# T04: Outliers and $l_1$ loss

## MATH 4432 Statistical Machine Learning

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Let's start by recalling linear regression!

# Least squares

Recall the least squares (LS) problem for linear regression

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} \sum_{i=1}^{N} (y_i - x_i\beta)^2 = \underset{\beta}{\operatorname{argmin}} \frac{1}{N} \sum_{i=1}^{N} (y_i - x_i\beta)^2.$$

A toy example | Code | Minimize the squared loss

- Suppose we know the ground truth of $f(\cdot) : f(x) = x$
- Now given $\{x_i\}_{i=1}^N$, we have a set of observations $\mathcal{D} = \{(x_i, y_i)\}_{i=1}^N$ with

$$y_i = f(x_i) + \epsilon_i,$$

where $\epsilon_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 0.1^2)$ is random noise.

# Least squares

Recall the least squares (LS) problem for linear regression

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} \sum_{i=1}^{N} (y_i - x_i\beta)^2 = \underset{\beta}{\operatorname{argmin}} \frac{1}{N} \sum_{i=1}^{N} (y_i - x_i\beta)^2.$$

A toy example | **Code** | Minimize the squared loss

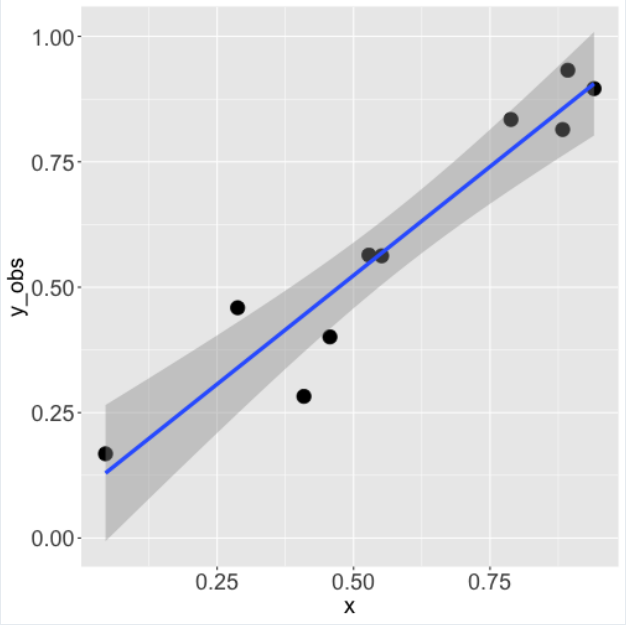
```
set.seed(123)
N <- 10 # Sample size
x <- runif(N, 0, 1)
y0 <- x # Ground truth
y_obs <- y0 + rnorm(N, mean = 0, sd = 0.1) # Add noise, observed data
ggplot(data = NULL, aes(x = x, y = y_obs)) +
  geom_point(aes(x = x, y = y_obs), size = 5) +
  geom_smooth(method = "lm", size = 1.5) +
  theme(
    text = element_text(size = 18),
    axis.text.y = element_text(size = 18),
    axis.text.x = element_text(size = 18)
    )
```

# Least squares

Recall the least squares (LS) problem for linear regression

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} \sum_{i=1}^{N} (y_i - x_i\beta)^2 = \underset{\beta}{\operatorname{argmin}} \frac{1}{N} \sum_{i=1}^{N} (y_i - x_i\beta)^2.$$

A toy example | Code | **Minimize the squared loss**



# What if there exists an outlier?

# Add an outlier

Add an outlier | Code | Minimize the squared loss

- We follow the above problem setting.
- But add an outlier

  $$(x, y, y_{\text{obs}}) = (0.9, 0.9, -2),$$

  of which the noise is extremely large. This situation is rare, but the probability is not zero!

