CounterExample_For_Least_Square_Model - Copy

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0.1 Is it possible to indicate that in a least square model:

$$y = \beta_0 + \sum_{i=1}^{p-1} \beta_i \implies \widehat{\beta_i} \perp\!\!\!\perp \widehat{\beta_j} \iff \sum_{k=1}^{n} x_{i,k} x_{j,k} = 0, \quad \text{and} \quad i \neq j. \tag{1}$$

0.2 We know that in a least square model the vector of coefficients estimated by below relation:

$$\begin{aligned}
&\hat{\beta} = (X'X)^{-1}X'y \implies \mathrm{Var}(\hat{\beta}) = \sigma^2 \times (X'X)^{-1} \implies \\
&\hat{\beta} \sim \mathcal{N}_p(\beta, \quad \sigma^2(X'X)^{-1}) \implies \\
&\text{if} \quad \mathbb{V} = (X'X)^{-1} \implies \mathbb{Cov}(\widehat{\beta_i}, \widehat{\beta_j}) = \sigma^2 \times \mathbb{V}[i,j] \implies \\
&\widehat{\beta_i} \perp\!\!\!\perp \widehat{\beta_j} \iff \mathbb{V}[i+1, j+1] = 0
\end{aligned}$$

0.3 I give an Counterexample for the relationship (1)!

let

$$x_1 = \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} \quad x_2 = \begin{bmatrix}2\\4\\7\\1\\-7\end{bmatrix} \implies$$

$$X = \begin{bmatrix}1 & 1 & 2\\1 & 2 & 4\\1 & 3 & 7\\4 & 1 & 1\\1 & 5 & -7\end{bmatrix}$$

$$\implies n = 5, \quad p = 3,$$

$$X'X = \begin{bmatrix}5 & 15 & 7\\15 & 55 & 0\\7 & 0 & 119\end{bmatrix} \implies \sum_{k=1}^{5} x_{1,k}x_{2,k} = 0$$

$$\mathbb{V} = (X'X)^{-1} = \begin{bmatrix}\frac{187}{93} & -\frac{17}{31} & -\frac{11}{93}\\-\frac{17}{31} & \frac{26}{155} & \frac{1}{31}\\-\frac{11}{93} & \frac{1}{31} & \frac{10}{651}\end{bmatrix} \implies$$

$$\mathbb{Cov}(\widehat{\beta_1}, \quad \widehat{\beta_2}) = \sigma^2 \times \mathbb{V}[2,3] = \sigma^2 \times \frac{1}{31} \neq 0$$

0.4 Using Python

[1]:
```python
from sympy import *
import numpy as np
```

[13]:
```python
x1 = [1, 2, 3, 4, 5]
x2 = [2, 4, 7, 1, -7]



X = Matrix(np.array([1, 1, 1, 1, 1, 1, 2, 3, 4, 5, 2, 4, 7, 1, -7]).reshape(3,
 5).T)
X
```

[13]:

$$\begin{bmatrix}1 & 1 & 2\\1 & 2 & 4\\1 & 3 & 7\\1 & 4 & 1\\1 & 5 & -7\end{bmatrix}$$

[14]:
```python
XX = X.T @ X
XX
```

[14]:

$$\begin{bmatrix} 5 & 15 & 7 \\ 15 & 55 & 0 \\ 7 & 0 & 119 \end{bmatrix}$$

[15]:

```
xxinverse = XX.inv()
xxinverse
```

[15]:

$$\begin{bmatrix} \frac{187}{93} & -\frac{17}{31} & -\frac{11}{93} \\ -\frac{17}{31} & \frac{26}{155} & \frac{1}{31} \\ -\frac{11}{93} & \frac{1}{31} & \frac{10}{651} \end{bmatrix}$$