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Report of Visiting Student Program HRI, Allahabad

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Abstract
In this report I present my workings and results obtained during my visit with the QIC group in HRI, Allahabad. I covered the fundamentals of Quantum Information Theory including Ensemble Theory, Measurement Theory, Quantum Entanglement and the basics of its Resource Theory, Informational Entropy and Quantum Communication Protocols including Dense Coding and Quantum Teleportation. During the later part of my visit, I covered the workings of Periodically Many-Body Closed Quantum systems especially Quantum Batteries and calculated the work statistics of the same.

Keywords
Quantum Information Theory — Quantum Entanglement — Quantum Communication — Quantum Battery

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Contents

1. Ensemble Theory in Quantum Mechanics[1]

The main interest here are the open systems where, the system is not perfectly isolated. It can be understood using the approach of considering S as the open system and E as the environment which together form a Closed i.e. Isolated System.

1.1 Axioms of Quantum Mechanics

Axiom 1: States A state is a complete description of a physical system, which is represented using a ray in a Hilbert space. A ray is an equivalence class of vectors ($|\psi\rangle$) that differ by a non-zero scalar complex

1. Hilbert space is a vector space over the Complex numbers.
2. An inner product is defined in this Hilbert space which basically maps ordered pairs of vectors to the Complex numbers.
3. It is complete in its Norm which is defined as the square root of Inner product of a vector(ket) and its dual vector (bra).

Axiom 2: Observables An observable is a property of a physical system that can be measured. It is described by a Self-Adjoint Operator.*Spectral Theorem* states that given a an

Observable **A**:

$$\mathbf{A} = \sum_n a_n E_n \tag{1}$$

$$\mathbf{A} = \sum_n |n\rangle\, a_n \langle n| \tag{2}$$

Where a_n are the Eigenvalues of **A** and E_n are the Eigenvectors corresponding to those Eigenvalues.

Axiom 3: Measurement A measurement is a process of retrieval of information of a state of a physical system by an observer. The measurement *collapses* the state into a eigenstate of the observable **A** and the observer obtains the eigenvalue corresponding eigenstate. The probability of obtaining an eigenstate is:

$$Prob(a_n) = ||E_n|\psi\rangle||^2 = \langle\psi|E_n|\psi\rangle \tag{3}$$

Normalized state of system after measurement given a_n is the eigenvalue obtained:

$$\frac{E_n|\psi\rangle}{||E_n|\psi\rangle||} \tag{4}$$

Axiom 4: Dynamics Dynamics as per *Schrodinger's Picture* say that the evolution of a state with time, given the constraint of a closed system is described by a *Unitary Operator*.

$$\left|\psi(t')\right\rangle = U(t',t)\,|\psi(t)\rangle \tag{5}$$

Infinitesimal time evolution is governed by the *Time Dependant Schrodinger's Equation*.

$$\frac{d}{dt}|\psi(t)\rangle = -i\mathbf{H(t)}|\psi(t)\rangle \tag{6}$$

Axiom 5: Composite Systems Given two systems **A** and **B** with Hilbert spaces H_A and H_B respectively then the composite systems AB are represented by the tensor product $H_A \bigotimes H_B$ and the states of the composite system is represented by $|\psi\rangle_A \bigotimes |\phi\rangle_B$. The properties of Hilbert space i.e. Inner product and operators are valid for this composite system Hilbert space as well.

1.2 Symmetry in Quantum Mechanics

A symmetry is defined as a transformation that acts on a state but doesn't change any observable properties of the system. Intuitively this is understood as rotation of both the state and the apparatus. Hence, corresponding to every symmetry operation (rotation of the state) there exists a Unitary Transformation (rotation of the apparatus). Considering the commutativity of this symmetry transformation ($\mathbf{HU}(R) = \mathbf{U}(R)\mathbf{H}$) with Hamiltonian **H** one can derive a *Generator* of the symmetry **Q** which satisfies the conservation Law;

$$[\mathbf{Q},\mathbf{H}] = 0 \tag{7}$$

1.3 Density Operator

Now, if we consider Open-systems (**AB**) the above Axioms will not be valid and

- States will not be represented by Rays.
- Measurements will not be orthonormal Projections.
- Evolution is not Unitary.

Hence we need a better form of representation of the states for our system. This is done by using *Density operator*. Suppose we consider a general operator acting on the system **AB** (Technically on **A**) i.e.

$$M_A \bigotimes I_B \tag{8}$$

The expectation value for this operator comes out to be of the form;

$$< M_A >= |a|^2 \langle 0|M_A|0\rangle + |b|^2 \langle 1|M_A|1\rangle \tag{9}$$

$$< M_A >= tr(M_A \rho_A) \tag{10}$$

$$\rho_A = |a|^2 |0\rangle\langle 0| + |b|^2 |1\rangle\langle 1| \tag{11}$$

Given the above equations (10) and (11) we can see that this density operator can be used to represent an *Ensemble* of possible states with prob. of $|0\rangle$ as $|a|^2$ and prob. of $|1\rangle$ as $|b|^2$. This ρ_A is called the density operator corresponding to the subsystem **A**. Some important properties of this Density operator are:

1. It is Self-Adjoint i.e. $\rho_A = \rho_A^\dagger$
2. It is positive i.e. to ensure positive eigenvalues hence positive Probabilities.
3. $tr(\rho_A) = 1$ i.e. due the assumption of a normalized state.

The state of a subsystem is obtained by performing a partial trace over the system i.e. essentially taking the inner product with the states that are orthonormal to the state *over* which the trace is being considered. For a Bipartite system this basically returns the density matrix of the other subsystem. Now, considering a general state of our subsystem is represented by a density operator even if the state of the larger system (of which our system is a part of), our subsystem may not be a Ray. If a system is a ray it suggests that the system is *Pure* while if this is not true it can be said that our system is *Mixed*. General form of a density operator is (Eigenbasis of ρ_A) :

$$\rho_A = \sum_a p_a |a\rangle\langle a| \tag{12}$$

If there are 2 or more terms in this summation then it is a *Mixed* state and if only 1 term is present then it is a pure state. Another way to check for this *Mixedness* is using the $tr(\rho_A^2)$ which is equal to 1 for pure states due idempotent nature which is absent in mixed states.

Bloch Sphere: Since, the 3 Pauli Matrices along with the identity matrix together form the basis for 2×2 matrices we can write any Density Matrix of a single Qubit state in terms of these under some constraints.

$$\rho(\vec{P}) = \frac{1}{2}(I+\vec{P}.\vec{\sigma}) \tag{13}$$

The constraints on $\vec{P}$ are due the positive eigenvalues of ρ and $tr(\rho)=1$. This ensures that the $\det\rho \geq 0$ i.e. $\vec{P}^2 \leq 1$. This inequality $|\vec{P}| \leq 1$ inherently represents the *Bloch Sphere* and hence the states on the surface of the sphere ($\det\rho = 0$ i.e. $\vec{P}^2 = 1$) are *Pure* states while the the ones inside are *Mixed*.

