T03: Probability

MATH 2411 Applied Statistics

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Counting and Operations Principles

Counting and operations principles

1. **Event:**
    - A
    - A^c — the Complement of an event A;
    - $\emptyset$ — the Empty Set;
    - S — the whole Sample Space
2. **Intersection of Events:** $A \cap B$
3. **Union of Events:** $A \cup B$
4. **Associative Law:**
    - $A \cup (B \cup C) = (A \cup B) \cup C$
    - $A \cap (B \cap C) = (A \cap B) \cap C$
5. **Distributive Law:**
    - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
    - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Counting and operations principles

1. **De Morgan's Law:** $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$

2. For **any events** A, B, and C:

$$
\begin{aligned}
P(A \cup B) =& P(A) + P(B) - P(A \cap B) \\
P(A \cup B \cup C) =& P(A) + P(B) + P(C) \\
& - P(A \cap B) - P(B \cap C) - P(C \cap A) \\
& + P(A \cap B \cap C)
\end{aligned}
$$

   We can calculate the probability of $P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n)$ by induction (inclusion–exclusion principle).

3. If events A, B, and C are **mutually exclusive**, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C).$$

   How about the probability of the union of an event family, say, $P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n)$ which has mutually exclusive members?

Counting and operations principles

1. If A and B are collectively exhaustive, then $P(A \cup B) = 1$, i.e., one of them must occur.

2. **The Factorial Notation:** $n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$.
   While $0! = 1$ by convention.

1. Number of permutations of r different objects taken from a given set of size n. Denote by r-permutations of n.

$$P(n, r) = \frac{n!}{(n-r)!} = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

1. Number of combinations of r different objects taken from a given set of size n. Denote by r-combinations of n.

$$C(n, r) = \frac{n!}{(n-r)!r!} = \binom{n}{r}$$

Remark: [number of ordered samples: $P(n, r)$] = [number of unordered samples] $\times$ [number of ways to order each sample $r!$]

Problem 1

[Evaluate the probability by counting techniques first. Then use **R** to mimic the random experiment, and get an "approximation" for the probability.]

If 3 balls are "randomly drawn" from a bowl containing 6 white and 5 black balls, what is the probability that one of the balls is white and the other two black?

```r
set.seed(20240923)
# Use 0 to denote white and 1 to denote black
all_balls <- c(rep(0, 6), rep(1, 5)) # 6 white balls and 5 black balls
n_rep <- 1000 # repeat the experiment 1000 times
n_success <- 0 # count the number of successes (one of the balls is whi
for (i in 1:n_rep) {
  sample_balls <- sample(x = all_balls, size = 3) # randomly draw 3 bal
  # cat("The sample balls are: ", sample_balls, "\n")
  if(sum(sample_balls) == 2) { # indicates that one of the balls is whi
    n_success <- n_success + 1
  }
}
n_success / n_rep
```

[1] 0.36

Conditional Probability

Conditional probability

1. Let A and B be two events in a sample space S with $P(B) > 0$. The **conditional probability** of A given B is defined as:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

2. Let A and B be two events in a sample space S. A and B are called **independent** if $P(A \cap B) = P(A)P(B)$.

3. Let $B_1, B_2, \ldots, B_n$ be a partition (both **exhaustive** and **mutually exclusive**) of the sample space S such that $P(B_i) \neq 0$ for all B_i. Then for any event A:

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A \mid B_i)P(B_i) \quad \text{(Law of total probability)}$$

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{P(A)} = \frac{P(A \mid B_i)P(B_i)}{\sum_{j=1}^{n} P(A \mid B_j)P(B_j)} \quad \text{(Bayes' Theorem)}$$

Problem 2

Let A and B be two events in the sample space S such that $P(B) > 0$. We know that the conditional probability that A will occur given that B has occurred is defined to be

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Show that the conditional probability satisfies the following properties (which implies "conditional probability is also a probability"):

i) $P(A|B) \geq 0$, for any event A.
ii) $P(S|B) = 1$.
iii) For any sequence of mutually exclusive events $A_1, A_2, \cdots$,

$$P\left(\bigcup_{k=1}^{\infty} A_k \,\middle|\, B\right) = \sum_{k=1}^{\infty} P(A_k|B).$$

Problem 3

A man has five coins, two of which are double-headed (i.e., *both faces are "Heads")*, one is double-tailed (i.e., *both faces are "Tails"*), and two are normal (i.e., *the coin has one "Head" and one "Tail"*). Consider the following scenarios successively, and answer each question.

i) The man shuts his eyes, picks a coin at random, and tosses it. What is the probability that the lower face of the coin is a head?

ii) The man then opens his eyes and sees that the coin is showing a head. What is the probability that the lower face is a head?

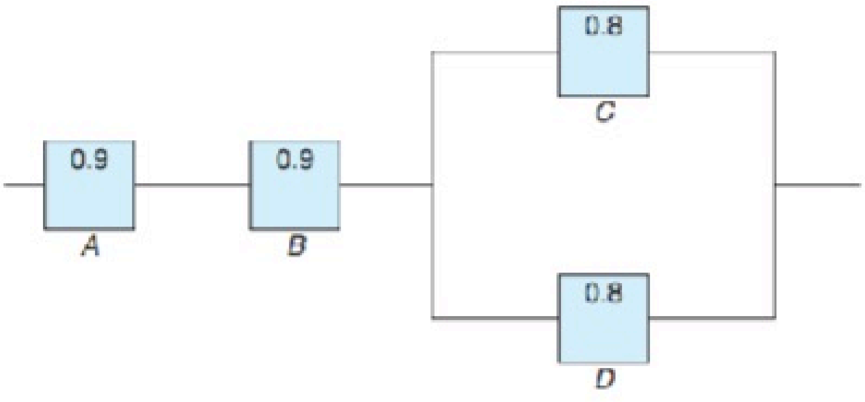
iii) The man shuts his eyes again, and tosses the coin again. What is the probability that the lower face is a head?

iv) The man opens his eyes and sees that the coin is showing a head. What is the probability that the lower face is a head?

Problem 4

An electrical system consists of four components as illustrated in the figure below. The reliability (probability of working) of each component is also shown in the figure. Assume that the four components work independently.

i) Find the probability that the entire system works.
ii) Find the probability that the component C does not work, given that the entire system works.



Thank you!

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