# Add an outlier

Add an outlier | Code | Minimize the squared loss

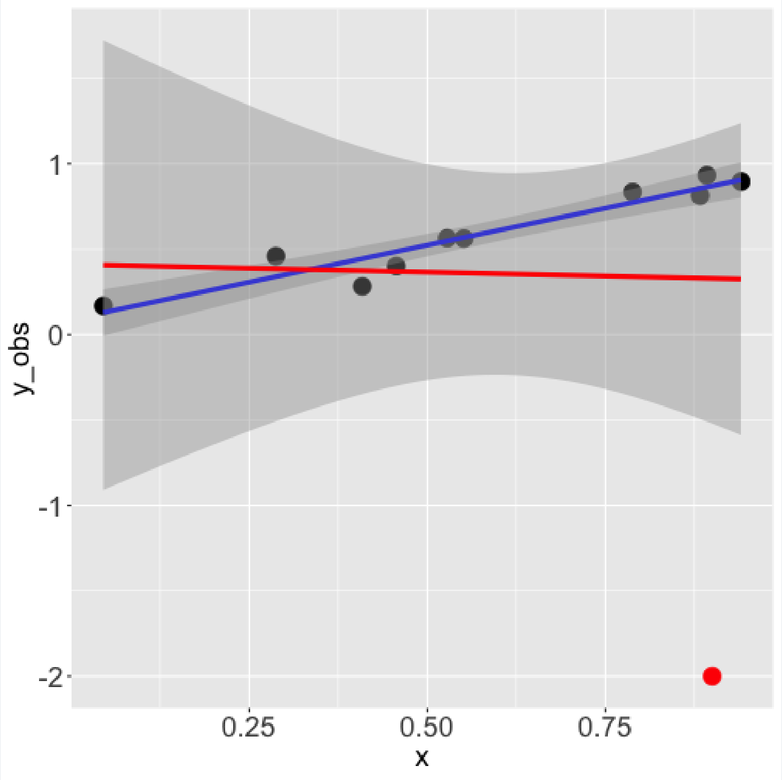
```
set.seed(123)

x_ol <- c(x, 0.9)
y0_ol <- c(y0, 0.9)
y_obs_ol <- c(y_obs, -2) # Add noise, observed data

ggplot(data = NULL) +
  geom_point(aes(x = x, y = y_obs), size = 5) +
  geom_point(aes(x = 0.9, y = -2), color = "red", size = 5) +
  geom_smooth(aes(x = x, y = y_obs), method = "lm", color = "blue", size
  geom_smooth(aes(x = x_ol, y = y_obs_ol), method = "lm", color = "red",
  theme(
    text = element_text(size = 18),
    axis.text.y = element_text(size = 18),
    axis.text.x = element_text(size = 18)
    )
```

# Add an outlier

Add an outlier | Code | **Minimize the squared loss**



# Brief summary of squared loss

Pros

- Natural, intuitive (Euclidean distance)
- Closed form solution

Con

- Not robust to outliers, equal weights to all data (assumes Gaussian distributed residual)

Let's consider a more robust loss function!

