \text{Pois}(\lambda_2)
\end{aligned}
$$

Example. S_1 is a comp. Poisson with Poisson parameter $\lambda^{(i)}$ $(i = 1, 2)$.

The secondary dist^n has pmf $\left\{f_k^{(i)}\right\}_{k\geq 0}$ and pgf

$$
P^{(i)}(t) = \sum_{k=0}^{\infty} t^k f_k^{(i)}
$$

Assuming that the dist^{n}s of S_1 and S_2 are independent, find the dist^n of $S_1 + S_2$.

Solution. $S_1 = \sum^{N_1} M_i$, $S_2 = \sum^{N_2} W_i$ ($N_1 \sim$ Pois$\left(\lambda^{(1)}\right)$, $N_2 \sim$ Pois$\left(\lambda^{(2)}\right)$, M_i : pmf $f_k^{(1)}$, W_i: pmf $f_k^{(2)}$)

Let $S = S_1 + S_2$

using the pgf of S, we have

$$
\begin{aligned}
P_S(t) = E\left(t^S\right) = E\left(t^{S_1+S_2}\right) &= E\left(t^{S_1}\right) E\left(t^{S_2}\right) \\
&= e^{\lambda^{(1)}\left(P^{(1)}(t)-1\right)} \times e^{\lambda^{(2)}\left(P^{(2)}(t)-1\right)} \\
&= e^{\lambda^{(1)}P^{(1)}(t)+\lambda^{(2)}P^{(2)}(t)-\left(\lambda^{(1)}+\lambda^{(2)}\right)} \\
&= e^{\left(\lambda^{(1)}+\lambda^{(2)}\right)(P(t)-1)}
\end{aligned}
$$

where

$$
P(t) = \frac{\lambda^{(1)}}{\lambda^{(1)} + \lambda^{(2)}} \underbrace{P^{(1)}(t)}_{\sum t^k f_k^{(1)}} + \frac{\lambda^{(2)}}{\lambda^{(1)} + \lambda^{(2)}} \underbrace{P^{(2)}(t)}_{\sum t^k f_k^{(2)}}
$$

one concludes that S is a comp Poisson with poisson parameter $\lambda^{(1)} + \lambda^{(2)}$ and the secondary dist^n with pmf $\{f_k\}_{k\geq 0}$ where

$$
f_k = \frac{\lambda^{(1)}}{\lambda^{(1)} + \lambda^{(2)}} f_k^{(1)} + \frac{\lambda^{(2)}}{\lambda^{(1)} + \lambda^{(2)}} f_k^{(2)}
$$

Note that this secondary dist^n is a mixture of the two original secondary dist^{n}s namely $\left\{f_k^{(1)}\right\}$ and $\left\{f_k^{(2)}\right\}$, with the mixing weights $\dfrac{\lambda^{(1)}}{\lambda^{(1)} + \lambda^{(2)}}$ and $\dfrac{\lambda^{(2)}}{\lambda^{(1)} + \lambda^{(2)}}$, respectively.

Therefore, the pmf of S can be calculated via Panjer recursion

$$\begin{aligned} g_k &= \left(\lambda^{(1)} + \lambda^{(2)}\right) \sum_{i=1}^{k} \left(\frac{i}{k}\right) \left(\frac{\lambda^{(1)}}{\lambda^{(1)} + \lambda^{(2)}} f_i^{(1)} + \frac{\lambda^{(2)}}{\lambda^{(1)} + \lambda^{(2)}} f_i^{(2)}\right) g_{k-1} \\ &= \sum_{i=1}^{k} \left(\frac{i}{k}\right) \left(\lambda^{(1)} f_i^{(1)} + \lambda^{(2)} f_i^{(2)}\right) g_{k-i}, \qquad k = 1, 2, \cdots \end{aligned}$$

The initial value is

$$g_0 = P_N(f_0) = \cdots = e^{\lambda^{(i)}\left(f_0^{(1)}-1\right)} \times e^{\lambda^{(2)}\left(f_0^{(2)}-1\right)}$$

$$f_k = \frac{\lambda^{(1)}}{\lambda^{(1)} + \lambda^{(2)}} f_k^{(1)} + \frac{\lambda^{(2)}}{\lambda^{(1)} + \lambda^{(2)}} f_k^{(2)} \to k = 0$$

(Panjer recursion 암기할 필요 없음. 혹시 써야할 일이 있으면 시험때 주어짐)

5.2.5.2 Panjer recursion for the $(a, b, 1)$ class

using a similar methodology, when N is a member of $(a, b, 1)$ class, then

$$g_k = \frac{\left[p_1 - (a+b)p_0\right] f_k + \sum_{i=1}^{k} \left(a + \frac{bi}{k}\right) f_i g_{k-i}}{1 - a f_0}, \quad k = 1, 2, \cdots$$

where

$$g_0 = P_N(f_0)$$

(DIY)

5.3 Effect of policy adjustments.

we examine two key factors that impact the dist^n of N.

