Handout 07: Review

We have now covered the three key types of tasks that we do in classical statistics: point estimation, confidence intervals, and hypothesis tests. We have derived these for the four most common cases of estimating the mean and variance of one sample as well as comparing the mean and variance across two samples. Let's review these concepts in the abstract case of estimating some unknown parameter θ.

A **point estimator** is a sample statistic $\hat{\theta}$ that is the best guess for the unknown θ. Typically, the first thing we would want to do is determine the expected value and variance of $\hat{\theta}$. The quantity $\mathbb{E}[\hat{\theta}] - \theta$ is called the bias, which we hope will be close to zero. We say that an estimator is consistent if both the bias and variance limit to zero in the limit as the sample size goes to infinity.[1]

[1] I gave the more formal definition on the original handout, but this description is equivalent as long as the variance is finite.

Confidence intervals and **hypothesis tests** both require starting by specifying a confidence level $1 - \alpha$. To form either, we typically start by constructing a pivot quantity G_θ that is (1) a function of the random sample and the unknown parameter θ, and (2) has a fixed distribution $\mathcal{G}$ that does not depend on θ. If we let g_α be the critical values of $\mathcal{G}$,[2] we can write the following:

[2] In other words, $\mathbb{P}[G > g_\alpha] = \alpha$.

$$\mathbb{P}\left[g_{1-\alpha/2} \leq G_\theta \leq g_{\alpha/2}\right] = 1 - \alpha.$$

To form a valid confidence interval, we plug in the formula for G_θ and try to re-arrange to inequality to get the quantity θ on its own. In other words, we get:

$$\mathbb{P}\left[L \leq \theta \leq U\right] = 1 - \alpha.$$

And that's exactly what we need. For a hypothesis test, we start with some null-hypothesis $H_0 : \theta = \theta_0$. Given this, we can compute the actual value G_{θ_0} given the random sample. We form the rejection region as follows:

$$R = \left\{G_{\theta_0} \leq g_{1-\alpha/2}\right\} \cup \left\{G_{\theta_0} \geq g_{\alpha/2}\right\}.$$

If G_{θ_0} is in the rejection region, we reject H_0; otherwise we retain the null hypothesis. The p-value is defined as the smallest value of α such that G_{θ_0} is in the rejection region.[3] We can compare the p-value to our desired α to determine whether to retain or reject H_0.

[3] Therefore, any confidence level where the p-value is less than α will result in G_{θ_0} being within the rejection region. Why? Because we would reject if α was equal to p, and larger values of α have a lower confidence level and will reject more liberally than small values.

Given a point estimator, you should be able to determine its bias and decide if it is consistent. Given a pivot quantity, you should be able to construct a confidence interval and set-up an hypothesis test. We do not know how to do yet is how to produce the point estimators and pivots in the first place. That will be the focus on the second unit of the course.

Statistical Tests

The following table provides a summary of the key results from the five tests we have to estimate the mean or variance of one, two, or many samples. All of the results assume normality. The two-sample and multiple-sample tests for the mean assume that all groups have the same variance. The distribution column provides the distribution of the pivot (in general) and the distribution of the test statistic (conditional on H_0). See the handouts for the details of the notation.

Type	**Parameter**	**P. Estimator**	**Pivot**	**Confidence Interval**	$\mathbf{H_0}$	**Test Statistic**	**Distribution**
One-Sample	μ_X	$\hat{\mu} = \bar{X}$	$\frac{\bar{X}-\mu_X}{\sqrt{S_X^2/n}}$	$\bar{X} \pm t_{\alpha/2} \times \sqrt{S_X^2/n}$	$\mu_X = \mu_0$	$\frac{\bar{X}-\mu_0}{\sqrt{S_X^2/n}}$	$T \sim t(n-1)$
	σ_X^2	$\hat{\sigma}^2 = S_X^2$	$\frac{(n-1)S_X^2}{\sigma_X^2}$	$\left[\frac{(n-1)S_X^2}{\chi^2_{1-\alpha/2}}, \quad \frac{(n-1)S_X^2}{\chi^2_{\alpha/2}}\right]$	$\sigma_X^2 = \sigma_0^2$	$\frac{(n-1)S_X^2}{\sigma_0^2}$	$C \sim \chi^2(n-1)$
Two-Sample	$\delta = \mu_X - \mu_Y$	$\hat{\delta} = \bar{X} - \bar{Y}$	$\frac{(\bar{X}-\bar{Y})-(\mu_X-\mu_Y)}{\sqrt{S_p^2 \cdot \left[\frac{1}{n}+\frac{1}{m}\right]}}$	$(\bar{X}-\bar{Y}) \pm t_{\alpha/2} \times \sqrt{S_p^2 \cdot \left[\frac{1}{n}+\frac{1}{m}\right]}$	$\delta = \delta_0$	$\frac{(\bar{X}-\bar{Y})-(\delta_0)}{\sqrt{S_p^2 \cdot \left[\frac{1}{n}+\frac{1}{m}\right]}}$	$T \sim t(n+m-2)$
	$\Delta = \frac{\sigma_Y^2}{\sigma_X^2}$	$\hat{\Delta} = \frac{S_Y^2}{S_X^2}$	$\frac{S_X^2}{S_Y^2} \times \frac{\sigma_Y^2}{\sigma_X^2}$	$\left[\frac{S_Y^2}{S_X^2} \cdot f_{1-\alpha/2}, \frac{S_Y^2}{S_X^2} \cdot f_{\alpha/2}\right]$	$\Delta = \Delta_0$	$\frac{S_X^2}{S_Y^2} \times \Delta_0$	$F \sim F(n-1, m-1)$
Many-Sample	.	.	.	.	$\forall k, \mu_k = \mu_0$	$\frac{\frac{1}{K}\sum_{j=1}^{K} n_j \cdot \left[\bar{X}_j - \bar{X}\right]^2}{\frac{1}{N-K}\sum_{j=1}^{k}(n_j-1)S_j^2}$	$F \sim F(K, N-K)$