

# T02: Bias-variance trade-off

MATH 4432 Statistical Machine Learning

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Let's start by recalling what we  
have learned in class!

# Bias–variance decomposition of squared error

When fitting a model, we want to minimize

$$\mathbb{E}_{\mathcal{D}} \left[ \left( f(X) - \hat{f}(X; \mathcal{D}) \right)^2 \right]$$

w.r.t  $\hat{f}$ .

If we add and subtract  $\mathbb{E}_{\mathcal{D}} \left[ \hat{f}(X; \mathcal{D}) \right]$  inside the brackets, we have

$$\begin{aligned} & \mathbb{E}_{\mathcal{D}} \left[ \left( f(X) - \hat{f}(X; \mathcal{D}) \right)^2 \right] \\ &= \underbrace{\left( f(X) - \mathbb{E}_{\mathcal{D}} \left[ \hat{f}(X; \mathcal{D}) \right] \right)^2}_{\text{Bias}^2} + \underbrace{\mathbb{E}_{\mathcal{D}} \left[ \left( \mathbb{E}_{\mathcal{D}} \left[ \hat{f}(X; \mathcal{D}) \right] - \hat{f}(X; \mathcal{D}) \right)^2 \right]}_{\text{Variance}} \end{aligned}$$

# Two sources of error

- **Bias** from erroneous assumptions in the learning algorithm.

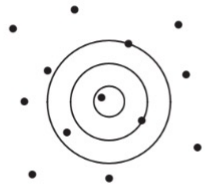
High bias  $\longrightarrow$  **underfitting**.

- **Variance** from sensitivity to small fluctuations in the training set.

High variance  $\longrightarrow$  **overfitting**.

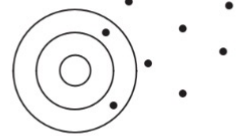


(a) High bias,  
Low variance  
(high precision)



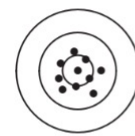
(b) Low bias,  
High variance  
(low precision)

Worst



(c) High bias,  
High variance  
(low precision)

Best



(d) Low bias  
Low variance  
(high precision)

# A toy example

# Problem setting

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Setting

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Code

Plot

Fit the model

Code

Plot

- Suppose we know the ground truth of  $f(\cdot) : f(x) = \sin(2\pi x)$
- Now given  $\{x_n\}_{n=1}^N$ , we have a set of observations  $\mathcal{D} = \{(x_n, y_n)\}_{n=1}^N$  with

$$y_n = f(x_n) + \epsilon_n,$$

where  $\epsilon_n \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$  is random noise.

# Problem setting

Setting

Code

Plot

Fit the model

Code

Plot

```
set.seed(123)

# Ground truth
x <- seq(0, 1, length.out = 100)
y <- sin(2 * pi * x)
# Observed data
N <- 50 # Sample size
X <- runif(N, 0, 1)
y0 <- sin(2 * pi * X)
y_obs <- y0 + rnorm(N, mean = 0, sd = 1) # Add noise

ggplot(data = NULL) +
  geom_line(aes(x = x, y = y), color = "red", size = 2) +
  geom_point(aes(x = X, y = y_obs), size = 5) +
  theme(
    text = element_text(size = 18),
    axis.text.y = element_text(size = 18),
    axis.text.x = element_text(size = 18)
  )
```