5.3.1 Effect of exposure

clearly, the larger the exposure or the portfolio (eg. # of insureds), the larger we expect N to be.

Assume that the current portfolio consists of a entities, each of which produce claims. Let N_j be the # of claims produced by the j^{th} entity . Then the total # of claims is

$$N = N_1 + N_2 + \cdots + N_a$$

Assuming that $\{N_j\}_{j=1}^{a}$ are iid with pgf $P_{N_1}(t)$, then

$$P_N(t) = E\left(t^N\right) = E\left(t^{N_1+\cdots+N_a}\right) = \left(P_{N_1}(t)\right)^a$$

Now suppose that the portfolio is expected to have $a^*(\neq a)$ entities in the following year. Let

$$N^* = N_1 + \cdots + N_{a^*}$$

be the total # of claims from a^*entities. Then

$$\begin{aligned} P_{N^*}(t) &= E\left(t^{N^*}\right) = \left(P_{N_1}(t)\right)^{a^*} \\ &= \left[\underbrace{\left(P_{N_1}(t)\right)^a}_{P_N(t)}\right]^{\frac{a^*}{a}} = \left(P_N(t)\right)^{a^*/a} \end{aligned}$$

If $P_N(t)$ has a property called the infinite divisibility.

we can use the same dist^n class for both N and N^*.

Definition. A discrete dist^n with pgf $P(t)$ is said to be infinitely divisible if for all $n = 1, 2, 3, \cdots$, the function $\left(P(t)\right)^{1/n}$ is the pgf of some random variable .

Example. $N \sim \text{Pois}\,(\lambda)$ Then

$$\begin{aligned} P_N(t) &= e^{\lambda(t-1)} \\ \Rightarrow \left(P(t)\right)^{\frac{1}{n}} &= e^{\frac{\lambda}{n}(t-1)}, \quad \text{pgf of Pois}\left(\frac{\lambda}{n}\right) \text{ for } n = 1, 2, 3, \cdots \end{aligned}$$

$\therefore$ Poisson dist^n is infinitely divisible.

11/27 (월)

Example. $N \sim NB(r, \beta)$ with $E(N) = r\beta$ and $Var(N) = r\beta(1+\beta)$. Then

$$P_N(t) = (1 + \beta - \beta t)^{-r}$$

which implies that

$$(P_N(t))^{1/n} = (1 + \beta - \beta t)^{-r/n}$$

is also the pgf of a NB rv with parameter $(\frac{r}{n}, \beta)$.

$\therefore$ NB is infinitely divisible.

Example. $N \sim \text{Bin}(m, q)$. with mean mq and variance $mq(1-q)$. Then

$$P_N(t) = (1 - q + qt)^m$$

which implies that

$$(P_N(t))^{1/n} = (1 - q + qt)^{m/n}$$

Note that $(P_N(t))^{1/n}$ is NOT necessarily binomial given that m/n may NOT be an integer.

In fact, let $n = 2m$, then

$$(P_N(t))^{1/n} = (1 - q + qt)^{m/2m} = (1 - q + qt)^{1/2}$$

Note that

$$(P_N(1))^{1/n} = (P_N(1))^{1/2m} = (1 - q + q \times 1)^{1/2} = 1$$

but

$$\underbrace{\frac{d^2}{dt^2} = (P_N(t))^{1/2m}\Big|_{t=0}}_{\Pr(N=2)} = -\frac{1}{4}q^2(1-q)^{-3/2} < 0 \quad \Leftarrow \text{impossible.}$$

$\therefore$ Bin rv is **NOT** infinitely divisible.

Example. Suppose that S is a discrete comp. rv with primary rv N and secondary rv M. Then

$$P_S(t) = P_N(P_M(t))$$

which implies that

$$(P_S(t))^{1/n} = (P_N\left(P_M(t)\right))^{1/n}$$

Thus, if the dist^{n}of N is infinitely divisible, so tis the dist^n of S. One concludes that comp Poisson and comp NB dist^{n}s are infinitely divisible.

5.3.2 Effect of severity

Let X be the ground-up loss and N_L be the # of losses. Suppose that the amt paid (either Y_L or Y_P) is an amt modified from X due to policy adjustments.

$$\begin{array}{ccc} & Y_L & Y_P \\ & \downarrow & \downarrow \\ N= & N_L & N_P \end{array}$$

There exists two common ways to define the loss model reflecting the total amt paid:

1. (per loss basis) Keep the frequency dist^n unchanged and use the modified severity on a per loss basis Y_L

$$S = \begin{cases} \sum_{i=1}^{N_L} Y_{L,i}, & N_L > 0 \\ 0, & N_L = 0 \end{cases}$$

   where $Y_{L,i}$ = amt paid on the i^{th} loss.