ASSIGNMENT 1 : APPENDIX A

1.4 Schmidt Decomposition[2]

Given a Bipartite system, we can expand a **Pure state** in the following form which is known as Schmidt decomposition:

$$|\psi\rangle_{AB} = \sum_i \sqrt{p_i}\,|i\rangle_A \bigotimes |i'\rangle_B \tag{14}$$

Here this $\sqrt{p_i}$ are the common eigenvalues between the subsystems, while $|i\rangle_A$ and $|i'\rangle_B$ represent the orthonormal Schmidt Basis for the subsystems A and B respectively.
If the two subsystems' density operators have non-degenerate eigenvalues then one can directly pair the eigenstates of the two systems with same eigenvalues to get the suitable Schmidt decomposition. If the eigenvalues are degenerate then while keeping one subsystem in the same orthonormal basis form we have to convert the other by finding out the suitable orthonormal basis using the same method as the proof of the fact that *There exists a pair of orthonormal basis for H_A and H_B* such that eqn. (14) is satisfied.

Proof:

$$|\psi\rangle_{AB} = \sum_{i,\mu} \psi_{i\mu}\,|i\rangle_A \bigotimes |\mu\rangle_B = \sum_i |i\rangle_A \bigotimes |i''\rangle_B \tag{15}$$

Now to convert the above equation in the form of the Schmidt Decomposition, we say that the basis $|i''\rangle_B$ may not be orthonormal. Now the state of the subsystem obtained from partial trace of the density matrix obtained from the state in (15) should be equal to the state: $\rho_A = \sum_i p_i |i\rangle\langle i|$. Hence equating the same;

$$\rho_A = tr_B(|\psi\rangle\langle\psi|) \tag{16}$$

$$= tr_B\Big(\sum_{i,j} |i\rangle\langle j| \bigotimes |i''\rangle\langle j''|\Big) \tag{17}$$

$$= \sum \langle j''|i''\rangle (|i\rangle\langle j|). \tag{18}$$

Comparing eqn. (19) and ρ_A we can see that the basis $|i''\rangle_B$ also turn out to be orthogonal and hence can be normalized now.
Hence it can be shown that a state can be written in the Schmidt decomposition form by choosing the correct orthonormal basis for its subsystem and and ensuring the partial trace condition which automatically leads to Schmidt Decomposition.

Example: Give the Schmidt Decomposition of the following State;

$$|\psi\rangle_{AB} = |00\rangle + |01\rangle + |02\rangle + |12\rangle \tag{19}$$

Now, considering the entire system AB we say that;

$$\rho_{AB} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0 & 0 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0 & 0 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0 & 0 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 0.25 & 0.25 & 0 & 0 & 0.25 \end{bmatrix}$$

And taking the partial Trace;

$$\rho_A = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.25 \end{bmatrix}$$

From this Density matrix of the subsystem we can write the it in the Spectral decomposition form using it expansion in the eigenbasis. By using this as the $|i\rangle_A$ basis we will get the Schmidt decomposition.

$$|a\rangle = (1-\sqrt{2})\,|0\rangle + |1\rangle\,; \tag{20}$$

$$|b\rangle = (1+\sqrt{2})\,|0\rangle + |1\rangle \tag{21}$$

Using the above Eigenbasis we write our state as;

$$|\psi\rangle_{AB} = \frac{|b\rangle \otimes (|0\rangle + |1\rangle + \sqrt{2}\,|2\rangle)}{2\sqrt{2}} + \frac{|a\rangle \otimes (-|0\rangle - |1\rangle + \sqrt{2}\,|2\rangle)}{2\sqrt{2}} \tag{22}$$

As it can be that the $|a\rangle$ and $|b\rangle$ are both orthogonal and in turn by considering the terms in the R.H.S. of the $\bigotimes$ i.e. subsystem B as the basis, we get due to their orthonormal nature, the Schmidt Decomposition.

Entanglement using Schmidt Decomposition: Using the number of terms in the Schmidt Decomposition i.e. the number of non-zero eigenvalues of the density operators of the subsystems, we can comment if a **Pure state** is Entangled or not. If there are two or more terms i.e. the *Schmidt Number* in the decomposition then we say the state is Entangled. Schmidt number being a measure of Quantum Entanglement, it can be said that Local operations can't increase the entanglement hence can't increase the Schmidt Number.

1.5 Ensemble preparation and Convexity

Convexity: Based on the 3 properties of density operators as mentioned above one can say that any density operator can be written as a convex linear sum of the form:

$$\rho(\lambda) = \lambda\rho_1 + (1-\lambda)\rho_2 \tag{23}$$

where λ is real and $0 \leq \lambda \leq 1$ and ρ_1 and ρ_2 are two density operators. Hence density operators are a *convex subset* of real

vector space of $d \times d$ Hermitian operators.
It can be easily proved that pure states can't be expressed as a convex sum of 2 density operators. These pure states actually are the *extremal points* of the convex set. Also, since any mixed state can be written as $\rho = \sum_i p_i |i\rangle\langle i|$ (Generalization of the definition used in (12)) in the diagonal basis, i.e. it can be written as a convex sum.
A Bloch sphere is the geometrical representation of the convex set of a single qubit where the boundary has the extremal points. Generalizing this to a $d \times d$ matrix i.e. a d - dimensional system, the density operators are a convex subset of $d^2 - 1$ dimensional set of $d \times d$ Hermitian matrices with unit trace where the extremal points are pure states. It is important to remember that for d = 2, the boundary was the extremal points as in the case for a 2×2 diagonal matrix with unit trace if one diagonal term is 0 then, it lies on the boundary and we are left with a single term as per the definition from (12). This may not be true for a $d > 2$ system, hence not all boundary points are pure states.

Ensemble Preparation: The physical interpretation of the above convexity of density is that while considering the expectation values of any observable over an *Ensemble* of two states ρ_1 and ρ_2 prepared with the probability of λ and $(1-\lambda$ we get the following:

$$\langle M \rangle = \lambda \langle M \rangle_1 + (1-\lambda) \langle M \rangle_2 \tag{24}$$
$$= \lambda tr(M\rho_1) + (1-\lambda) tr(M\rho_2) \tag{25}$$
$$= tr(M\rho(\lambda)). \tag{26}$$

This suggests that we construct any state ρ using ρ_1 and ρ_2 if we know how to prepare them. Also it can be seen that there infinitely many ways to prepare this state ρ given it is a mixed state. If it is pure state then there is only one unique method of preparation of the state. Also every pure state actually represents an eigenstate of a unique observable and its measurement gives a probability 1 for the given pure state.
This non - unique nature of the Ensemble preparation makes this probabilistic method different from the unique probabilistic method of preparation of classical states.

