Assignment 1 MATH2411

- Complete ALL questions. Show ALL your working steps when you solve each question.
- Unless otherwise specified, numerical answers should be EITHER exact OR correct to 4 decimal places.
- For questions involving **R**, provide your **R** codes and a snapshot of the output. Handwritten codes are NOT accepted.
- **Submit your assignment (in pdf format) to Canvas before the deadline: 11:30pm on Oct 6, 2024 (Sun). NO late submission will be accepted.**

Q1: Descriptive Statistics

[Use the definitions in our course materials to answer Q1.]

Use your own Student ID number to create the data set in the following way:

Step 1: Get one-digit each time by splitting your Student ID;

Step 2: Read your ID from left to right, each time with two digits; If the two-digit number is in the form of "0x", then "x" will be taken;

Step 3: Put all the numbers together.

For example, for a student with ID Number 20812300, the final data set will be

2, 0, 8, 1, 2, 3, 0, 0, 20, 8, 81, 12, 23, 30, 0

i) Find the mean, median, sample standard deviation and IQR of the data.
ii) Are there any outliers in this data set? If yes, what are they?
iii) Construct a boxplot to present the data without using any software.

Q2: Probability

[Evaluate the probability by counting techniques first. Then use **R** to mimic the random experiment, and get an "approximation" for the probability.]

If 3 balls are "randomly drawn" from a bowl containing 6 white and 5 black balls, what is the probability that one of the balls is white and the other two black?

Q3: Probability

Let A and B be two events in the sample space S such that $P(B) > 0$. We know that the conditional probability that A will occur given that B has occurred is defined to be

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Show that the conditional probability satisfies the following properties (which implies "conditional probability is also a probability"):

i) $P(A|B) \geq 0$, for any event A.

ii) $P(S|B) = 1$.

iii) For any sequence of mutually exclusive events $A_1, A_2, \cdots$,

$$P\left(\bigcup_{k=1}^{\infty} A_k \middle| B\right) = \sum_{k=1}^{\infty} P(A_k|B).$$

Q4: Probability

A man has five coins, two of which are double-headed (i.e., *both faces are "Heads"),* one is double-tailed (i.e., *both faces are "Tails"*), and two are normal (i.e., *the coin has one "Head" and one "Tail"*). Consider the following scenarios successively, and answer each question.

i) The man shuts his eyes, picks a coin at random, and tosses it. What is the probability that the lower face of the coin is a head?

ii) The man then opens his eyes and sees that the coin is showing a head. What is the probability that the lower face is a head?

iii) The man shuts his eyes again, and tosses the coin again. What is the probability that the lower face is a head?

iv) The man opens his eyes and sees that the coin is showing a head. What is the probability that the lower face is a head?

Q5: Probability

If A, B, C are mutually independent, then is each of the following statements true or not? Justify your answer. (If true, prove it; If false, give a counterexample.)

i) $A \cap B$ and C are independent.

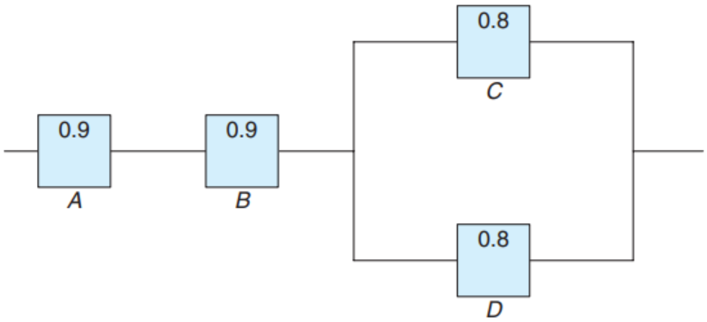
ii) $A \cup B$ and C are independent.

Q6: Probability

An electrical system consists of four components as illustrated in the figure below. The reliability (probability of working) of each component is also shown in the figure. Assume that the four components work independently.

i) Find the probability that the entire system works.

ii) Find the probability that the component C does not work, given that the entire system works.



