STA0054: Reliability Theory **Spring 2024**

Lecture 02: Failure models

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2.1 Failure models

2.1.1 History of reliability theory

- Reliability theory has been studied using a variety of approaches in recent years, as shown below. But in this lecture, we will only discuss the survival function approach.

(1) Failure models for components (1930s);

- Survival function;
- Hazard & failure rate;
- IFR/DFR.

(2) Tools to analyze system structure

- Structure function (1960s);
- Signature (1980s);
- Survival signature (2010s).

(3) Recent research

- Topological inference and Bayesian posterior predictive system lifetime (2012).
- Bayesian Network (2020s)

2.1.2 Introduction

We will now introduce several quantitative measures for the reliability of a nonrepairable item. This item can be anything from a small component to a large system. When we classify an item as nonrepairable, we are only interested in studying the item until the first failure occurs. In some cases the item may be literally nonrepairable, meaning that it will be discarded by the first failure. **In other cases, the item may be repaired**, but we are not interested in what is happening with the item after the first failure.

First we will introduce four important measures for the reliability of a nonrepairable item. These are:

- The reliability (survivor) function $R(t)$

- The failure rate function $z(t)$
- The mean time to failure (MTTF)
- The mean residual life (MRL) – skip?

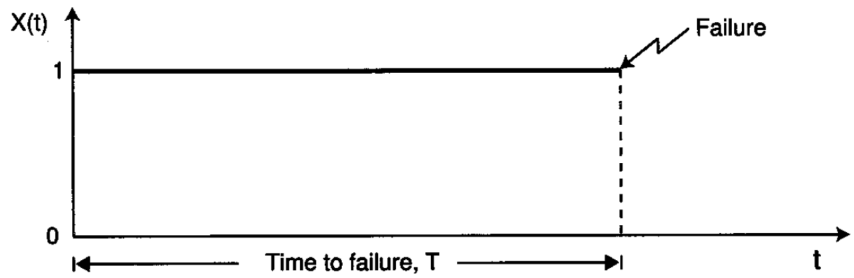

Figure 2.1: The state variable and the time ot failure of an item.

2.1.3 Component lifetimes

A component is defined to be any part of a system, meaning no constituent parts of the unit are modelled directly, only the unit as a whole. Note that a "component" may itself be a system.

Thus, the lifetime T of a nonrepairable component is modelled directly by some lifetime (probability) distribution, typically with **non-negative support** $[0, \infty)$.

- Exponential
- Gamma
- Log-normal
- Weibull
- Gompertz
- Coxian
- ...

2.1.4 State variable

The state of the item at time t may be described by the state variable $X(t)$:

$$X(t) = \begin{cases} 1 & \text{if the item is functioning at time } t \\ 0 & \text{if the item is in a failed state at time } t \end{cases} \tag{2.1}$$

The state variable of a nonrepairable item is illustrated in Fig. 2.1 and will generally be a random variable.

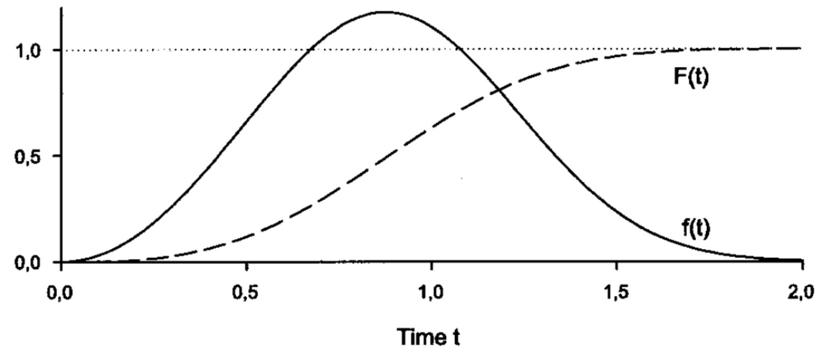

Figure 2.2: Distribution function $F(t)$ and probability density function $f(t)$.

2.1.5 Time to failure

"The time to failure" of an item means the time elapsing from when the item is put into operation until it fails for the first time. We set $t = 0$ as the starting point. At least to some extent the time to failure is subject to chance variations. It is therefore natural to interpret the time to failure as a **random variable**, T. The connection between the state variable $X(t)$ and the time to failure T is illustrated in Fig. 2.1.

Note that the time to failure T is not always measured in **calendar time**. It may also be measured by more indirect time concepts, such as:

- Number of times a switch is operated
- Number of kilometers driven by a car
- Number of rotations of a bearing
- Number of cycles for a periodically working item

From these examples, we notice that time to failure may often be a **discrete variable**. However, a discrete variable can be approximated by a continuous variable. Here, unless stated otherwise, we will assume that the time to failure T is continuously distributed with probability density function (pdf) $f(t)$ and (cumulative) distribution function (cdf) $F(t)$:

$$F(t) = \Pr(T \leq t) = \int_0^t f(u)du \quad \text{for } t > 0. \tag{2.2}$$

$F(t)$ thus denotes the probability that the item fails within the time interval $(0, t]$. The probability density function $f(t)$ is defined as

$$f(t) = \frac{d}{dt}F(t) = \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Pr(t < T \leq t + \Delta t)}{\Delta t}$$

This implies that when Δt is small,

$$\Pr(t < T \leq t + \Delta t) \approx f(t) \cdot \Delta t$$

The distribution function $F(t)$ and the probability density function $f(t)$ are illustrated is Fig. 2.2.

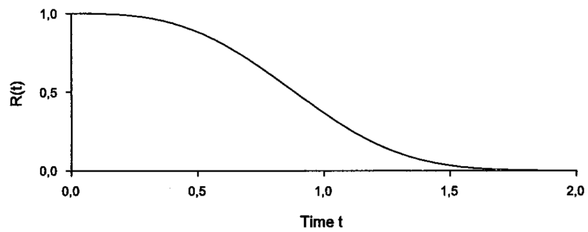

Figure 2.3: The reliability (survival) function $R(t)$.

2.1.6 Reliability function

The reliability function of an item is defined by

$$R(t) = 1 - F(t) = \Pr(T > t) \quad \text{for } t > 0 \tag{2.3}$$

or equivalently

$$R(t) = 1 - \int_0^t f(u)du = \int_t^\infty f(u)du \tag{2.4}$$

Hence $R(t)$ is the probability that the item **does not fail** in the time interval $(0, t]$, or, in other words, the probability that the item **survives** the time interval $(0, t]$ and is still functioning at time t. The reliability function $R(t)$ is also called the survivor function and is illustrated in Fig. 2.3.

