A Simple Method for Computing Some Pseudo-Elliptic Integrals in Terms of Elementary Functions

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DRAFT version: 16/04/2020:12:54:48

Abstract

We introduce a method for computing some pseudo-elliptic integrals in terms of elementary functions. The method is simple and fast in comparison to the algebraic case of the Risch-Trager-Bronstein algorithm[13][15][1]. This method can quickly solve many pseudo-elliptic integrals, which other well-known computer algebra systems (CAS) either fail, return an answer in terms of special functions, or require more than 20 seconds of computing time. Unlike the symbolic integration algorithms of Risch[13], Davenport[3], Trager[15], Bronstein[1] and Miller[9]; our method is not a decision process. The implementation of this method is less than 200 lines of Mathematica code and can be easily ported to other CAS that can solve systems of linear equations.

1 Introduction

The problem of finding elementary solutions to integrals of algebraic functions has challenged mathematicians for centuries. In 1905, Hardy conjectured that the problem may be unsolvable[7]. Sophisticated algorithms have been developed, including the famed Risch algorithm[13] and its modern variants by Davenport[3], Trager[15], Bronstein[1] and Miller[9]. However, it is known that the implementation of these algorithms are highly complex and sometimes fail, are incomplete or hang[5].

We will be describing a seemingly new method for computing pseudo-elliptic integrals.

Definition 1. For our purposes, a pseudo-elliptic integral is of the form

$$\int \frac{a(x)}{b(x)} p(x)^{n/m} dx,$$

where $a(x), b(x), p(x)$ are polynomials, $\deg_x(p(x)) > 2$, $\gcd(n, m) = 1$.

Before we describe our method, we begin with some background and motivating examples.

The *derivative divides* method is a substitution method that finds all composite functions, $u = g(x)$, of the integrand $f(x)$, and tests if $f(x)$ divided by the derivative of u is independent of x after the substitution of $u = g(x)$. In other words, up to a constant factor the derivative divides method simplifies integrals of the form $\int f(g(x))g'(x)dx$ to $\int f(u)\,du$. The following integral illustrates the derivative divides method

$$\int x^2\sqrt{1+x^3}dx = \int \frac{\sqrt{u}}{3}\,du,$$

where $u = 1 + x^3$ and $du = 3x^2dx$. This method was first implemented in Moses symbolic integrator, SIN, in 1967[10] and is used in some CAS prior to calling more advanced algorithms[6, pp. 473-474].

A slightly more difficult example where the derivative divides method fails is

$$\int \sqrt{x-1+\sqrt{x-1}}\,dx.$$

In this case we make the substitution $u = \sqrt{x-1}$, then $2udu = dx$. Furthermore, we need to express $x - 1$ in terms of u, which is $u^2 = x - 1$. Then the integral becomes

$$\int 2u\sqrt{u^2+u}\,du.$$

While this integral was somewhat more difficult than the previous example, we could still pick a composite function, $\sqrt{x-1}$, of our integrand, $\sqrt{x-1+\sqrt{x-1}}$, to use as our u substitution. In the following well-known example, this is not immediately possible

$$\int \frac{x^2-1}{(x^2+1)\sqrt{1+x^4}}\,dx.$$

A common approach to solve this integral is rearranging the integrand into

$$\int \frac{x^2\left(1-1\,/x^2\right)}{x(x+1/x)\sqrt{x^2\left(x^2+1\,/x^2\right)}}\,dx = \int \frac{1-1\,/x^2}{(x+1/x)\sqrt{(x+1/x)^2-2}}\,dx.$$

Then the substitution $u = x + 1/x$, $du = (1 - 1\,/x^2\,)\,dx$ yields the integral

$$\int \frac{du}{u\sqrt{u^2-2}},$$

which can be transformed into a rational function using the Euler substitution[4].

Moses devised methods for integrating specific classes of integrals, these included

$$\int R(x)\exp(p(x))dx,$$

where $R(x)$ is a rational function of x and $p(x)$ is a polynomial in x[10, pp. 85]. Shortly after, these methods were phased-out with the work of Risch. However in most CAS the algebraic case of the Risch algorithm is either partially implemented, not implemented, or contains computational bottlenecks that result in long computations. So for algebraic functions a domain-specific approach still has merit.