Q1: i) $x = \{2, 0, 8, 1, 2, 3, 0, 0, 20, 8, 81, 12, 23, 30, 0\}$

mean: $\bar{x} = \frac{1}{15}\sum_{i=1}^{15} x_i = \frac{190}{15} \approx 12.67$

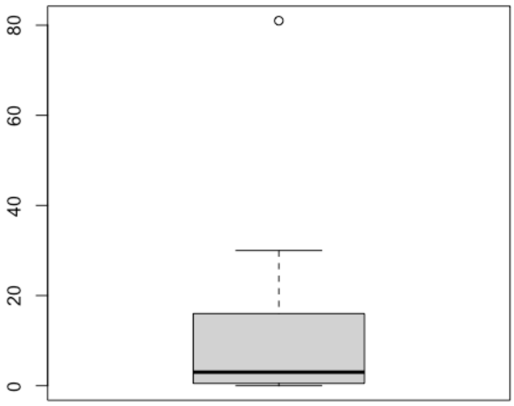
median: $\tilde{x} = x_{(8)} = 3$

sample sd: $S_{n-1} = \sqrt{\frac{1}{15-1}\sum_{i=1}^{15}(x_i - \bar{x})^2} = 21.17$

IQR: $IQR = Q_3 - Q_1 = 16 - 0.5 = 15.5$

ii) Yes. 81

iii)



