"Joint Distribution of X and $|X|$ when $X \sim \mathcal{N}(0, \sigma^2)$

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We want to calculate Joint Distribution of X and $|X|$ When $X \sim \mathcal{N}(0, \sigma^2)$

$$X \sim \mathcal{N}(0, \sigma^2), \quad Y = |X| \implies$$
$$P^* = \mathcal{P}(X \leq x, Y \leq y) = F_{X,Y}(x, y) = ?$$
$$\mathcal{P}(X \leq x, Y \leq y) = \mathcal{P}(X \leq x, Y \leq y, X > 0) + \mathcal{P}(X \leq x, Y \leq y, X < 0)$$
$$\mathcal{P}(0 < X \leq x,\ X \leq y) + \mathcal{P}(X \leq \min(0, x),\ 0 < -X < y) \implies$$
$$P^* = \mathcal{P}(0 < X \leq \min(x, y)) + \mathcal{P}(X \leq \min(0, x), -y < X < 0) \implies$$
$$P^* = \begin{cases} x > 0: & \mathcal{P}(-y \leq X \leq \min(x, y)) \\ x < 0: & \begin{cases} x > -y: & \mathcal{P}(-y < X < x) \\ x < -y: & 0. \end{cases} \end{cases}$$

0.0.1 To show that the way we followed was the correct method. We also estimate the results based on the simulation and compare it to the above proposition.

python code

[66]:
```python
# load require libraries
from scipy import stats
import numpy as np
import pandas as pd

class get_prob():
    def __init__(self, nsim):
        ## define number of simulation
        self.nsim = nsim

        # define method for get x and y
    def get_xy(self, x = 1, y = 2, sigg = 2, seed = 123):
        self.x = x
        self.y = y
        self.sigg = sigg
        self.seed = seed
```

```
    ## define method for implement simulation
    def sim_main(self):
        sim_res = stats.norm.rvs(loc = 0, scale = self.sigg, size = self.
↪nsim, random_state = self.seed)
        temp1 = list(map(lambda u: 1 if (u <= self.x and np.abs(u) <=␣
↪self.y) else 0, sim_res))
        return np.mean(temp1)

    ## define method for exactly value
    def exact_prob(self):
        rvss = stats.norm(loc = 0, scale = self.sigg)
        if self.x > 0:
            temp = (rvss.cdf(np.min([self.x, self.y])) - rvss.cdf(-self.
↪y))
        elif self.x <= -self.y:
            temp = 0
        else:
            temp = (rvss.cdf(self.x) - rvss.cdf(-self.y))
        return temp

    # return results
    def output_result(self):
        temp2 = {"nsim": self.nsim, "x": self.x, "y": self.y, "Sig": self.
↪sigg,
                 "sim_result": self.sim_main(), "Exact_result": self.
↪exact_prob()}
        temp3 = pd.DataFrame(temp2, index = [1])
        return temp3
```

[67]:
```
nList = list(map(lambda x: int(x), [1e+4, 1e+5]))
xList = [-2, -1, 1, 2]
yList = [.5, 1.5, 2.5, 3.5]
sigList = [0.1, 1.5, 4]
Result = []
for i in sigList:
    for j in xList:
        for k in yList:
            for m in nList:
                Prob = get_prob(nsim = m)
                Prob.get_xy(x = j, y = k, sigg = i)
                tempp = Prob.output_result()
                Result.append(tempp)
```

