T04: Random Variables

MATH 2411 Applied Statistics

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Two common types of r.v.'s

Discrete Random Variable

Let X be a discrete random variable:

- $p(x)$, the Probability Mass Function (PMF) of X, is the probability that event $X = x$ will occur for each x in the range of X, i.e., $p(x) = P(X = x)$.

- $F(x)$, the Cumulative Distribution Function (CDF) of X, is defined as $F(x) = P(X \leq x)$.

- $E(X)$, the Expectation of X, is defined as

$$E(X) = \sum_{x \in \mathrm{Range}(X)} [x \cdot P(x)]$$

- $\mathrm{Var}(X)$, the Variance of X, is defined as

$$\mathrm{Var}(X) = \sum_{x \in \mathrm{Range}(X)} [(x - E(X))^2 \cdot P(x)] = E((X - E(X))^2)$$

Properties of population mean and variance

When it exists, the mathematical expectation E satisfies the following properties: Suppose X, Y are random variables and a and b are two constants. Then

- $E(b) = b$
- $E(aX) = aE(X)$
- $E(aX + b) = aE(X) + b$
- $E(X + Y) = E(X) + E(Y)$

When it exists, the population variance satisfies the following properties: Suppose X and Y are random variables and a and b are two constants. Then

- $\mathrm{Var}(b) = 0$
- $\mathrm{Var}(aX) = a^2\mathrm{Var}(X)$
- $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$
- If X and Y are independent random variables, then $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.

Problem 1.1

Let W be a random variable giving the number of heads minus the number of tails in four tosses of a coin. List the elements of the sample space S for the four tosses of the coin and to each sample point assign a value w of W.

Solution Let 'H' denote head and 'T' denote tail. The sample space S and the random variable W are as follows:

S	W
HHHH	4
HHHT, HHTH, HTHH, THHH	2
HHTT, HTHT, HTTH, THHT, THTH, TTHH	0
HTTT, THTT, TTHT, TTTH	-2
TTTT	-4

Problem 1.2

A coin is flipped until 2 heads occur in succession. List only those elements of the sample space that require 6 or fewer tosses. Is this a discrete sample space? Explain.

Solution Let 'H' denote head and 'T' denote tail. The possible elements are:

- HH
- THH
- TTHH, HTHH
- HTTHH, TTTTH, THTHH
- TTTTHH, HTTTHH, THTTHH, TTHTHH, HTHTHH

The sample space is discrete since it contains finite elements.

Problem 1.3

Let X be a random variable with the following probability distribution:

x	**2**	**3**	**4**
$p(x)$	$\frac{4}{15}$	$\frac{4}{15}$	$\frac{7}{15}$

a. Find Expected Value $E(X)$

b. Find Variance $\text{Var}(X)$

c. Find Standard Deviation $\sigma(Y)$, where $Y = 2X - 1$ (Hint: $\sigma(X) = \sqrt{\text{Var}(X)}$)

d. Find the CDF of X and plot the graph of the CDF.

Problem 1.4

Given that the cdf (cumulative distribution function) of a discrete random variable X is

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{4}, & 1 \leq x < 3 \\ \frac{3}{4}, & 3 \leq x < 5 \\ 1, & x \geq 5 \end{cases}$$

i) Find the pmf (probability mass function) of X.

ii) Draw the graphs of cdf and pmf of X.

iii) Evaluate $P(X > 3)$ and $P(1.5 < X < 5)$.

iv) Let $Y = 3X - 1$, find the population mean and population variance of Y.

Problem 2.1

The probability that a patient recovers from a delicate heart operation is 0.8. What is the probability that

(a) exactly 2 of the next 3 patients who have this operation survive?

Solution

Let $P(A_i) = 0.8$ for $i = 1, 2, 3$. The probability that a patient does not survive is $P(A_i^c) = 0.2$.

The probability that exactly 2 out of 3 patients survive is:

$$\binom{3}{2} \cdot (0.8)^2 \cdot (0.2) = 3 \cdot 0.64 \cdot 0.2 = 0.384$$

(b) all of the next 3 patients who have this operation survive?

Solution

The probability that all 3 patients survive is:

$$(0.8)^3 = 0.512$$

Problem 2.2

There are 8 female students and 24 male students in MATH2411 Tutorial. For the next 5 weeks, every week after class one student will be chosen randomly to invite everybody for tea. Let X be the total number of female students chosen.

a. Find the probability that the 'paying gender' sequence will be FMMFM.

b. Find the probability that the 'paying gender' sequence will be MMFFM.

c. In how many ways 2 out of the 5 places can be chosen?

d. What is the probability that 2 out of the 5 weeks will be paid by female students?

e. What is the probability that no more than 1 week will be paid by female students?

f. What is the probability that at least 1 week will be paid by a male student?

g. What is the mean of X?

Thank you!

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