# $l_1$ loss

- $l_2$ norm and $l_1$ norm

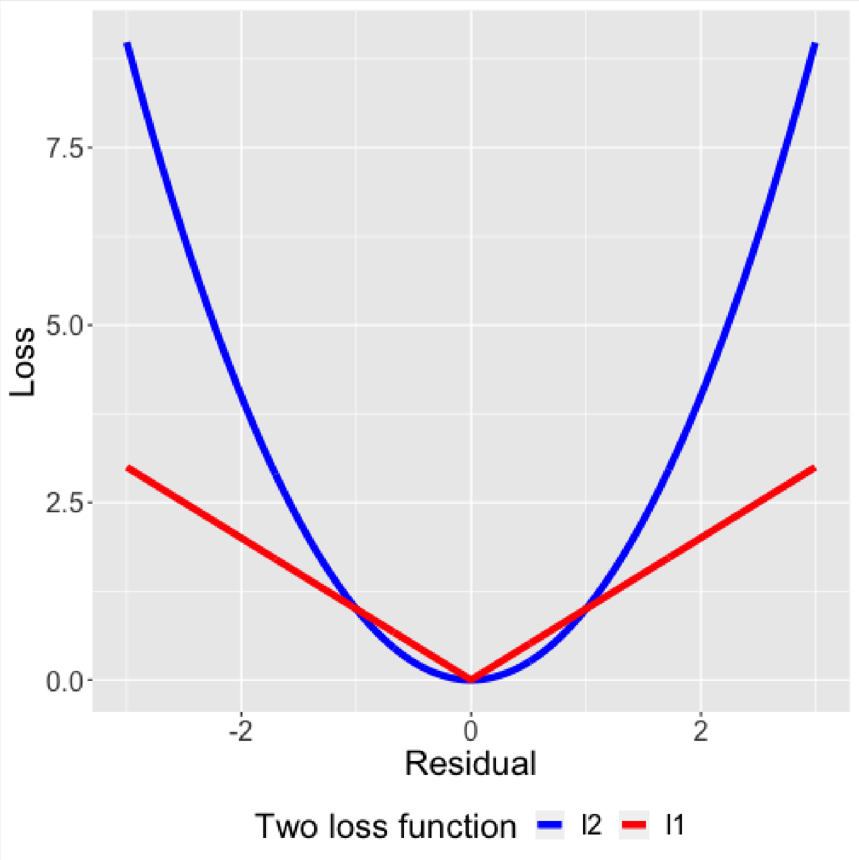
$$||x||_2 = \left(\sum_{j=1}^{p} x_j^2\right)^{\frac{1}{2}}, \quad ||x||_1 = \sum_{j=1}^{p} |x_j|.$$

- $l_2$ loss and $l_1$ loss

$$\mathcal{L}_2 = \sum_{i=1}^{N} (y_i - x_i\beta)^2 = ||y - X\beta||_2^2,$$

$$\mathcal{L}_1 = \sum_{i=1}^{N} |y_i - x_i\beta| = ||y - X\beta||_1.$$

# Why is $l_1$ more robust?




However, the bad news is that $l_1$ loss function is not differentiable :(

# Let's relax it!

# MM algorithm *

The "MM" stands for “Majorization-Minimization” or “Minorization-Maximization”.
In the following "MM" refers to "Majorization-Minimization".

- Consider the following presumably difficult optimization problem

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \; & f(\mathbf{x}) \\ \text{subject to} \; & \mathbf{x} \in \mathcal{X}, \end{aligned}$$

  with $\mathcal{X}$ being the feasible set and $f(\mathbf{x})$ being continuous.
- Idea: successively minimize a more managable surrogate function $u\left(\mathbf{x}, \mathbf{x}^k\right)$

$$\mathbf{x}^{k+1} = \arg\min_{\mathbf{x} \in \mathcal{X}} u\left(\mathbf{x}, \mathbf{x}^k\right),$$

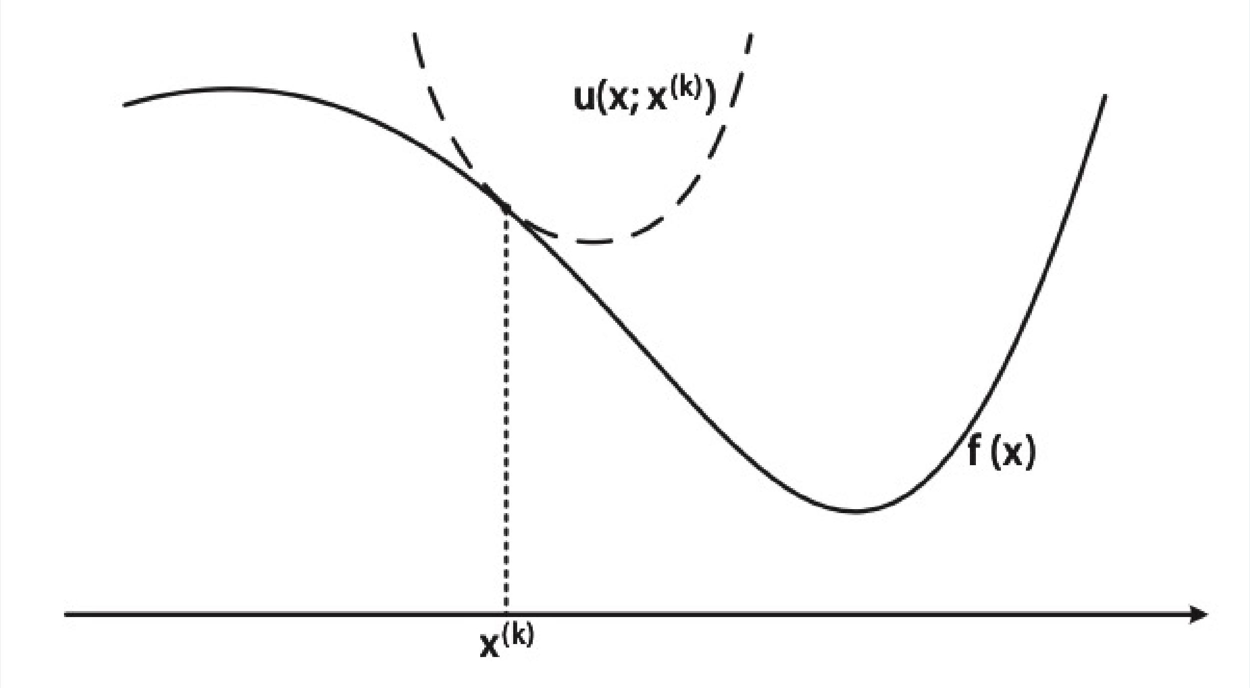
  hoping the sequence of minimizers $\left\{\mathbf{x}^k\right\}$ will converge to optimal $\mathbf{x}^\star$.