2.1.7 Failure rate function

The probability that an item will fail in the time interval $(t, t+\Delta t]$ when we know that the item is functioning at time t is

$$\Pr(t < T \leq t + \Delta t \mid T > t) = \frac{\Pr(t < T \leq t + \Delta t)}{\Pr(T > t)} = \frac{F(t + \Delta t) - F(t)}{R(t)}$$

By dividing this probability by the length of the time interval, Δt, and letting $\Delta t \to 0$, we get the failure rate function $z(t)$ of the item

$$\begin{aligned} z(t) &= \lim_{\Delta t \to 0} \frac{\Pr(t < T \leq t + \Delta t \mid T > t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t} \frac{1}{R(t)} = \frac{f(t)}{R(t)} \end{aligned} \tag{2.5}$$

This implies that when Δt is small,

$$\Pr(t < T \leq t + \Delta t \mid T > t) \approx z(t) \cdot \Delta t$$

Remark: Note the similarity and the difference between the probability density function $f(t)$ and the failure rate function $z(t)$.

$$\Pr(t < T \leq t + \Delta t) \approx f(t) \cdot \Delta t \tag{2.6}$$

$$\Pr(t < T \leq t + \Delta t \mid T > t) \approx z(t) \cdot \Delta t \tag{2.7}$$

Table 2.1: Relationship between the Functions $F(t), f(t), R(t)$, and $z(t)$

Expressed by	$F(t)$	$f(t)$	$R(t)$	$z(t)$
$F(t) =$	-	$\int_0^t f(u)du$	$1 - R(t)$	$1 - \exp\left(-\int_0^t z(u)du\right)$
$f(t) =$	$\frac{d}{dt}F(t)$	-	$-\frac{d}{dt}R(t)$	$z(t) \cdot \exp\left(-\int_0^t z(u)du\right)$
$R(t) =$	$1 - F(t)$	$\int_t^\infty f(u)du$	-	$\exp\left(-\int_0^t z(u)du\right)$
$z(t) =$	$\frac{dF(t)/dt}{1-F(t)}$	$\frac{f(t)}{\int_t^\infty f(u)du}$	$-\frac{d}{dt}\ln R(t)$	

Say that we start out with a new item at time $t = 0$ and at time $t = 0$ ask: "What is the probability that this item will fail in the interval $(t, t + \Delta t]$?" According to (2.6) this probability is approximately equal to the probability density function $f(t)$ at time t multiplied by the length of the interval Δt. Next consider an item that has survived until time t, and ask: "What is the probability that this item will fail in the next interval $(t, t + \Delta t]$?" This (conditional) probability is according to (2.7) approximately equal to the failure rate function $z(t)$ at time t multiplied by the length of the interval, Δt.

If we put a large number of identical items into operation at time $t = 0$, then $z(t) \cdot \Delta t$ will roughly represent the relative proportion of the items still functioning at time t, failing in $(t, t + \Delta t]$. Since

$$f(t) = \frac{d}{dt}F(t) = \frac{d}{dt}(1 - R(t)) = -R'(t)$$

then

$$z(t) = -\frac{R'(t)}{R(t)} = -\frac{d}{dt}\ln R(t) \tag{2.8}$$

Since $R(0) = 1$, then

$$\int_0^t z(t)dt = -\ln R(t) \tag{2.9}$$

and

$$R(t) = \exp\left(-\int_0^t z(u)du\right) \tag{2.10}$$

The reliability (survivor) function $R(t)$ and the distribution function $F(t) = 1 - R(t)$ are therefore uniquely determined by the failure rate function $z(t)$. From (2.5) and (2.10) we see that the probability density function $f(t)$ can be expressed by

$$f(t) = z(t) \cdot \exp\left(-\int_0^t z(u)du\right) \text{ for } t > 0 \tag{2.11}$$

The relationships between the functions $F(t), f(t), R(t)$, and $z(t)$ are presented in Table 2.1. From (2.10) we see that the reliability (survivor) function $R(t)$ is uniquely determined by the failure rate function $z(t)$.

To determine the form of $z(t)$ for a given type of items, the following experiment may be carried out: Split the time interval $(0, t)$ into disjoint intervals of equal length Δt. Then put n identical items into operation at time $t = 0$. When an item fails, note the time and leave that item out. For each interval record:

- The number of items $n(i)$ that fail in interval i.

- The functioning times for the individual items $(T_{1i}, T_{2i}, \ldots, T_{ni})$ in interval i. Hence T_{ji} is the time item j has been functioning in time interval i. T_{ji} is therefore equal to 0 if item j has failed before interval i, where $j = 1, 2, \ldots, n$.

Thus $\sum_{j=1}^{n} T_{ji}$ is the total functioning time for the items in interval i. Now

$$z(i) = \frac{n(i)}{\sum_{j=1}^{n} T_{ji}}$$

which shows the number of failures per unit functioning time in interval i is a natural estimate of the "failure rate" in interval i for the items that are functioning at the start of this interval.

2.1.8 Mean time to failure

The mean time to failure (MTTF) of an item is defined by

$$\mathrm{MTTF} = E(T) = \int_0^\infty t f(t) dt \tag{2.12}$$

Since $f(t) = -R'(t)$,

$$\mathrm{MTTF} = -\int_0^\infty t R'(t) dt$$

By partial integration

$$\mathrm{MTTF} = -[tR(t)]_0^\infty + \int_0^\infty R(t) dt$$

If MTTF $< \infty$, it can be shown that $[tR(t)]_0^\infty = 0$. In that case

$$\mathrm{MTTF} = \int_0^\infty R(t) dt \tag{2.13}$$

It is often easier to determine MTTF by (2.13) than by (2.12). The mean time to failure of an item may also be derived by using Laplace transforms. The Laplace transform of the survivor function $R(t)$ is

$$R^*(s) = \int_0^\infty R(t) e^{-st} dt$$

When $s = 0$, we get

$$R^*(0) = \int_0^\infty R(t) dt = \mathrm{MTTF} \tag{2.14}$$

The MTTF may thus be derived from the Laplace transform $R^*(s)$ of the survivor function $R(t)$, by setting $s = 0$.

Median Lifetime

The MTTF is only one of several measures of the "center" of a life distribution. An alternative measure is the median life t_{m}, defined by

$$R(t_{\mathrm{m}}) = 0.50 \tag{2.15}$$

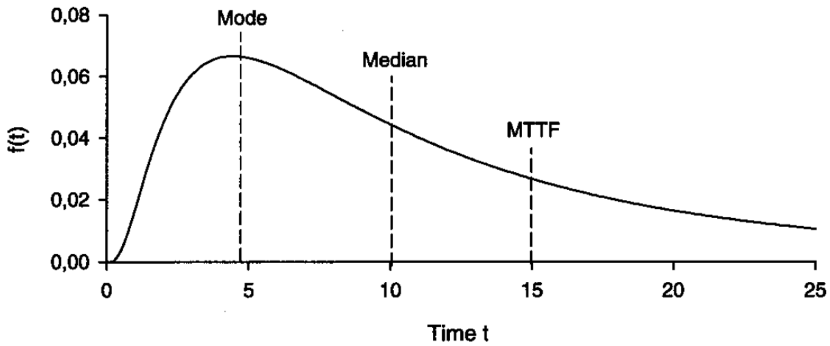

Figure 2.4: Location of the MTTF, the median life, and the mode of a distribution.

The median divides the distribution in two halves. The item will fail before time t_{m} with 50% probability, and will fail after time t_{m} with 50% probability.

Mode

The mode of a life distribution is the most likely failure time, that is, the time t_{mode} where the probability density function $f(t)$ attains its maximum:

$$f\left(t_{\text{mode}}\right) = \max_{0 \leq t < \infty} f(t) \tag{2.16}$$

Fig. 2.4 shows the location of the MTTF, the median life t_{m}, and the mode t_{mode} for a distribution that is skewed to the right.

