Worksheet 09 (Solutions)

1. Find the MLE estimator for the estimation of the parameter λ from i.i.d. observations of an exponentialy distributed random variable.

Solution: We have the following log-likelihood:

$$\begin{aligned} l(\lambda; x_1, \ldots, x_n) &= \sum_{i=1}^{n} \log\left[\lambda \cdot e^{-\lambda x_i}\right] \\ &= \sum_{i=1}^{n} \left[\log(\lambda) - \lambda x_i\right] \end{aligned}$$

The derivative with respect to λ is:

$$\frac{\partial}{\partial \lambda} l(\lambda; x_1, \ldots, x_n) = \sum_{i=1}^{n} \left[\frac{1}{\lambda} - x_i\right]$$

Setting this equal to zero (and putting a hat on the parameter), gives:

$$\begin{aligned} \sum_{i=1}^{n} \frac{1}{\hat{\lambda}} &= \sum_{i=1}^{n} x_i \\ \frac{n}{\hat{\lambda}} &= \sum_{i=1}^{n} x_i \\ \frac{n}{\sum_{i=1}^{n} x_i} &= \hat{\lambda} \\ \frac{1}{\frac{1}{n} \cdot \sum_{i=1}^{n} x_i} &= \hat{\lambda} \end{aligned}$$

In other words, the MLE is just one divided by the sample mean. That makes a lot of sense (but, again, not maybe very interesting) given that λ is the inverse of the mean for the exponential distribution.

2. Find the MLE estimator for the estimation of the variance from i.i.d. observations of an exponentialy distributed random variable. Hint: This is easily derived from the previous result. Should not require any new derivatives.

Solution: We know that the variance of an exponentially distributed random variable is λ^{-2}. We already have the MLE for λ, so the MLE of the variance is just this value to the -2 power:

$$\begin{aligned} \text{MLE} &= \left[\frac{1}{n} \cdot \sum_{i=1}^{n} x_i\right]^2 \\ &= \bar{X}^2. \end{aligned}$$

Notice that this **is** quite different than the typical estimator that we use for estimating the variance of a sample (S_X^2), taking into account the special structure of the exponential distribution.

3. Find the MLE estimator for the estimation of the parameter p from i.i.d. observations of a Bernoulli distributed random variable. Hint: When you set the derivative equal to zero, multiple by $\frac{1}{n}$ to write the equation in terms of just $\bar{X}$ and $\hat{p}$.

Solution: We have the following log-likelihood:

$$\begin{aligned} l(p; x_1, \ldots, x_n) &= \sum_{i=1}^{n} \log \left[p^{x_i} \cdot (1-p)^{1-x_i} \right] \\ &= \sum_{i=1}^{n} \left[x_i \cdot \log(p) + (1-x_i) \cdot \log(1-p) \right] \end{aligned}$$

The derivative with respect to p is:

$$\frac{\partial}{\partial p} l(p; x_1, \ldots, x_n) = \sum_{i=1}^{n} \left[\frac{x_i}{p} + \frac{(-1) \cdot (1-x_i)}{1-p} \right]$$

Setting this equal to zero (and putting a hat on the parameter), gives the following

$$\frac{1}{\hat{p}} \cdot \sum_{i=1}^{n} x_i = \frac{1}{1-\hat{p}} \cdot \sum_{i} (1-x_i)$$

Dividing both side by n as in the hint gives:

$$\frac{1}{\hat{p}} \cdot \bar{x} = \frac{1}{1-\hat{p}} \cdot (1-\bar{x})$$

And then, solving gives:

$$\begin{aligned} \bar{x}(1-\hat{p}) &= \hat{p}(1-\bar{x}) \\ \bar{x} - \bar{x} \cdot \hat{p} &= \hat{p} - \bar{x} \cdot \hat{p} \\ \bar{x} &= \hat{p} \end{aligned}$$

And again, we see that the MLE is just the sample mean.

4. Find the MLE estimator for the estimation of the parameters μ and σ^2 from i.i.d. observations of a normally distributed random variable. Hint: We want to think of σ^2 as a single parameter (not the square of a parameter). I recommend using $v = \sigma^2$ to keep this clear. Also, find $\hat{\mu}$ first. You can find the MLE for the mean without knowing the MLE of the variance.

Solution: This is where things get a bit more interesting. We have the following log-likelihood:

$$
\begin{aligned}
l(p; x_1, \ldots, x_n) &= \sum_{i=1}^{n} \log \left[\frac{1}{\sqrt{2\pi v}} \cdot e^{-\frac{1}{2v}[x_i - \mu]^2} \right] \\
&= \sum_{i=1}^{n} [x_i \cdot \log(p) + (1 - x_i) \cdot \log(1 - p)]
\end{aligned}
$$

5. What is the bias of the MLE estimator for the variance from a normal distribution with unknown mean and variance? Hint: Use what we know about S_X^2 to make this relatively easy.

Solution: TODO