[*] Not required in this course. Materials are form ELEC 5470 / IEDA 6100A Convex Optimization, Prof. Daniel P. Palomar, ECE, HKUST.

# Iterative algorithm

Suppose $x_0$ is the initial point, in $k$-th step, we want $x_{k-1} \to x_k$.

After k-th step | (k+1)-th step | (k+2)-th step



# Iterative algorithm

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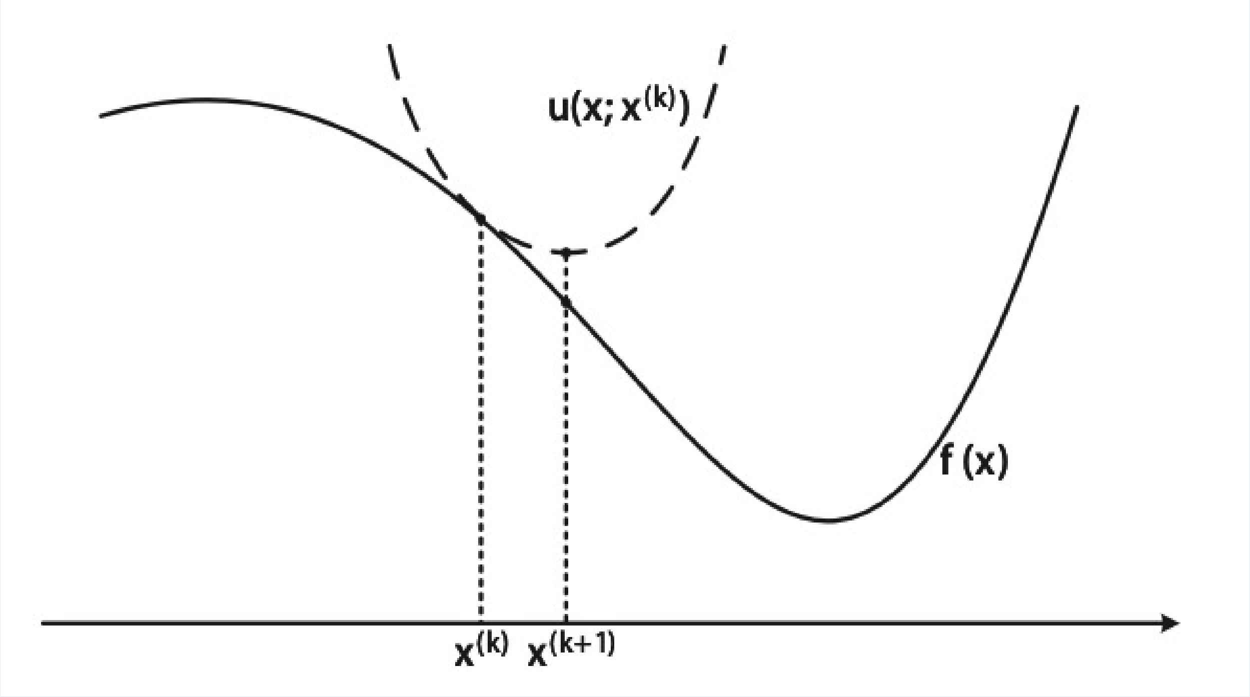