Superluminal Communication: This paradox arises due to the property of entanglement of 2 systems. Suppose the systems exist in the bell state $|\phi^+\rangle$. Suppose Alice has many copies of one qubit of this bipartite state and Bob has the other and they are space-like separated and share the state:

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|1_z\rangle_A |1_z\rangle_B + |0_z\rangle_A |0_z\rangle_B) \tag{27}$$

Since a Unitary transformation can be applied to both the basis terms of the Schmidt decomposition form without affecting the state $\psi_A B$, one can write;

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|1_n\rangle_A |1_{n'}\rangle_B + |0_n\rangle_A |0_{n'}\rangle_B) \tag{28}$$

This means that by measuring the qubit B (or A) in any basis one can prepare the qubit A (or B) in ensemble of the same basis. This can be used to communicate information instantaneously. Now, if Alice moves hundreds of light-years away and Bob stays on earth and Bob measures his qubit spin using σ_1 or σ_3 to communicate his message (1-bit) to Alice. Now Alice immediately measures her spins to interpret the message, hence message transferred instantaneously over hundreds of light-years.
However there is an issue with this protocol. When Alice measures her spins how can she interpret the spins to get the message? That is how will she know in which basis did Bob performed his measurements. Bob measuring his qubit can prepare Alice's state in his required basis, but if he doesn't classically share his measurement results there is no way for Alice to compare her measurement results to his to distinguish between the Ensemble preparation method (Basis of measurement).
For Eg.: If Bob and Alice have 10 Entangled spin states (analogous to above system) and Bob's measurement results are 1110000101 (using σ_1). Now Alice performs her measurement and gets the result 1110000101 (using σ_1). Without the knowledge of Bob's measurement result (transmitted classically) there is no way possible to know if both of their Ensemble preparation methods were the same because as far as Alice is concerned Bob's result could have been 1011010000 i.e. Bob might have performed his measurement using σ_3.

1.6 Uhlmann's Theorem and Fidelity

Purification: Given a single qubit mixed state $\rho_A = \sum_i p_i |\phi\rangle\langle\phi|$ we can prepare what is called a *Purification* of the state i.e. a bipartite pure state of the form; $\Phi_{AB} = \sum_i \sqrt{p_i} |\Phi_i\rangle_A \bigotimes |\alpha_i\rangle_B$. We can get the subsystem A using the Partial Trace of the system AB.

Fidelity: Distinguishability of two *pure* states can be quantified using the overlap $|\langle\phi|\psi\rangle|^2$ which is called *Fidelity*. For density matrices, it is defined as;

$$F(\rho,\sigma) = (tr\sqrt{\rho^{1/2}\sigma\rho^{1/2}})^2 \tag{29}$$

Fidelity basically means how well two states are distinguishable and are different from each other under various types of Norms i.e. in a loose sense how differently they evolve when acted upon by the operators. Due to this, the definition of fidelity varies from situation to situation and many times depends on what norm is being used (Which basically act as distance relation between states).
Ulhmann's Theorem defines fidelity of two density operators as the maximal possible overlap of all their purifications.

$$F(\rho,\sigma) = (tr\sqrt{\rho^{1/2}\sigma\rho^{1/2}})^2 = \max_{V,W} |\langle \Phi_\sigma(W) | \Phi_\rho(V) \rangle|^2 \tag{30}$$

Based on Ulhmann's Fidelity we can infer the monotonicity of fidelity wrt to the purifications i.e.

$$F(\rho_{AB},\rho_{AB}) \leq F(\rho_A,\rho_A) \tag{31}$$

This basically means that the distinguishability of a system remains the same or infact increases if throw away a subsystem.

2. Measurement and Evolution[1]

2.1 Orthogonal Measurements

In case of Closed systems, we say that the measurements are performed using orthogonal projectors i.e. outlets that are exactly distinguishable. This is called Projective Measurement. The probability of the *ith* outlet being chosen is given by:

$$p_i = tr(P_i\rho) \tag{32}$$

and the output state is given by:

$$\frac{P_i\rho P_i}{tr(P_i\rho)} \tag{33}$$

Conditions satisfied by these Projective measurements are:

$$\sum_i P_i = I \tag{34}$$

$$P_i P_{i'} = \delta_{ii'} P_i \tag{35}$$

We then define what is called *Generalized measurements* i.e. suppose our system is part of a bigger system on which orthogonal measurements are performed, then it may be the case that the measurements are not orthogonal on our system *S*. In this case we use an *Ancilla A* i.e. a system to mainly couple to our system on which we perform orthogonal measurements. Due to its coupling the orthogonal measurements (Differentiability between the states of A) it allows us to differentiate and measure the eigenstates of the our system S. We to consider some conditions for properly executing the above measurement protocol. Considering **M** as the observable, due to Uncertainty principle a minimal uncertainty needs to be taken into consideration. Majorly the concept is that, the non-orthogonal measurements of the system S bring about a *shift* in the Eigenvalues of the Ancilliary system on which the orthogonal measurements are performed.

Example of Measurement using Position: Considering the Hamiltonian s.t. $[M,H_0] = 0$ and $H = \lambda(t)M\bigotimes P$. The evolution operator due to this Hamiltonian is:

$$U(T) \approx e^{(-\iota\lambda TM\bigotimes P)} \tag{36}$$

This actually brings about a translation on the Ancilliary system based on the value of the

$$M = \sum_a |a\rangle M_a \langle a| \tag{37}$$

Hence we get;

$$U(T) = \sum_a |a\rangle e^{(-\iota\lambda TM_a P)} \langle a| \tag{38}$$

So now Applying this Unitary operator on the un-entangled (initially) State:

$$U(T)(\sum_a \alpha_a |a\rangle \bigotimes |\psi(x)\rangle) = \sum_a \alpha_a |a\rangle \bigotimes |\psi(x-\lambda TM_a\rangle \tag{39}$$

Example of Stern Gerlach Apparatus Considering the application of a Magnetic field in the *z direction*

$$B = \lambda z \tag{40}$$

And the Magnetic Moment as $\mu\vec{\sigma}$ the coupling Hamiltonian becomes:

$$H = -\lambda\mu z\sigma_3 \tag{41}$$

This Hamiltonian has the *Generator of Momentum* in z-direction and hence upon applying this coupling Hamiltonian, an impulse is imparted to the electrons which is possibly imparted in 2 directions only based and hence we project the direction so the spin by measuring the position of the particle w.r.t. its original path.
A more abstract way of representing the *Orthogonal Measurements* can be done using the fact that we entangle a pointer/Ancilla (with a fiducial orthonormal basis) with our system using a Unitary operator;

$$U : |\Psi\rangle = |\psi\rangle \bigotimes |0\rangle \rightarrow |\Psi'\rangle = \sum_a E_a |\psi\rangle \bigotimes |a\rangle \tag{42}$$

Here the unitary introduces a set of *Orthonormal Projectors* that satisfy the properties:

$$E_a E_b = \delta_{ab} E_a \tag{43}$$

$$E_a = E_a{}^{\dagger} \tag{44}$$

$$\sum_a E_a = I \tag{45}$$

This entangling of Ancilla with the system the state of our pure system actually becomes a Mixed state given by;

$$Prob(a) = \langle\Psi'| (I \bigotimes |a\rangle\langle a|) |\Psi'\rangle = \langle\psi| E_a |\psi\rangle \tag{46}$$

$$\sum_a Prob(a) \frac{E_a |\psi\rangle\langle\psi| E_a}{\langle\psi| E_a |\psi\rangle} = \sum_a E_a |\psi\rangle\langle\psi| E_a \tag{47}$$

Now, it can be said that if if we measure the pointer but do not acquire the information the system occurs in the state as mentioned above i.e. in a mixture of states. However, if we measure using the pointer and acquire the information, we get the following normalized states (of system A) with the probability mentioned above in eqn. (47);

$$\frac{E_a |\psi\rangle}{||E_a |\psi\rangle||} \tag{48}$$

Here the Orthogonality of the measurement is due to the Orthogonality of the Projectors and not the pointer.