Example 2.1 *Consider an item with reliability (survivor) function*

$$R(t) = \frac{1}{(0.2t+1)^2} \text{ for } t \geq 0$$

where the time t is measured in months. The probability density function is

$$f(t) = -R'(t) = \frac{0.4}{(0.2t+1)^3}$$

and the failure rate function is from (2.5)*:*

$$z(t) = \frac{f(t)}{R(t)} = \frac{0.4}{0.2t+1}$$

The mean time to failure is from (2.13)*:*

$$\text{MTTF} = \int_0^\infty R(t)dt = 5 \text{ months}$$

The functions $R(t), f(t)$, and $z(t)$ are illustrated in Fig. 2.5.

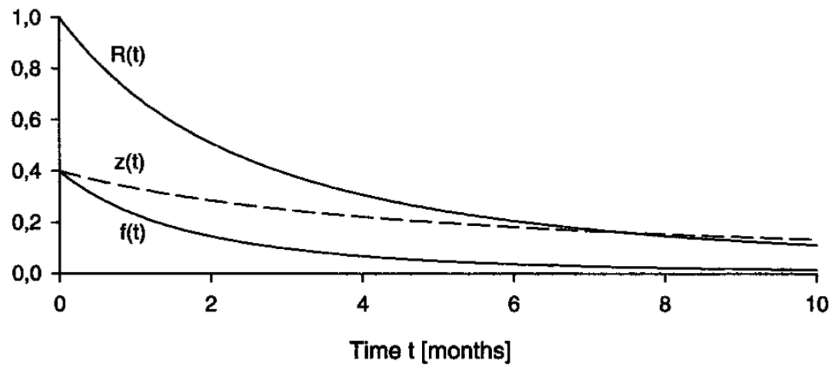

Figure 2.5: The survivor function $R(t)$, the probability density function $f(t)$, and the failure rate function $z(t)$ (dashed line) in Example 2.1.

2.1.9 Mean residual life

Consider an item with time to failure T that is put into operation at time $t = 0$ and is still functioning at time t. The probability that the item of age t survives an additional interval of length x is

$$R(x \mid t) = \Pr(T > x + t \mid T > t) = \frac{\Pr(T > x + t)}{\Pr(T > t)} = \frac{R(x + t)}{R(t)} \tag{2.17}$$

$R(x \mid t)$ is called the conditional survivor function of the item at age t. The mean residual (or, remaining) life, MRL (t), of the item at age t is

$$\mathrm{MRL}(t) = \mu(t) = \int_0^\infty R(x \mid t)dx = \frac{1}{R(t)} \int_t^\infty R(x)dx \tag{2.18}$$

When $t = 0$, the item is new, and we have $\mu(0) = \mu = \mathrm{MTTF}$. It is sometimes of interest to study the function

$$g(t) = \frac{\mathrm{MRL}(t)}{\mathrm{MTTF}} = \frac{\mu(t)}{\mu} \tag{2.19}$$

When an item has survived up to time t, then $g(t)$ gives the MRL(t) as a percentage of the initial MTTF. If, for example, $g(t) = 0.60$, then the mean residual lifetime, MRL(t) at time t, is 60% of mean residual lifetime at time 0 .

By differentiating $\mu(t)$ with respect to t it is straightforward to verify that the failure rate function $z(t)$ can be expressed as

$$z(t) = \frac{1 + \mu'(t)}{\mu(t)} \tag{2.20}$$

Example 2.2 *Consider an item with failure rate function $z(t) = t/(t + 1)$. The failure rate function is increasing and approaches 1 when $t \to \infty$. The corresponding survivor function is*

$$R(t) = \exp\left(-\int_0^t \frac{u}{u+1}du\right) = (t+1)e^{-t}$$

and

$$\text{MTTF} = \int_0^\infty (t+1)e^{-t}dt = 2$$

The conditional survival function is

$$R(x \mid t) = \Pr(T > x+t \mid T > t) = \frac{(t+x+1)e^{-(t+x)}}{(t+1)e^{-t}} = \frac{t+x+1}{t+1}e^{-x}$$

The mean residual life is

$$\text{MRL}(t) = \int_0^\infty R(x \mid t)dx = 1 + \frac{1}{t+1}$$

We see that $\text{MRL}(t)$ *is equal to* $2(=\text{MTTF})$ *when* $t=0$, *that* $\text{MRL}(t)$ *is a decreasing function in* t, *and that* $\text{MRL}(t) \to 1$ *when* $t \to \infty$.

2.1.10 The binomial and geometric distributions

The binomial distribution is one of the most widely used discrete distributions in reliability engineering. The distribution is used in the following situation:

(1) We have n independent trials.

(2) Each trial has two possible outcomes A and A^*.

(3) The probability $\Pr(A) = p$ is the same in all the n trials.

This situation is called a binomial situation, and the trials are sometimes referred to as Bernoulli trials. Let X denote the number of the n trials that have outcome A. Then X is a discrete random variable with distribution

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{for } x = 0, 1, \ldots, n \tag{2.21}$$

where $\binom{n}{x}$ is the binomial coefficient

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

The distribution (2.21) is called the binomial distribution (n, p), and we sometimes write $X \sim \text{bin}(n, p)$. The mean value and the variance of X are

$$E(X) = np \tag{2.22}$$

$$\text{var}(X) = np(1-p) \tag{2.23}$$

Assume that we carry out a sequence of Bernoulli trials, and want to find the number Z of trials until the first trial with outcome A. If $Z = z$, this means that the first $(z-1)$

trials have outcome A^*, and that the first A will occur in trial z. The distribution of Z is

$$\Pr(Z = z) = (1-p)^{z-1}p \quad \text{for } z = 1, 2, \ldots$$

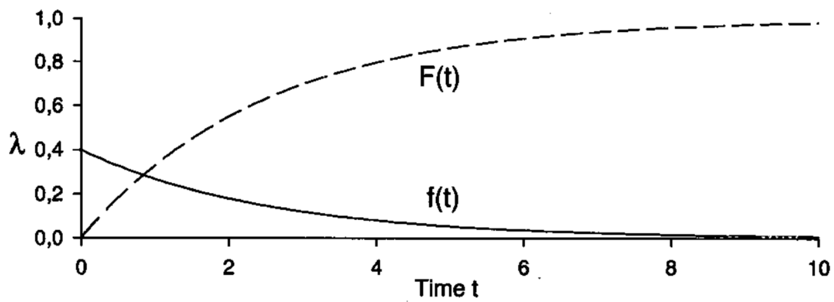

Figure 2.6: Exponential distribution ($\lambda = 1$).

