

## Worksheet 20 (Solutions)

1. Find the Jeffreys prior for estimating the mean of a normal distribution with a known variance  $\sigma^2$ . You can assume we have only one observation  $X$ . What is the corresponding Bayesian point estimator and how does it compare to the MLE?

*Solution:* We have, from the results last time, the following:

$$\sqrt{\mathcal{I}(p)} = \sqrt{\sigma^{-2}} \propto 1.$$

So, this is the improper distribution of a uniform distribution over the entire real line. Therefore, the Bayesian estimator is just equal to the MLE.

2. Find the Jeffreys prior for estimating the parameter  $p$  from a Binomial with a known value  $n$ . What is the corresponding Bayesian point estimator? What does this mean in the case when  $n = 1$  and  $X = 0$  and in the case when  $n = 1$  and  $X = 1$ ?

*Solution:* We have, from the results last time, the following:

$$\sqrt{\mathcal{I}(p)} = \sqrt{p(1-p)} = p^{1/2}(1-p)^{1/2} = p^{1-1/2}(1-p)^{1-1/2}.$$

This is proportional to a  $Beta(1/2, 1/2)$ . Therefore, from our previous notes, the point estimator is:

$$\hat{p}_{Bayes} = \frac{X + 1/2}{1/2 + 1/2 + n} = \frac{X + 1/2}{1 + n}.$$

So, when  $n = 1$  and  $X = 0$  we guess  $p = 0.25$ , and when  $X = 1$  we guess  $p = 0.75$ .

3. Find the Jeffreys prior for estimating the parameter  $\lambda$  from a Poisson. Write down a formula that gives, up to a constant, the posterior distribution. Note that you will not be able to relate this to a known distribution on our chart.

*Solution:* We have, from the results last time, the following:

$$\sqrt{\mathcal{I}(\lambda)} = \sqrt{\lambda^{-1}} = \lambda^{-1/2}.$$

This implies that the posterior distribution is:

$$f(\lambda|x) \propto \frac{\lambda^{x-1/2} e^{-\lambda}}{x!}$$

This should be a proper distribution, but I am not aware of a way to analytically find the normalizing constant. I believe that numerical techniques such as Gibbs sampling are needed to find the Bayes point estimator.

4. The Fisher information for the geometric distribution is  $\mathcal{I}(p) = \frac{(1-p)}{p^2}$ . Find the Jeffreys prior for estimating the parameter  $p$  from a geometric distribution. What is, more-or-less, this distribution?<sup>1</sup> What is the corresponding Bayesian point estimator? Using previous results, you should be able to do this for a sample of size  $n$ .

*Solution:* We have, from the results last time, the following:

$$\sqrt{\mathcal{I}(p)} = \sqrt{\frac{(1-p)}{p^2}} = p^{-1}(1-p)^{-1/2}.$$

This is mathematically equivalent to the distribution of a  $Beta(0, 1/2)$ , however this is not a proper distribution when  $\alpha = 0$ . Not a problem though! We can still use our formula for the posterior that  $p|X \sim Beta(a + n, b + \sum_i x_i - n)$ . In this case, we have  $p|X \sim Beta(n, 1/2 + \sum_i x_i - n)$ , which gives a Bayesian point estimator of:

$$\hat{p} = \frac{n}{1/2 + \sum_i x_i}.$$

<sup>1</sup> It should line up with one of the results on the table, but the hyper-parameter is out of bounds. That's okay though. It just means we have an improper prior. All of the results still hold.