Interacting Particle Systems

Diploma Thesis Presentation

Stavros Birmpilis

National Technical University of Athens

July 7, 2017

Outline

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Interacting Particle Systems?

- Large-scale systems of components interacting with each other governed by stochastic dynamics
- Several applications on
  - natural sciences: reaction diffusion, gas particles systems...
  - social sciences: traffic flow, opinion dynamics, spread of epidemics...

Goal?

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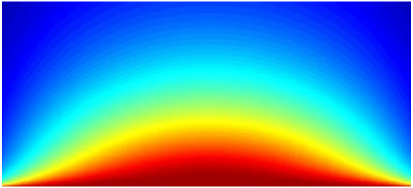
Deduce macroscopic behavior



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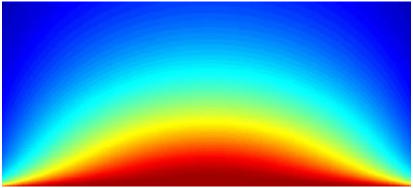
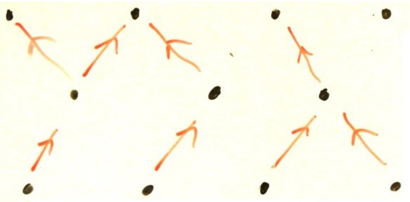


from microscopic interactions.



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- Markov Property: $\mathbb{P}^\eta(\eta_{t+s} \in A|\mathcal{F}_t) = \mathbb{P}^{\eta_t}(\eta_s \in A)$
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  "Memorylessness"
- Markov Chain: Markov Process on countable now X characterized by *transition rates* $c(\eta, \eta')$

Invariant Measures

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- Phase transitions, i.e. existence of more that one invariant measures, occur only in infinite systems.

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Space

- $\mathbb{Z}^d$ the d-dimensional integer lattice
- $\mathbb{T}_L = \mathbb{Z}_L = \{0, 1, ..., L-1\}$ the torus with L points and $\mathbb{T}_L^d = (\mathbb{T}_L)^d$
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- L represents the inverse of the distance between the points of $\mathbb{T}_L^d$, namely the particle sites
- Transition probability $p(x, y) = p(0, y-x) =: p(y-x)$ for some $p(\cdot)$ on $\mathbb{Z}^d$ (*elementary transition probability*)

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Thus, there is only one invariant measure.

Poisson Measures

Poisson distribution of parameter $\alpha > 0$ is the probability measure

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- under $\nu^L_{\rho(\cdot)}$ the random variables $(\eta(x) : x \in \mathbb{T}^d_L)$ must be independent,
- for every fixed site $x \in \mathbb{T}^d_L$,

$$\nu^L_{\rho(\cdot)}(\eta(x) = k) = p_{\rho(x),k}.$$

Poisson Measures

Theorem

If particles are initially distributed according to a Poisson measure associated to a constant function equal to α *then the distribution at time* t *is exactly the same Poisson measure* ν_α^L.

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Note that

$$E_{\nu_\alpha^L}(\eta(x)) = \sum_{k\geq 0} e^{-\alpha}\frac{\alpha^k}{k!}k = \alpha.$$

The Poisson measures are in this way naturally parametrized by the density of particles.

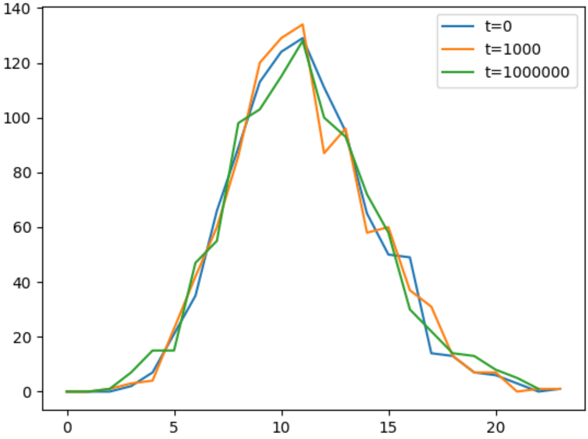

Figure: Distribution at time t of IRW on $\mathbb{T}^1_{1000}$ with $\alpha = 10$

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- Is that the same profile?

Macroscopic Profile

- We have different space scales, $\mathbb{T}^d$ and $L^{-1}\mathbb{T}_L^d$.
- Why not distinguish between two different time scales?
- A microscopic time t and a macroscopic time infinitely large with respect to t.

