Floating Point

- Representation for non-integral numbers
  - Including very small and very large numbers
- Like scientific notation
  - -2.34×10^{56} ← normalized
  - $+0.002 \times 10^{-4}$ ← not normalized
  - $+987.02 \times 10^{9}$ ← not normalized
- In binary
  - $\pm 1.xxxxxxx_2 \times 2^{yyyy}$
- Types `float` and `double` in C

Floating Point Standard

- Defined by IEEE Std 754-1985
- Developed in response to divergence of representations
  - Portability issues for scientific code
- Now almost universally adopted
- Two representations
  - Single precision (32-bit)
  - Double precision (64-bit)

IEEE Floating-Point Format

	single: 8 bits double: 11 bits	single: 23 bits double: 52 bits
S	Exponent	Fraction

$$x = (-1)^S \times (1 + \text{Fraction}) \times 2^{(\text{Exponent} - \text{Bias})}$$

- S: sign bit (0 $\Rightarrow$ non-negative, 1 $\Rightarrow$ negative)
- Normalize significand: 1.0 ≤ |significand| < 2.0
  - Always has a leading pre-binary-point 1 bit, so no need to represent it explicitly (hidden bit)
  - Significand is Fraction with the “1.” restored
- Exponent: excess representation: actual exponent + Bias
  - Ensures exponent is unsigned
  - Single: Bias = 127; Double: Bias = 1023

Single-Precision Range

- Exponents 00000000 and 11111111 reserved
- Smallest value
  - Exponent: 00000001
    $\Rightarrow$ actual exponent = 1 – 127 = –126
  - Fraction: 000…00 $\Rightarrow$ significand = 1.0
  - $\pm 1.0 \times 2^{-126} \approx \pm 1.2 \times 10^{-38}$
- Largest value
  - exponent: 11111110
    $\Rightarrow$ actual exponent = 254 – 127 = +127
  - Fraction: 111…11 $\Rightarrow$ significand ≈ 2.0
  - $\pm 2.0 \times 2^{+127} \approx \pm 3.4 \times 10^{+38}$

Double-Precision Range

- Exponents 0000…00 and 1111…11 reserved
- Smallest value
  - Exponent: 00000000001
    $\Rightarrow$ actual exponent = 1 – 1023 = –1022
  - Fraction: 000…00 $\Rightarrow$ significand = 1.0
  - $\pm 1.0 \times 2^{-1022} \approx \pm 2.2 \times 10^{-308}$
- Largest value
  - Exponent: 11111111110
    $\Rightarrow$ actual exponent = 2046 – 1023 = +1023
  - Fraction: 111…11 $\Rightarrow$ significand ≈ 2.0
  - $\pm 2.0 \times 2^{+1023} \approx \pm 1.8 \times 10^{+308}$

Floating-Point Precision

- Relative precision
  - all fraction bits are significant
  - Single: approx 2^{-23}
    - Equivalent to $23 \times \log_{10}2 \approx 23 \times 0.3 \approx 6$ decimal digits of precision
  - Double: approx 2^{-52}
    - Equivalent to $52 \times \log_{10}2 \approx 52 \times 0.3 \approx 16$ decimal digits of precision

Floating-Point Example

- Represent –0.75
  - $-0.75 = (-1)^1 \times 1.1_2 \times 2^{-1}$
  - S = 1
  - Fraction = $1000\ldots00_2$
  - Exponent = –1 + Bias
    - Single: –1 + 127 = 126 = 01111110_2
    - Double: –1 + 1023 = 1022 = 01111111110_2
- Single: 1011111101000…00
- Double: 1011111111101000…00

Floating-Point Example

- What number is represented by the single-precision float

  11000000101000…00
  - S = 1
  - Fraction = $01000\ldots00_2$
  - Exponent = $10000001_2 = 129$
- $x = (-1)^1 \times (1 + 0.01_2) \times 2^{(129 - 127)}$

  $= (-1) \times 1.25 \times 2^2$

  $= -5.0$

Denormal Numbers

- Exponent = 000...0 $\Rightarrow$ hidden bit is 0

$$x = (-1)^S \times (0 + \text{Fraction}) \times 2^{-\text{Bias}}$$

- Smaller than normal numbers
  - allow for gradual underflow, with diminishing precision
- Denormal with fraction = 000...0

$$x = (-1)^S \times (0 + 0) \times 2^{-\text{Bias}} = \pm 0.0$$

Two representations of 0.0!