2.2 Generalized Measurements[2]

Now, if we consider the case where the entangling Unitary operator acting on the system and the Ancilla does the following:

$$U : |\psi\rangle_A \bigotimes |0\rangle_B \to \sum_a M_a |\psi\rangle_A \bigotimes |a\rangle_B \tag{49}$$

Such that the M_a are measurement operators that are not orthogonal, though the fiducial pointer basis are orthonormal. In principle we had defined a unitary providing us a basis of the pointer by correlating it to the *Projective measurements* of the system, which we are not having in this case.
Given these type of measurements we can say;

$$Prob(a) = \| M_a |\psi\rangle \|^2 \tag{50}$$

$$|\psi\rangle = \frac{M_a |\psi\rangle}{\| M_a |\psi\rangle \|} \tag{51}$$

$$\tag{52}$$

Also, if we perform a measurement and again perform a measurement, then the results may not be the same i.e. idempotent nature of these operators is not ensured, however the following is true;

$$Prob(b|a) = \frac{\| M_b M_a |\psi\rangle \|^2}{\| M_a |\psi\rangle \|^2} \tag{53}$$

$$\sum_a M_a^\dagger M_a = I \tag{54}$$

Based on these M_a we define operators that act on the density operators s.t.

$$E_a = M_a^\dagger M_a \tag{55}$$

$$Prob(a) = tr(\rho E_a) \tag{56}$$

This type of measurement satisfies certain properties:

1. *Hermiticity* i.e. $E_a = E_a^\dagger$
2. *Positivity* i.e. the eigenvalues of these measurement operators are non-negative and in-turn the expectation values of all possible states are also non-negative.
3. *Completeness* i.e. $\sum_a E_a = I$

This Type of Measurements are called **Positive Operator Valued Measurements i.e. POVMs**.

2.3 Quantum Channels

Looking from a new perspective at our discussion about the approach of the measurements, where we took an Ancilliary system and entangle it with our system i.e. perform a unitary like in eqn.(50) what if we consider only the effect of the unitary on our system by forgetting that the Ancilliary ever existed. We can consider that the system represented by a Density Matrix is subjected to a *Linear Map*;

$$\varepsilon(\rho) = \sum_a M_a \rho M_a^\dagger \tag{57}$$

This Linear map is equivalent to saying that we consider an initial system A, entangle it to Ancilla B using eqn.(50) and trace out B to get back system A from the new closed system. Here, the operators M_a are called *Kraus Operators* and this Linear map is called *Quantum Channel* or a *Super-operator* (as it operates on a matrix).These operators follow the eqns. (56) and (57) as well.
We also call this map as a *Completely Positive Trace Preserving Map*. This is because;

1. It is Linear i.e. $\varepsilon(a\rho 1 + b\rho 2) = a\varepsilon(\rho 1) + b\varepsilon(\rho 2)$
2. It preserves Hermiticity i.e. if $\rho = \rho^\dagger$ then, $\varepsilon(\rho) = \varepsilon(\rho)^\dagger$
3. It Preserves Trace i.e. $tr(\varepsilon(\rho)) = tr(\rho)$
4. It Preserves Positivity i.e. $\rho \geq 0 \implies \varepsilon(\rho) \geq 0$

This map being analogous to the introduction of an Ancilliary System implies the *Kraus Operators* are not to be unique i.e. since we trace out the system B, we have the control of the basis in which we decide to perform the partial trace. Hence considering in terms of a Rotated basis;

$$|a\rangle = \sum_\mu |\mu\rangle V_{\mu a} \tag{58}$$

$$\sum_a M_a |\psi\rangle_A \bigotimes |\mu\rangle_B V_{\mu a} = \sum_\mu N_\mu |\psi\rangle_A \bigotimes |\mu\rangle_B \tag{59}$$

$$N_\mu = \sum_a V_{\mu a} M_a \tag{60}$$

2.4 Quantum Channels in Heisenberg Picture

In the Schrodinger picture we evolved the density matrices using Unitary Evolution, while in the Heisenberg picture we evolve the Operators rather than the density matrices themselves. We do this in such a way that the expectation values obtained after evolution wrt the observable remains the same in both the pictures.

We will follow a similar approach in the case of quantum channels too.
Given we have the following relation

$$\rho' = \varepsilon(\rho) = \sum_a M_a \rho M_a^\dagger \tag{61}$$

we use the alternative description as the *Dual* or *Adjoint* of the ε

$$A' = \varepsilon^*(A) = \sum_a M_a^\dagger A M_a \tag{62}$$

This satisfies the condition of equal expectation values in both the pictures;

$$tr(A\varepsilon(\rho)) = tr(\varepsilon^*(A)\rho) \tag{63}$$

This Dual of a channel may not be a channel at all as it may not preserve trace because considering the cyclic property of trace, we can use completeness relation of Kraus operators to preserve trace which may not be the case given the definitions (62) and (63).

Unital Map: Due to the Completeness property of the Kraus Operators we can say That

$$\varepsilon^*(I) = I \tag{64}$$

i.e. it preserves the Identity and hence is a Unital Mapping. So, a dual of a channel is always a Unital but a Quantum Channel is Unital only if;

$$\sum_a M_a^\dagger M_a = I = \sum_a M_a M_a^\dagger \tag{65}$$

We can say that a Maximally mixed state is preserved under a Unital Channel.

2.5 Quantum Operations

Until now we have considered cases where we perform Measurements i.e. measure/acquire complete information about the meter which is entangled to our system and Quantum channels in which we don't acquire any information at all about the Ancilliary so as to observe an Evolution in our system.
However in case of Quantum Operations, we don't acquire the complete information of our system but rather partial information.

$$\sum_{a,\mu} M_{a\mu}^\dagger M_{a\mu} = I \tag{66}$$

So here, we acquire information about say 'a' and forget completely about 'μ' to get;

$$\varepsilon_a(\rho) = \sum_\mu M_{a\mu}\rho M_{a\mu}^\dagger \tag{67}$$

Completeness relation

$$\sum_\mu M_{a\mu}^\dagger M_{a\mu} \leq I \tag{68}$$

This is a more general case than the Measurement in which μ is completely known and the Quantum channel in which 'a' takes only one value so as to satisfy the Completeness relation as per eqn. (55). This evolution of a state may not be trace preserving but is linear (Which can be proved using the ensemble theory of a quantum state). In order to ensure the unit trace condition we have to invoke a non-linearity condition by renormalizing the state after evolution.

$$\rho \rightarrow \frac{\varepsilon_a(\rho)}{tr(\varepsilon_a(\rho))} \tag{69}$$

2.6 Complete Positivity

This is a stronger condition than positivity i.e. according to this condition, the positivity of a state should be maintained even if the quantum channel is being applied on a subsystem of a larger system. This condition is important because if this is not true then can say that positivity is not maintained in the 'world'. Not every positive map is however completely positive;

Example: Considering the Transposition map for say d-dimensional system which acts as a SWAP gate if we consider its application on a subsystem of bipartite system; which have -1 as an eigenvalue and hence is not a positive map for the bigger system.