# Problem setting

Setting

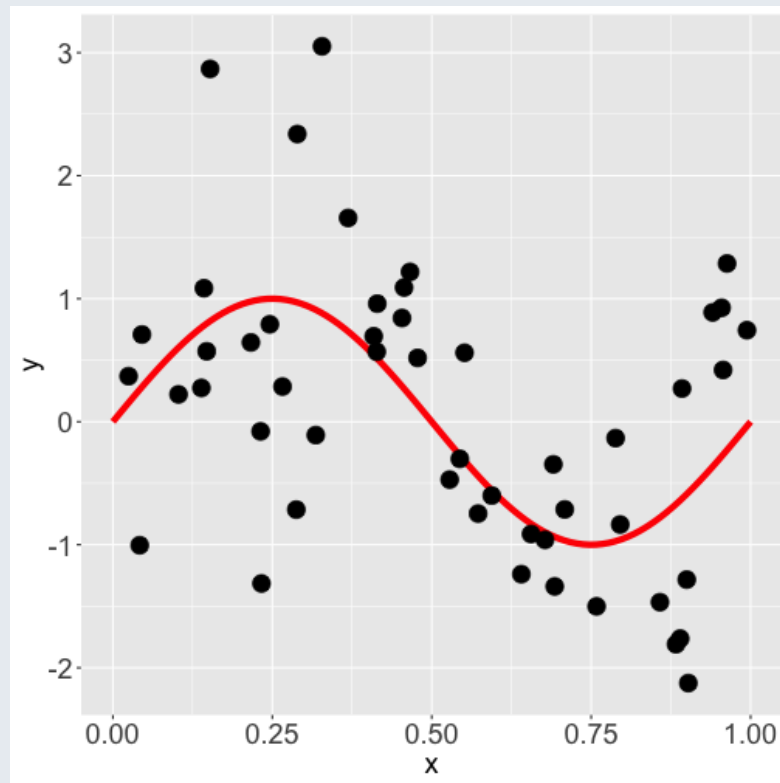
Code

**Plot**

Fit the model

Code

Plot





# Problem setting

Setting

Code

Plot

Fit the model

Code

Plot

For a given  $\mathcal{D}$ , we can fit a model  $\hat{f}(X; \mathcal{D})$ , e.g., linear regression.

# Problem setting

Setting

Code

Plot

Fit the model

Code

Plot

```
set.seed(123)

ggplot(data = NULL) +
  geom_line(aes(x = x, y = y), color = "red", size = 2) +
  geom_point(aes(x = X, y = y_obs), size = 5) +
  geom_smooth(aes(x = X, y = y_obs), method = "lm") + # Linear regression
  theme(legend.position = "none") +
  theme(
    text = element_text(size = 18),
    axis.text.y = element_text(size = 18),
    axis.text.x = element_text(size = 18)
  )
```

# Problem setting

Setting

Code

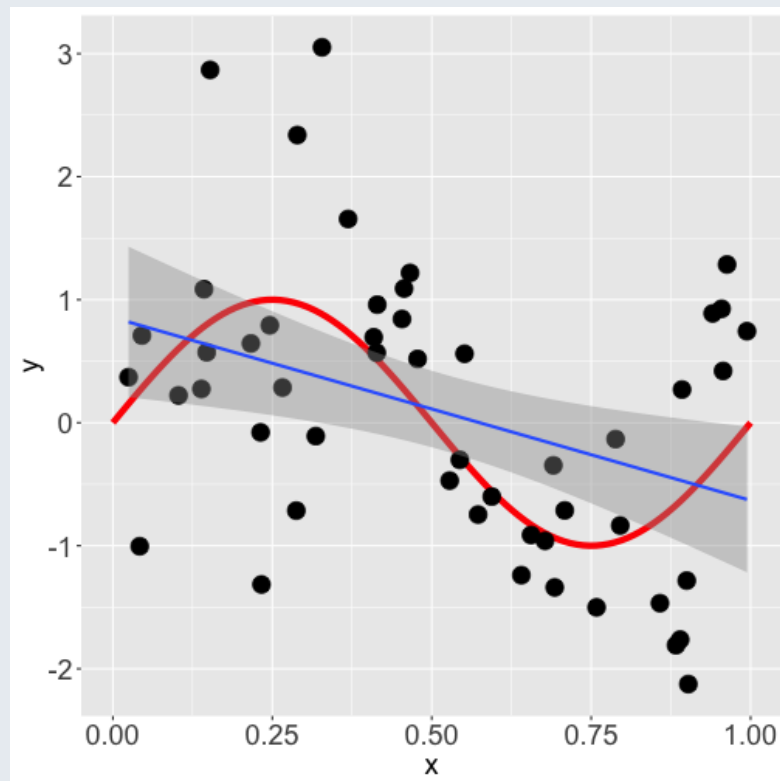
Plot

Fit the model

Code

Plot

```
#> `geom_smooth()` using formula 'y ~ x'
```



# Smoothing spline\*

Assume  $f(\cdot)$  is some unknown **smooth** function, to estimate  $f(\cdot)$ , a smoothing spline minimizes the penalized least squares functional

$$f_\lambda = \min_{f \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^n (y_i - f(x_i))^2 + \lambda J_m(f),$$

where  $J_m(f) = \int |f^{(m)}(z)|^2 dz$  is a penalty term that quantifies the lack of parsimony of the function estimate, and  $\lambda > 0$  is the smoothing parameter that controls the influence of the penalty.

Note that  $f^{(m)}(\cdot)$  denotes the  $m$ -th derivative of  $f(\cdot)$ , and  $\mathcal{H} = \{f : J_m(f) < \infty\}$  is the space of functions with square integrable  $m$ -th derivative.