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- Let $m := \sum xp(x) \in \mathbb{R}^d$, the expectation of $p(\cdot)$.
- Introducing time scale tL:

$$\lim_{L\to\infty} p^L_{tL}([uL]) = \rho_0(u - mt) =: \rho(t, u)$$

- Now the profile did change. We observe a new macroscopic profile: the original one translated by mt.

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- Scaling limits $\rightarrow$ Hydrodynamic description

$$\partial_t \rho + m \cdot \nabla \rho = 0.$$

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- Scaling limits $\to$ Hydrodynamic description

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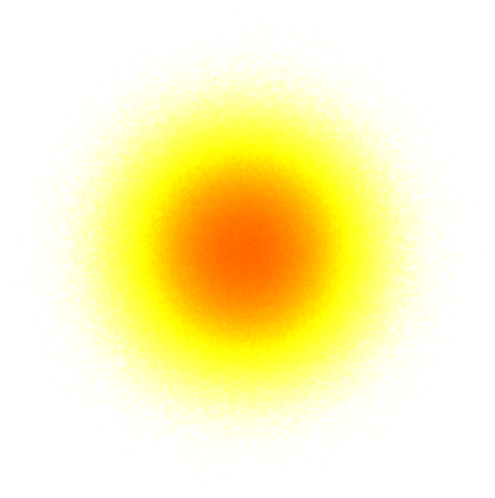
- However, if the random walk is not asymmetric, then again the profile remains the same.
- Still, if we consider a larger time scale, times of order L^2, even when $m = 0$, we can observe an interesting time evolution.



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Model

- First of all, SEP allows at most one particle per site.
- State space: $\{0,1\}^{\mathbb{T}_L^d}$
- $\eta^{x,y}$: configuration from η letting a particle jump from x to y

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if } z \neq x, y, \\ \eta(x) - 1 & \text{if } z = x, \\ \eta(y) + 1 & \text{if } z = y \end{cases}$$

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- Each particle in x jumps to y at rate

$$\eta(x)(1 - \eta(y))p(y - x)$$

Bernoulli Measures

Again, there is a unique invariant measure for the process.

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Let $(\nu_\alpha : 0 \leq \alpha \leq 1)$, the *Bernoulli measure* on $\{0,1\}^{\mathbb{T}_L^d}$ of parameter α, satisfying that:

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Again, the hydrodynamic description given by the heat equation

$$\partial_t p = (1/2)\Delta\rho$$

ASEP with step initial condition

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- Let $\gamma = q - p$.

Main quantity of interest: the position of the mth particle from the left at time t

$$x_m(t), \text{ with } x_m(0) = m.$$

Marginal particle

Behavior of $|x_1|$, i.e. the distance that the marginal particle has covered on a given time t?

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Assume that

$$E(|x_1(t)|) \sim ct^a,$$

which transforms to

$$\log E(|x_1(t)|) \sim a \log t + \log c.$$

Expected value

Performing experiments for $\gamma = 0.5$:

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It is true that, in general,

$$E(|x_1(t/\gamma)|) \sim t$$

for every $\gamma \in (0, 1]$.

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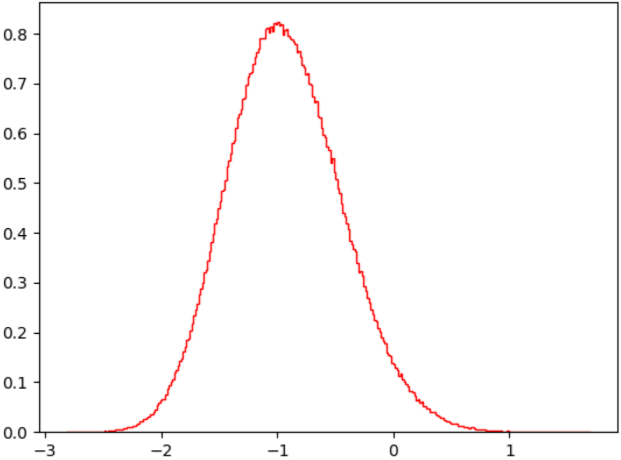
Therefore, what about

$$\frac{|x_1(t)| - \gamma t}{t^{0.6}}$$

?

Tracy-Widom distribution!



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So, distribution of the largest eigenvalue λ_{max}?