MK MORGAN KAUFMANN

Infinities and NaNs

- Exponent = 111...1, Fraction = 000...0
  - ±Infinity
  - Can be used in subsequent calculations, avoiding need for overflow check
- Exponent = 111...1, Fraction ≠ 000...0
  - Not-a-Number (NaN)
  - Indicates illegal or undefined result
    - e.g., 0.0 / 0.0
  - Can be used in subsequent calculations

Floating-Point Addition

- Consider a 4-digit decimal example
  - $9.999 \times 10^{1} + 1.610 \times 10^{-1}$
- Assumption: we can only store 4 digits
- 1. Align decimal points
  - Shift number with smaller exponent
  - $9.999 \times 10^{1} + 0.016 \times 10^{1}$
- 2. Add significands
  - $9.999 \times 10^{1} + 0.016 \times 10^{1} = 10.015 \times 10^{1}$
- 3. Normalize result & check for over/underflow
  - 1.0015×10^{2}
- 4. Round and renormalize if necessary
  - 1.002×10^{2}

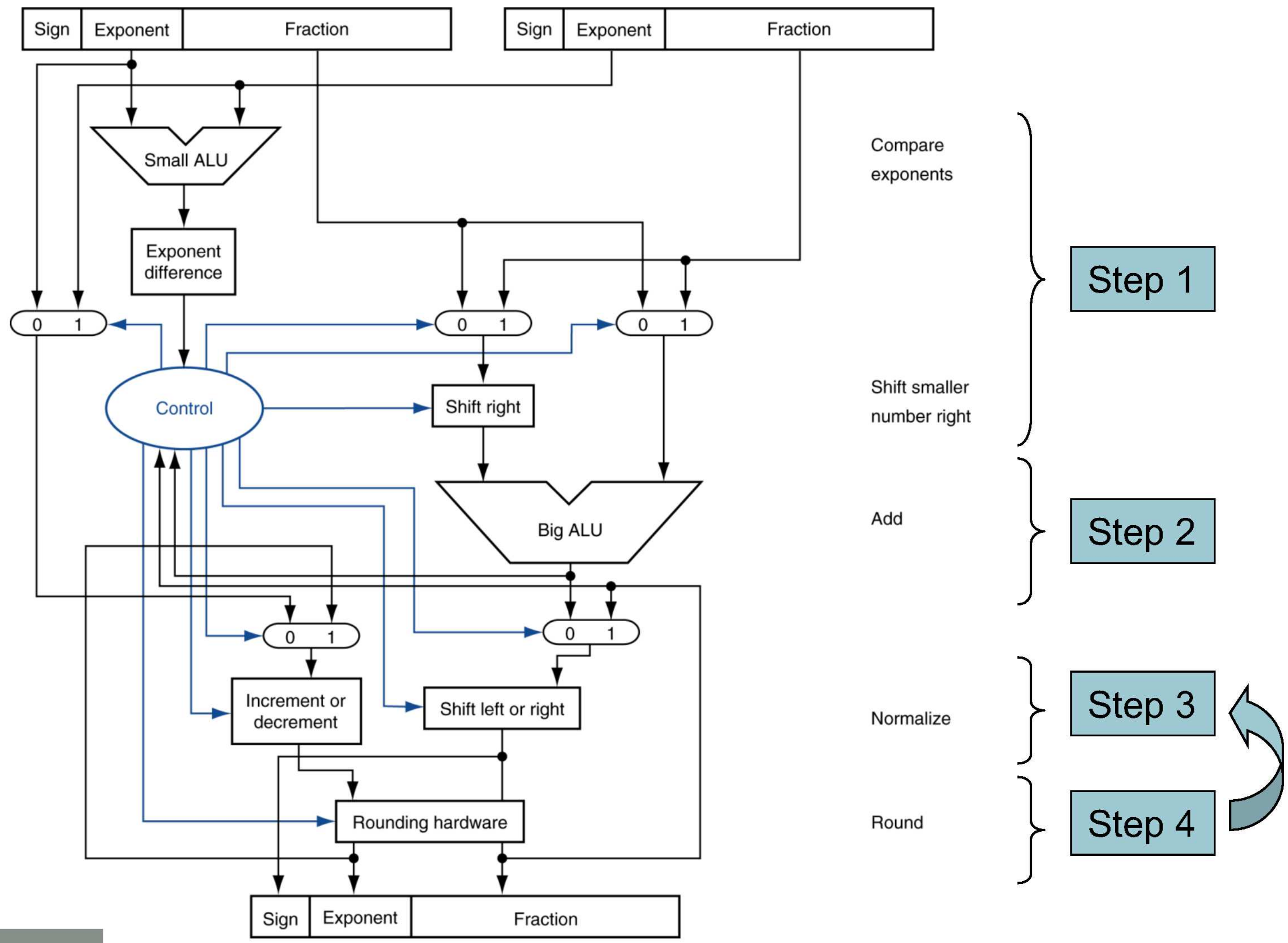
Floating-Point Addition

- Now consider a 4-digit binary example
  - $1.000_2 \times 2^{-1} + -1.110_2 \times 2^{-2}$ (0.5 + –0.4375)
  - Again: just 4 digits precision
- 1. Align binary points
  - Shift number with smaller exponent
  - $1.000_2 \times 2^{-1} + -0.111_2 \times 2^{-1}$
- 2. Add significands
  - $1.000_2 \times 2^{-1} + -0.111_2 \times 2–1 = 0.001_2 \times 2^{-1}$
- 3. Normalize result & check for over/underflow
  - $1.000_2 \times 2^{-4}$, with no over/underflow
- 4. Round and renormalize if necessary
  - $1.000_2 \times 2^{-4}$ (no change) = 0.0625

FP Adder Hardware

- Much more complex than integer adder
- Doing it in one clock cycle would take too long
  - Much longer than integer operations
  - Slower clock would penalize all instructions
- FP adder usually takes several cycles
  - Can be pipelined

FP Adder Hardware



Floating-Point Multiplication

- Consider a 4-digit decimal example
  - $1.110 \times 10^{10} \times 9.200 \times 10^{-5}$