2 A canonical form for algebraic functions

We require that the integrand is of the form $\frac{p(x)}{q(x)}r(x)^{n/m}$, where $p(x), q(x), r(x) \in \mathbb{Z}[x]$ and $\gcd(n, m) = 1$. If the integrand is not in this form then we attempt to write it in this form plus a rational function of x. For example,

$$\frac{x^2+1+x\sqrt[4]{x^4+1}}{(x^2+1)\sqrt[4]{x^4+1}} = \frac{y^3}{x^4+1} + \frac{x}{x^2+1},$$

where $y^4 = x^4 + 1$. The rational part, $\frac{x}{x^2+1}$, is integrated using known algorithms for rational function integration[2]. An algorithmic way to create this representation is given in Trager[15], however it requires an integral basis and is beyond the scope of this paper.

3 A method for solving some pseudo-elliptic integrals

Following on from our example integral $\int \frac{x^2-1}{(x^2+1)\sqrt{1+x^4}}\,dx$, where making the substitution $u = x+1/x$ resulted in the integral $\int \frac{du}{u\sqrt{u^2-2}}$. We would like to generalise this method, however the difficulty was in the choice of the algebraic manipulation to the form $\int \frac{1-1/x^2}{(x+1/x)\sqrt{(x+1/x)^2-2}}\,dx$ in order to discover a rational substitution, which simplifies the integral. Consequently our approach does not directly rely on such an algebraic manipulation of the integrand.

Our method attempts to parameterise constants a_0, a_1, a_2, polynomials $a(u), b(u)$ and a substitution of the form

$$u = \frac{s(x)}{x^k}$$

such that

$$\int \frac{p(x)}{q(x)}r(x)^{n/m}dx = \int \frac{a(u)}{b(u)}\left(a_2u^2 + a_1u + a_0\right)^{n/m} du, \tag{3.1}$$

where $\deg_x(r(x)) > 2$ and $\gcd(n, m) = 1$. Consequently, our method does not directly compute the integral, and requires a recursive call to an algebraic integrator[1]. We note that a reduction to this form does not guarantee an elementary solution (for example, when $m > 2, a_2 \neq 0$ an elementary form is often not possible).

Our method is broken into two parts. The first part is computing the *radicand part of the integral*, which is a parameterisation of a_0, a_1, a_2 and the u substitution. The second part is computing the *rational part of the integral*, which is a parameterisation of $a(u)$ and $b(u)$.

The radicand part of the integral. Clearly, if we cannot parameterise the radicand $r(x)$ to the form $a_2u^2 + a_1u + a_0$ for a given substitution, then we cannot find a parameterisation of

[1] If such an integrator is not available then a reasonable implementation could call a lookup table of algebraic forms[11] followed by a rational function integrator[2].

(3.1). Thus, we begin by computing the radicand part of the integral, which requires solving

$$r(x) = \mathrm{num}\left(a_2 u^2 + a_1 u + a_0\right),$$

for the constants a_0, a_1, a_2 and the substitution $u = s(x)/x^k$. We do this by iterating over $0 < d \leq N$ such that $s(x) = \sum_{i=0}^{d} c_i\, x^i$, where for each d we iterate over $0 < h \leq M$ such that $u = s(x)/x^h$. Given a candidate u, we then iterate over the radicands

$$r_1(u) = a_1 u + a_0, \quad r_2(u) = a_2 u^2 + a_0, \quad r_3(u) = a_2 u^2 + a_1 u + a_0,$$

where for each radicand we require $m \mid \deg_x(\mathrm{den}(r_i(u)))$ (otherwise we would have a fractional power in the denominator of $r_i(u)^{n/m}$), then we solve

$$r(x) = \mathrm{num}\left(r_i(u)\right). \tag{3.2}$$

If $\deg_x(r(x)) \neq \deg_x(\mathrm{num}(r_i(u)))$, then a solution does not exist (as we have already considered lower-degree polynomials). Otherwise we solve (3.2) by equating coefficients of x and solving the system of equations for the unknowns a_0, a_1, a_2, c_0, c_1, $\cdots$, c_d. If a solution (or multiple solutions) exists we move to computing the rational part of the integral, otherwise we move onto the next radicand or candidate substitution. If no solution exists to the radicand part of the integral for any candidate u-substitutions, then our method fails to compute the integral.

