Worksheet 03 (Solutions)

1. Let $X_1, \ldots, X_n \sim N(\mu_X, \sigma_X^2)$ be a random sample. Using the results from the handout, construct a pivot statistic T as a function of $\bar{X}$, S_X^2, μ_X, and σ_X^2 that has a distribution of $t(n-1)$. Do not simplify.

Solution: We have the following, by simply plugging in the results from the handout:

$$T = \frac{\frac{\mu_X - \bar{X}}{\sqrt{\sigma_X^2/n}}}{\sqrt{\frac{(n-1)S_X^2}{\sigma_X^2 \cdot (n-1)}}}$$

2. Simplify the form of the T statistic. It should no longer have any σ_X^2 terms (in fact this is the whole point of this specific form). Try to write the solution with $(\mu - \bar{X})$ in the numerator and everything else in the denominator.

Solution: Simplifying, we see that:

$$\begin{aligned} T &= \frac{\frac{\mu_X - \bar{X}}{\sqrt{\sigma_X^2/n}}}{\sqrt{\frac{(n-1)S_X^2}{\sigma_X^2 \cdot (n-1)}}} \\ &= \frac{\mu_X - \bar{X}}{\sqrt{S_X^2/n}} \end{aligned}$$

As desired.

3. Let $t_\alpha(k)$ be the tail probability of a T-distribution with k degrees of freedom, just as we had with z_α on the handout. Following the same procedure with the example on the handout, construct a confidence interval with confidence level $(1-\alpha)$ for μ_X. Write the solution as $\bar{X} \pm \Delta$ for some Δ.

Solution: Starting with the pivot statistic, we have:

$$\begin{aligned}
\mathbb{P}\left[-t_\alpha(n-1) \le T \le t_\alpha(n-1)\right] &= 1-\alpha \\
\mathbb{P}\left[-t_\alpha(n-1) \le \frac{\mu_X - \bar{X}}{\sqrt{S_X^2/n}} \le t_\alpha(n-1)\right] &= 1-\alpha \\
\mathbb{P}\left[-t_\alpha(n-1)\cdot\sqrt{S_X^2/n} \le (\mu_X - \bar{X}) \le t_\alpha(n-1)\cdot\sqrt{S_X^2/n}\right] &= 1-\alpha \\
\mathbb{P}\left[\bar{X} - t_\alpha(n-1)\cdot\sqrt{S_X^2/n} \le \mu_X \le \bar{X} + t_\alpha(n-1)\cdot\sqrt{S_X^2/n}\right] &= 1-\alpha
\end{aligned}$$

Which we can write as the following:

$$\bar{X} \pm t_\alpha(n-1)\cdot\sqrt{\frac{S_X^2}{n}}$$

4. We will go back to the story about the fish. Say we sample 25 fish and have a sample mean of 12.1 centimeters and a sample variance of 6 centimeters squared. Given that $t_{0.01/2}(24)$ is approximately equal to 2.797, derive the confidence interval for the mean.

Solution: We have:

$$\begin{aligned}
&12.1 \pm \left[2.797\cdot\sqrt{\frac{6}{24}}\right] \\
&12.1 \pm 1.3985
\end{aligned}$$

5. Now, let's build a confidence interval for the variance. The chi-squared distribution is not symmetric, so we need to start with a more general form of the equation with the pivot statistic (the last equation on the handout). Namely, we have:

$$\mathbb{P}\left[\chi_{1-\alpha/2}(n-1) \le \frac{(n-1)S_X^2}{\sigma_X^2} \le \chi_{\alpha/2}(n-1)\right]$$

To manipulate this into a confidence interval, first take the (multiplicative) inverse of all three terms. Note that for positive numbers, taking the inverse of both sides of an inequality reverses the sign of the inequality. Then, simplify to get a confidence interval of σ_X^2.

Solution: Starting with the pivot statistic, we have:

$$\mathbb{P}\left[\frac{1}{\chi_{1-\alpha/2}(n-1)} \geq \frac{\sigma_X^2}{(n-1)S_X^2} \geq \frac{1}{\chi_{\alpha/2}(n-1)}\right]$$
$$\mathbb{P}\left[\frac{(n-1)S_X^2}{\chi_{1-\alpha/2}(n-1)} \geq \sigma_X^2 \geq \frac{(n-1)S_X^2}{\chi_{\alpha/2}(n-1)}\right]$$

And we now have a confidence interval for the variance. Not too bad, right?

6. Given that $\chi^2_{0.01/2}(24) \approx 45.56$ and $\chi^2_{1-0.01/2}(24) \approx 9.88$, what is the 99% confidence interval for the variance of the lengths of the fish from our example data?

Solution: The lower bound is:

$$\frac{(n-1)S_X^2}{\chi_{\alpha/2}(n-1)} = \frac{24 \cdot 6}{9.88} = 3.16$$

And the upper bound is:

$$\frac{(n-1)S_X^2}{\chi_{\alpha/2}(n-1)} = \frac{24 \cdot 6}{45.56} = 14.57$$

So the confidence interval is $[3.16, 14.57]$.