2.7 Revised Axioms

After analyzing the open systems with various possibilities of mixed states, Generalized Measurements etc. we can restate the axioms as;

1. **States** of a Quantum Mechanical system is represented using Density Operators i.e. a non-negative Hermitian operator in the Hilbert space with Unit Trace.
2. **Measurement** is actually represented by POVMs i.e. a set of partition of Unity.
3. **Dynamics** of a Quantum state can is described by using Completely Positive Trace Preserving (CPTP) Maps also known as Quantum Channels.

2.8 Major Causes of Error using Quantum channels

2.8.1 Depolarizing Channel

There can be 3 types of errors that can occur i.e.Quantum

1. Bit Flip Error($\sigma 1$): $|0\rangle \rightarrow |1\rangle ; |1\rangle \rightarrow |0\rangle$
2. Phase Flip Error($\sigma 2$): $|0\rangle \rightarrow |0\rangle ; |1\rangle \rightarrow -|1\rangle$
3. Both($\sigma 3$): $|0\rangle \rightarrow +i|1\rangle ; |1\rangle \rightarrow -i|0\rangle$

Assuming that the probability that any type of error can occur be p and hence the probability of no error occurring be 1-p. This occurrence of error can be considered to during the *time traversal* of a state i.e. using a quantum channel. So to describe, this one can attach an *Environment* (Ancilla) to our system and perform a Unitary (Say an isometry mapping) on the the complete system in such a way that the environment can be used to detect which error has occurred.

$\mathbf{U_{A\rightarrow AE}}$:

$$|\psi\rangle_A \rightarrow \sqrt{1-p}\,|\psi\rangle_A \bigotimes |0\rangle_E + \sqrt{\frac{p}{3}}[\sigma 1\,|\psi\rangle_A \bigotimes |1\rangle_E + \sigma 2\,|\psi\rangle_A \bigotimes |2\rangle_E + \sigma 3\,|\psi\rangle_A \bigotimes |3\rangle_E] \tag{70}$$

We can calculate the Kraus operators by taking partial trace about the orthogonal basis of the environment as described in the above equation.

$$M_0 = (\sqrt{1-p})I, \tag{71}$$

$$M_1 = \sqrt{\frac{p}{3}}\sigma 1, \tag{72}$$

$$M_2 = \sqrt{\frac{p}{3}}\sigma 2, \tag{73}$$

$$M_3 = \sqrt{\frac{p}{3}}\sigma 3 \tag{74}$$

Considering the initial state as the maximally entangled 2-qubit state, we define a term called Entanglement fidelity;

$$F_e = \langle \phi^+ | \rho | \phi^+ \rangle \tag{75}$$

which turns out to be equal to 1-p i.e. hence we interpret it as the probability of no error occurred. Now to understand why exactly why the term 'Depolarizing' is used, we do the Bloch sphere representation of the changed Density matrix wrt original matrix in the form of eqn. (13) which leads to;

$$\vec{P}' = (1 - \frac{4}{3}p)\vec{P} \tag{76}$$

As it can be observed that the Polarization decreases (Bloch sphere actually shrinks) with the increasing value of p. This depolarizing is irreversible because any mapping corresponding to increasing polarization will lead to negative eigenvalues of the density operator.

2.8.2 Dephasing Channel

Considering a quantum channel in which the change occurs mainly in the environment i.e. it *scatters* off of the qubit with a probability p;

$$\begin{aligned} |0\rangle_A &\to \sqrt{1-p}\,|0\rangle_A \bigotimes |0\rangle_E + \sqrt{p}\,|0\rangle_A \bigotimes |1\rangle_E \\ |1\rangle_A &\to \sqrt{1-p}\,|1\rangle_A \bigotimes |0\rangle_E + \sqrt{p}\,|1\rangle_A \bigotimes |2\rangle_E \end{aligned} \tag{77}$$

Again taking the partial trace we calculate the Kraus Operators as;

$$M_0 = \sqrt{1-p}\mathbf{I}, \tag{78}$$

$$M_1 = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{79}$$

$$M_2 = \sqrt{p}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{80}$$

They can be however also represented as;

$$M_0 = \sqrt{p}(\mathbf{I} + \sigma 3) \tag{81}$$

$$M_1 = \sqrt{p}(\mathbf{I} - \sigma 3) \tag{82}$$

And hence can be written in terms of the following channel;

$$\varepsilon(\rho) = (1 - \frac{p}{2})\rho + \frac{p}{2}\sigma_3 \rho \sigma_3 \tag{83}$$

Applying this on a random Density operator of the following gives the result as;

$$\varepsilon \begin{bmatrix} \rho_0 0 & \rho_0 1 \\ \rho_1 0 & \rho_1 1 \end{bmatrix} = \begin{bmatrix} \rho_0 0 & (1-p)\rho_0 1 \\ (1-p)\rho_1 0 & \rho_1 1 \end{bmatrix} \tag{84}$$

As it can be clearly seen that this Quantum channel actually makes the off-diagonal terms 0 as p increases i.e. it *Decoheres* the state.

3. Entanglement[3]

3.1 Separable vs Entangled states

A pure state is a called a *product state* if the local sub states of that state are also pure in nature;

$$|\psi_{AB}\rangle = |\psi_A\rangle \bigotimes |\psi_B\rangle \tag{85}$$

if the states that can't be written in this form, then those states are called *entangled* states.
As mentioned earlier the Schmidt Decomposition of a pure state helps us check the entanglement in a system by looking at the number of terms in the decomposition. It is important to understand that product states can be locally prepared without any interaction between the two systems. Hence, to check if a given pure bipartite state is entangled or not, we just need to check their respective reduced density matrices to be pure or not, which is equivalent to saying that there is only one Schmidt coefficient in the Schmidt decomposition.