The distribution (2.1.10) is called the geometric distribution. We have that

$$\Pr(Z > z) = (1-p)^z$$

The mean value and the variance of Z are

$$E(Z) = \frac{1}{p} \tag{2.24}$$

$$\mathrm{var}(X) = \frac{1-p}{p^2} \tag{2.25}$$

2.1.11 Exponential distribution

Consider an item that is put into operation at time $t = 0$. The time to failure T of the item has probability density function

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for } t > 0, \lambda > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.26}$$

This distribution is called the exponential distribution with parameter λ, and we sometimes write $T \sim \exp(\lambda)$. The reliability (survivor) function of the item is

$$R(t) = \Pr(T > t) = \int_t^\infty f(u)du = e^{-\lambda t} \text{ for } t > 0 \tag{2.27}$$

The probability density function $f(t)$ and the survivor function $R(t)$ for the exponential distribution are illustrated in Fig. 2.6. The mean time to failure is

$$\mathrm{MTTF} = \int_0^\infty R(t)dt = \int_0^\infty e^{-\lambda t}dt = \frac{1}{\lambda} \tag{2.28}$$

and the variance of T is

$$\mathrm{var}(T) = \frac{1}{\lambda^2}$$

The probability that an item will survive its mean time to failure is

$$R(\mathrm{MTTF}) = R\left(\frac{1}{\lambda}\right) = e^{-1} \approx 0.3679$$

The failure rate function is

$$z(t) = \frac{f(t)}{R(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda \tag{2.29}$$

Accordingly, the failure rate function of an item with exponential life distribution is constant (i.e., independent of time). We see that this indicates that the exponential distribution may be a realistic life distribution for an item during its useful life period, at least for certain types of items.

The results (2.28) and (2.29) compare well with the use of the concepts in everyday language. If an item on the average has $\lambda = 4$ failures/year, the MTTF of the item is $1/4$ year.

Consider the conditional survivor function (2.17)

$$\begin{aligned} R(x \mid t) = \Pr(T > t + x \mid T > t) &= \frac{\Pr(T > t + x)}{\Pr(T > t)} \\ &= \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x} = \Pr(T > x) = R(x) \end{aligned} \tag{2.30}$$

The survivor function of an item that has been functioning for t time units is therefore equal to the survivor function of a new item. A new item, and a used item (that is still functioning), will therefore have the same probability of surviving a time interval of length t. The MRL for the exponential distribution is

$$\mathrm{MRL}(t) = \int_0^\infty R(x \mid t)dx = \int_0^\infty R(x)dx = \mathrm{MTTF}$$

The MRL (t) of an item with exponential life distribution is hence equal to its MTTF irrespective of the age t of the item. The item is therefore as good as new as long as it is functioning, and we often say that the exponential distribution has no memory. Therefore, an assumption of exponentially distributed lifetime implies that

- A used item is stochastically as good as new, so there is no reason to replace a functioning item.
- For the estimation of the reliability function, the mean time to failure, and so on, it is sufficient to collect data on the number of hours of observed time in operation and the number of failures. The age of the items is of no interest ni this connection.

The exponential distribution is the most commonly used life distribution in applied reliability analysis. The reason for this is its mathematical simplicity and that it leads to realistic lifetime models for certain types of items.

Example 2.3 *A rotary pump has a constant failure rate* $\lambda = 4.28 \cdot 10^{-4}$ *hours* $^{-1}$ *(data from OREDA 2002). The probability that the pump survives one month* ($t = 730$ *hours) in continuous operation is*

$$R(t) = e^{-\lambda t} = e^{-4.28 \cdot 10^{-4} \cdot 730} \approx 0.732$$

The mean time to failure is

$$MTTF\ = \frac{1}{\lambda} = \frac{1}{4.28 \cdot 10^{-4}}\ hours\ \approx 2336\ hours\ \approx 3.2\ months$$

Suppose that the pump has been functioning without failure during its first 2 months ($t_1 = 1460$ hours) in operation. The probability that the pump will fail during the next month ($t_2 = 730$ hours) is

$$\Pr(T \le t_1 + t_2 \mid T > t_1) = \Pr(T \le t_2) = 1 - e^{-4.28 \cdot 10^{-4} .730} \approx 0.268$$

since the pump is as good as new when it is still functioning at time t_1.

Example 2.4 *Consider a system of two independent components with failure rates λ_1 and λ_2, respectively. The probability that component 1 fails before component 2 is*

$$\begin{aligned} \Pr(T_2 > T_1) &= \int_0^\infty \Pr(T_2 > t \mid T_1 = t) f_{T_1}(t) dt \\ &= \int_0^\infty e^{-\lambda_2 t} \lambda_1 e^{-\lambda_1 t} dt \\ &= \lambda_1 \int_0^\infty e^{-(\lambda_1 + \lambda_2) t} dt = \frac{\lambda_1}{\lambda_1 + \lambda_2} \end{aligned}$$

This result can easily be generalized to a system of n independent components with failure rates $\lambda_1, \lambda_2, \ldots, \lambda_n$. The probability that component j is the first component to fail is

$$\Pr(\textit{ component } j \textit{ fails first }) = \frac{\lambda_j}{\sum_{i=1}^n \lambda_i}$$

Example 2.5 (Mixture of Exponential Distributions) *Assume that the same type of items are produced at two different plants. The items are assumed to be independent and have constant failure rates. The production process is slightly different at the two plants, and the items will therefore have different failure rates. Let λ_i denote the failure rate of the items coming from plant i, for $i = 1, 2$. The items are mixed up before they are sold. A fraction p is coming from plant 1 , and the rest $(1-p)$ is coming from plant 2. If we pick one item at random, the survival function of this item is*

$$R(t) = p \cdot R_1(t) + (1-p) \cdot R_2(t) = p e^{-\lambda_1 t} + (1-p) e^{-\lambda_2 t}$$

The mean time to failure is

$$MTTF = \frac{p}{\lambda_1} + \frac{1-p}{\lambda_2}$$

and the failure rate function is

$$z(t) = \frac{p\lambda_1 e^{-\lambda_1 t} + (1-p)\lambda_2 e^{-\lambda_2 t}}{p e^{-\lambda_1 t} + (1-p) e^{-\lambda_2 t}}$$

The failure rate function, which is illustrated in Fig. 2.7, is seen to be decreasing. If we assume that $\lambda_1 > \lambda_2$, early failures should have a failure rate close to λ_1. After a while all the "weak" components have failed, and we are left with components with a lower failure rate λ_2.

2.1.12 Gamma distribution

Consider an item that is exposed to a series of shocks, where the time intervals $T_1, T_2, \ldots$, between consecutive shocks are then independent and exponentially distributed with parameter λ. Assume that the item fails exactly at shock k, and not earlier. The time to failure of the item is then

$$T = T_1 + T_2 + \cdots + T_k$$

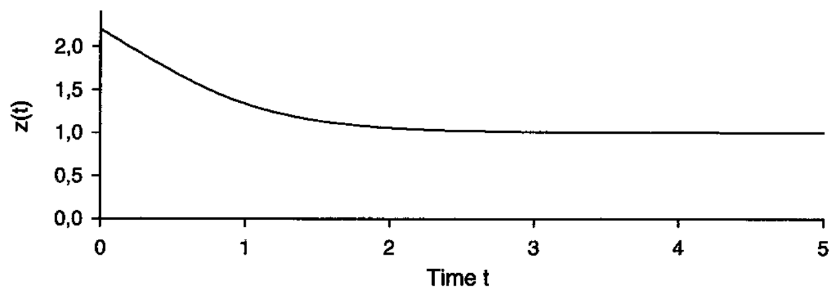

Figure 2.7: The failure rate function of the mixture of two exponential distributions in Example 2.5 ($\lambda_1 = 1, \lambda_2 = 3$, and $p = 0.4$).