[\*] See [Wikipedia](#) or [Smoothing Spline Regression in R](#) for more details if you are interested. However, this is not the point of this course!

# Smoothing parameter

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Smoothing parameter influence	Code	Plot
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- As  $\lambda \rightarrow 0$  the penalty has less influence on the penalized least squares functional. So, for very small values of  $\lambda$ , the function estimate  $f_\lambda$  essentially minimizes the residual sum of squares.
- As  $\lambda \rightarrow \infty$  the penalty has more influence the penalized least squares functional. So, for very large values of  $\lambda$ , the function estimate  $f_\lambda$  is essentially constrained to have a zero penalty, i.e.,  $J_m(f_\lambda) \approx 0$ .
- As  $\lambda$  increases from 0 to  $\infty$ , the function estimate  $f_\lambda$  is forced to be smoother with respect to the penalty functional  $J_m(\cdot)$ . The goal is to find the  $\lambda$  that produces the "correct" degree of smoothness for the function estimate.

# Smoothing parameter

Smoothing parameter influence

Code

Plot

```
library(ggformula)

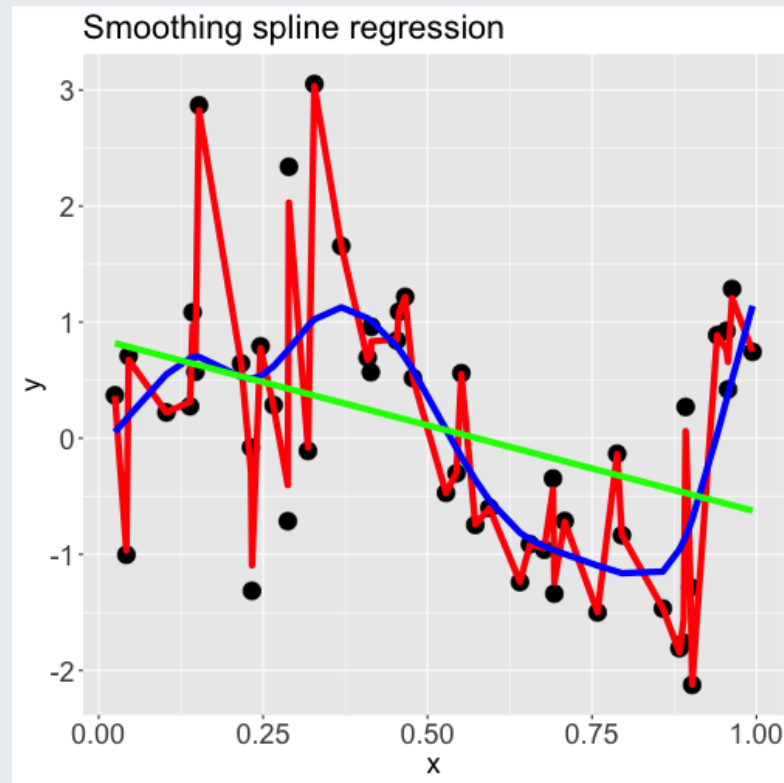
ggplot(data = NULL, aes(x = X, y = y_obs)) +
  geom_point(size = 5) +
  geom_spline(aes(x = X, y = y_obs), spar = 1e-2, colour = "red", size = 1) +
  geom_spline(aes(x = X, y = y_obs), cv = TRUE, colour = "blue", size = 1) +
  geom_spline(aes(x = X, y = y_obs), spar = 2, colour = "green", size = 1) +
  xlab("x") +
  ylab("y") +
  ggtitle("Smoothing spline regression") +
  theme(
    text = element_text(size = 18),
    axis.text.y = element_text(size = 18),
    axis.text.x = element_text(size = 18)
  )
```

# Smoothing parameter

Smoothing parameter influence

Code

Plot



# How do we evaluate the bias-variance trade-off in this example?

We need  $\mathbb{E}_{\mathcal{D}}(\hat{f}(X; \mathcal{D}))$ , therefore, we need to have multiple datasets.