It should be noted that for mixed states the entanglement detection is much more complex and its concrete detection methods are yet to be discovered.
A state ρ_{AB} is called *Separable* if and only if it can be represented as a convex combination if the product of projectors of local states;

$$\rho_{AB} = \sum_{i=1}^{K} p_i |e_i\rangle\langle e_i| \bigotimes |f_i\rangle\langle f_i| \tag{86}$$

LOCC: Local Operations and Classical Communication (LOCC) as the name suggests refers to performing local operations on subsystems based on some classical communication.
Eg. Alice and Bob share a bipartite state ρ_{AB} and first Alice performs a set of quantum operations locally on her part of the shared state and sends her measurement results to Bob. Based on her classical communication of the results he also performs local operation (in their respective Hilbert Spaces) on his part of the system and shares his results to Alice. This can continue and these type of operations on the shared system are termed as LOCC.
It is important to note that any states can't be entangled only performing LOCC. The two states need to interact with one another or have to undergo a Entanglement Swapping.

3.2 Operation Entanglement Detection Criteria

3.2.1 Partial Transposition[4]

As per this criteria the entanglement of a bipartite state ρ_{AB} can be detected by taking the partial transposition and check its positivity. If the following is true then the given state is separable.

$$\begin{aligned} \rho_{AB}^{T_A} &\geq 0 \\ \rho_{AB}^{T_B} &\geq 0 \end{aligned} \tag{87}$$

Proof:

$$\rho_{AB} = \sum_{i=1}^{K} p_i\,|e_i\rangle\langle e_i| \bigotimes |f_i\rangle\langle f_i| \geq 0$$
$$\rho_{AB}^{T_A} = rho_{AB} = \sum_{i=1}^{K} p_i(|e_i\rangle\langle e_i|)^{T_A} \bigotimes |f_i\rangle\langle f_i|$$
$$\rho_{AB}^{T_A} = rho_{AB} = \sum_{i=1}^{K} p_i(|e_i^*\rangle\langle e_i^*|) \bigotimes |f_i\rangle\langle f_i| \geq 0 \tag{88}$$

This can be understood by using the property that $A^\dagger = (A^*)_A^T$ and since here the eigenvalues are positive, we get the above condition as true.

3.2.2 Majorization[5]

This is a criteria independent of Partial Tranposition but is not stronger that it. Say we have two probability distributions, $X = x_1, x_2, x_3, ...x_n$ and $Y = y_1, y_2, y_3, ...y_n$ arranged in the decreasing order then we define *X is majorized by Y* i.e. $Y \succ X$ as;

$$\sum_{i=1}^{l} x_i \leq \sum_{i=1}^{l} y_i$$

where $l = 1, 2, 3, ...d-1$ and equality holds at $l = d$

Now wrt Entanglement, a state is said to be separable then;

$$\lambda(\rho_A) \succ \lambda(\rho_{AB}) \tag{89}$$
$$\lambda(\rho_B) \succ \lambda(\rho_{AB}) \tag{90}$$

Here, $\lambda(\rho_{AB})$ is the set of eigenvalues for a ρ_{AB} and similarly for the reduced density matrices. A Bipartite entangled state violates the above equations.
Following these equations, we get define a new relation between the Quantum Analog of Shannon entropy i.e. Von-neumann entropy which states that given a separable state;

$$S(\rho_{AB}) \geq S(\rho_A) \tag{91}$$
$$S(\rho_{AB}) \geq S(\rho_B) \tag{92}$$

This basically states that the amount of entropy i.e. disorder of a larger system (global) can't be smaller than the entropy of the smaller systems (local) which also a classically known fact for two probability distributions. However entangled states violate this which further strengthens our belief that entanglement has no classical analog.

3.3 Non-operational Entanglement Criteria

The non-operational behavior of the following criteria is based on the fact that these aren't state independent.

3.3.1 Entanglement Witness

Considering S as a convex compact set in finite dimensional Banach space and a point ρ in the space s.t. $\rho \notin S$, then there exists a Hyperplane that separates ρ from S.
If we consider S as the set of all separable states,then by defining a suitable orthogonal vector to the Hyperplane i.e. an Entanglement Witness W (Hermitian Operator) i.e.

$$\exists\rho \text{ s.t. } tr(\rho W) < 0 \text{ while } \forall\sigma \in S, tr(W\sigma) \geq 0$$

This basically means we are projecting ρ on to this W and checking its location w.r.t. the set S.
The convexity of the set ensures that no two separable states can lie on opposite side of the hyperplane i.e.

$$\forall EW, tr(W\sigma) \geq 0 \tag{93}$$

3.4 CHSH Inequality and Entanglement[2]

Considering the assumptions of *Local Realism* i.e. valid as per the classical concepts, the following equality stands given that A and B are 2 independent Particles;

$$E(a,b) = \int A_a(\lambda) B_b(\lambda) \rho(\lambda) d\lambda \tag{94}$$
$$-2 \leq E(a,b) + E(a,b') + E(a',b) - E(a',b') \leq 2 \tag{95}$$

Here, λ is the Hidden variable corresponding to the assumption of realism i.e. Measurement doesn't cause a value to appear, it merely uncovers the value. E(a,b) is the correlation function correspond to the possible outcomes of A and B in different directions. The assumption of Locality states that the measurement of one observable on 1 particle can't affect the state of the other particle instantaneously.
Later it was shown that by considering an operator of the same form as seen in above equation and getting its expectation value on the maximally entangled state $|\psi^-\rangle$ the inequality is violated, i.e. bipartite entangled particles may violate the CHSH inequality i.e. *Quantum Mechanics is not Local Realist*.

$$B_{CHSH} = \sigma_a.\sigma_b + \sigma_a.\sigma_b' + \sigma_a'.\sigma_b - \sigma_a'.\sigma_b' \tag{96}$$

One possible case (maximum value) can be;

$$|B_{CHSH}| = 2\sqrt{2} \tag{97}$$

It can be noted that this Operator can actually act as an Entanglement Witness for the singlet state with some suitable scaling.

3.5 Quantification of Entanglement

As shown in [6] that entangled states when used to perform Quantum Protocols such as Teleportation and Dense Coding provide efficiencies greater than that of unentangled states. Hence Entanglement should be treated as resource which needs to be quantified.
Generalizing what properties should be there in an entanglement quantification theory;

- Non-negativity i.e. $\varepsilon(\rho) \geq 0$ for any bipartite state.
- Entanglement vanishes for separable states i.e. $\varepsilon(\rho) = 0$ if ρ is separable.
- Invariance under Local Unitary Transformations
- Entanglement cannot increase under the Local Operations and Classical Communication, i.e. for a given LOCC λ,
  $\varepsilon(\lambda(\rho)) \leq \varepsilon(\rho)$

As per Resource Theory the quantification of a resource must be done by a quantity that doesn't increase under the free operations and any state created by free operations will have the value as 0. Free operations in our case are LOCC and thus the entanglement measures can't increase under LOCC and separable state must have 0 entanglement measure. A quantity satisfying these conditions is an entanglement measure or *entanglement monotone*.

