Worksheet 03 (Solutions)

1. Let $X_1, \ldots, X_n \sim N(\mu_X, \sigma_X^2)$ be a random sample. Using the results from the handout, construct a pivot statistic T as a function of $\bar{X}$, S_X^2, μ_X, and σ_X^2 that has a distribution of $t(n-1)$. Do not simplify.

Solution: We have the following, by simply plugging in the results from the handout:

$$T = \frac{\frac{\mu_X - \bar{X}}{\sqrt{\sigma_X^2/n}}}{\sqrt{\frac{(n-1)S_X^2}{\sigma_X^2 \cdot (n-1)}}}$$

2. Simplify the form of the T statistic. It should no longer have any σ_X^2 terms (in fact this is the whole point of this specific form). Try to write the solution with $(\mu - \bar{X})$ in the numerator and everything else in the denominator.

Solution: Simplifying, we see that:

$$\begin{aligned} T &= \frac{\frac{\mu_X - \bar{X}}{\sqrt{\sigma_X^2/n}}}{\sqrt{\frac{(n-1)S_X^2}{\sigma_X^2 \cdot (n-1)}}} \\ &= \frac{\mu_X - \bar{X}}{\sqrt{S_X^2/n}} \end{aligned}$$

As desired.

3. Let $t_\alpha(k)$ be the tail probability of a T-distribution with k degrees of freedom, just as we had with z_α on the handout. Following the same procedure with the example on the handout, construct a confidence interval with confidence level $(1-\alpha)$ for μ_X. Write the solution as $\bar{X} \pm \Delta$ for some Δ.

Solution: Starting with the pivot statistic, we have:

$$\begin{aligned}
\mathbb{P}\left[-t_\alpha(n-1) \leq T \leq t_\alpha(n-1)\right] &= 1-\alpha \\
\mathbb{P}\left[-t_\alpha(n-1) \leq \frac{\mu_X - \bar{X}}{\sqrt{S_X^2/n}} \leq t_\alpha(n-1)\right] &= 1-\alpha \\
\mathbb{P}\left[-t_\alpha(n-1)\cdot\sqrt{S_X^2/n} \leq (\mu_X - \bar{X}) \leq t_\alpha(n-1)\cdot\sqrt{S_X^2/n}\right] &= 1-\alpha \\
\mathbb{P}\left[\bar{X} - t_\alpha(n-1)\cdot\sqrt{S_X^2/n} \leq \mu_X \leq \bar{X} + t_\alpha(n-1)\cdot\sqrt{S_X^2/n}\right] &= 1-\alpha
\end{aligned}$$

Which we can write as the following:

$$\bar{X} \pm t_\alpha(n-1)\cdot\sqrt{\frac{S_X^2}{n}}$$

4. We will go back to the story about the fish. Say we sample 25 fish and have a sample mean of 12.1 centimeters and a sample variance of 6 centimeters squared. Given that $t_{0.01/2}(24)$ is approximately equal to 2.797, derive the confidence interval for the mean.

Solution: We have:

$$\begin{aligned}
&12.1 \pm \left[2.797\cdot\sqrt{\frac{6}{24}}\right] \\
&12.1 \pm 1.3985
\end{aligned}$$

5. Let $C_k \sim \chi^2(k)$ for every integer k. Use Chebychev's Inequality to show that for any $\epsilon > 0$, we have:

$$\lim_{k\to\infty} \mathbb{P}\left[|C_k/k - 1| \geq \epsilon\right] = 0$$

In this case we say that C_k limits in probability to 1, written as $C_k \to_P 1$.

Solution: TODO

6. Let $Y_n \to_P y$ for a constant y, f is a real-valued function that is invertable around the neighborhood of y, and X is another random variable. Then, Slutsky's Theorem says that $g(Y_n)\cdot X$ limits in probability to $g(y)\cdot X$. Use this to show that the T distribution limits to the standard normal as the degrees of freedom limit to infinity.

Solution: TODO