2. (per payment basis) use the modified severity on a per payment basis Y_P in conjunction with a modified frequency dist^n N_P which counts the # of losses resulting in a non-zero payment

$$S = \begin{cases} \sum_{i=1}^{N_P} Y_{P,i}, & N_P > 0 \\ 0, & N_P = 0 \end{cases}$$

   where $Y_{P,i}$= amt paid on the i^{th} non-zero payment.

Remark. Note that N_P itself is a comp rv.

$$N_P = \begin{cases} \sum_{i=1}^{N_L} I_i, & N_L > 0 \\ 0, & N_L = 0 \end{cases}$$

where

$$I_i = \begin{cases} 1, & \text{if the } i^{\text{th}} \text{ loss results in a non-zero payment.} \\ 0, & \text{if the } i^{\text{th}} \text{ loss results in a zero payment.} \end{cases}$$

Assume that $\Pr(I_i = 0) = \Pr(Y_L = 0) = 1 - \alpha$. Then

$$P_{N_P}(t) = P_{N_L}(P_I(t)) = P_{N_L}(1 - \alpha + \alpha t)$$

Example. $N_L \sim \text{Pois}(\lambda)$. Find the dist^n of N_P given that

$$\Pr(Y_L = 0) = 1 - \alpha$$

Solution. $P_{N_P}(t) = P_{N_L}(1 - \alpha + \alpha t) = e^{\lambda((1-\alpha+\alpha t)-1)} = e^{\lambda\alpha(t-1)}$

$\therefore N_P \sim \text{Pois}(\lambda\alpha)$

Example. $N_L \sim \text{Bin}(m, q)$. Find the dist^n of N_P given that

$$\Pr(Y_L = 0) = 1 - \alpha$$

Solution.

$$\begin{aligned} P_{N_P}(t) = P_{N_L}(1 - \alpha + \alpha t) &= (1 - q + q(1 - \alpha + \alpha t))^m \\ &= (1 - q\alpha + q\alpha t)^m \end{aligned}$$

$\therefore N_P \sim \text{Bin}(m, \alpha q)$

Example. N_L is a ZM Pois rv with Poisson parameter λ and som p_0^M. Show that the dist^n of N_P given that $\Pr(Y_L = 0) = 1 - \alpha$ is also a ZM poisson.

Solution. Recall that

$$\begin{aligned} P_{N_L}^M(t) &= p_0^M + \frac{1 - p_0^M}{1 - p_0}\left[P_{N_L}(t) - p_0\right] \quad (N_L = N) \\ &= p_0^M + \frac{1 - p_0^M}{1 - e^{-\lambda}}\left[e^{\lambda(t-1)} - e^{-\lambda}\right] \quad (N_L \sim \text{Pois}(\lambda)) \end{aligned}$$

It follows that

$$
\begin{aligned}
P_{N_P}^M(t) = P_{N_L}^M(1-\alpha+\alpha t) &= p_0^M + \frac{1-p_0^M}{1-e^{-\lambda}}\left[e^{\lambda((1-\alpha+\alpha t)-1)} - e^{-\lambda}\right] \\
&= p_0^M + \frac{1-p_0^M}{1-e^{-\lambda}}\left[e^{\alpha\lambda(t-1)} - e^{-\alpha\lambda} + e^{-\alpha\lambda} - e^{-\lambda}\right] \\
&= p_0^M + \frac{\left(1-e^{-\lambda\alpha}\right)\frac{1-p_0^M}{1-e^{-\lambda}}}{1-e^{-\lambda\alpha}}\left[e^{\alpha\lambda(t-1)} - e^{-\alpha\lambda} + e^{-\alpha\lambda} - e^{-\lambda}\right] \\
&= \cdots \\
&= \left(p_0^M\right)^* + \frac{1-(p_0^M)^*}{1-e^{-\lambda\alpha}}\left(e^{\alpha\lambda(t-1)} - e^{-\alpha\lambda}\right),
\end{aligned}
$$

where

$$
\left(p_0^M\right)^* = p_0^M + \frac{\left(1-e^{-\lambda\alpha}\right)\frac{1-p_0^M}{1-e^{-\lambda}}}{1-e^{-\lambda\alpha}}\left[e^{-\alpha\lambda} - e^{-\lambda}\right]
$$

Thus N_P is also a ZM Poisson but with prob mass at 0 given by $(p_0^M)^*$ (rahter than p_0^M) and poisson parameter $\alpha\lambda$ (rather than λ).

6 The collective risk model

6.1 Introduction

The collective risk model is defined as a random sum of the severity amts

$$
S = \begin{cases} \sum_{i=1}^{N} X_i, & N > 0 \\ 0, & N = 0 \end{cases}
$$

where

- $\{X_i\}_{i\geq 1}$ are iid severity rv's (discrete or continuous)
- $N = \#$ of losses (discrete)
- $S =$ aggregate (total) claim amt rv.