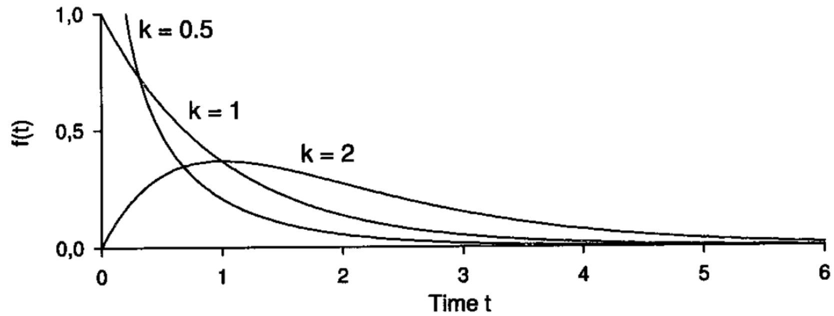

Figure 2.8: The gamma probability density, $\lambda = 1.0$.

and T is gamma distributed (k, λ), and we sometimes write $T \sim \text{gamma}(k, \lambda)$. The probability density function is

$$f(t) = \frac{\lambda}{\Gamma(k)}(\lambda t)^{k-1}e^{-\lambda t} \tag{2.31}$$

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$ denotes the gamma function, $t > 0, \lambda > 0, \alpha > 0$, and k is a positive integer. The probability density function $f(t)$ is sketched in Fig. 2.8 for selected values of k.

The parameter λ denotes the rate (frequency) of shocks and is an external parameter for the item. The integer k may be interpreted as a measure of the ability to resist the shocks and will from now on generally not be restricted to integer values but be a positive constant. Equation (2.31) will still be a probability density function. From (2.31) we find that

$$\text{MTTF} = \frac{k}{\lambda} \tag{2.32}$$

$$\text{var(T)} = \frac{k}{\lambda^2} \tag{2.33}$$

For integer values of k the reliability function is given by

$$R(t) = 1 - F(t) = \sum_{n=0}^{k-1} \frac{(\lambda t)^n}{n!}e^{-\lambda t} \tag{2.34}$$

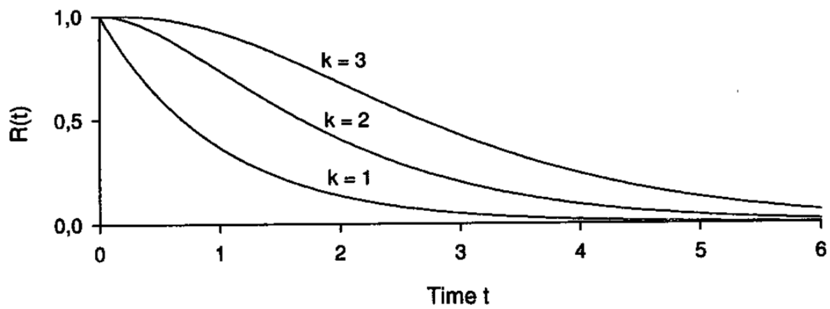

Figure 2.9: Reliability function for the gamma distribution, $\lambda = 1.0$.

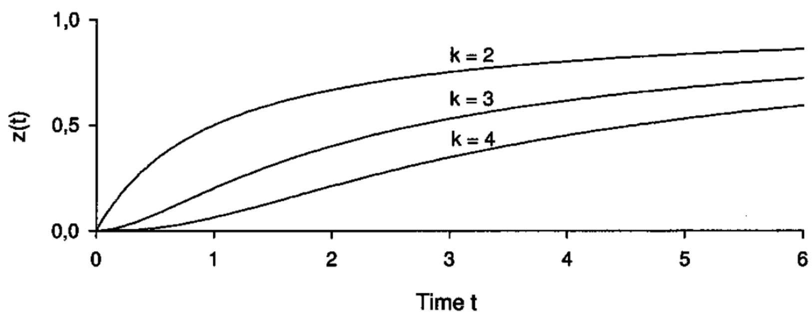

Figure 2.10: Failure rate function of the gamma distribution, $\lambda = 1$.

A sketch of $R(t)$ is given in Fig. 2.9 for some values of k. The corresponding failure rate function is

$$z(t) = \frac{f(t)}{R(t)} = \frac{\lambda(\lambda t)^{k-1}e^{-\lambda t}/\Gamma(k)}{\sum_{n=0}^{k-1}(\lambda t)^n e^{-\lambda t}/n!} \tag{2.35}$$

For $k = 2$ the failure rate function is

$$z(t) = \frac{\lambda^2 t}{1 + \lambda t} \tag{2.36}$$

and the distribution in Example 2.2 is therefore a gamma distribution with $k = 2$ and $\lambda = 1$.

When k is not an integer, we have to use the general formulas (2.4) and (2.8) to find the reliability function $R(t)$ and the failure rate function $z(t)$, respectively. It may be shown (e.g., see Cocozza-Thivent, 1997, p. 10) that

$$\begin{aligned} &\lim_{t\to 0} z(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} z(t) = \lambda \text{ when } 0 < k < 1 \\ &\lim_{t\to 0} z(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} z(t) = \lambda \text{ when } k > 1 \end{aligned} \tag{2.37}$$

The failure rate function $z(t)$ is illustrated in Fig. 2.10 for some integer values of k. Let $T_1 \sim \text{gamma}\,(k_1, \lambda)$ and $T_2 \sim \text{gamma}\,(k_2, \lambda)$ be independent. It is then easy to show that $T_1 + T_2 \sim \text{gamma}\,(k_1 + k_2, \lambda)$. Gamma distributions with a common λ are therefore closed under addition.

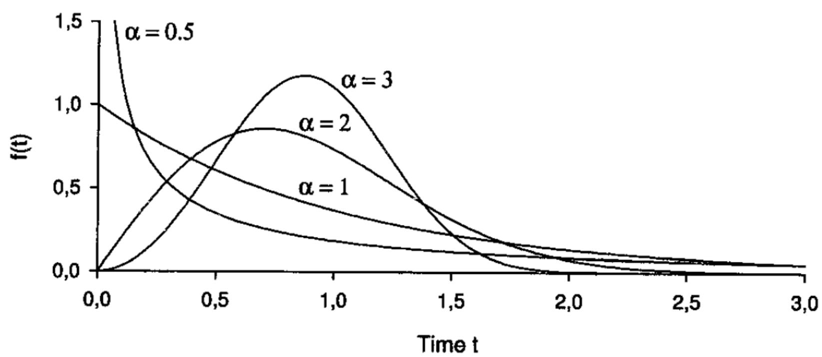

Figure 2.11: The probability density function of the Weibul distribution for selected values of the shape parameter α ($\lambda = 1$)

2.1.13 The Weibull distribution

The Weibull distribution is one of the most widely used life distributions in reliability analysis. The distribution is named after the Swedish professor Waloddi Weibull (1887-1979) who developed the distribution for modeling the strength of materials. The Weibull distribution is very flexible, and can, through an appropriate choice of parameters, model many types of failure rate behaviors.