# Iterative algorithm

Suppose $x_0$ is the initial point, in $k$-th step, we want $x_{k-1} \to x_k$.

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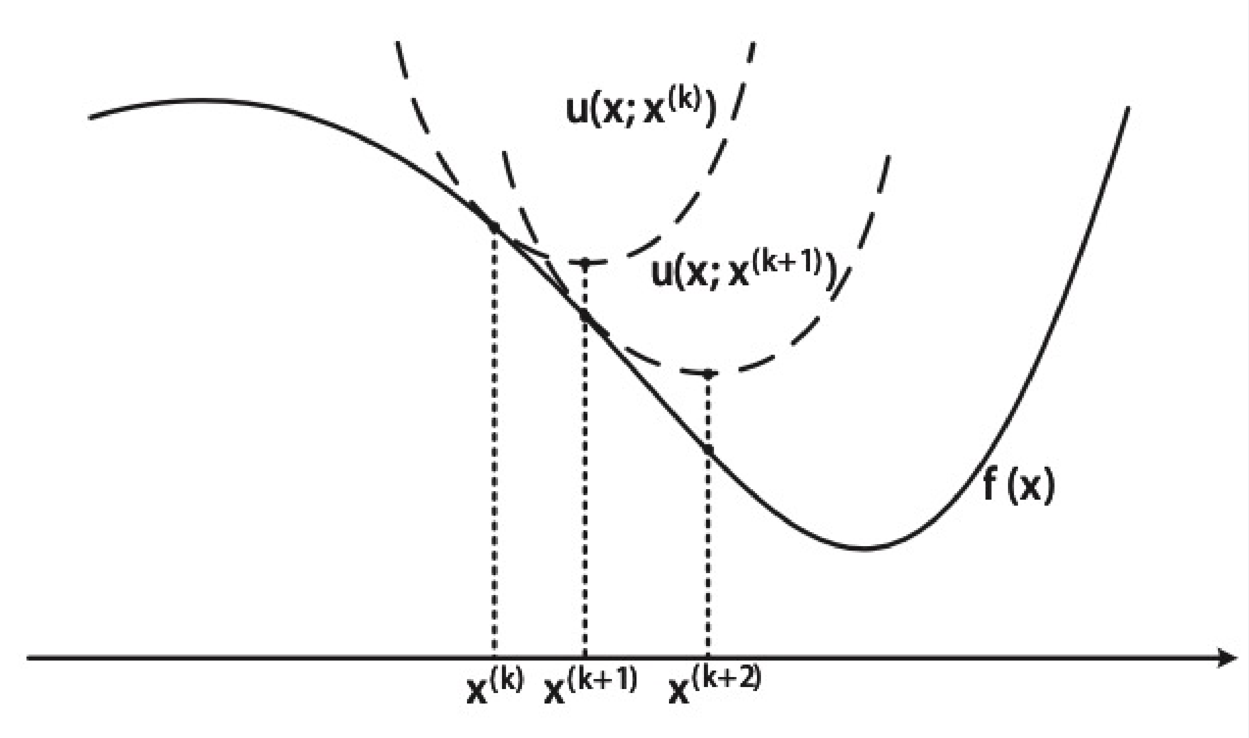
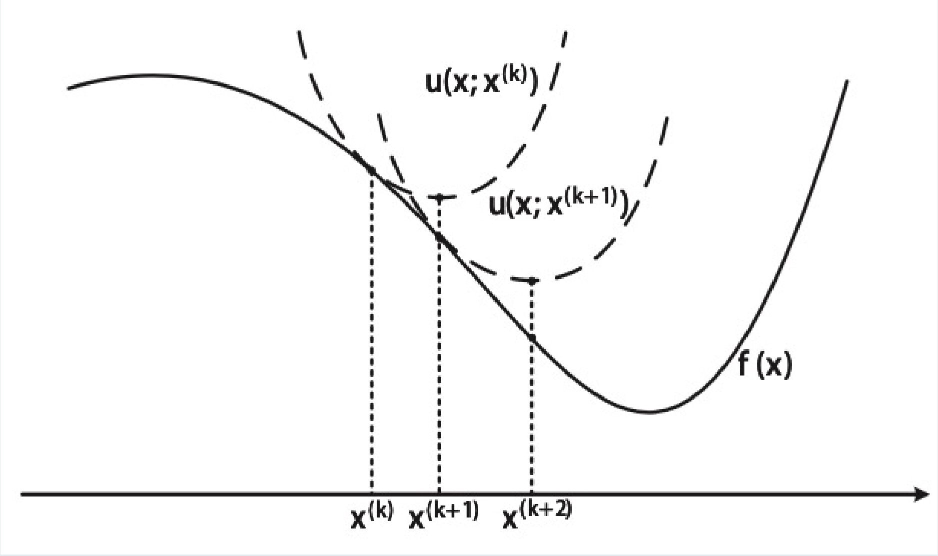