- For  $l = 1, \dots, L$ ,
  - generate  $\mathcal{D}^l = \{(x_n, y_n^l)\}_{n=1}^N$ , where  $y_n^l = f(x_n) + \epsilon_n^l$
  - fit the  $l$ -th model  $\hat{f}(X^l; \mathcal{D}^l)$  and denote the predicted value as  $y^l(x_n) = \hat{f}(x_n; \mathcal{D}^l)$

**Note that  $f(x_n)$  is fixed across  $l$  while the observed values  $y_n^l$  are varying due to random noise  $\epsilon_n^l$ .**

- Estimate  $\mathbb{E}_{\mathcal{D}}[\hat{f}(X; \mathcal{D})]$  by  $\bar{y}(x) = \frac{1}{L} \sum_{l=1}^L y^l(x)$
- Compute squared bias:  $\frac{1}{N} \sum_{n=1}^N (\bar{y}(x_n) - f(x_n))^2$
- Compute variance:  $\frac{1}{N} \sum_{n=1}^N \frac{1}{L} \sum_{l=1}^L (y^l(x_n) - \bar{y}(x_n))^2$



# Experiments

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Experiments setting

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Implementation

We take the above example with  $N = 20$ ,  $L = 500$  and use smoothing spline regression with  $\lambda \in [1 \times 10^{-6}, 10]$ .

```
set.seed(123)

trial <- 500 # Number of experiment trials
N <- 20 # Number of samples for each trial
lambda_list <- exp(seq(log(1e-6), log(10), length.out = 100)) # Parameter list
model_list <- list() # Model list
biasSQ <- variance <- vector(mode = "numeric", length = length(lambda_list))

X <- runif(N, 0, 1) # Predictor
y0 <- sin(2 * pi * X) # True values of y
```

# Experiments

Experiments setting

Implementation

```
for(i in 1 : length(lambda_list)){
  model_list_i <- list()
  y_hat <- matrix(0, nrow = N, ncol = trial) # Predicted values

  for(j in 1 : trial){
    y <- y0 + rnorm(N, mean = 0, sd = 1) # Add noise
    fit_ss <- smooth.spline(x = X, y = y, lambda = lambda_list[i]) # Smooth
    model_list_i <- c(model_list_i, list(fit_ss)) # Save the model for trial j
    y_hat[, j] <- predict(fit_ss, X)$y # Predicted values
  }

  model_list <- c(model_list, list(model_list_i)) # Save the model list
  y_bar <- rowMeans(y_hat) # Mean of predicted values,  $E(\hat{f})$ 
  biasSQ[i] <- mean((y0 - y_bar)^2) # Bias square,  $E[(f - E(\hat{f}))^2]$ 
  variance[i] <- mean((y_hat - y_bar)^2) # Variance,  $E[(E(\hat{f}) - f)^2]$ 
}
```

# Visualization

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Bias and variance

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Code

Plot

Let's first take a look at how the two sources of error change as the parameter  $\lambda$  changes.

# Visualization

Bias and variance

Code

Plot

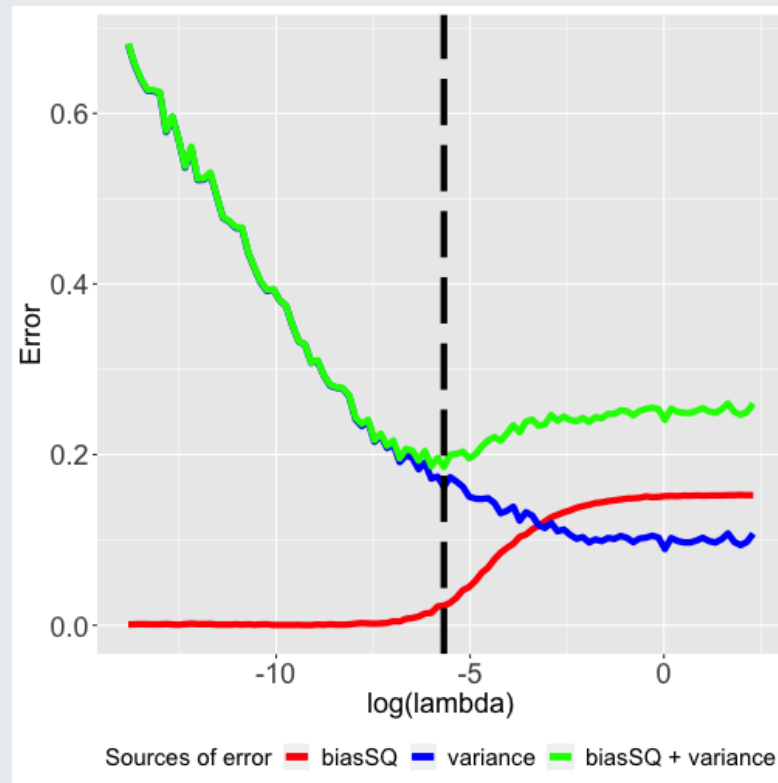
```
ggplot(data = NULL) +  
  geom_line(aes(x = log(lambda_list), y = biasSQ, color = "biasSQ"), size = 1.5) +  
  geom_line(aes(x = log(lambda_list), y = variance, color = "variance"), size = 1.5) +  
  geom_line(aes(x = log(lambda_list), y = biasSQ + variance, color = "biasSQ + variance"), size = 1.5) +  
  geom_vline(xintercept = log(lambda_list)[which.min(biasSQ + variance)], color = "black", size = 1.5) +  
  xlab("log(lambda)") +  
  ylab("Error") +  
  scale_color_manual(name = "Sources of error",  
                     breaks = c("biasSQ", "variance", "biasSQ + variance"),  
                     values = c("biasSQ" = "red", "variance" = "blue", "biasSQ + variance" = "black")) +  
  theme(  
    text = element_text(size = 18),  
    axis.text.y = element_text(size = 18),  
    axis.text.x = element_text(size = 18),  
    legend.title = element_text(size = 15),  
    legend.text = element_text(size = 15),  
    legend.position = "bottom"  
  )
```