The time to failure T of an item is said to be Weibull distributed with parameters $\alpha(>0)$ and $\lambda(>0)[T \sim$ Weibull $(\alpha, \lambda)]$ if the distribution function is given by

$$F(t) = \Pr(T \leq t) = \begin{cases} 1 - e^{-(\lambda t)^\alpha} & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.38}$$

The corresponding probability density is

$$f(t) = \frac{d}{dt}F(t) = \begin{cases} \alpha\lambda^\alpha t^{\alpha-1} e^{-(\lambda t)^\alpha} & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.39}$$

where λ is a scale parameter, and α is referred to as the shape parameter. Note that when $\alpha = 1$, the Weibull distribution is equal to the exponential distribution. The probability density function $f(t)$ is illustrated in Fig. 2.11 for selected values of α.

The survivor function is

$$R(t) = \Pr(T > 0) = e^{-(\lambda t)^\alpha} \text{ for } t > 0 \tag{2.40}$$

and the failure rate function is

$$z(t) = \frac{f(t)}{R(t)} = \alpha\lambda^\alpha t^{\alpha-1} \text{ for } t > 0 \tag{2.41}$$

When $\alpha = 1$, the failure rate is constant; when $\alpha > 1$, the failure rate function is increasing; and when $0 < \alpha < 1, z(t)$ is decreasing. When $\alpha = 2$, the resulting distribution is known as the Rayleigh distribution. The failure rate function $z(t)$ of the Weibull distribution is illustrated in Fig. 2.12 for some selected values of α. The Weibull distribution is seen to be flexible and may be used to model life distributions, where the failure rate function is decreasing, constant, or increasing.

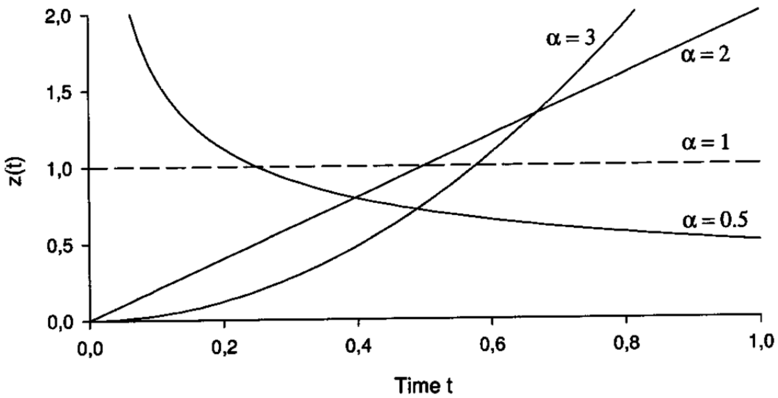

Figure 2.12: Failure rate function of the Weibull distribution, $\lambda = 1$.

Remark: Notice that the failure rate function is discontinuous as a function of the shape parameter α at $\alpha = 1$. It is important to be aware of this discontinuity in numerical calculations, since, for example, $\alpha = 0.999, \alpha = 1.000$, and $\alpha = 1.001$ will give significantly different failure rate functions for small values of t.

From (2.40) it follows that

$$R\left(\frac{1}{\lambda}\right) = \frac{1}{e} \approx 0.3679 \text{ for all } \alpha > 0$$

Hence $\Pr(T > 1/\lambda) = 1/e$, independent of α. The quantity $1/\lambda$ is sometimes called the characteristic lifetime. The mean time to failure is

$$\mathrm{MTTF} = \int_0^\infty R(t)dt = \frac{1}{\lambda}\Gamma\left(\frac{1}{\alpha} + 1\right) \tag{2.42}$$

The median life t_{m} is

$$R\left(t_{\mathrm{m}}\right) = 0.50 \Rightarrow t_{\mathrm{m}} = \frac{1}{\lambda}(\ln 2)^{1/\alpha} \tag{2.43}$$

The variance of T is

$$\operatorname{var}(T) = \frac{1}{\lambda^2}\left(\Gamma\left(\frac{2}{\alpha} + 1\right) - \Gamma^2\left(\frac{1}{\alpha} + 1\right)\right) \tag{2.44}$$

Note that MTTF $/\sqrt{\operatorname{var}(T)}$ is independent of λ. The Weibull distribution also arises as a limit distribution for the smallest of a large number of independent, identically distributed, nonnegative random variables. The Weibull distribution is therefore often called the weakest link distribution.

The Weibull distribution has been widely used in reliability analysis of semiconductors, ball bearings, engines, spot weldings, biological organisms, and so on.

Example 2.6 *The time to failure T of a variable choke valve is assumed to have a Weibull distribution with shape parameter $\alpha = 2.25$ and scale parameter $\lambda = 1.15 \cdot 10^{-4}$ hours $^{-1}$. The valve will survive 6 months ($t = 4380$ hours) in continuous operation with probability:*

$$R(t) = e^{-(\lambda t)^\alpha} = e^{-\left(1.15 \cdot 10^{-4} \cdot 8760\right)^{2.25}} \approx 0.808$$

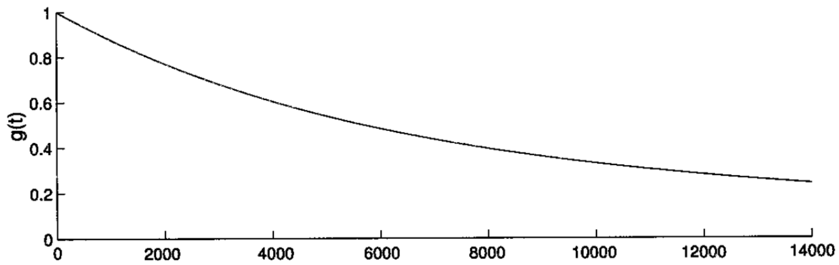

Figure 2.13: The scaled mean residual lifetime function $g(t) = \mathrm{MRL}(t)/\mathrm{MTTF}$ for the Weibull distribution with parameters $\alpha = 2.25$ and $\lambda = 1.15 \cdot 10^{-4}$ hours $^{-1}$.

The mean time to failure is

$$MTTF = \frac{1}{\lambda}\Gamma\left(\frac{1}{\alpha}+1\right) = \frac{\Gamma(1.44)}{1.15\cdot 10^{-4}} \text{ hours} \approx 7706 \text{ hours}$$

and the median life is

$$t_{\mathrm{m}} = \frac{1}{\lambda}(\ln 2)^{1/\alpha} \approx 7389 \text{ hours}$$

A valve that has survived the first 6 months ($t_1 = 4380$ hours), will survive the next 6 months ($t_2 = 4380$ hours) with probability

$$R(t_1+t_2 \mid t_1) = \frac{R(t_1+t_2)}{R(t_1)} = \frac{e^{-(\lambda(t_1+t_2))^{\alpha}}}{e^{-(\lambda t_1)^{\alpha}}} \approx 0.448$$

that is, significantly less than the probability that a new valve will survive 6 months. The mean residual life when the valve has been functioning for 6 months ($t = 4380$ hours) is

$$\mathrm{MRL}(t) = \frac{1}{R(t)}\int_0^{\infty} R(t+x)dx \approx 4730 \text{ hours}$$

The MRL (t) cannot be given a closed form in this case and was therefore found by using a computer. The function $g(t) = \mathrm{MRL}(t)/\mathrm{MTTF}$ is illustrated in Fig. 2.17.

