Question 2

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \qquad y = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 3 \end{bmatrix}$$

1. Apply the matrix form to calculate the least squares estimates of B_0, B_1

$$\tilde{X}'\tilde{X} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}$$

$2 \times \underbrace{4 \cdot 4}_{} \times 1$

$$(\tilde{X}'\tilde{X})^{-1} = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}^{-1}$$

$$= \frac{1}{20} \begin{bmatrix} 14 & -6 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} 14/20 & -6/20 \\ -6/20 & 4/20 \end{bmatrix}$$

$$(\tilde{X}'\tilde{X})^{-1} = \begin{bmatrix} 0.7 & -0.3 \\ -0.3 & 0.2 \end{bmatrix}$$

$$\tilde{X}'\tilde{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 13 \end{bmatrix}$$

$2 \times \underbrace{4 \cdot 4}_{} \times 1$

$$\hat{B} = (\tilde{X}'\tilde{X})^{-1} \tilde{X}'y$$

$$= \begin{bmatrix} 0.7 & -0.3 \\ -0.3 & 0.2 \end{bmatrix} \begin{bmatrix} 11 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} \hat{B}_0 \\ \hat{B}_1 \end{bmatrix} = \begin{bmatrix} 3.8 \\ -0.7 \end{bmatrix} \qquad \hat{y} = 3.8 - 0.7x_1$$

$\therefore$ The least squares estimator of $\hat{B}_0 = 3.8$ and $\hat{B}_1 = -0.7$

2. $SSRes = \sum y_i^2 - \hat{\beta}' \underset{\sim}{X}' \underset{\sim}{y}$

$$y^2 = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 3 \end{bmatrix}^2 = \begin{bmatrix} 16 \\ 9 \\ 1 \\ 9 \end{bmatrix} \qquad \sum y_i^2 = 35$$

$$\hat{\beta}' \underset{\sim}{X}' \underset{\sim}{y} = \underset{1\times 2 \cdot 2\times 1}{\begin{bmatrix} 3.8 & -0.7 \end{bmatrix} \begin{bmatrix} 11 \\ 13 \end{bmatrix}} = 32.7$$

$SSRes = 35 - 32.7 = 2.3$

$\underline{\underline{SSRes = 2.3}}$

$\bar{y} = 2.75$

$SSR = \hat{\beta}' \underset{\sim}{X}' \underset{\sim}{y} - n\bar{y}^2$

$= 32.7 - 4(2.75)^2$

$\underline{\underline{SSR = 2.45}}$

$SST = SSR + SSRes$

$= 2.45 + 2.3$

$\underline{\underline{SST = 4.75}}$

$R^2 = \frac{SSR}{SST} = \frac{2.45}{4.75} = 0.5158$

$\therefore \underline{\underline{R^2 = 0.5158}}$

4. 90% prediction interval for y_0 given $x_0 = [1, 2]$

3. $\hat{y}_0 = 3.8(1) - 0.7(2) = \underline{2.4} \rightarrow y_0$ given $x_0 = [1,2] = 2.4$

$\alpha = 0.1$

$\hat{\sigma} = \sqrt{MSRes}$

$= \sqrt{\frac{SSRes}{n-2}}$

$= \sqrt{\frac{2.3}{2}}$

$\hat{\sigma} = 1.0724$

$$\hat{y}_0 \pm t_{\alpha/2}^{(n-k)} \hat{\sigma} \sqrt{1 + \underbrace{x_0' (\underset{\sim}{X}'\underset{\sim}{X})^{-1} \underset{\sim}{x_0}}_{h_{00}}}$$

$$\underset{\sim}{x_0}' (\underset{\sim}{X}'\underset{\sim}{X})^{-1} = \underset{1\times 2}{\begin{bmatrix} 1 & 2 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 0.7 & -0.3 \\ -0.3 & 0.2 \end{bmatrix}} = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}$$

$$h_{00} = \underset{\sim}{x_0} (\underset{\sim}{X}'\underset{\sim}{X})^{-1} \underset{\sim}{x_0} = \underset{1\times 2 \cdot 2\times 1}{\begin{bmatrix} 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \begin{bmatrix} 0.3 \end{bmatrix}$$

$$\hat{y}_0 \pm t_{\alpha/2}^{(n-k)} \hat{\sigma}\sqrt{1+h_{00}}$$

$$= 2.4 \pm t_{0.05}^{2} \times 1.0724 \times \sqrt{1+0.3}$$

$$= 2.4 \pm 2.919986 \times 1.0724 \times \sqrt{1.3}$$

$$= 2.4 \pm 3.5703$$

$$= (-1.1703, 5.9703)$$

$\therefore$ A 90% prediction interval for y_0 given $x_0 = [1, 2]$ is $(-1.1703, 5.9703)$