

Handout 10: Likelihood-Ratio Test

In addition to the nice point-estimator properties of MLE estimators, they also come along with a built-in general procedure for building hypothesis tests. This process is called the **likelihood-ratio test**; while it only generates asymptotically correct tests in the limit of large data, it often performs very well even for small datasets.

It will be helpful for us to start with a slightly more general description of the test than we needed for the introductory methods we covered in the first weeks. Namely, we will extend the idea of a simple hypothesis, where H_0 is a single value of the parameters, to complex hypotheses where H_0 is a general subset of all possibilities. Note that all of the elements we have already described, such p-values and rejection regions, still apply. Those concepts just need to hold for any value in the null hypothesis. Assume that we are working with a distribution that has k unknown parameters. We are going to define a null-hypothesis as a subset of all possible configurations of the unknown parameters, which is in turn a subset of \mathbb{R}^k . Symbolically, we have $\Theta_0 \subset \Theta \subseteq \mathbb{R}^k$.

Given a null hypothesis that $\theta \in \Theta_0$, we can define the quantity Λ as the following:

$$\Lambda = -2 \cdot \log \left[\frac{\sup_{\theta \in \Theta_0} \mathcal{L}(\theta)}{\sup_{\theta \in \Theta} \mathcal{L}(\theta)} \right].$$

The value inside of the brackets must be between zero and one, and therefore Λ will always be positive. If we define θ_0 as the argmax of the numerator and $\hat{\theta}$ as the argmax of the denominator, this can be simplified in terms of the log-likelihood $l(\cdot)$:

$$\Lambda = 2 \cdot [l(\hat{\theta}) - l(\theta_0)].$$

There is an important (but not easy to prove) result called **Wilks' Theorem** that establishes that Λ will be approximately distributed as a χ^2 . The degrees of freedom will be the difference in dimensionality between Θ and Θ_0 . It is therefore an asymptotic pivot quantity.