$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$$
$$\to \mathcal{E}(|X_i - \bar{X}|) = ?$$
$$\text{if } Y_i = |X_i - \bar{X}| \implies$$
$$f_{Y_i}(u) = f_{Y_j}(u), \quad 0 \le Y_i \le \infty, \quad i, j \in \{1, 2, \ldots, n\}, \quad \text{It can be easily proven.}$$
$$\implies \mathcal{E}(Y_i) = \mathcal{E}(Y_j) \to$$
$$\mathcal{P}(Y_i \le y) = \mathcal{P}(|X_i - \bar{X}| \le y) = \mathcal{P}(-y \le X_i - \bar{X} \le y)$$
$$X_i - \bar{X} = X_i - \frac{1}{n}\sum_{j=1}^{n} X_j = \overbrace{\frac{n-1}{n}X_i - \frac{1}{n}\sum_{j=1, j\ne i}^{n} X_j}^{U} \implies$$
$$U = \text{Linear combination of normal independent variables} \implies U \sim N(\mu_U, \sigma_U^2)$$
$$\mathcal{E}(U) = \mathcal{E}(X_i - \bar{X}) = \mu - \mu = 0,$$
$$\sigma_U^2 = Var(X) + Var(\bar{X}) - 2Cov(X_i, \bar{X}) = \sigma^2 + \frac{\sigma^2}{n} - 2\frac{\sigma^2}{n} = \frac{n-1}{n}\sigma^2$$
$$\implies f_U(u) = \frac{1}{\sqrt{2\pi\frac{n-1}{n}\sigma^2}} \exp\left\{-\frac{1}{2\frac{n-1}{n}\sigma^2}u^2\right\} \implies$$
$$F_{Y_i}(y) = \mathcal{P}(Y_i \le y) = \mathcal{P}(-y \le U \le y) = \int_{-y}^{y} \frac{1}{\sqrt{2\pi\frac{n-1}{n}\sigma^2}} \exp\left\{-\frac{1}{2\frac{n-1}{n}\sigma^2}u^2\right\} du$$
$$\overset{\text{Even function}}{=} 2 \times \int_{0}^{y} \frac{1}{\sqrt{2\pi\frac{n-1}{n}\sigma^2}} \exp\left\{-\frac{1}{2\frac{n-1}{n}\sigma^2}u^2\right\} du$$
$$\implies f_{Y_i}(y) = \frac{\partial F_{Y_i}(y)}{\partial y}$$
$$f_{Y_i}(y) \overset{\text{Leibniz's method}}{=} 2 \times \frac{1}{\sqrt{2\pi\frac{n-1}{n}\sigma^2}} \exp\left\{-\frac{1}{2\frac{n-1}{n}\sigma^2}y^2\right\}, \quad y \ge 0$$
$$\mathcal{E}(Y_i) = \int_0^\infty y f_{Y_i}(y) dy =$$

In [ ]:
```
from sympy import integrate, exp, log, Integral, sqrt, pi

from sympy.abc import y, n, sigma
```

In [ ]:
```
ey = 2 * y * (1/sqrt(2*pi*(n-1)/n * sigma**2)) * exp(1/(2*(n-1)/n * sigma**2) * (-y**2))
```

```
In [ ]: from sympy import simplify


        res = integrate(ey, y)
        simplify(res)
```

Out[ ]:

$$\begin{cases} - \frac{\sqrt{2} \sqrt{\frac{\sigma^{2} (n - 1)}{n}} e^{- \frac{n y^{2}}{2 \sigma^{2} (n - 1)}}}{\sqrt{\pi}} & \text{for}\: \pi \sigma^{2} \left(n - 1\right) \neq 0 \\ \frac{\sqrt{2} y^{2}}{2 \sqrt{\pi} \sqrt{\frac{\sigma^{2} (n - 1)}{n}}} & \text{otherwise} \end{cases}$$

$$\implies \mathcal{E}(Y_i) = 0 - (-\sigma \times \frac{\sqrt{2\frac{n-1}{n}}}{\sqrt{\pi}}) =$$

$$\sigma\sqrt{2} \times \frac{\sqrt{n-1}}{\sqrt{n\pi}} \implies$$

$$\mathcal{E}\left(\sum_{i=1}^{n} |X_i - \bar{X}|\right) = \mathcal{E}\left(\sum_{i=1}^{n} Y_i\right) =$$

$$n \times \mathcal{E}(Y_i) = \sigma \times \frac{\sqrt{2n(n-1)}}{\sqrt{\pi}} \implies$$

$$a \times \sigma \times \frac{\sqrt{2n(n-1)}}{\sqrt{\pi}} = \sigma \implies$$

$$a = \sqrt{\frac{\pi}{2n(n-1)}}$$