Lab00-Proof

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2021.

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1. Prove that for any integer $n > 2$, there is a prime p satisfying $n < p < n!$. (Hint: consider a prime factor p of $n! - 1$ and prove by contradiction)

   Proof. If $n! - 1$ is a prime, there is a prime $p = n! - 1$ satisfying $n < p < n!$.

   If $n! - 1$ isn't a prime, we consider prime factors $\{p_n\}$ of $n! - 1$.($p_i \in N$, $1 < p_i < n!$)

   Supposing that $\exists p_i$, $p_i \leq n$.

   Then we have $p_i \mid n!$, which means $p_i \nmid n! - 1$. This contradicts the assumption that $p_i \mid n! - 1$.

   So $\forall p_i$,we have $n < p_i < n!$.

   To sum up, for any integer $n > 2$, there is a prime p satisfying $n < p < n!$.

   □

2. Use the minimal counterexample principle to prove that for any integer $n \geq 7$, there exists integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 2 + j_n \times 3$.

   Proof. If $P(n) = i_n \times 2 + j_n \times 3$ is not true for every $n \geq 7$, then there are values of n for which $P(n)$ is false, and there must be a smallest such value, say $n = k$.

   Since $P(7) = 2 \times 2 + 1 \times 3$ and $P(8) = 4 \times 2 + 0 \times 3$, we have $k \geq 9$ and $k - 2 \geq 7$.

   Since k is the smallest value for which $P(k)$ is false, $P(k-2)$ is true. Thus $\exists i_0, \exists j_0$ s.t. $k - 2 = i_0 \times 2 + j_0 \times 3$.

   However, we have

$$\begin{aligned} k = (k-2) + 2 &= i_0 \times 2 + j_0 \times 3 + 2 \\ &= (i_0 + 1) \times 2 + j_0 \times 3 \\ &= i_1 \times 2 + j_1 \times 3 \end{aligned}$$

   Thus, $\exists i_1 = i_0 + 1$ and $\exists j_1 = j_0$, s.t. $k = i_1 \times 2 + j_1 \times 3$. We have derived a contradiction, which allows us to conclude that our original assumption is false.

   To sum up, we can alaways find integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 2 + j_n \times 3$. □

3. Suppose the function f be defined on the natural numbers recursively as follows: $f(0) = 0$, $f(1) = 1$, and $f(n) = 5f(n-1) - 6f(n-2)$, for $n \geq 2$. Use the strong principle of mathematical induction to prove that for all $n \in N$, $f(n) = 3^n - 2^n$.

   Proof. For $n = 0$, $f(0) = 3^0 - 2^0 = 0$.

   For $n = 1$, $f(1) = 3^1 - 2^1 = 1$.

   Supposing $k \in N$, $k \geq 0$, $f(k) = 3^k - 2^k$, $f(k+1) = 3^{k+1} - 2^{k+1}$.

   By the condition, $f(k+2) = 5f(k+1) - 6f(k) = 3^k \times (5 \times 3 - 6) - 2^k(5 \times 2 - 6) = 3^{k+2} - 2^{k+2}$.

   According to the strong principle of mathematical induction, $\forall n \in N$, $f(n) = 3^n - 2^n$. □

4. An n-team basketball tournament consists of some set of $n \geq 2$ teams. Team p beats team q iff q does not beat p, for all teams $p \neq q$. A sequence of distinct teams p_1, p_2,..., p_k, such that team p_i beats team p_{i+1} for $1 \leq i < k$ is called a ranking of these teams. If also team p_k beats team p_1, the ranking is called a *k-cycle*.

   Prove by mathematical induction that in every tournament, either there is a "champion" team that beats every other team, or there is a 3-cycle.

   Proof. For $n = 2$, there are only two possible situations where team 1 beats team 2 or team 2 beats team 1. And in both cases there is a "champion" team.

   For $n = 3$, either one team defeats the other two team, or there is a 3-cycle.

   Supposing $k \in N$, either there is a "champion" team that beats every other team, or there is a 3-cycle.

   Case there is a 3-cycle:

   For $n = k + 1$, we can still find this 3-cycle after adding a new team.

   Case there is a "champion" team:

   For $n = k + 1$, if the new team beats every other team, it become a "champion" team.

   If the previous "champion" team beats the new team, the previous one goes on to be the champion.

   If the new team beats the previous "champion" team but is defeated by another team, the three teams form a 3-cycle.

   According to the strong principle of mathematical induction, $\forall k \in N$, either there is a "champion" team that beats every other team, or there is a 3-cycle.

   □

Remark: You need to include your .pdf and .tex files in your uploaded .rar or .zip file.