3.5.1 Von Neumann Entropy and Entanglement[2]

Similar to the Shannon entropy for probability distribution in classical systems, the Von Neumann Entropy was defined for the Density Operators in Quantum Mechanics;

$$S(\rho) = -tr(\rho \log(\rho)) \tag{98}$$

$$S(\rho) = -tr(\lambda_x \log(\lambda_x)) \tag{99}$$

A property that Von Neumann Entropy doesn't follow contrary to the Shannon Entropy is that the Von Neumann Entropy of a subsystem can be higher than that of its system which is always present in a bipartite Entangled State i.e. the Conditional Entropy of the system is negative;

$$S(A|B) = S(A,B) - S(A) \tag{100}$$

It was observed that singlet states i.e. pure bipartite entangled states can be converted from and to other entangled states (however not with full fidelity i.e. equal number of states before and after) and also perform most efficiently.
Hence, the quantification of the pure quantum state in by the *Von Neumann Entropy* of its reduced density matrices;

$$\varepsilon_E(|\psi_{AB}\rangle) = S(\rho_A) = S(\rho_B) \tag{101}$$

For maximally entangled states the entanglement entropy of the subsystems is 1.

3.5.2 Entanglement of Formation

One can extend the entanglement to mixed states in a using the following approach also known as a Convex Roof optimization;

$$\varepsilon_{EoF}(\rho_{AB}) = \min_{(pi,|AB\rangle)} \sum_i p_i \varepsilon_E(|\psi^i_{AB}\rangle) \tag{102}$$

Here, we basically select an ensemble of pure states i.e. $\rho_{AB} = \sum_i p_i |\psi^i_{AB}\rangle\langle\psi^i_{AB}|$ and then with probability p_i choose a pure state and calculate the average of their entanglement of entropy using the same probability of choosing these states from the ensemble. We try to minimise this value by choosing new ensembles of pure states. The values of individual entanglement entropies of pure states can be done using the fact that these states can be obtained from singlet states via LOCC.

3.5.3 Concurrence

For a 2 qubit pure state $|\psi_{AB}\rangle$ we have defined concurrence as;

$$C(\psi_{AB}) = \left\| \langle \psi_{AB} | \psi'_{AB} \rangle \right\| \tag{103}$$

where $|\psi_{AB}\rangle = \sigma_y \otimes \sigma_y |\psi^*_{AB}\rangle$

$$C(\rho_{AB}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 \lambda_4\} \tag{104}$$

Here the $\lambda_i's$ are the eigenvalues (in decreasing ordered) of;

$$R = \sqrt{\sqrt{\rho_{AB}} \rho^*_{AB} \sqrt{\rho_{AB}}} \tag{105}$$

3.5.4 Entanglement Cost

Given a state, ρ_{AB} we can define entanglement cost as a quantification of entanglement based on the rate of conversion of 'm' singlets to the copies of a states using only LOCC. This is an asymptotic approach hence we consider required states to tend to infinity and minimize the rate over the LOCC Protocols 'P' of conversion;

$$\varepsilon_C(\rho_{AB}) = \min_P \{\lim_{n\to\infty} \frac{m}{n}\} \tag{106}$$

such that the state obtained is σ_n where $D(\rho_{AB}^{\otimes n}, \sigma_n) \to 0$ as $n \to \infty$ given 'D' is a distance measure between states.

3.5.5 Distillable Entanglement

This Quantification can be considered as the dual of Entanglement cost because as the name suggests, we 'distill' the given states to singlets and quantify using this rate of distillation. The term 'Distillation' is used because since the entanglement entropy of any state is less than or equal to singlet states and the conversion of many copies of such states to singlets is being considered. So, given 'n' copies of a state ρ_{AB} we prepare the state σ_n s.t. $D(|\psi^-\rangle\langle\psi^-|, \sigma_n) \to 0$ as $n \to \infty$ and the Distillable entanglement is defined as

$$\varepsilon_D(\rho_{AB}) = \max_P \{\lim_{n\to\infty} \frac{m}{n}\} \tag{107}$$

The maximization can be understood clearly as we wish to maximize our singlet output over all the possible LOCC conversions.

$$\varepsilon_C \geq \varepsilon_D \tag{108}$$

The above inequality is mainly can be understood based on the very nature of how entanglement is defined as a resource and a free operation can't increase the quantification of a resource. If the above inequality fails we can get a situation where more singlet states can be distilled out of the state produced by a state which was produced by less number of singlets.

3.5.6 Negativity

The partial transposition can be used to define another Entanglement Quantification based on the absolute sum of the number of negative eigenvalues of the partially transposed Density Matrix known as negativity;

$$N(\rho_{AB}) = \frac{\left\|\rho_{AB}^{T_B}\right\| - 1}{2} = \frac{\left\|\rho_{AB}^{T_A}\right\| - 1}{2} \tag{109}$$

Here we are considering the L2 norm. $N(\rho)$ doesn't satisfy additive property so we define a quantity that follows convexity and is additive;

$$\varepsilon_{LN}(\rho_{AB}) = \log_2(2N(\rho_{AB}+1) \tag{110}$$

This quantity is known as Logarithmic Negativity.

ASSIGNMENT 2: APPENDIX B

4. Quantum Communication[6]

Communication is the process of information transfer from the sender to a receiver. Quantum Communication i.e. the use of Quantum physics to execute communication protocols rather than classical ones is an important application as it often provides Advantages over its classical analog. Quantum entanglement can be considered a *Resource* for these Quantum Communication Protocols. Maximally entangled states are often considered to be the most important resource states of such protocols.

4.1 Dense Coding

In this protocol we transmit classical information via quantum states over classical channels hence making superluminal transmissions impossible. The advantage of using this protocol over its classical analog is achieved by exploiting the entanglement of a state.

Protocol: Say we wish to communicate a 2 bit data (for eg.) i.e. 4 possibilities, then using this protocol we use 2 qubits that are entangled. Both the sender and the receiver take one qubit each. Since performing a local operation on one subsystem of the state doesn't break the entanglement the sender does Local operations on their subsystem to basically store the information in the bipartite state. Now the sender transmit their qubit (single) classically to the receiver and the receiver measures the bipartite state to decode the information. This allows a single qubit transfer to transmit 2 bit of data.

4.1.1 Holevo Bound[2][3]

This is basically a bound on how much maximum information can the receiver extract upon measurement of the bipartite state after a local operation has been performed on it.
Say Alice (sender) performs U_i operation on her qubit to send the classical information i to Bob (receiver) which has a probability of p_i i.e. the ensemble $\{\rho_i, p_i\}$ obtained by Bob is;

$$\rho_i = (U_i \bigotimes I)\rho_{AB}(U_i^\dagger \bigotimes I) \tag{111}$$

Now suppose Bob measures thus bipartite state and obtains result m with probability q_m . So, considering the post measurement ensemble $\{p_{i|m}, \rho_{i|m}\}$. The mutual information (Shannon Entropy) between the information i and m ;

$$I(i:m) = H(\{p_i\}) - \sum_m q_m H(\{p_i|m\}) \tag{112}$$

We wish to maximize this mutual information extracted post measurement. The upper bound to the maximization of this mutual information is called Holevo bound i.e.;

$$I_acc \leq \chi(\{p_i, \rho_i\}) \equiv S(\rho') - \sum_i p_i S(\rho_i) \tag{113}$$

where $\rho' = \sum_i p_i \rho_i$ i.e. the ensemble average state.