The rational part of the integral. Given the substitution and solution set of the radicand part of the integral, we now look to solve the rational part of the integral, which is given by

$$\frac{p(x)}{q(x)} = \frac{a(u)}{b(u)} \frac{u'(x)}{\mathrm{den}\left(a_2 u^2 + a_1 u + a_0\right)^{n/m}}, \tag{3.3}$$

where a_0, a_1, a_2, $u(x)$ are known and $a(u)$, $b(u)$ are unknown. The degree bound estimate of $a(u)$ and $b(u)$ is given by $\mathcal{D} = \deg_x(u(x)) + \deg_x(u'(x)) + \max\left(\deg_x(p(x)), \deg_x(q(x))\right)$. We solve (3.3) by increasing the degree, d, of $a(u)$ and $b(u)$ from 1 to the degree bound, $\mathcal{D}$, where for each iteration we solve

$$p(x)\, b(u(x))\, \mathrm{den}\left(a_2 u^2 + a_1 u + a_0\right)^{n/m} - q(x)\, a(u(x))\, u'(x) = 0,$$

where $a(u) = \sum_{i=0}^{d} v_i u^i$, $b(u) = \sum_{i=0}^{d} v_{d+i+1} u^i$, and $a(u(x))$, $b(u(x))$ are rational functions in x after replacing u with the candidate substitution. As before, we equate powers of x and solve for the unknowns v_0, v_1, $\cdots$, v_{2d-1}. If a solution is found, then we have a complete solution to (3.1) and we stop. Otherwise if we have iterated up to the degree bound, and iterated through all solution sets from the radicand part of the integral and we have not computed a solution, then the candidate substitution is rejected and must return to the radicand part of the integral to try the next substitution.

Example 3.1. We will apply the method detailed above to compute the following integral

$$\int \frac{(x^3-2)\sqrt{x^3-x^2+1}}{(x^3+1)^2}\,dx.$$

This integral is already in our canonical form.

The radicand part of the integral. We find the substitution $u = (c_1x^3+c_0)/x^2$ yields a solution to the radicand part of the integral, and is parameterised as follows

$$x^3-x^2+1 = \operatorname{num}\left(a_2u^2+a_1u+a_0\right) = a_2c_1^2x^6+a_1c_1x^5+a_0x^4+2a_2c_0c_1x^3+a_1c_0x^2+a_2c_0^2.$$

Then, equating coefficients of powers of x yields the system of equations

$$\begin{aligned} a_2c_0^2 &= 1\\ a_1c_0 &= -1\\ 2a_2c_0c_1 &= 1\\ a_0 &= 0\\ a_1c_1 &= 0\\ a_2c_1^2 &= 0, \end{aligned}$$

which has no solution, so we use a linear radicand in u as follows

$$x^3-x^2+1 = \operatorname{num}(a_1u+a_0) = a_1c_1x^3+a_0x^2+a_1c_0.$$

Again, we equate coefficients of powers of x and we have the system of equations

$$\begin{aligned} a_1c_0 &= 1\\ a_0 &= -1\\ a_1c_1 &= 1, \end{aligned}$$

which has the solution $a_0=-1, a_1=1, c_0=1, c_1=1$. Thus, the radicand part of the integral is $u-1$, where $u=(1+x^3)/x^2$.

