Worksheet 01 (Solutions)

1. Assume we have a random sample of size $n = 5$ with the following data: $x_1 = 2$, $x_1 = 6$, $x_1 = 1$, $x_1 = 0$, $x_1 = 6$. What is the observered sample mean $\bar{x}$? [1]

[1] I am using the standard convention that we replace upper-case random variable names with lower-case variables when we have specific observations of them.

Solution: We have:

$$\bar{x} = \frac{2+6+1+0+6}{5} = \frac{15}{5} = 3.$$

2. Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{G}$ be a random sample from a distribution with mean μ_X and variance σ_X^2. What is the expected value of the sample mean $\bar{X}$?[2] Does this imply that $\bar{X}$ is an unbiased estimator of μ_X?

[2] I gave the answer on the handout. Make sure that you can justify the result.

Solution: We have:

$$\begin{aligned}
\mathbb{E}\bar{X} &= \mathbb{E}\left[\frac{1}{n} \times \sum_{i=1}^{n} X_i\right] \\
&= \frac{1}{n} \times \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] \\
&= \frac{1}{n} \times \left[\sum_{i=1}^{n} \mathbb{E}X_i\right] \\
&= \frac{1}{n} \times \left[\sum_{i=1}^{n} \mu_X\right] \\
&= \frac{1}{n} \times n \cdot \mu_X = \mu_X.
\end{aligned}$$

So by the definition of unbiased, $\bar{X}$ is an unbiased estimator of μ_X.

3. Using the same set-up as the previous question, what is $Var(\bar{X})$?

Solution: We have:

$$\begin{aligned}
\mathrm{Var}[\bar{X}] &= \mathrm{Var}\left[\frac{1}{n} \times \sum_{i=1}^{n} X_i\right] \\
&= \frac{1}{n^2} \times \mathrm{Var}\left[\sum_{i=1}^{n} X_i\right] \\
&= \frac{1}{n^2} \times \left[\sum_{i=1}^{n} \mathrm{Var}X_i\right] \\
&= \frac{1}{n^2} \times \left[\sum_{i=1}^{n} \sigma_X^2\right] \\
&= \frac{1}{n^2} \times n \cdot \sigma_X^2 = \frac{\sigma_X^2}{n}.
\end{aligned}$$

As given on the handout.

4. Let Y be a random variable with mean m and variance v. Chebyshev's Inequality tells us that if for any $a > 0$,

$$\mathbb{P}[|Y - m| \geq a] \leq \frac{v}{a^2}.$$

Use this result to show that $\bar{X}$ is a consistent estimator of μ_X.

Solution: Apply Chebyshev's inequality with $Y = \bar{X}$ and $a = \epsilon$, to get that for any ϵ we have (using the two previous results):

$$\mathbb{P}[|\bar{X} - \mu_X| \geq \epsilon] \leq \frac{\sigma_X^2}{\epsilon^2 n}.$$

Since ϵ and σ_X^2 are assumed to be fixed, we have that:

$$\lim_{n\to\infty} \mathbb{P}[|\bar{X} - \mu_X| \geq \epsilon] = 0.$$

By definition, then, $\bar{X}$ is a consistent estimator of μ_X.

5. Assume that $\mathcal{G}$ has a normal distribution. Define the following:

$$Z = \frac{\mu_X - \bar{X}}{\sqrt{\sigma_X^2/n}}$$

What is the distribution of Z?

Solution: We see that Z is a scaled version of a normal distribution, and therefore Z must also be normal. All that is left is to determine its mean and variance. These are:

$$\begin{aligned}
\mathbb{E}Z &= \mathbb{E}\left[\frac{\mu_X - \bar{X}}{\sqrt{\sigma_X^2/n}}\right] \\
&= \frac{\mu_X - \mathbb{E}\bar{X}}{\sqrt{\sigma_X^2/n}} \\
&= \frac{\mu_X - \mu_X}{\sqrt{\sigma_X^2/n}} \\
&= 0.
\end{aligned}$$

And

$$\begin{aligned}
\mathrm{Var}Z &= \mathrm{Var}\left[\frac{\mu_X - \bar{X}}{\sqrt{\sigma_X^2/n}}\right] \\
&= \mathrm{Var}\left[\frac{\bar{X}}{\sqrt{\sigma_X^2/n}}\right] \\
&= \frac{1}{\sqrt{\sigma_X^2/n}} \times \mathrm{Var}\bar{X} \\
&= \frac{\frac{\sigma_X^2}{n}}{\sqrt{\sigma_X^2/n}} \\
&= 1.
\end{aligned}$$

So, $Z \sim N(0,1)$, a standard normal.