# Construction rule of the surrogate / majorizer function *

$$u(\mathbf{y},\mathbf{y}) = f(\mathbf{y}), \forall \mathbf{y} \in \mathcal{X}$$
$$u(\mathbf{x},\mathbf{y}) \geq f(\mathbf{x}), \forall \mathbf{x},\mathbf{y} \in \mathcal{X}$$
$$u'(\mathbf{x},\mathbf{y};\mathbf{d})\big|_{\mathbf{x}=\mathbf{y}} = f'(\mathbf{y};\mathbf{d}), \forall \mathbf{d} \text{ with } \mathbf{y}+\mathbf{d} \in \mathcal{X}$$
$$u(\mathbf{x},\mathbf{y}) \text{ is continuous in } \mathbf{x} \text{ and } \mathbf{y}$$



Question: how to construct $u\left(\mathbf{x}, \mathbf{x}^k\right)$ ?

Answer: that's more like an art :)

Luckily, the MM algorithm for $l_1$-norm minimization has been well established!

# Majorizer for $l_1$-norm

- Consider the following quadratic majorizer of $f(t) = |t|$ for $t \neq 0$ (for simplicity we ignore this case)

$$u\left(t, t^k\right) = \frac{1}{2\left|t^k\right|}\left(t^2 + \left(t^k\right)^2\right).$$