The rational part of the integral. Now we see if a solution exists to the rational part of the integral. The degree bound on the solution to the rational part is 3. When the degree is 1, we have no solution. When the degree is 2, we have

$$\frac{a(u)}{b(u)} = \frac{v_2u^2+v_1u+v_0}{v_5u^2+v_4u+v_3}.$$

For the rational part, we are solving the following equation

$$\frac{(x^3-2)}{(x^3+1)^2} = \left(\frac{v_2u^2+v_1u+v_0}{v_5u^2+v_4u+v_3}\right)\operatorname{den}(u-1)^{-1/2}\,u'(x),$$

where $\operatorname{den}(u-1)^{-1/2}=1/x$. After replacing u with $\left(1+x^3\right)/x^2$ and $u'(x)$ with $\left(x^3-2\right)/x^3$ we have

$$\frac{x^3-2}{\left(x^3+1\right)^2}=\frac{\left(x^3-2\right)\left(u^2v_2+uv_1+v_0\right)}{x^4\left(u^2v_5+uv_4+v_3\right)}=\frac{\left(x^3-2\right)\left(x^4v_0+x^2v_1+x^5v_1+v_2+2x^3v_2+x^6v_2\right)}{x^4\left(x^4v_3+x^2v_4+x^5v_4+v_5+2x^3v_5+x^6v_5\right)},$$

which after clearing denominators is a polynomial equation in x, given by

$$\begin{aligned}&-v_2x^{15}-v_1x^{14}+\left(v_5-v_0\right)x^{13}+\left(v_4-2v_2\right)x^{12}+\left(v_3-v_1\right)x^{11}+\left(2v_2-v_4\right)x^9+\\&\left(3v_1-2v_3\right)x^8+\left(3v_0-3v_5\right)x^7+\left(8v_2-2v_4\right)x^6+5v_1x^5+\left(2v_0-2v_5\right)x^4+7v_2x^3+2v_1x^2+2v_2=0,\end{aligned}$$

which we solve for the undetermined coefficients v_0,v_1,v_2,v_3,v_4,v_5. Then equating coefficients of powers of x yields the system of equations

$$\begin{aligned}2v_2&=0\\2v_1&=0\\7v_2&=0\\2v_0-2v_5&=0\\5v_1&=0\\8v_2-2v_4&=0\\3v_0-3v_5&=0\\3v_1-2v_3&=0\\2v_2-v_4&=0\\-v_1+v_3&=0\\-2v_2+v_4&=0\\-v_0+v_5&=0\\-v_1&=0\\-v_2&=0,\end{aligned}$$

which has the solution $v_0=v_5,v_1=0,v_2=0,v_3=0,v_4=0$. Thus, the rational part of the integral is

$$\frac{v_0}{v_0u^2}=\frac{1}{u^2}$$

and the integral is given by

$$\begin{aligned}&\int\frac{\left(x^3-2\right)\sqrt{x^3-x^2+1}}{\left(x^3+1\right)^2}\,dx=\int\frac{\sqrt{u-1}}{u^2}\,du\\&\quad=-\frac{\sqrt{u-1}}{u}+\tan^{-1}\left(\sqrt{u-1}\right)=-\frac{x\sqrt{x^3-x^2+1}}{x^3+1}+\tan^{-1}\left(\frac{\sqrt{x^3-x^2+1}}{x}\right).\end{aligned}$$

Our implementation in Mathematica took 0.085 seconds to compute this integral.

4 Generalisations of the method

This method can be generalised to include reductions of the form

$$\int \frac{p(x)}{q(x)} r(x)^{n/m} dx = \int \frac{a(u)}{b(u)} \left(a_2 u^{2k} + a_1 u^k + a_0\right)^{n/m} du$$

and

$$\int \frac{p(x)}{q(x)} r(x)^{n/m} dx = \int \frac{a(u)}{b(u)} \left(a_1 u^k + a_0\right)^{n/m} du.$$

For both these forms we use essentially the same procedure as in the linear or quadratic reduction. However, in terms of efficiency, obviously the more forms we try to solve for the radicand part of the integral, the longer the process takes to iterate through these possible forms.