Characteristics of S

(a) cdf :

$$
\begin{aligned}
F_S(x) = \Pr(S \leq x) &= \sum_{n=0}^{\infty} \Pr(S \leq x \mid N = n)\Pr(N = n) \\
&= \sum_{n=0}^{\infty} \Pr(X_1 + \cdots + X_n \leq x) \underbrace{p_n}_{\text{pmf of } N} = \sum_{n=0}^{\infty} p_n F_X^{*n}(x)
\end{aligned}
$$

where $F_X^{*n}(x)$ is a cdf of the n-fold convolution of X itself.

For a continuous X,

$$\begin{aligned} F_X^{*n}(x) &= \Pr\left(X_1 + \cdots + X_n \le x\right) \\ &= \int_0^x \Pr\left(X_1 + \cdots + X_n \le x \mid X_n = y\right) f_X(y)dy \\ &= \int_0^x \Pr\left(X_1 + \cdots + X_{n-1} \le x - y\right) f_X(y)dy \\ &= \int_0^x F_x^{*n}(x-y) \times f_X(y)dy \end{aligned}$$

(b) pgf:

$$P_S(t) = P_N\left(P_X(t)\right)$$

(c) mgf:

$$M_S(t) = P_N\left(M_X(t)\right)$$

(d) Mean:

$$E(S) = E\left(E(S|N)\right) = E(N)E(X)$$

(e) Variance:

$$\begin{aligned} Var(S) &= E\left[Var(S|N)\right] + Var\left[E(S|N)\right] \\ &= E\left[N \cdot Var(X)\right] + Var\left[N \cdot E(X)\right] \\ &= E(N)Var(X) + Var(N) \cdot E(X)^2 \end{aligned}$$

Example.

S	$E(S)$	$Var(S)$
Comp Pois	$\lambda E(X)$	$\lambda E(X^2)$
Comp NB	$r\beta E(X)$	$r\beta E(X^2) + r\beta^2 E(X)^2$
Comp Bin	$mqE(X)$	$mqE(X^2) - mq^2 E(X)^2$

11/29 (수)

Remark. If policy adjustments are applied to a ground-up loss X, the aggregate payment by the insurer is

$$S' = \begin{cases} \sum_{i=1}^{N_L} Y_{L,i}, & N_L > 0 \\ 0, & N_L = 0 \end{cases}$$

or

$$S' = \begin{cases} \sum_{i=1}^{N_P} Y_{P,i}, & N_P > 0 \\ 0, & N_P = 0 \end{cases}$$

Consequently, all the above results ((a) - (e)) also hold for S'

6.2 Some analytic results

How to determine the dist^n of S ?

A. Suppose that X is discrete with probability only at the set of non-negative integers.

- S is a discrete rv with pmf $\Pr(S = s) = f_S(s)$
- S is a compound rv with discrete primary dist^n and discrete secondary dist^n
- use the results of the section "compounding discrete rv's" in **Chap 5**.
  For $N \in (a, b, 1)$, the pmf of S can be found via

$$f_S(s) = \frac{1}{1 - af_X(0)} \left[(p_1 - (a+b)p_0) f_X(x) + \sum_{y=1}^{x} \left(a + \frac{by}{x} \right) f_X(y) f_S(x-y) \right],$$

$$\text{for} x = 1, 2, 3, \cdots, \text{ with } f_S(0) = P_N(f_X(0))$$

B. Suppose now that X is conbinuous with density $f_X(x)$ in general, given that $\Pr(N = 0) > 0$, S is a mixed rv (because of the mass point of S at 0)

Example. $N \sim NB(r, p)$, $X \sim \text{Exp}(1/\theta)$ Find the dist^n of the comp. NB rv S, assuming an integer r.

Solution. Note that

$$M_X(t) = \frac{1}{1 - \theta t}, \quad t < \frac{1}{\theta}$$

and

$$P_N(t) = (1 - \beta(t-1))^{-r}$$

Hence

$$M_S(t) = P_N(M_X(t)) = P_N\left(\frac{1}{1-\theta t}\right) = \left(1-\beta\left(\frac{1}{1-\theta t}-1\right)\right)^{-r}$$