Q2: $P = \frac{C(6,1)\,C(5,2)}{C(6+5,\,3)} \approx 0.36$

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```{r}
set.seed(20240923)
# Use 0 to denote white and 1 to denote black
all_balls <- c(rep(0, 6), rep(1, 5)) # 6 white balls and 5 black balls
n_rep <- 1000 # repeat the experiment 1000 times
n_success <- 0 # count the number of successes (one of the balls is white and
the other two black)
for (i in 1:n_rep) {
  sample_balls <- sample(x = all_balls, size = 3) # randomly draw 3 balls
  # cat("The sample balls are: ", sample_balls, "\n")
  if(sum(sample_balls) == 2) { # indicates that one of the balls is white and
the other two black
    n_success <- n_success + 1
  }
}
n_success / n_rep
```

[1] 0.36
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Q3 i) $P(A|B) = \dfrac{P(A\cap B)}{P(B)} \geq 0$

since $P(A\cap B) \geq 0$ and $P(B) > 0$

ii) $P(S|B) = \dfrac{P(S\cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

iii) $$P\left(\bigcup_{k=1}^{\infty} A_k \mid B\right) = \frac{P\left(\left(\bigcup_{k=1}^{\infty} A_k\right) \cap B\right)}{P(B)}$$

$$= \frac{P\left(\bigcup_{k=1}^{\infty}(A_k \cap B)\right)}{P(B)}$$

$$= \frac{\sum_{k=1}^{\infty} P(A_k \cap B)}{P(B)} = \sum_{k=1}^{\infty} P(A_k \mid B)$$

Q4: i) $P(\text{lower head}) = P(\text{lower head} \cap \text{double-headed})$
$+ P(\text{lower head} \cap \text{double-tailed})$
$+ P(\text{lower head} \cap \text{normal})$

$= P(\text{lower head} \mid \text{double-headed})\, P(\text{double-headed})$
$+ P(\text{lower head} \mid \text{double-tailed})\, P(\text{double-tailed})$
$+ P(\text{lower head} \mid \text{normal})\, P(\text{normal})$

$= 1 \times \frac{2}{5} + 0 \times \frac{1}{5} + \frac{1}{2} \times \frac{2}{5} = \frac{3}{5}$

ii) $P(\text{lower head} \mid \text{see head})$

$$= \frac{P(\text{lower head} \cap \text{see head})}{P(\text{see head})}$$

$$= P((\text{lower head} \cap \text{see head}) \cap \text{double-headed})$$
$$+ P((\text{lower head} \cap \text{see head}) \cap \text{double-tailed})$$
$$+ P((\text{lower head} \cap \text{see head}) \cap \text{normal})$$

$$= P(\text{lower head} \cap \text{see head} \mid \text{double-headed})\, P(\text{double-headed})$$
$$+ P(\text{lower head} \cap \text{see head} \mid \text{double-tailed})\, P(\text{double-tailed})$$
$$+ P(\text{lower head} \cap \text{see head} \mid \text{normal})\, P(\text{normal})$$

$$= P(\text{see head} \cap \text{double-headed})$$
$$+ P(\text{see head} \cap \text{double-tailed})$$
$$+ P(\text{see head} \cap \text{normal})$$
$$= P(\text{see head} \mid \text{double-headed})\, P(\text{double-headed})$$
$$+ P(\text{see head} \mid \text{double-tailed})\, P(\text{double-tailed})$$
$$+ P(\text{see head} \mid \text{normal})\, P(\text{normal})$$

$$= \frac{1 \times \frac{2}{5} + 0 \times \frac{1}{5} + 0 \times \frac{2}{5}}{1 \times \frac{2}{5} + 0 \times \frac{1}{5} + \frac{1}{2} \times \frac{2}{5}} = \frac{\frac{2}{5}}{\frac{3}{5}} = \frac{2}{3}$$

iii) $P(\text{lower head} \mid \text{(ii)})$ sees that the coin is showing a head

$$= \frac{P(\text{lower head} \cap \text{(ii)})}{P(\text{(ii)})}$$

$P(\text{(ii)}) = 1 \times \frac{2}{5} + 0 \times \frac{1}{5} + \frac{1}{2} \times \frac{2}{5}$ as in (ii)

$P(\text{lower head} \cap \text{(ii)} \cap \text{double-headed})$

$+ P(\text{lower head} \cap \text{(ii)} \cap \text{double-tailed})$

$+ P(\text{lower head} \cap \text{(ii)} \cap \text{normal})$

$= P(\text{lower head} \cap \text{(ii)} \mid \text{double-headed})\, P(\text{double-headed})$

$+ P(\text{lower head} \cap \text{(ii)} \mid \text{double-tailed})\, P(\text{double-tailed})$

$+ P(\text{lower head} \cap \text{(ii)} \mid \text{normal})\, P(\text{normal})$

$= 1 \times 1 \times \frac{2}{5} + 0 \times 0 \times \frac{1}{5} + \frac{1}{2} \times \frac{1}{2} \times \frac{2}{5} = \frac{1}{2}$

Conditional independence:

Given the coin type, the two tosses are independent

$$= \frac{\frac{1}{2}}{\frac{3}{5}} = \frac{5}{6}$$

iv) P(lower head | see head twice)

$$= \frac{P(\text{lower head} \cap \text{see head twice})}{P(\text{see head twice})}$$

$= P((\text{lower head} \cap \text{see head twice}) \cap \text{double-headed})$

$+ P((\text{lower head} \cap \text{see head twice}) \cap \text{double-tailed})$

$+ P((\text{lower head} \cap \text{see head twice}) \cap \text{normal})$

$= P(\text{lower head} \cap \text{see head twice} \mid \text{double-headed})\, P(\text{double-headed})$

$+ P(\text{lower head} \cap \text{see head twice} \mid \text{double-tailed})\, P(\text{double-tailed})$

$+ P(\text{lower head} \cap \text{see head twice} \mid \text{normal})\, P(\text{normal})$

$= P(\text{see head twice} \cap \text{double-headed})$

$+ P(\text{see head twice} \cap \text{double-tailed})$

$+ P(\text{see head twice} \cap \text{normal})$

$= P(\text{see head twice} \mid \text{double-headed})\, P(\text{double headed})$

$+ P(\text{see head twice} \mid \text{double-tailed})\, P(\text{double-tailed})$

$+ P(\text{see head twice} \mid \text{normal})\, P(\text{normal})$

$$= \frac{1 \times \frac{2}{5} + 0 \times \frac{1}{5} + 0 \times \frac{2}{5}}{1 \times \frac{2}{5} + 0 \times \frac{1}{5} + \left(\frac{1}{2}\right)^2 \times \frac{2}{5}} = \frac{\frac{2}{5}}{\frac{1}{2}} = \frac{4}{5}$$

Q5. i) True. $P((A\cap B)\cap C) = P(A\cap B\cap C)$

$= P(A)P(B)P(C)$

$= (P(A)P(B))P(C)$

$= P(A\cap B)P(C)$

ii) True. $P((A\cup B)\cap C) = P((A\cap C)\cup(B\cap C))$

$= P(A\cap C) + P(B\cap C) - P(A\cap B\cap C)$

$= P(A)P(C) + P(B)P(C) - P(A)P(B)P(C)$

$= (P(A) + P(B) - P(A)P(B))P(C)$

$= P(A\cup B)P(C)$

Q6. i) $P(\text{entire } \checkmark) = P(A\cap B\cap (C \text{ or } D))$

$= P(A)P(B)(1 - P(C^c\cap D^c))$

$= 0.9\times 0.9 \times (1 - 0.2\times 0.2)$

$= 0.7776$

ii) $P(C\times \mid \text{entire } \checkmark)$

$$= \frac{P(C\times \cap \text{entire } \checkmark)}{P(\text{entire } \checkmark)}$$ $= P(A)P(B)P(C^c)P(D)$

$$= \frac{0.9\times 0.9\times 0.2\times 0.8}{0.9\times 0.9\times (1-0.2\times 0.2)} \approx 0.1667$$