Example 2.7 *Consider a series system of n components. The times to failure $T_1, T_2, \ldots, T_n$ of the n components are assumed to be independent and Weibull distributed:*

$$T_i \sim \text{ Weibull } (\alpha, \lambda_i) \text{ for } i = 1, 2, \ldots, n$$

A series system fails as soon as the first component fails. The time to failure of the system, T_s is thus

$$T_s = \min\{T_1, T_2, \ldots, T_n\}$$

The survivor function of this system becomes

$$\begin{aligned} R_s(t) &= \Pr(T_s > t) = \Pr\left(\min_{1\le i\le n} T_i > t\right) = \prod_{i=1}^{n} \Pr(T_i > t) \\ &= \prod_{i=1}^{n} \exp\left(-(\lambda_i t)^{\alpha}\right) = \exp\left(-\sum_{i=1}^{n}(\lambda_i t)^{\alpha}\right) = \exp\left(-\left[\sum_{i=1}^{n}\lambda_i^{\alpha}\right]t^{\alpha}\right) \end{aligned}$$

Hence a series system of independent components with Weibull life distribution with the same shape parameter α again has a Weibull life distribution, with scale parameter $\lambda_s = \left(\sum_{i=1}^{n} \lambda_i^{\alpha}\right)^{1/\alpha}$ and with the shape parameter being unchanged.

When all the n components have the same distribution, such that $\lambda_i = \lambda$ for $i = 1, 2, \ldots, n$, then the series system has a Weibull life distribution with scale parameter $\lambda \cdot n^{1/\alpha}$ and shape parameter α.

The Weibull distribution we have discussed so far is a two-parameter distribution with shape parameter $\alpha > 0$ and scale parameter $\lambda > 0$. A natural extension of this distribution is the three-parameter Weibull distribution (α, λ, ξ) with distribution function

$$F(t) = \Pr(T \leq t) = \begin{cases} 1 - e^{-[\lambda(t-\xi)]^{\alpha}} & \text{for } t > \xi \\ 0 & \text{otherwise} \end{cases}$$

The corresponding density is

$$f(t) = \frac{d}{dt} F(t) = \alpha\lambda[\lambda(t-\xi)]^{\alpha-1} e^{-[\lambda(t-\xi)]^{\alpha}} \text{ for } t > \xi$$

The third parameter ξ is sometimes called the guarantee or threshold parameter, since a failure occurs before time ξ with probability 0 (e.g., see Mann et al. 1974, p. 185).

Since $(T - \xi)$ obviously has a two-parameter Weibull distribution (α, λ), the mean and variance of the three-parameter Weibull distribution (α, λ, ξ) follows from (2.42) and (2.44).

$$\begin{aligned} MTTF &= \xi + \frac{1}{\lambda}\Gamma\left(\frac{1}{\alpha} + 1\right) \\ \operatorname{var}(T) &= \frac{1}{\lambda^2}\left(\Gamma\left(\frac{2}{\alpha} + 1\right) - \Gamma^2\left(\frac{1}{\alpha} + 1\right)\right) \end{aligned}$$

In statistical literature, reference to the Weibull distribution usually means the twoparameter family, unless otherwise specified.

2.1.14 The normal distribution

The most commonly used distribution in statistics is the normal (Gaussian [1]) distribution. A random variable T is said to be normally distributed with mean v and variance τ^2, $T \sim \mathcal{N}(\nu, \tau^2)$, when the probability density of T is

$$f(t) = \frac{1}{\sqrt{2\pi} \cdot \tau} e^{-(t-\nu)^2/2\tau^2} \text{ for } -\infty < t < \infty$$

The $\mathcal{N}(0, 1)$ distribution is called the standard normal distribution. The distribution function of the standard normal distribution is usually denoted by $\Phi(\cdot)$. The probability density of the standard normal distribution is

$$\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

The distribution function of $T \sim \mathcal{N}(\nu, \tau^2)$ may be written as

$$F(t) = \Pr(T \leq t) = \Phi\left(\frac{t - v}{\tau}\right)$$

[1] Named after the German mathematician Johann Carl Friedrich Gauss (1777-1855).

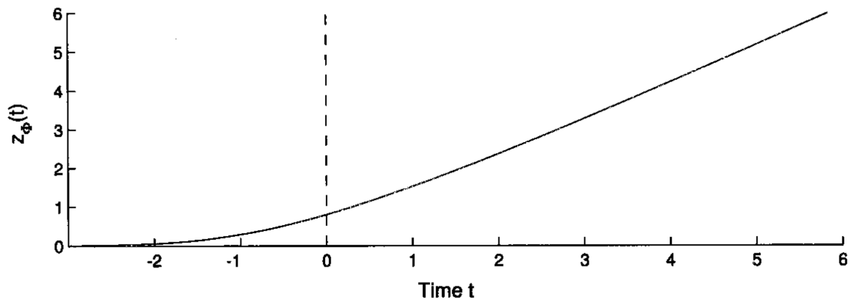

Figure 2.14: Failure rate function of the standard normal distribution $\mathcal{N}(0,1)$.

The normal distribution is sometimes used as a lifetime distribution, even though it allows negative values with positive probability. The survivor function is

$$R(t)=1-\Phi\left(\frac{t-v}{\tau}\right)$$

The failure rate function of the normal distribution is

$$z(t)=-\frac{R'(t)}{R(t)}=\frac{1}{\tau}\cdot\frac{\phi((t-v)/\tau)}{1-\Phi((t-v)/\tau)}$$

If $z_\Phi(t)$ denotes the failure rate function of the standard normal distribution, the failure rate function of $\mathcal{N}\left(\nu,\tau^2\right)$ is seen to be

$$z(t)=\frac{1}{\tau}\cdot z_\Phi\left(\frac{t-v}{\tau}\right)$$

The failure rate function of the standard normal distribution, $\mathcal{N}(0,1)$, is illustrated in Fig. 2.14. The failure rate function is increasing for all t and approaches $z(t)=t$ when $t\to\infty$.

When a random variable has a normal distribution but with an upper bound and/or a lower bound for the values of the random variable, the resulting distribution is called a truncated normal distribution. When there is only a lower bound, the distribution is said to be left truncated. When there is only an upper bound, the distribution is said to be right truncated. Should there be an upper as well as a lower bound, it is said to be doubly truncated.

A normal distribution, left truncated at 0 , is sometimes used as a life distribution. This left truncated normal distribution has survivor function

$$R(t)=\Pr(T>t\mid T>0)=\frac{\Phi((\nu-t)/\tau)}{\Phi(\nu/\tau)}\text{ for }t\geq 0$$

The corresponding failure rate function becomes

$$z(t)=\frac{-R'(t)}{R(t)}=\frac{1}{\tau}\cdot\frac{\phi((t-v)/\tau)}{1-\Phi((t-v)/\tau)}\text{ for }t\geq 0$$

Note that the failure rate function of the left truncated normal distribution is identical to the failure rate function of the (untruncated) normal distribution when $t\geq 0$.