Simple manipulations yield

$$\begin{aligned}
M_S(t) &= \left(1-\beta\frac{\theta t}{1-\theta t}\right)^{-r} = \left(\frac{1-(\beta+1)\theta t}{1-\theta t}\right)^{-r} \\
&= \left(\frac{1-\theta t}{1-(\beta+1)\theta t}\right)^{r} = \left(\frac{1}{\beta+1}\times\frac{(\beta+1)(1-\theta t)}{1-(\beta+1)\theta t}\right)^{r} \\
\text{where } & (\beta+1)(1-\theta t) = 1-(\beta+1)\theta t + \beta \\
&= \left(\frac{1}{\beta+1}+\frac{\beta}{\beta+1}\left(\frac{1}{1-(\beta+1)\theta t}\right)\right)^{r}
\end{aligned}$$

using the binomial expansion,

$$\begin{aligned}
M_S(t) &= \sum_{j=0}^{r}\binom{r}{j}\left(\frac{1}{\beta+1}\right)^{r-j}\left(\frac{\beta}{\beta+1}\right)^{j}\left(\frac{1}{1-(\beta+1)\theta t}\right)^{j} \\
&= \alpha_{o,r} + \sum_{j=1}^{r}\alpha_{j,r} \quad \times \underbrace{\left(\frac{1}{1-(\beta+1)\theta t}\right)^{j}}_{\text{mgf of gamma(Erlang) } (J \text{ is interger})} \\
\text{where } \alpha_{j,r} &= \binom{r}{j}\left(\frac{1}{\beta+1}\right)^{r-j}\left(\frac{\beta}{\beta+1}\right)^{j}
\end{aligned}$$

$\Rightarrow$ S is a mixture of r Erlang densities and one degenerate rv. Each Erlang density has shape parameter j ($j = 1, \cdots, r$) and scale parameter $(\beta+1)\theta$. Then it follows that

$$\begin{aligned}
\Pr(S \le s) &= \alpha_{0,r} + \sum_{j=1}^{r}\alpha_{j,r}\times\Pr(Y_J \le s) \\
\text{where } P(Y_j \le s) &= \int_0^s f_{Y_j}(x)dx \underbrace{=}_{\text{Done this before}} 1-\sum_{i=0}^{j-1}\frac{e^{-x/\theta^*}\left(\frac{x}{\theta^*}\right)^i}{i!}
\end{aligned}$$

Thus

$$\begin{aligned}
F_S(s) = \Pr(S \le s) &= \alpha_{0,r} + \sum_{j=1}^{r}\alpha_{j,r}\left(1-\sum_{i=0}^{j-1}\frac{e^{-x/\theta^*}\left(\frac{x}{\theta^*}\right)^i}{i!}\right) \\
&= 1 - e^{-x/\theta^*}\sum_{j=1}^{r}\alpha_{j,r}\sum_{i=0}^{j-1}\frac{\left(\frac{x}{\theta^*}\right)^i}{i!}
\end{aligned}$$

12/04 (월)

Example. Consider the comp. rv S with primary dist^n having pmf $\{p_n\}_{n\geq 0}$ and an exponential secondary dist^n with mean θ. Find the cdf of S.

Solution. In general, the cdf of S is

$$F_S(x) = p_0 + \sum_{n=1}^{\infty} p_n F_X^{*n}(x)$$

when claim sizes are exponential,

$$M_{X_1+\cdots+X_n}(t) = E\left[e^{t(X_1+\cdots+X_n)}\right] = \left(\frac{1}{1-\theta t}\right)^n$$

which implies that $X_1+\cdots+X_n$ is Erlang-n with scale parameter θ. Hence, the n-fold convolution of the exponential density is an Erlang-n density:

$$F_X^{*n}(x) = \int_0^x \frac{y^{n-1}e^{-x/\theta}}{\theta^n \Gamma(n)} dy = 1 - \sum_{i=0}^{n-1} \frac{e^{-x/\theta}\left(\frac{x}{\theta}\right)^i}{i!},$$

which implies that

$$\begin{aligned} F_S(x) &= p_0 + \sum_{n=1}^{\infty} p_n \left(1 - \sum_{i=0}^{n-1} \frac{e^{-x/\theta}\left(\frac{x}{\theta}\right)^i}{i!}\right) \\ &= 1 - \sum_{n=1}^{\infty} p_n \sum_{i=0}^{n-1} \frac{e^{-x/\theta}\left(\frac{x}{\theta}\right)^i}{i!} \end{aligned}$$

In general, only a few comp rv's have an analytic (close-form) expression for their cdf. As a consequence, various approx techniques have been proposed to evaluate the cdf of S.

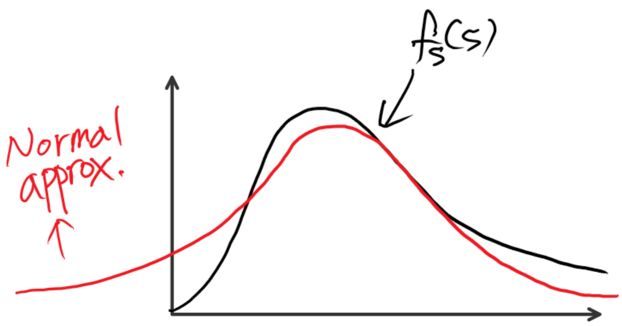
6.3 Approximation methods

6.3.1 Normal approximation

Idea is to approximate the dist^n of $\dfrac{S-E(S)}{\sqrt{Var(S)}}$ by Z, where $Z \sim N(0,1)$.