4.2 No Cloning Theorem

This theorem basically states that an arbitrary state can't be cloned.

Proof: Lets assume an arbitrary state (unknown α and β) of the form $\psi = \alpha|0\rangle + \beta|1\rangle$ and consider the following "xerox machine"

$$|\psi\rangle|\beta\rangle \to |\psi\rangle|\psi\rangle \tag{114}$$

So lets consider the situation;

$$|0\rangle|B\rangle|M\rangle \to |0\rangle|0\rangle|M'\rangle \tag{115}$$

$$|1\rangle|B\rangle|M\rangle \to |1\rangle|1\rangle|M''\rangle \tag{116}$$

And

$$(\alpha|0\rangle + \beta|1\rangle)|B\rangle|M\rangle \to (\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)|M'''\rangle \tag{117}$$

Adding the above equations we get the condition that $\alpha = 0$ or $\beta = 0$ i.e. the machine can copy only known states.

4.3 Quantum State Teleportation

In this protocol the entanglement between states is exploited by sharing this entanglement to an additional state (Target State) which needs to be teleported to the receiver (Bob) by the sender (Alice) to whom the state is known. The term *Teleportation* is being used as the state itself isn't being transported, rather in a way it is being destroyed at the sender's end created at the receiver's end. As shown above copying of states is not possible and giving the information of the entire state classically isn't possible, so oe needs to either transfer the state over a channel or use this protocol.
Suppose we have a singlet state and we send the sub-states to the receiver and the sender each. Now at the sender's end the target state and the singlet state are entangled together using Hadamard and CNOT gates. Then Alice performs a measurement on the 2 states that she has she transmits her measurement results classically to Bob. Based on the outcomes Bob performs some local Unitary Operations on his state which transforms his state to the Target state.

5. Quantum Batteries

Quantum batteries are based on the fundamental principle of storage of energy by a system upon action of some Hamiltonian. The batteries are charged i.e. gain energy wrt their initial energy state and get discharged i.e. lose the stored energy. The Quantum analog to these classical batteries is mainly by exploiting entanglement between the particles of the system which may provide a Quantum in advantage in analysis of certain work statistics such as power extracted, ergotropy wrt various Hamiltonians used for charging.
I studied the batteries created by the systems defined by the Quantum Spin Chain Models as explained in [7] which can be charged by the Time periodic Hamiltonians hence the solutions to the Schrodinger equation invoke the Floquet Theorem. I reproduced the results as shown in the **FIG. 2** and **FIG 4.** of [7] in which the maximum power extracted by charging the Transverse XY Model by a local transverse charging field. The initial state is considered to be the Canonical equilibrium state of wrt an initial Hamiltonian involving a field term to remove the degeneracies of the ground state and an interacting term for the short range interactions. The power extracted by this system is calculate using the difference between the energy of the states before and after charging by optimizing the time over which the Power is extracted. The general Quantum spin chain systems have initial Hamiltonian of the form;

5.1 Periodically Driving of Quantum Systems

The concept of Periodic driving basically involves using time periodic Hamiltonians to evolve our systems. This leads to the use of Floquet Theorem [8][9][10] which states that given a first order Linear Differential Equation with Time periodic coefficients (which in our case is the Time periodic Hamiltonian) of the form;

$$i\frac{d\psi(t)}{dt} = H(t)\psi(t) \tag{118}$$

s.t. $H(t) = H(t+T)$ where 'T' is the time periodicity and $\hbar = 1$
then the wavefunction can be evolved using the following Unitary operator:

$$U(t) = P(t)e^{-itH_F} \tag{119}$$

where H_F is the Floquet Hamiltonian corresponding our given Hamiltonian. Using such Hamiltonians if one evolves a multi-particle system over the stroboscopic instants i.e. the time periodicity of Hamiltonian then the evolution can be performed using Unitary operator corresponding to the Floquet Hamiltonian i.e. a time independent Hamiltonian and we can avoid the evolution using the Time propagator due to time dependant Hamiltonian. However the calculation of the Floquet Hamiltonians is tough and is usually approximated using Floquet-Magnus expansion [11][9].

5.2 Quantum Advantage

Periodically Driven Quantum Batteries[12][13][14] combine the above two concepts and can sometimes provide a Quantum Advantage . The quantum Advantage as suggested in the paper [15][16] [12]is that by globally charging the system with Hamiltonians that are non-local/parallel i.e. entangling in nature (Not entanglement necessary though) may increase the energy of the system faster than a classically charged battery with same number of particles i.e. more possible extraction of power. There can be also what is called superlinear Scaling [12] of power with system size i.e. power extracted by a system of say N-particles may scale power that is non-linearly (at max quadratically though) as N increases contrary to the classical situation where linearity is present.

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6. Appendix (Assignment Problems)

6.1 Appendix A

PROBLEM: Write a Program to randomly choose the components of $\vec{P}$ from *U(-1, 1)* and generate their corresponding density matrices. Then find the percentage of such matrices that follow the property of Positivity.

ANALYTICAL SOLUTION: We get non-zero probabilities of matrices not satisfying positivity condition (i.e. not a density matrix) because, based on the general definition of Density matrices we get the value of $|\vec{P}| = \sqrt{(P_x^2 + P_y^2 + P_z^2)}$ as lying between -1 and 1. While we have considered individual P_i as lying between -1 and 1. So, considering uniform probability distribution we can geometrically prove that the probability of success is the ratio of vol. of a sphere (Bloch sphere) of radius = 1 and vol. of cube (Uniform Distribution) of length = 2 in the P_i phase space i.e.

$$\frac{V(sphere)}{V(cube)} = \frac{\frac{4\pi 1^3}{3}}{2^3} = \frac{\pi}{6} \approx 0.523598 \tag{120}$$

PYTHON PROGRAM: https://github.com/stavyapuri/Quantum-Information-codes

6.2 Appendix B

PROBLEM: Plot logarithmic negativity wrt 'p' of:

$$a)\rho 1 = p\left|\phi^-\right\rangle\left\langle\phi^-\right| + (1-p)\left|\phi^+\right\rangle\left\langle\phi^+\right|$$
$$b)\rho 1 = p\left|\phi\right\rangle\left\langle\phi\right| + (1-p)\frac{I}{4}$$

where $\phi = \cos(a/2)\left|01\right\rangle + \sin(a/2)\left|10\right\rangle$; $0 \leq a \leq \pi$

PYTHON PROGRAM: https://github.com/stavyapuri/Quantum-Information-codes

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दिनांक/Date : ....

CERTIFICATE

This is to certify that Mr. Stavya Puri from Bits Pilani has visited and worked with the QIC Group at Harish-Chandra Research Institute, Prayagraj from 4th June – 31st July 2023.

Aditi Sen(De)

Prof. Aditi Sen De
Coordinator, QIC Group,
HRI, Prayagraj (Allahabad)

हरीश-चन्द्र अनुसंधान संस्थान, प्रयागराज
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