- 1. Add exponents
  - For biased exponents, subtract bias from sum
  - New exponent = 10 + –5 = 5
- 2. Multiply significands
  - $1.110 \times 9.200 = 10.212 \Rightarrow 10.212 \times 10^{5}$
- 3. Normalize result & check for over/underflow
  - 1.0212×10^{6}
- 4. Round and renormalize if necessary
  - 1.021×10^{6}
- 5. Determine sign of result from signs of operands
  - $+1.021 \times 10^{6}$

Floating-Point Multiplication

- Now consider a 4-digit binary example
  - $1.000_2 \times 2^{-1} \times -1.110_2 \times 2^{-2}$ (0.5×-0.4375)
- 1. Add exponents
  - Unbiased: $-1 + -2 = -3$
  - Biased: $(-1 + 127) + (-2 + 127)-127 = -3 + 254 - 127 = -3 + 127$
- 2. Multiply significands
  - $1.000_2 \times 1.110_2 = 1.1102 \Rightarrow 1.110_2 \times 2^{-3}$
- 3. Normalize result & check for over/underflow
  - $1.110_2 \times 2^{-3}$ (no change) with no over/underflow
- 4. Round and renormalize if necessary
  - $1.110_2 \times 2^{-3}$ (no change)
- 5. Determine sign: $+\text{ve} \times -\text{ve} \Rightarrow -\text{ve}$
  - $-1.110_2 \times 2^{-3} = -0.21875$

FP Arithmetic Hardware

- FP multiplier is of similar complexity to FP adder
  - But uses a multiplier for significands instead of an adder
- FP arithmetic hardware usually does
  - Addition, subtraction, multiplication, division, reciprocal, square-root
  - FP $\leftrightarrow$ integer conversion
- Operations usually takes several cycles
  - Can be pipelined

FP Instructions in RISC-V

- Separate FP registers: f0, …, f31
  - double-precision
  - single-precision values stored in the lower 32 bits
- FP instructions operate only on FP registers
  - Programs generally don't do integer ops on FP data, or vice versa
  - More registers with minimal code-size impact
- FP load and store instructions
  - `flw, fld`
  - `fsw, fsd`