5 A comparison with major CAS and algebraic integration packages

We will compare our method with the Mathematica (12.1.0), Maple (2018.1), REDUCE (5286, 1-Mar-20), and FriCAS (version 1.2.6) computer algebra systems. We will also include in the comparison a table lookup package, RUBI[12], which has been ported to a number of computer algebra systems and compares favourably with most built-in integrators on a large suite of problems[14]. We have also included an experimental algebraic integration package developed in Mathematica by Manuel Kauers[8]. Within the Kauers package we have replaced the calls to Singular in favour of Mathematica's built-in Groebner basis routine.

We have included results from Maple twice. Once with a call of `int(integrand, x)` and once with `int(convert(integrand, RootOf),x)`. This is because the default behaviour of Maple is to not use the Risch-Trager-Bronstein integration algorithm[15][1] for algebraic functions unless the radicals in the integrand are converted to the Maple `RootOf` notation.

Within REDUCE we have used the algint package by James Davenport[3].

Our test suite is 190 integrals that can be found on github[16]. All the integrals in the suite have a solution in terms of elementary functions.

We will show the results from all the systems and packages for one integral from the test suite. It is intriguing to see the variety of forms for this integral.

Our method returns:

$$\int \frac{\left(x^4-1\right)\sqrt{x^4+1}}{x^8+1} dx = -\frac{1}{2\sqrt[4]{2}} \tan^{-1}\left(\frac{\sqrt[4]{2}x}{\sqrt{x^4+1}}\right) - \frac{1}{2\sqrt[4]{2}} \tanh^{-1}\left(\frac{\sqrt[4]{2}x}{\sqrt{x^4+1}}\right)$$

FriCAS returns:

$$
\begin{aligned}
&\int \frac{\left(x^4-1\right)\sqrt{x^4+1}}{x^8+1}\,dx = \\
&\frac{1}{8\sqrt[4]{2}}\log\left(\frac{1}{x^8+1}\left(4x^6+4x^2+\sqrt{2}\left(x^8+4x^4+1\right)-\sqrt{x^4+1}\left(2^{3/4}\left(2x^5+2x\right)+4\sqrt[4]{2}x^3\right)\right)\right)- \\
&\frac{1}{8\sqrt[4]{2}}\log\left(\frac{-1}{x^8+1}\left(4x^6+4x^2+\sqrt{2}\left(x^8+4x^4+1\right)+\sqrt{x^4+1}\left(2^{3/4}\left(2x^5+2x\right)+4\sqrt[4]{2}x^3\right)\right)\right)+ \\
&\frac{1}{2\sqrt[4]{2}}\tan^{-1}\left(\frac{-4x^6-4x^2+\sqrt{2}\left(x^8+4x^4+1\right)}{\sqrt{2}\left(-x^8-1\right)+\sqrt{x^4+1}\left(2^{3/4}\left(2x^5+2x\right)-4\sqrt[4]{2}x^3\right)}\right)
\end{aligned}
$$

Kauer's algorithm returns:

$$
\int \frac{\left(x^4-1\right)\sqrt{x^4+1}}{x^8+1}\,dx = \sum_{512\alpha^4-1=0}\alpha\log\left(4\alpha\sqrt{x^4+1}-x\right)
$$

Maple (default) returns:

$$
\int \frac{\left(x^4-1\right)\sqrt{x^4+1}}{x^8+1}\,dx = \frac{1}{2\sqrt[4]{2}}\tan^{-1}\left(\frac{\sqrt{x^4+1}}{\sqrt[4]{2}x}\right)-\frac{1}{4\sqrt[4]{2}}\log\left(\frac{\frac{\sqrt{x^4+1}}{\sqrt{2}x}+\frac{1}{\sqrt[4]{2}}}{\frac{\sqrt{x^4+1}}{\sqrt{2}x}-\frac{1}{\sqrt[4]{2}}}\right)
$$