Example 2.8 *A specific type of car tires has an average wear-out "time" T of* 50000 km, *and* 5% *of the tires last for at least* 70000 km. *We will assume that T is normally distributed with mean $v=50000$ km, and that $\operatorname{Pr}(T>70000)=0.05$. Let τ denote the standard deviation of T. The variable $(T-50000)/\tau$ then has a standard normal distribution. Standardizing, we get*

$$\operatorname{Pr}(T>70000)=1-P\left(\frac{T-50000}{\tau} \leq \frac{70000-50000}{\tau}\right)=0.05$$

Therefore

$$\Phi\left(\frac{20000}{\tau}\right)=0.95 \approx \Phi(1.645)$$

and

$$\frac{20000}{\tau} \approx 1.645 \Rightarrow \tau \approx 12158$$

The probability that a tire will last more than 60000 km *is now*

$$\begin{aligned}
\operatorname{Pr}(T>60000) &=1-P\left(\frac{T-50000}{12158} \leq \frac{60000-50000}{12158}\right) \\
& \approx 1-\Phi(0.883) \approx 0.188
\end{aligned}$$

The probability of a "negative" life length is in this case

$$\operatorname{Pr}(T<0)=P\left(\frac{T-50000}{12158}<\frac{-50000}{12158}\right) \approx \Phi(-4.11) \approx 0$$

The effect of using a truncated normal distribution instead of a normal distribution is therefore negligible.

2.1.15 The lognormal distribution

The time to failure T of an item is said to be lognormally distributed with parameters v and $\tau^2, T \sim$ lognormal $\left(v, \tau^2\right)$, if $Y=\ln T$ is normally (Gaussian) distributed with mean v and variance τ^2 [i.e., $Y \sim \mathcal{N}\left(\nu, \tau^2\right)$]. The probability density function of T is

$$f(t)= \begin{cases}\frac{1}{\sqrt{2 \pi} \tau t} e^{-(\ln t-v)^2 / 2 \tau^2} & \text { for } t>0 \\ 0 & \text { otherwise }\end{cases}$$

The lognormal probability density is sketched in Fig. 2.15 for selected values of v and τ. The mean time to failure is

$$\mathrm{MTTF}=e^{\nu+\tau^2 / 2}$$

the median time to failure [satisfying $R\left(t_{\mathrm{m}}\right)=0.5$] is

$$t_{\mathrm{m}}=e^v$$

and the mode of the distribution is

$$t_{\text {mode }}=e^{\nu-\tau^2}$$

Notice that the MTTF may be written

$$\mathrm{MTTF}=t_{\mathrm{m}} \cdot e^{\tau^2 / 2}$$

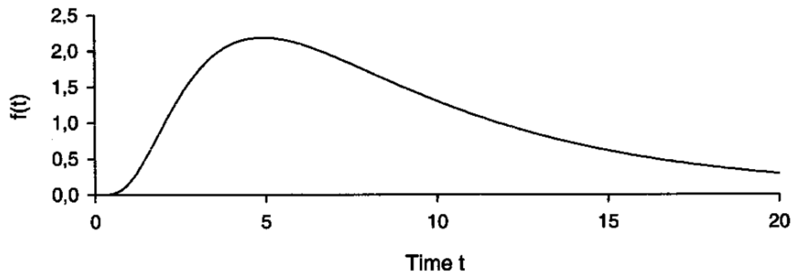

Figure 2.15

and that the mode may be written

$$t_{\text{mode}} = t_{\text{m}} \cdot e^{-\tau^2}$$

It is therefore easy to see that

$$t_{\text{mode}} < t_{\text{m}} < \text{ MTTF for } \tau > 0$$

The variance of T is

$$\operatorname{var}(T) = e^{2\nu} \left(e^{2\tau^2} - e^{\tau^2} \right)$$

The reliability (survivor) function becomes

$$\begin{aligned} R(t) &= \Pr(T > t) = \Pr(\ln T > \ln t) \\ &= P\left(\frac{\ln T - v}{\tau} > \frac{\ln t - v}{\tau} \right) = \Phi\left(\frac{v - \ln t}{\tau} \right) \end{aligned}$$

where $\Phi(\cdot)$ is the distribution function of the standard normal distribution. The failure rate function of the lognormal distribution is

$$z(t) = -\frac{d}{dt}\left(\ln \Phi\left(\frac{\nu - \ln t}{\tau} \right) \right) = \frac{\phi((\nu - \ln t)/\tau))/\tau t}{\Phi((\nu - \ln t)/\tau)/\tau)}$$

where $\phi(t)$ denotes the probability density of the standard normal distribution. The shape of $z(t)$ which is illustrated in Fig. 2.16 is discussed in detail by Sweet (1990) who describes an iterative procedure to compute the time t for which the failure rate function attains its maximum value. He also proves that $z(t) \to 0$ when $t \to \infty$.

Let $T_1, T_2, \ldots, T_n$ be independent and lognormally distributed with parameters v_i and τ_i^2 for $i = 1, 2, \ldots, n$. The product $T = \prod_{i=1}^{n} T_i$ is then lognormally distributed with parameters $\sum_{i=1}^{n} v_i$ and $\sum_{i=1}^{n} \tau_i^2$.

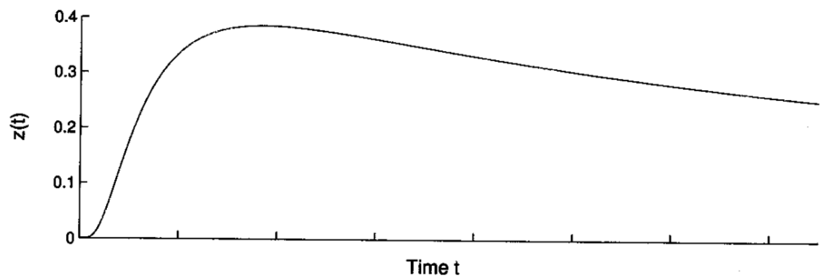

Figure 2.16

Table 2.2: Summary of Life Distributions and Parameters

Distribution	Probability density $f(t)$	Survivor function $R(t)$	Failure rate $z(t)$	MTTF
Exponential	$\lambda e^{-\lambda t}$	$e^{-\lambda t}$	λ	$1/\lambda$
Gamma	$\frac{\lambda}{\Gamma(k)}(\lambda t)^{k-1}e^{-\lambda t}$	$\sum_{x=0}^{k-1}\frac{(\lambda t)^x}{x!}e^{-\lambda t}$	$\frac{f(t)}{R(t)}$	k/λ
Weibull	$\alpha\lambda(\lambda t)^{\alpha-1}e^{-(\lambda t)^{\alpha}}$	$e^{-(\lambda t)^{\alpha}}$	$\alpha\lambda(\lambda t)^{\alpha-1}$	$\frac{1}{\lambda}\Gamma\left(\frac{1}{\alpha}+1\right)$
Lognormal	$\frac{1}{\sqrt{2\pi}}\frac{1}{\tau}\frac{1}{t}e^{-(\ln t-v)^2/2\tau^2}$	$\Phi\left(\frac{\nu-\ln t}{\tau}\right)$	$\frac{f(t)}{R(t)}$	$e^{\nu+\tau^2/2}$