# Variance Reduction for Experiments: CUPED, Ratio Metrics, and CURE

Kane Luo

## 1 Introduction

Reducing variance is a fundamental goal in experimentation, as lower variance directly translates into higher statistical power and faster decision making. Techniques that leverage pre-experiment information are particularly attractive because they preserve unbiasedness while improving sensitivity. This document reviews the classical CUPED framework, its extension to ratio metrics, and a further generalization—CURE—that incorporates additional covariates.

## 2 Summary of the Current CUPED

The original CUPED technique [1] was proposed to accelerate experiments by reducing variance using pre-experiment data. The key insight is that, given experimental outcomes, any auxiliary signal computed from pre-treatment data that satisfies a mean-zero (non-bias) condition can be used to reduce variance without affecting the expectation of the estimator.

Let $Y$ denote the post-treatment outcome and $X$ a pre-experiment covariate. The CUPED-adjusted estimator is

$$\hat{Y} = Y - \theta(X - \mathbb{E}[X]) \tag{1}$$

The variance-minimizing coefficient is

$$\theta = \frac{\mathrm{Cov}(Y, X)}{\mathrm{Var}(X)} \tag{2}$$

The resulting variance is

$$\mathrm{Var}(Y_{\mathrm{cv}}) = \mathrm{Var}(Y - \theta X) = \mathrm{Var}(Y) + \theta^2 \mathrm{Var}(X) - 2\theta\,\mathrm{Cov}(Y, X). \tag{3}$$

## 3 Extension of CUPED to Ratio Metrics

Many commonly used metrics, such as click-through rate, are ratios of two quantities. Let

$$R = \frac{Y}{N}, \tag{4}$$

where $Y$ is the numerator (e.g., clicks) and $N$ the denominator (e.g., impressions).

With pre-experiment quantities $X$ and $M$, a CUPED-style adjustment is

$$\widehat{\frac{Y}{N}}_{cv} = \frac{Y}{N} - \theta\left(\frac{X}{M} - \mathbb{E}\left[\frac{X}{M}\right]\right). \tag{5}$$

The optimal coefficient and variance are obtained via a delta-method–style approximation that explicitly preserves numerator–denominator covariance,

$$\theta = \frac{\operatorname{Cov}\left(\frac{Y}{N}, \frac{X}{M}\right)}{\operatorname{Var}\left(\frac{X}{M}\right)} \tag{6}$$

$$\operatorname{Var}(\frac{Y}{N}_{\text{cv}}) = \operatorname{Var}\left(\frac{Y}{N} - \theta\frac{X}{M}\right) = \operatorname{Var}(\frac{Y}{N}) + \theta^2\operatorname{Var}(\frac{X}{M}) - 2\theta\operatorname{Cov}(\frac{Y}{N}, \frac{X}{M}). \tag{7}$$

Adjusting per-unit ratios can discard covariance information; therefore, adjustment is performed on aggregated quantities. Estimating $\mathbb{E}[\frac{Y}{N}]$ is also nontrivial because the empirical mean of a ratio does not need to be an unbiased estimator in finite samples.

We use a relative approach to this problem: realizing that $\mathbb{E}[\frac{Y}{N}]$ is the same for both test and control groups, we are able to calculate the difference between the adjusted average. Assuming control mean's change with CUPED is trivial, we get:

$$\widehat{\frac{Y_c}{N_c}}_{cv} = \frac{Y_c}{N_c}. \tag{8}$$

$$\widehat{\frac{Y_t}{N_t}}_{cv} = \frac{Y_t}{N_t} - \theta\left(\frac{X_t}{M_t}\right) + \theta\left(\frac{X_c}{M_c}\right). \tag{9}$$

## 4 CURE: Using Pre-Experiment Data and Covariates

When variance reduction based solely on the metric's own pre-period value is insufficient—e.g., noisy metrics, new-user experiments, or the need for rapid decisions—additional covariates can help.

CURE (Covariate-Adjusted CUPED) generalizes CUPED to a multivariate regression. With post-treatment outcome $Y$, pre-experiment metric $X$, and covariates $c_1, \ldots, c_k$,

$$\hat{Y} = Y - \theta_0(X - \mathbb{E}[X]) - \sum_{j=1}^{k} \theta_j(c_j - \mathbb{E}[c_j]). \tag{10}$$

Define the covariate vector

$$A = \begin{bmatrix} X & c_1 & c_2 & \cdots & c_k \end{bmatrix}^\top. \tag{11}$$

The optimal coefficients are

$$\boldsymbol{\theta} = \frac{\operatorname{Cov}(Y, A)}{\operatorname{Var}(A)}. \tag{12}$$

Mean-centering of $A$ and $Y$ is not required; a proof is provided in the appendix.

## 5 CURE for Ratio Metrics

CURE extends naturally to ratio metrics. With ratios $Y/N$ and $X/M$ and covariates $c_1, \ldots, c_k$,

$$\widehat{\frac{Y}{N}} = \frac{Y}{N} - \theta_0 \left( \frac{X}{M} - \mathbb{E}! \left[ \frac{Y}{N} \right] \right) - \sum_{j=1}^{k} \theta_j (c_j - \mathbb{E}[c_j]). \tag{13}$$

Again, we use the technique to shift the control mean back to the unadjusted value, plus the adjustment that comes from the covariates.

$$\widehat{\frac{Y_c}{N_c}}_{cv} = \frac{Y_c}{N_c} - \theta_0 \left( \frac{X_c}{M_c} \right) - \sum_{j=1}^{k} \theta_j (c_j^c - \mathbb{E}[c_j]). \tag{14}$$

$$\widehat{\frac{Y_t}{N_t}}_{cv} = \frac{Y_t}{N_t} - \theta_0 \left( \frac{X_t}{M_t} \right) + \theta \left( \frac{X_c}{M_c} \right) - \sum_{j=1}^{k} \theta_j (c_j^t - \mathbb{E}[c_j]). \tag{15}$$

Variance estimation must again preserve numerator–denominator covariance and joint covariance with covariates. Instead of using the complex equation, we use the $R^2$ from the regression to shrink the variance of each group.

$$\mathrm{Var}(\widehat{\frac{Y}{N}}) = \mathrm{Var}(\frac{Y}{N})(1 - R^2). \tag{16}$$

## 6 Appendix

Comparison of the theta using raw values versus mean-centered values:

$$\boldsymbol{\theta} = \frac{\mathrm{Cov}(Y, X)}{\mathrm{Var}(X)}. \tag{17}$$

$$\boldsymbol{\theta} = \frac{\mathrm{Cov}(Y - \bar{Y}, X - \bar{X})}{\mathrm{Var}(X - \bar{X})}. \tag{18}$$

By the properties of variance and covariance, the above equations are equivalent.

## References


[1] Deng, A., Xu, Y., Kohavi, R., Walker, T. (2013). *Improving the Sensitivity of Online Controlled Experiments by Utilizing Pre-Experiment Data.* [`https://exp-platform.com/Documents/2013-02-CUPED-ImprovingSensitivityOfControlledExperiments.pdf`](https://exp-platform.com/Documents/2013-02-CUPED-ImprovingSensitivityOfControlledExperiments.pdf)