[68]:
```
for k in Result:
    print(k, "\n\n\n")
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -2  0.5  0.1         0.0             0
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -2  0.5  0.1         0.0             0
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -2  1.5  0.1         0.0             0
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -2  1.5  0.1         0.0             0
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -2  2.5  0.1         0.0  2.753624e-89
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -2  2.5  0.1         0.0  2.753624e-89
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -2  3.5  0.1         0.0  2.753624e-89
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -2  3.5  0.1         0.0  2.753624e-89
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -1  0.5  0.1         0.0             0
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -1  0.5  0.1         0.0             0
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -1  1.5  0.1         0.0  7.619853e-24
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -1  1.5  0.1         0.0  7.619853e-24
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -1  2.5  0.1         0.0  7.619853e-24
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -1  2.5  0.1         0.0  7.619853e-24
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -1  3.5  0.1         0.0  7.619853e-24
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -1  3.5  0.1         0.0  7.619853e-24
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  1  0.5  0.1         1.0      0.999999
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  1  0.5  0.1         1.0      0.999999
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  1  1.5  0.1         1.0           1.0
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  1  1.5  0.1         1.0           1.0
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  1  2.5  0.1         1.0           1.0
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  1  2.5  0.1         1.0           1.0
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  1  3.5  0.1         1.0           1.0
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  1  3.5  0.1         1.0           1.0
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  2  0.5  0.1         1.0      0.999999
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  2  0.5  0.1         1.0      0.999999
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  2  1.5  0.1         1.0           1.0
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  2  1.5  0.1         1.0           1.0
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  2  2.5  0.1         1.0           1.0
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  2  2.5  0.1         1.0           1.0
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  2  3.5  0.1         1.0           1.0
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  2  3.5  0.1         1.0           1.0
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -2  0.5  1.5         0.0             0
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -2  0.5  1.5         0.0             0
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -2  1.5  1.5         0.0             0
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -2  1.5  1.5         0.0             0
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -2  2.5  1.5      0.0485      0.043421
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -2  2.5  1.5     0.04401      0.043421
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -2  3.5  1.5      0.0836      0.081396
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -2  3.5  1.5     0.08112      0.081396
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -1  0.5  1.5         0.0             0
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -1  0.5  1.5         0.0             0
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -1  1.5  1.5      0.0951      0.093837
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -1  1.5  1.5     0.09471      0.093837
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -1  2.5  1.5      0.2078      0.204702
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -1  2.5  1.5     0.20657      0.204702
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -1  3.5  1.5      0.2429      0.242677
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -1  3.5  1.5     0.24368      0.242677
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  1  0.5  1.5      0.2626      0.261117
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  1  0.5  1.5     0.26085      0.261117
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  1  1.5  1.5      0.5877      0.588852
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  1  1.5  1.5     0.58863      0.588852
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  1  2.5  1.5      0.7004      0.699717
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  1  2.5  1.5     0.70049      0.699717
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  1  3.5  1.5      0.7355      0.737692
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  1  3.5  1.5      0.7376      0.737692
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  2  0.5  1.5      0.2626      0.261117
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  2  0.5  1.5     0.26085      0.261117
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  2  1.5  1.5      0.6843      0.682689
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  2  1.5  1.5     0.68124      0.682689
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  2  2.5  1.5      0.8641      0.860998
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  2  2.5  1.5     0.86187      0.860998
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  2  3.5  1.5      0.8992      0.898973
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  2  3.5  1.5     0.89898      0.898973
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -2  0.5    4         0.0             0
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -2  0.5    4         0.0             0
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -2  1.5    4         0.0             0
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -2  1.5    4         0.0             0
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -2  2.5    4      0.0426      0.042552
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -2  2.5    4     0.04187      0.042552
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -2  3.5    4      0.1181      0.117751
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -2  3.5    4     0.11763      0.117751
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -1  0.5    4         0.0             0
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -1  0.5    4         0.0             0
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -1  1.5    4      0.0437      0.047463
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -1  1.5    4     0.04703      0.047463
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -1  2.5    4      0.1294      0.135308
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -1  2.5    4     0.13405      0.135308
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000 -1  3.5    4      0.2049      0.210507
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000 -1  3.5    4     0.20981      0.210507
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  1  0.5    4      0.0973      0.099476
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  1  0.5    4     0.09946      0.099476
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  1  1.5    4      0.2418      0.244876
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  1  1.5    4     0.24445      0.244876
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  1  2.5    4      0.3275      0.332721
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  1  2.5    4     0.33147      0.332721
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  1  3.5    4       0.403      0.407919
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  1  3.5    4     0.40723      0.407919
```

```
    nsim  x    y  Sig  sim_result  Exact_result
1  10000  2  0.5    4      0.0973      0.099476
```

```
     nsim  x    y  Sig  sim_result  Exact_result
1  100000  2  0.5    4     0.09946      0.099476
```

	nsim	x	y	Sig	sim_result	Exact_result
1	10000	2	1.5	4	0.2913	0.29234

	nsim	x	y	Sig	sim_result	Exact_result
1	100000	2	1.5	4	0.29223	0.29234

	nsim	x	y	Sig	sim_result	Exact_result
1	10000	2	2.5	4	0.4193	0.425477

	nsim	x	y	Sig	sim_result	Exact_result
1	100000	2	2.5	4	0.42348	0.425477

	nsim	x	y	Sig	sim_result	Exact_result
1	10000	2	3.5	4	0.4948	0.500676

	nsim	x	y	Sig	sim_result	Exact_result
1	100000	2	3.5	4	0.49924	0.500676