FP Instructions in RISC-V

- Single-precision arithmetic
  - `fadd.s, fsub.s, fmul.s, fdiv.s, fsqrt.s`
    - e.g., `fadds.s f2, f4, f6`
- Double-precision arithmetic
  - `fadd.d, fsub.d, fmul.d, fdiv.d, fsqrt.d`
    - e.g., `fadd.d f2, f4, f6`
- Single- and double-precision comparison
  - `feq.s, flt.s, fle.s`
  - `feq.d, flt.d, fle.d`
  - Result is 0 or 1 in integer destination register
    - Use beq, bne to branch on comparison result
- Branch on FP condition code true or false
  - `B.cond`

FP Example: °F to °C

- C code:

```
float f2c (float fahr) {
  return ((5.0/9.0)*(fahr - 32.0));
}
```

  - fahr in f10, result in f10, literals in global memory space
- Compiled RISC-V code:

```
f2c:
 flw    f0,const5(x3)  // f0 = 5.0f
 flw    f1,const9(x3)  // f1 = 9.0f
 fdiv.s f0, f0, f1  // f0 = 5.0f / 9.0f
 flw    f1,const32(x3) // f1 = 32.0f
 fsub.s f10,f10,f1  // f10 = fahr – 32.0
 fmul.s f10,f0,f10  // f10 = (5.0f/9.0f) * (fahr–32.0f)
 jalr   x0,0(x1)    // return
```

FP Example: Array Multiplication

- $C = C + A \times B$
  - All 32 × 32 matrices, 64-bit double-precision elements
- C code:

```
void mm (double c[][],
         double a[][], double b[][]) {
  size_t i, j, k;
  for (i = 0; i < 32; i = i + 1)
    for (j = 0; j < 32; j = j + 1)
      for (k = 0; k < 32; k = k + 1)
        c[i][j] = c[i][j]
                  + a[i][k] * b[k][j];
}
```

  - Addresses of `c`, `a`, `b` in x10, x11, x12, and `i`, `j`, `k` in x5, x6, x7

FP Example: Array Multiplication

- RISC-V code:

```
mm:...
        li    x28,32        // x28 = 32 (row size/loop end)
        li    x5,0          // i = 0; initialize 1st for loop
L1:     li    x6,0          // j = 0; initialize 2nd for loop
L2:     li    x7,0          // k = 0; initialize 3rd for loop
        slli  x30,x5,5      // x30 = i * 2**5 (size of row of c)
        add   x30,x30,x6    // x30 = i * size(row) + j
        slli  x30,x30,3     // x30 = byte offset of [i][j]
        add   x30,x10,x30   // x30 = byte address of c[i][j]
        fld   f0,0(x30)     // f0 = c[i][j]
L3:     slli  x29,x7,5      // x29 = k * 2**5 (size of row of b)
        add   x29,x29,x6    // x29 = k * size(row) + j
        slli  x29,x29,3     // x29 = byte offset of [k][j]
        add   x29,x12,x29   // x29 = byte address of b[k][j]
        fld   f1,0(x29)     // f1 = b[k][j]
```

FP Example: Array Multiplication

…

```
slli   x29,x5,5      // x29 = i * 2**5 (size of row of a)
add    x29,x29,x7    // x29 = i * size(row) + k
slli   x29,x29,3     // x29 = byte offset of [i][k]
add    x29,x11,x29   // x29 = byte address of a[i][k]
fld    f2,0(x29)     // f2 = a[i][k]
fmul.d f1, f2, f1    // f1 = a[i][k] * b[k][j]
fadd.d f0, f0, f1    // f0 = c[i][j] + a[i][k] * b[k][j]
addi   x7,x7,1       // k = k + 1
bltu   x7,x28,L3     // if (k < 32) go to L3
fsd    f0,0(x30)     // c[i][j] = f0
addi   x6,x6,1       // j = j + 1
bltu   x6,x28,L2     // if (j < 32) go to L2
addi   x5,x5,1       // i = i + 1
bltu   x5,x28,L1     // if (i < 32) go to L1
```