Maple (RootOf) returns:

$$
\begin{aligned}
\int \frac{\left(x^4-1\right)\sqrt{x^4+1}}{x^8+1}\,dx = \frac{1}{4\sqrt[4]{2}}\log\left(\frac{2\times 2^{3/4}x^4-8\sqrt{x^4+1}x+4\sqrt[4]{2}x^2+2\times 2^{3/4}}{-2x^4+2\sqrt{2}x^2-2}\right)+ \\
\frac{i}{4\sqrt[4]{2}}\log\left(\frac{2\times 2^{3/4}ix^4-8\sqrt{x^4+1}x-4i\sqrt[4]{2}x^2+2\times 2^{3/4}i}{2x^4+2\sqrt{2}x^2+2}\right)
\end{aligned}
$$

Mathematica returns:

$$
\begin{aligned}
\int \frac{\left(x^4-1\right)\sqrt{x^4+1}}{x^8+1}\,dx = \frac{1}{2}\sqrt[4]{-1}\Big(-2F\left(i\sinh^{-1}\left(\sqrt[4]{-1}x\right)\middle|-1\right)+ \\
\Pi\left(-\sqrt[4]{-1};i\sinh^{-1}\left(\sqrt[4]{-1}x\right)\middle|-1\right)+\Pi\left(\sqrt[4]{-1};i\sinh^{-1}\left(\sqrt[4]{-1}x\right)\middle|-1\right)+ \\
\Pi\left(-(-1)^{3/4};i\sinh^{-1}\left(\sqrt[4]{-1}x\right)\middle|-1\right)+\Pi\left((-1)^{3/4};i\sinh^{-1}\left(\sqrt[4]{-1}x\right)\middle|-1\right)\Big)
\end{aligned}
$$

where F is the elliptic integral of the first kind, and Π is the incomplete elliptic integral.

REDUCE (using algint package) returns:

$$
\int \frac{\left(x^4-1\right)\sqrt{x^4+1}}{x^8+1}\,dx = \int \frac{x^4\sqrt{x^4+1}}{x^8+1}\,dx-\int \frac{\sqrt{x^4+1}}{x^8+1}\,dx
$$

RUBI returns[2]:

$$\int \frac{(x^4-1)\sqrt{x^4+1}}{x^8+1}\,dx = -\frac{\tan^{-1}\left(\frac{\sqrt[4]{2}x}{\sqrt{x^4+1}}\right)}{2\sqrt[4]{2}} - \frac{\tanh^{-1}\left(\frac{\sqrt[4]{2}x}{\sqrt{x^4+1}}\right)}{2\sqrt[4]{2}} +$$

$$\frac{(x^2+1)\sqrt{\frac{x^4+1}{(x^2+1)^2}}F\left(2\tan^{-1}(x)|\frac{1}{2}\right)}{2\sqrt{x^4+1}} + \frac{\left((-1-i)-i\sqrt{2}\right)(x^2+1)\sqrt{\frac{x^4+1}{(x^2+1)^2}}F\left(2\tan^{-1}(x)|\frac{1}{2}\right)}{8\sqrt{x^4+1}} +$$

$$\frac{\left(\sqrt{2}+(-1+i)\right)i\,(x^2+1)\sqrt{\frac{x^4+1}{(x^2+1)^2}}F\left(2\tan^{-1}(x)|\frac{1}{2}\right)}{8\sqrt{x^4+1}} +$$

$$\frac{\left(\sqrt{2}+(1+i)\right)i\,(x^2+1)\sqrt{\frac{x^4+1}{(x^2+1)^2}}F\left(2\tan^{-1}(x)|\frac{1}{2}\right)}{8\sqrt{x^4+1}} -$$

$$\frac{\left(\frac{1}{8}-\frac{i}{8}\right)\left(1+(-1)^{3/4}\right)(x^2+1)\sqrt{\frac{x^4+1}{(x^2+1)^2}}F\left(2\tan^{-1}(x)|\frac{1}{2}\right)}{\sqrt{x^4+1}}$$

where F is the elliptic integral of the first kind, and Π is the incomplete elliptic integral.

The table below summarises the comparison between all systems on the test suite of integrals. The expression size is the leaf count(`LeafCount`) in Mathematica.

Table 1: A comparison of our method with major CAS and algebraic integration packages.