Its application results in

$$\begin{aligned} F_S(s) = \Pr(S \le s) &= \Pr\left(\underbrace{\frac{S-E(S)}{\sqrt{Var(S)}}}_{=Z} \le \frac{s-E(S)}{\sqrt{Var(S)}}\right) \\ &= \Phi\left(\frac{s-E(S)}{\sqrt{Var(S)}}\right) \end{aligned}$$

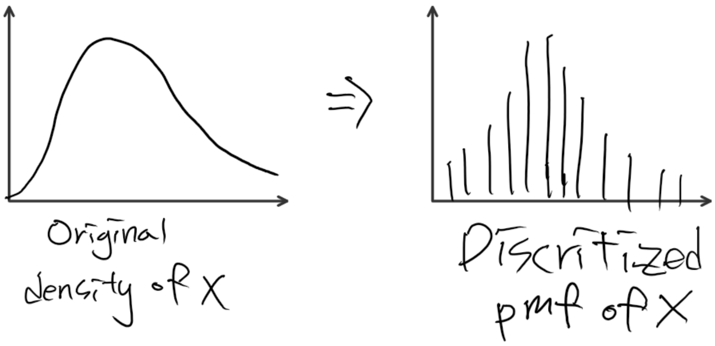


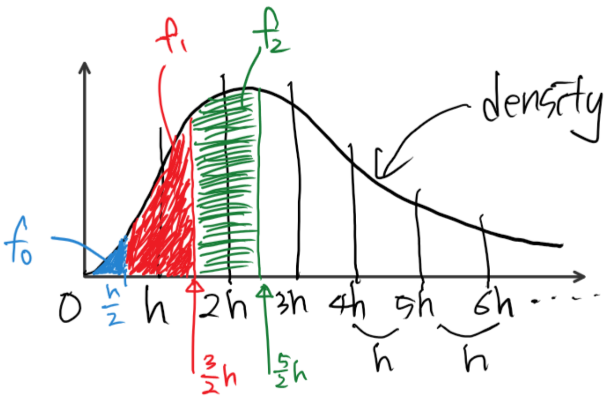
6.3.2 Method of rounding (Discretization method)

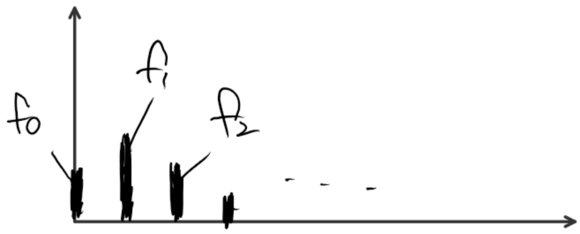
Idea is to onstruct a discrete dist^n from the continuous severity dist^n on multiple of a convenient unit of measurement so that the Panjer recursion can be used. However, we want to do so in such a way to preserve some properties of the original severity dist^n. Here is the suggested method to discretize a mixed or continuous dist^n for X_i.

- Let the span h be a suitably chosen positive number which will be our convenient choice for this unit of measurement.
- Define f_j as the probability at jh, $j = 0, 1, \cdots$
  - set $f_0 = \Pr\left(X < \frac{h}{2}\right) = F_X\left(\frac{h}{2}\right)$
  - for $j = 1, 2, \cdots$:

$$\begin{aligned} f_j &= \Pr\left(jh - \frac{h}{2} \le X < jh + \frac{h}{2}\right) \\ &= F_X\left(jh + \frac{h}{2}\right) - F_X\left(jh - \frac{h}{2}\right) \end{aligned}$$







Example. $N \sim \text{Pois}(3)$. $X \sim \text{Pareto}(\alpha = 4, \theta = 10)$. Find the discretized dist^n of $S = X_1 + \cdots + X_n$. with $h = 2.5$

Solution.

$$
\begin{aligned}
f_0 &= \Pr\left(X < \frac{2.5}{2}\right) = F_X(1.25) = 1 - \left(\frac{10}{10 + 1.25}\right)^4 = 0.3757 \\
f_1 &= \Pr\left(2.5 - \frac{2.5}{2} \le X < 2.5 + \frac{2.5}{2}\right) = F_X(3.75) - F_X(1.25) \\
&= \left(\frac{10}{10 + 1.25}\right)^4 - \left(\frac{10}{10 + 3.75}\right)^4 = 0.34453 \\
f_2 &= F_X(6.25) - F_X(3.75) = \cdots = 0.13635 \\
&\vdots
\end{aligned}
$$