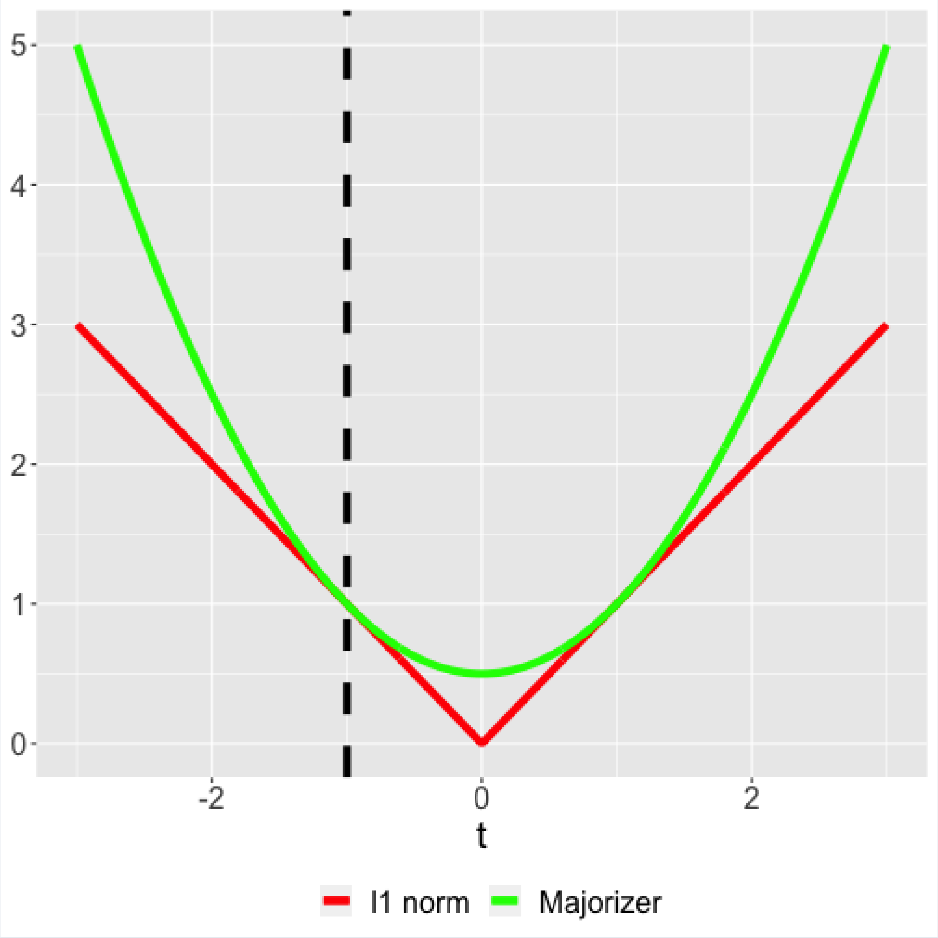
- It is a valid majorizer since it is continuous, and

$$u\left(t, t^k\right) \geq f(t),$$

$$u\left(t^k, t^k\right) = f(t),$$

$$\frac{d}{dt}u\left(t^k, t^k\right) = \frac{d}{dt}f\left(t^k\right).$$

# Majorizer for $l_1$-norm



# Reweighted LS for $l_1$-norm minimization

- Now we can apply it to the $\ell_1$-norm: a quadratic majorizer of $f(\beta) = \|X\beta - y\|_1$ is

$$u\left(\beta, \beta^k\right) = \sum_{i=1}^{N} \frac{1}{2\left|[X\beta^k - y]_i\right|}\left([X\beta - y]_i^2 + \left(\left[X\beta^k - y\right]_i\right)^2\right)$$

- Now that we have the majorizer, we can write the MM iterative algorithm for $k = 0, 1, \ldots$ as

$$\underset{\beta}{\text{minimize}}\, ||(X\beta - y) \odot w^k||_2^2$$

where $w_i^k = \sqrt{\frac{1}{2\left|[X\beta^k - y]_i\right|}}$.

# Algorithm

- Set $k = 0$ and initialize with a feasible point $\beta^0$
- **repeat**
    - $w_i^k \leftarrow \sqrt{\frac{1}{2\left|[X\beta^k - y]_i\right|}}$
    - Update $\beta^{k+1} \leftarrow \underset{\beta}{\operatorname{argmin}} \left\|(X\beta - y) \odot w^k\right\|_2^2$
    - $k \leftarrow k + 1$
- **until** convergence
- **return** $\beta^k$

# Implementation

Initialize | Call lm() | Call geom_smooth() | ggplot

```
df <- data.frame(y = y_obs_ol, x = x_ol, weight = rep(0, length(x_ol)))
#beta_0 <- rnorm(1) # Initialize intercept
#beta_1 <- rnorm(1) # Initialize slope
beta_0 <- -3 # To demostrate, we fit beta_0 = -3
beta_1 <- 1 # To demostrate, we fit beta_1 = 1
```

# Implementation

Initialize | Call lm() | Call geom_smooth() | ggplot

```
opar <- par()
par(mfrow = c(2, 2))

for (k in 0:3) { # Here we only repeat 4 times
  fit_l1 <- lm(y ~ x, data = df, weights = sqrt(1 / 2 / abs(beta_0 + x *

  beta_0 <- coef(fit_l1)[1] # Update intercept
  beta_1 <- coef(fit_l1)[2] # Update slope

  # Visualize
  plot(x_ol, y_obs_ol, pch = 16,
       xlab = "x", ylab = "y",
       cex = 1.5, cex.axis = 1.5, cex.lab = 1.5)
  abline(fit_l1, col = "blue", lwd = 3,
         cex = 1.5, cex.axis = 1.5, cex.lab = 1.5)
}
```