CAS /package	Elementary forms [%]	Contains $\int dx$ [%]	Contains special functions [%]	Timed-out (>20s) [%]	Median time [s]	Expression size - string length
new	100.0	0.0	0.0	0.0	0.26	81–118
Maple (`RootOf`)	91.6	2.0	0.0	6.3	4.32	136–238
FriCAS	70.0	4.7	0.0	25.3	0.26	. –129
Kauers	62.6	7.9	0.0	29.4	0.40	88–116
RUBI	13.7	60.0	16.3	10.0	0.34	244–405
Maple	11.6	55.3	33.2	0.0	0.34	436–943
Mathematica	9.5	44.7	43.2	2.63	1.21	574–1019
REDUCE (algint)	6.3	65.3	0.0	28.4	1.16	.

In regards to this table: The median time excludes integrals that timed-out. For FriCAS and REDUCE we did not find a built-in routine to compute the leaf count.

[2] After posting a preprint of this comparison on the sci.math.symbolic newsgroup, Albert Rich (the creator of RUBI) devised a general rule for integrals of this form which will be included in the next release of RUBI:

$$\int \frac{(f+gx^4)\sqrt{d+ex^4}}{a+bx^4+cx^8} = \frac{e^2 f}{2cd\sqrt[4]{2de-\frac{be^2}{c}}}\tan^{-1}\left(\frac{x\sqrt[4]{2de-\frac{be^2}{c}}}{\sqrt{d+ex^4}}\right) + \frac{e^2 f}{2cd\sqrt[4]{2de-\frac{be^2}{c}}}\tanh^{-1}\left(\frac{x\sqrt[4]{2de-\frac{be^2}{c}}}{\sqrt{d+ex^4}}\right),$$

when $ef+dg=0$ and $cd^2-ae^2=0$.

Mathematica is far more comfortable in returning an answer in terms of elliptic functions, but these results are far from concise. Computing the integral in terms of elliptic functions takes considerable time.

Maple fares much better when it uses the algebraic case of the Risch-Trager-Bronstein algorithm. When the integrand is converted to `RootOf` notation in Maple, the time required increases significantly, however the results improve significantly. We are then left wondering why Maple does not make this conversion internally if the initial algebraic integration routines fail?

FriCAS is a fork of AXIOM. FriCAS inherits the most complete implementation of the Risch-Trager-Bronstein algorithm for integration in finite (elementary) terms. The results suggest that the FriCAS (version 1.2.6) integrator is either fast, hangs, or returns an error. We have reported these issues to the FriCAS developers and most issues will be fixed in the next release[18]. Thus, the performance of FriCAS on our test suite of integrals will be much better, with the exception of incomplete parts of the Risch-Bronstein-Trager algorithm for algebraic functions, for example after 16 seconds of computing time FriCAS (version 1.2.6) returns

```
(1) -> integrate(((4+x^6)*(-4+x^4+2*x^6)*
                       (32-14*x^4-32*x^6-4*x^8+7*x^10+8*x^12)^(1/2))/(x^9*(-2+x^6)),x)

   >> Error detected within library code:
   integrate: implementation incomplete (residue poly has multiple non-linear factors)
```

When Kauer's heuristic does not time out, then its fast, with only 11 problems taking longer than 2 seconds. It also often returns a concise form. Miller[9] has extended Kauer's heuristic to a complete algorithm for the logarithmic part of the integral of a mixed algebraic-transcendental function. Unfortunately we did not have access to an implementation for comparison.

6 Conclusions

We have shown that we can efficiently solve some pseudo-elliptic integrals. Our method compares favourably with major CAS and algebraic integration packages.

The computational burden of our method is low, as the core computational routine requires solving multiple systems of linear equations, and consequently our method should be tried before the more computationally expensive algorithms of Trager[15], Bronstein[1], Kauers[8], or Miller[9].

Our method, relative to the algebraic case of the Risch-Bronstein-Trager algorithm, is very simple to implement. Our exemplar implementation in Mathematica is only a couple of hundred lines of code and relies heavily on `SolveAlways` for solving polynomial equations with undetermined coefficients. The implementation is available on github[17].

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