Given that $N \in (a, b, 0)$ class, we can use the panjer recursion to compute pmf of the discretized dist^n S^*

$$g_k = \sum_{j=1}^{k} \underbrace{\lambda}_{=3} \frac{j}{k} f_j \, g_{k-j} \quad \longleftarrow \text{ recursion for possion}$$

Now

$$\begin{aligned}
g_0 &= \Pr(S^* = 0) = P_N(f_0) = e^{3(f_0-1)} = 0.15368 \quad (\because f_0 = 0.3757,\ 3 = \lambda) \\
g_1 &= \Pr(S^* = h) = 3 f_1 g_0 = 3 \times 0.34453 \times 0.15368 = 0.15884 \\
g_2 &= \Pr(S^* = 2h) = \lambda \left(\frac{1}{2} f_1 g_1 + \frac{2}{2} f_2 g_0 \right) \\
&= 3 \left(\frac{1}{2} \times 0.34453 \times 0.15884 + \frac{2}{2} \times 0.13635 \times 0.15368 \right) \\
&= 0.14495 \\
&\vdots
\end{aligned}$$

when losses are subject to policy modifications, there exists two representations for the aggregate payment.

$$S' = \begin{cases} \sum_{i=1}^{N_L} Y_{L,i}, & N_L > 0 \\ 0, & N_L = 0 \end{cases} \quad \longleftarrow \text{ per loss basis}$$

$$S' = \begin{cases} \sum_{i=1}^{N_P} Y_{P,i}, & N_P > 0 \\ 0, & N_P = 0 \end{cases} \quad \longleftarrow \text{ per payment basis}$$

Example. $N \sim \text{Pois}(3)$, $X \sim \text{Pareto}(\alpha = 4, \theta = 10)$. An individual loss limit of 11 and a deductible of 6 are applied to each loss. Determine the discretized dist^n of S' (per payment basis) using a span of 2.5

Solution. The prob of non-zero payment is

$$\alpha = 1 - F_X(6) = 1 - \left\{ 1 - \left(\frac{10}{10+6} \right)^4 \right\} = 0.15259$$

Given that $N_L \sim \text{Pois}(3)$, N_P is again Poisson with mean $3 \times \alpha = 0.4578$.

For the amount paid per payment, we have seen that

$$F_{Y_P}(y) = \begin{cases} \dfrac{F_X(y+d) - F_X(d)}{1 - F_X(d)}, & 0 < y \le u - d \\ 1, & y > u - d \end{cases}$$
$$= \begin{cases} \dfrac{\left(\frac{10}{16}\right)^4 - \left(\frac{10}{16+y}\right)^4}{\left(\frac{10}{16}\right)^4}, & 0 < y \le 5 \\ 1, & y > 5 \end{cases}$$

Let Y_P^* be the discretized version of Y_P with span $h = 2.5$. Then

$$f_0 = \Pr(Y_P^* = 0) = \Pr\left(Y_P \le \frac{2.5}{2}\right) = 1 - \frac{\left(\frac{10}{16}\right)^4 - \left(\frac{10}{16+1.25}\right)^4}{\left(\frac{10}{16}\right)^4}$$
$$= 0.25984$$
$$f_1 = \Pr(Y_P^* = h) = \Pr\left(h - \frac{h}{2} < Y_P \le h + \frac{h}{2}\right)$$
$$= \Pr(1.25 < Y_P \le 3.75) = F_{Y_P}(3.75) - F_{Y_P}(1.25)$$
$$= 0.30942$$
$$f_2 = \Pr(Y_P^* = 2h) = \Pr(3.75 < Y_P \le 6.25)$$
$$= \underbrace{F_{Y_P}(6.25)}_{=1} - F_{Y_P}(3.75) = 1 - \left\{1 - \frac{\left(\frac{10}{16+3.75}\right)^4}{\left(\frac{10}{16}\right)^4}\right\}$$
$$= 0.43074$$

Let $(S')^*$ be the approx of S'

$$(S')^* = Y_{P,1}^* + \cdots + Y_{P,N_P}^*$$

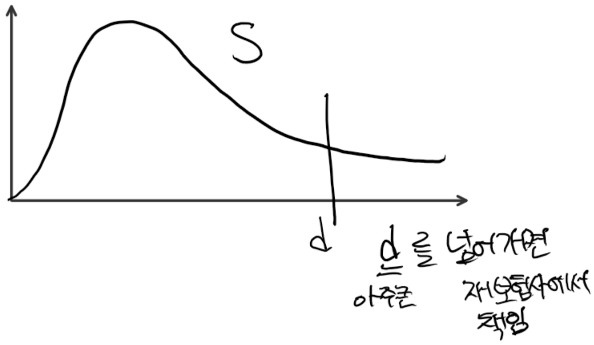
having pmf

$$\Pr\left((S')^* = 0\right) = P_{N_P}(f_0) = e^{0.4578(0.25984-1)} = 0.71259$$
$$\underbrace{\Pr\left((S')^* = 2.5k\right)}_{g_k} = \sum_{j=1}^{k} 0.4578 \times \frac{j}{k} \times f_j \times \underbrace{\Pr\left((S')^* = 2.5(k-j)\right)}_{g_{k-j}}$$
$$\text{for } k = 1, 2, \cdots$$