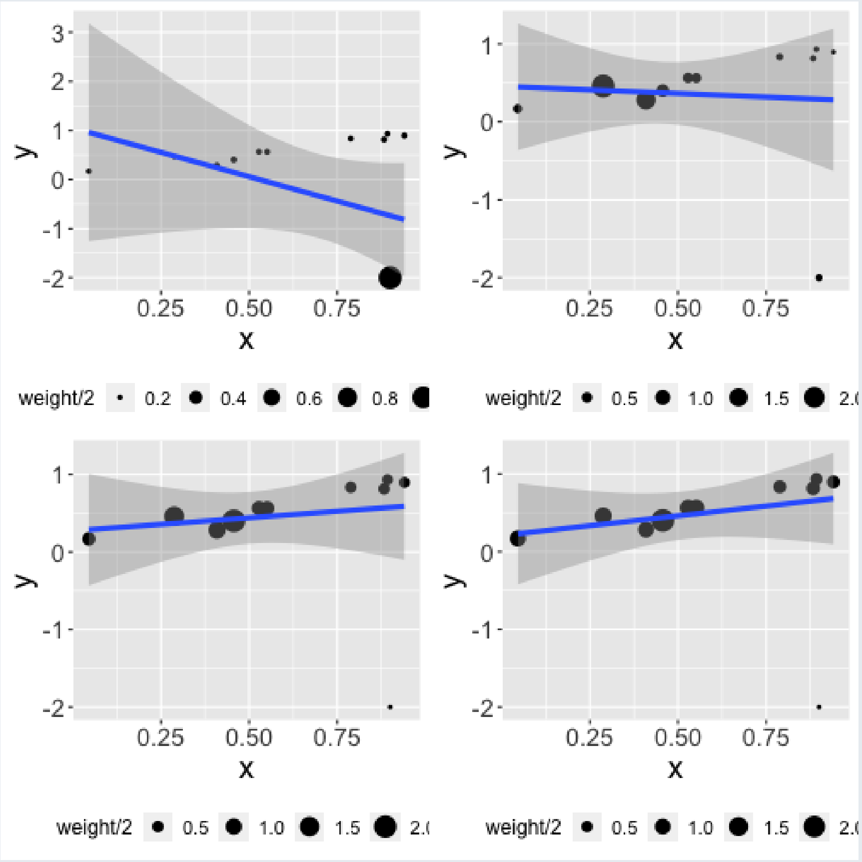
# Implementation

Initialize | Call lm() | **Call geom_smooth()** | ggplot

```
p_list <- list() # Figure list
for (k in c(0:3)) { # Here we only repeat 4 times
  df$weight <- sqrt(1 / 2 / abs(beta_0 + x_ol * beta_1 - y_obs_ol))
  fit_l1 <- lm(y ~ x, data = df, weights = weight) # Reweighted LS for
  beta_0 <- coef(fit_l1)[1] # Update intercept
  beta_1 <- coef(fit_l1)[2] # Update slope
  p <- ggplot(df, aes(x = x, y = y, size = weight / 2)) +
    geom_point(shape = 16) +
    geom_smooth(method = "lm", aes(weight = weight), size = 1.5, show.le
    theme(
      text = element_text(size = 18),
      legend.title = element_text(size = 12),
      legend.text = element_text(size = 11),
      legend.position = "bottom"
    )
  p_list <- c(p_list, list(p))
}
```

# Implementation

Initialize | Call lm() | Call geom_smooth() | **ggplot**



# Good night!

Slides created via Yihui Xie's R package **xaringan**.

Theme customized via Garrick Aden-Buie's R package **xaringanthemer**.

Tabbed panels created via Garrick Aden-Buie's R package **xaringanExtra**.

The chakra comes from remark.js, **knitr**, and R Markdown.