Appearance of the distribution

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- But... in 1999, the same distribution was found in the length of the longest increasing subsequence of random permutations.
- It started to appear in models all over physics and mathematics.
- Especially in systems with interacting components.

Universality

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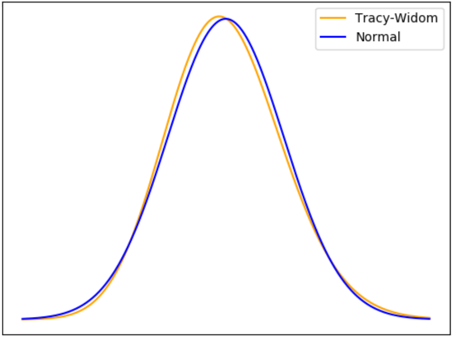
- Tracy-Widom complex cousin of the familiar bell curve...



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- Central Limit Theorem: natural observations and other uncorrelated variables form a Normal distribution. (rigorous about a century ago)
- Tracy-Widom from strongly correlated variables, such as interacting species, stock prices, matrix eigenvalues...
- Tracy-Widom universally proved to hold for certain classes of random matrices.
- Looser handle in counting problems, random walk problems, growth models...

Maybe an explanation?

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In general,

- left tail: all components act in concert, (unstable)
- right tail: the components act alone. (stable)

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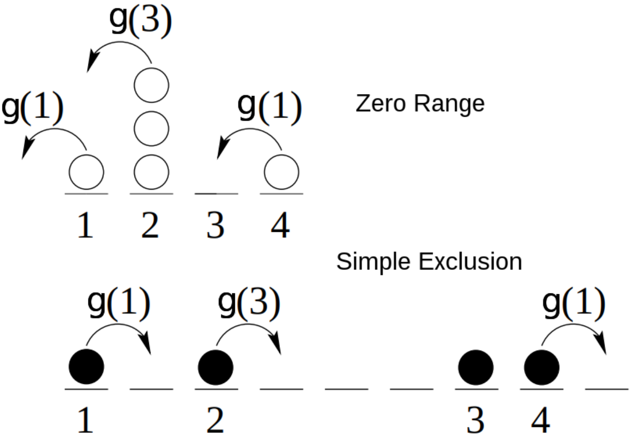
- Non-linear (parabolic) hydrodynamic description

$$\partial_t \rho = (1/2)\Delta(\Phi(\rho))$$

where $\Phi(\cdot)$ is the inverse of $\rho(\cdot)$ until a critical density ρ_c.

Duality



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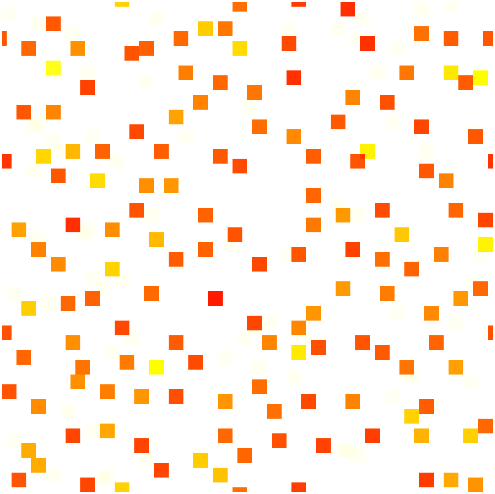
$$g(k) = \begin{cases} 0, & \text{if } k = 0 \\ 1 + \frac{b}{k}, & \text{if } k \geq 1 \end{cases}$$

- For $b > 2$, there is a critical density

$$\rho_c = \frac{1}{b-2}$$



Order of time needed to reach equilibrium?

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Performing experiments, for 3 initial states,

- totally asymmetric zero range process:

$$T_{eq} = \mathcal{O}(L^2)$$

- symmetric zero range process:

$$T_{eq} = \mathcal{O}(L^3)$$

(diffusion without a drift)

Tagged Particles

Three classes $\rightarrow$ a jump occurs from the tagged particle's site x,

- First class: the tagged will jump,
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Distance covered by the tagged particle in a given time:
(space scale L, time scale L^2)

- First class: $X_{tag}(tL^{-2})/L = \mathcal{O}(1)$,
- Random: $X_{tag}(tL^{-2})/L = \mathcal{O}(\sqrt{t})$,
- Second class: $X_{tag}(tL^{-2})/L = \mathcal{O}(\sqrt{t})$.

Thank you very much!