

Worksheet 01 (Solutions)

1. Assume we have a random sample of size $n = 5$ with the following data: $x_1 = 2, x_1 = 6, x_1 = 1, x_1 = 0, x_1 = 6$. What is the observed sample mean \bar{x} ?¹

Solution: We have:

$$\bar{x} = \frac{2 + 6 + 1 + 0 + 6}{5} = \frac{15}{5} = 3.$$

¹ I am using the standard convention that we replace upper-case random variable names with lower-case variables when we have specific observations of them.

2. Let $X_1, \dots, X_n \stackrel{iid}{\sim} \mathcal{G}$ be a random sample from a distribution with mean μ_X and variance σ_X^2 . What is the expected value of the sample mean \bar{X} ?² Does this imply that \bar{X} is an unbiased estimator of μ_X ?

Solution: We have:

$$\begin{aligned} \mathbb{E}\bar{X} &= \mathbb{E}\left[\frac{1}{n} \times \sum_{i=1}^n X_i\right] \\ &= \frac{1}{n} \times \mathbb{E}\left[\sum_{i=1}^n X_i\right] \\ &= \frac{1}{n} \times \left[\sum_{i=1}^n \mathbb{E}X_i\right] \\ &= \frac{1}{n} \times \left[\sum_{i=1}^n \mu_X\right] \\ &= \frac{1}{n} \times n \cdot \mu_X = \mu_X. \end{aligned}$$

So by the definition of unbiased, \bar{X} is an unbiased estimator of μ_X .

3. Using the same set-up as the previous question, what is $\text{Var}(\bar{X})$?

Solution: We have:

$$\begin{aligned} \text{Var}[\bar{X}] &= \text{Var}\left[\frac{1}{n} \times \sum_{i=1}^n X_i\right] \\ &= \frac{1}{n^2} \times \text{Var}\left[\sum_{i=1}^n X_i\right] \\ &= \frac{1}{n^2} \times \left[\sum_{i=1}^n \text{Var}X_i\right] \\ &= \frac{1}{n^2} \times \left[\sum_{i=1}^n \sigma_X^2\right] \\ &= \frac{1}{n^2} \times n \cdot \sigma_X^2 = \frac{\sigma_X^2}{n}. \end{aligned}$$

² I gave the answer on the handout. Make sure that you can justify the result.

As given on the handout.

4. Let Y be a random variable with mean m and variance v . Chebyshev's Inequality tells us that if for any $a > 0$,

$$\mathbb{P}[|Y - m| \geq a] \leq \frac{v}{a^2}.$$

Use this result to show that \bar{X} is a consistent estimator of μ_X .

Solution: Apply Chebyshev's inequality with $Y = \bar{X}$ and $a = \epsilon$, to get that for any ϵ we have (using the two previous results):

$$\mathbb{P}[|\bar{X} - \mu_X| \geq \epsilon] \leq \frac{\sigma_X^2}{\epsilon^2 n}.$$

Since ϵ and σ_X^2 are assumed to be fixed, we have that:

$$\lim_{n \rightarrow \infty} \mathbb{P}[|\bar{X} - \mu_X| \geq \epsilon] = 0.$$

By definition, then, \bar{X} is a consistent estimator of μ_X .

5. Assume that \mathcal{G} has a normal distribution. Define the following:

$$Z = \frac{\mu_X - \bar{X}}{\sqrt{\sigma_X^2/n}}$$

What is the distribution of Z ?

Solution: We see that Z is a scaled version of a normal distribution, and therefore Z must also be normal. All that is left is to determine its mean and variance. These are:

$$\begin{aligned} \mathbb{E}Z &= \mathbb{E} \left[\frac{\mu_X - \bar{X}}{\sqrt{\sigma_X^2/n}} \right] \\ &= \frac{\mu_X - \mathbb{E}\bar{X}}{\sqrt{\sigma_X^2/n}} \\ &= \frac{\mu_X - \mu_X}{\sqrt{\sigma_X^2/n}} \\ &= 0. \end{aligned}$$

And

$$\begin{aligned}
 \text{Var}Z &= \text{Var} \left[\frac{\mu_X - \bar{X}}{\sqrt{\sigma_X^2/n}} \right] \\
 &= \text{Var} \left[\frac{\bar{X}}{\sqrt{\sigma_X^2/n}} \right] \\
 &= \frac{1}{\sqrt{\sigma_X^2/n}} \times \text{Var}\bar{X} \\
 &= \frac{\frac{\sigma_X^2}{n}}{\sqrt{\sigma_X^2/n}} \\
 &= 1.
 \end{aligned}$$

So, $Z \sim N(0,1)$, a standard normal.