# Visualization

Bias and variance

Code

Plot



# More details

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More details

Code

Plot

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Then we chose three different values for the parameter  $\lambda$  (too small, suitable, too large) and visualize for more performance details.

# More details

More details

Code

Plot

```
par(mfrow = c(3, 2))

N <- 100
lambda_idx_list <- c(1, which.min(biasSQ + variance), 100)
x <- seq(0, 1, length.out = N)
y0 <- sin(2 * pi * x)

for(lambda_idx in lambda_idx_list){
  plot(x, y0, col = "green", type = "l", ylim = c(-3, 3), ylab = "y", ce

  y_hat <- matrix(0, nrow = N, ncol = 20)
  for(j in 1 : 20){
    y_hat[, j] <- predict(model_list[[lambda_idx]][[j]], x)$y
    lines(x, y_hat[, j], col = "red", ylim = c(-3, 3))
  }

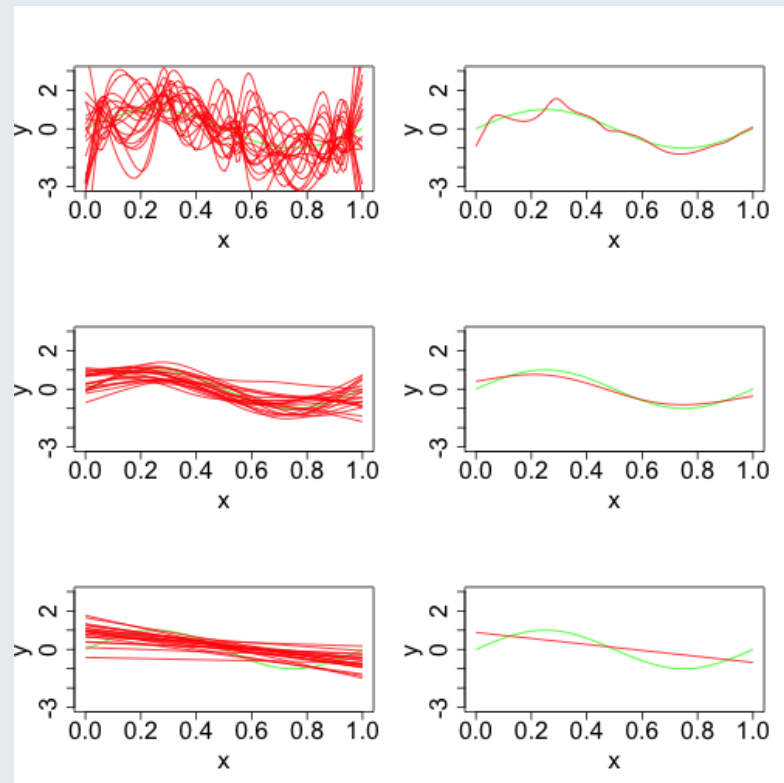
  plot(x, y0, col = "green", type = "l", ylim = c(-3, 3), ylab = "y", ce
  lines(x, rowMeans(y_hat), col = "red", ylim = c(-3, 3))
}
```

# More details

More details

Code

Plot





# Thank you!

Slides created via Yihui Xie's R package [xaringan](#).

Theme customized via Garrick Aden-Buie's R package [xaringanthemmer](#).

Tabbed panels created via Garrick Aden-Buie's R package [xaringanExtra](#).

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