12/6 (수)

6.4 Stop-loss insurance

An insurer has a portfolio of risks producing aggregate loss S. Suppose the insurer pays a reinsurer a premium to $\underbrace{\text{assume}}_{=\text{transfer}}$ all losses in excess of an amount d.



- Amount paid by reinsurer:

$$I_d = \max(S-d, 0) = (S-d)_+ = \begin{cases} S-d, & S > d \\ 0, & S \le d \end{cases}$$

- Amount retained by the insurer:

$$S - I_d = \min(S, d) = \begin{cases} d, & S > d \\ S, & S \le d \end{cases}$$

Net stop-loss premium (or net reinsurance premium)

$$E[I_d] = \int_d^\infty (x-d)\, \underbrace{dF_S(x)}_{=f_S(x)dx}$$

Integration by parts gives

$$\begin{aligned} E[I_d] &= -(x-d)\overline{F}_S(x)\Big|_{x-d}^{\infty} + \int_d^\infty \overline{F}_S(x)dx \\ &= \int_d^\infty \overline{F}_S(x)dx \end{aligned}$$

For S an integer-valued rv and d integer,

$$E\left[I_d\right] = \sum_{x=d+1}^{\infty} (x-d) f_S(x) = \sum_{x=d+1}^{\infty} x f_S(x) - d\overline{F}_S(d),$$

where

$$\overline{F}_S(d) = \sum_{x=d+1}^{\infty} f_S(x)$$

Given that $f_S(x) = \overline{F}_S(x-1) - \overline{F}_S(x)$, we have

$$\begin{aligned} E\left[I_d\right] &= \sum_{x=d+1}^{\infty} x\left[\overline{F}_S(x-1) - \overline{F}_S(x)\right] - dF_S(d) \\ &= \sum_{x=d+1}^{\infty} (x-1+1)\overline{F}_S(x-1) - \sum_{x=d+1}^{\infty} x\overline{F}_S(x) - d\overline{F}_S(d) \qquad (7) \\ &= d\overline{F}_S(d) + \sum_{x=d+1}^{\infty} \overline{F}_S(x-1) - d\overline{F}_S(d) \\ &= \sum_{x=d}^{\infty} \overline{F}_S(x) \end{aligned}$$

(7) 의 첫 항에서 $y = x - 1$ 로 두면,

$$\sum_{y=d}^{\infty} (y+1)\overline{F}_S(y) = \sum_{y=d}^{\infty} y\overline{F}_S(y) + \sum_{y=d}^{\infty} \overline{F}_S(y)$$

Consequently, the net stop-loss premium can be computed recursively via

$$E\left[I_d\right] = \overline{F}_S(d) + \sum_{x=d+1}^{\infty} \overline{F}_S(x) = \overline{F}_S(d) + E\left[I_{d+1}\right],\ d = 0, 1, \cdots$$

where the starting point is

$$E\left[I_0\right] = E\left[\max(S, 0)\right] = E(S) = E(N)E(X)$$

Example. S is comp Poisson with $\lambda = 5$ and secondary dist^n $f_1 = 0.8$ and $f_2 = 0.2$, Compute $E\left[I_2\right]$.

Solution. Let $\{f_S(x)\}_{x \geq 0}$ be the pmf of S.

$$f_S(0) = e^{-5}$$

and

$$f_S(x) = \sum_{j=1}^{x} \lambda \cdot \frac{j}{k} \cdot f_j \cdot f_S(x-j), \quad x = 1, 2, \cdots$$

with

$$f_S(1) = 5 \cdot f_1 \cdot f_S(0) = 5 \times 0.8 \times e^{-5} = 4e^{-5}$$

using the stop-loss recursion,

$$\begin{aligned}
E\left[I_0\right] &= E(S) = 5\left(1 \times 0.8 + 2 \times 0.2\right) = 6 \\
E\left[I_1\right] &= E\left[I_0\right] - \overline{F}_S(0) = 6 - \left(1 - \underbrace{f_S(0)}_{=e^{-5}}\right) = 5.0067 \\
E\left[I_2\right] &= E\left[I_1\right] - \overline{F}_S(1) \\
&= E\left[I_1\right] - \left(1 - \underbrace{f_S(0)}_{=e^{-5}} - \underbrace{f_S(1)}_{=4e^{-5}}\right) \\
&= 4.0404
\end{aligned}$$