Simulation results

General model:

$$\mu(t|Z) = f(t, \beta_0^\top Z) g(\gamma_0^\top Z),$$

where for fixed $x \in \mathbb{R}$, $f(\cdot, x)$ is an unspecified density function on $[0, \tau]$, and $g(x)$ is unknown but monotone in x. We assume $\|\beta_0\| = \|\gamma_0\| = 1$.

Simulation settings used in the paper:

- Z is generated from a multivariate truncated normal distribution satisfying $Z \sim N_2(0, I_2)$ and $\|Z\| \leq 1$.
- Censoring time is an exponential distribution with mean $10 \cdot (1 + |z_1|)$.
- Recurrent event times are generated from Poisson process with rate functions:

  M1: $\mu(t|Z) = \mu_0(t) \exp(\gamma_0^\top Z)$
  - $\mu_0(t) = \frac{2}{1+t}$.
  - $\beta_0 = (\beta_1, \beta_2) = (0, 0)$, $\gamma_0 = (\gamma_1, \gamma_2) = (0.28, 0.96)$.

  M2: $\mu(t|Z) = \mu_0(t) + \alpha_0^\top Z$
  - $\mu_0(t) = e^{0.1t}$.
  - $\beta_0 = \gamma_0 = \alpha_0 = (0.6, 0.8)$.

  M3: $\mu(t|Z) = \mu_0\{t \exp(\alpha_0^\top Z)\}$
  - $\mu_0(t) = e^{-t}$.
  - $\beta_0 = \gamma_0 = \alpha_0 = (0.6, 0.8)$.

  M4: $\mu(t|Z) = \mu_0\{t, \exp(\beta_0^\top Z)\} \exp(\gamma_0^\top Z)$
  - $\mu_0(t, x) = \frac{t \cdot (1-t)^{1+x}}{B(2, 1+x)}$
  - $\beta_0 = (0.6, 0.8)$, $\gamma_0 = (0.28, 0.96)$
- Set $\tau = 10$ for **M1, M2, M3** and $\tau = 1$ for **M4**.
- **M1-ind** solves γ_0 under shape-independence.

Table 1: Point estimator (PE), empirical standard error (ESE) and asymptotic standard error (ASE) for **M1**–**M4** with 1000 replications.

	$n=50$			$n=100$			$n=200$			$n=500$		
	PE	ESE	ASE	PE	ESE	ASE	PE	ESE	ASE	PE	ESE	ASE
Scenario **M1**; without assuming shape-independent.												
β_1	0.302	0.626	0.488	0.196	0.662	0.502	0.151	0.667	0.515	0.102	0.679	0.520
β_2	0.300	0.654	0.493	0.276	0.670	0.515	0.264	0.681	0.526	0.152	0.712	0.539
γ_1	0.273	0.114	0.133	0.276	0.082	0.087	0.274	0.076	0.060	0.279	0.037	0.036
γ_2	0.955	0.032	0.038	0.957	0.025	0.025	0.958	0.029	0.018	0.960	0.010	0.011
Scenario **M1**; assuming shape-independent.												
γ_1	0.264	0.118	0.141	0.272	0.077	0.089	0.276	0.054	0.059	0.275	0.033	0.035
γ_2	0.957	0.031	0.038	0.959	0.021	0.023	0.960	0.015	0.016	0.961	0.009	0.010
Scenario **M2**.												
β_1	−0.194	0.655	0.566	−0.435	0.456	0.522	−0.592	0.197	0.379	−0.597	0.073	0.099
β_2	−0.296	0.669	0.567	−0.606	0.486	0.520	−0.743	0.245	0.396	−0.797	0.054	0.094
γ_1	0.591	0.116	0.132	0.599	0.086	0.092	0.593	0.076	0.067	0.597	0.051	0.047
γ_2	0.792	0.099	0.109	0.793	0.068	0.078	0.800	0.052	0.053	0.797	0.085	0.036
Scenario **M3**.												
β_1	−0.080	0.670	0.576	−0.376	0.513	0.545	−0.584	0.172	0.407	−0.602	0.057	0.088
β_2	−0.304	0.673	0.570	−0.594	0.492	0.528	−0.775	0.170	0.388	−0.795	0.042	0.083
γ_1	−0.415	0.493	0.531	−0.561	0.220	0.397	−0.594	0.083	0.150	−0.599	0.047	0.050
γ_2	−0.567	0.514	0.522	−0.765	0.225	0.371	−0.798	0.055	0.140	−0.797	0.056	0.039
Scenario **M4**.												
β_1	−0.070	0.672	0.569	−0.312	0.548	0.547	−0.556	0.269	0.445	−0.597	0.063	0.132
β_2	−0.271	0.686	0.570	−0.534	0.564	0.533	−0.743	0.259	0.419	−0.798	0.047	0.125
γ_1	0.251	0.217	0.231	0.266	0.142	0.153	0.274	0.094	0.102	0.280	0.059	0.062
γ_2	0.941	0.068	0.078	0.953	0.041	0.042	0.957	0.025	0.029	0.958	0.017	0.019

Table 2: Summary of rejection proportions based on the simulation scenarios **M1**–**M4**. The rejection proportions are computed based on $n = 50, 100, 200, 500$ observations with 1000 replications at $\alpha = 0.05$. The resampling size is 200.

	n			
	50	100	200	500
M1	0.031	0.037	0.046	0.052
M2	0.281	0.700	0.960	1.000
M3	0.382	0.831	0.996	1.000
M4	0.287	0.686	0.963	1.000

Rank estimating equation with induced smoothing The rank estimating equation

$$U_1(\beta) = \frac{1}{n(n-1)} \sum_{i \neq j} I(\beta^\intercal Z_i > \beta^\intercal Z_j) \int_0^{C_{ij}} \int_0^{C_{ij}} I(u > t) dN_i(u) dN_j(t),$$

is sensitive to initial value; local maximums are often detected instead of the global maximum.

The motivation of the induced smoothing technique is to overcome computational difficulty caused by non-smooth estimating equations. The basic idea is to replace the estimating equation $U_1(\beta)$ with a smooth version $E(\beta + \Gamma Z)$, where Z is a p-dimensional standard normal random vector, $\Gamma^\top \Gamma = \Sigma$ is the a working covariance matrix of $\hat{\beta}_n$, and the expectation is taken with respect to Z. The smoothed version of $U_1(\beta)$ is then

$$\tilde{U}_1(\beta) = \frac{1}{n(n-1)} \sum_{i \neq j} \Phi\left(\frac{\beta^\intercal Z_i - \beta^\intercal Z_j}{r_{ij}}\right) \int_0^{C_{ij}} \int_0^{C_{ij}} I(u > t) dN_i(u) dN_j(t), \tag{1}$$

where $r_{ij}^2 = (X_i - X_j)^\top \Gamma_n (X_i - X_j)$. In my other papers, we tried different choices of Γ but all yields similar results, so I usually set Γ to be the identity matrix.

In Figure 1, I plotted the **negative** $U_1(\cdot)$ and $\tilde{U}_1(\cdot)$ values against θ for one simulated data set with $n = 100$ from scenario M2. (so we are looking to find the global minimum here). There are many local minimums for the unsmoothed estimating equation, causing the `optim` and `spg` to converge to one of the local minimums. However, we don't have such problem with the smoothed version. Simulation results that shows the β estimates by optimizing the smoothed estimating equation based on 200 replications is presented in Table 3.

Figure 1: Comparison between $U_1(\beta)$ and $\tilde{U}_1(\beta)$.

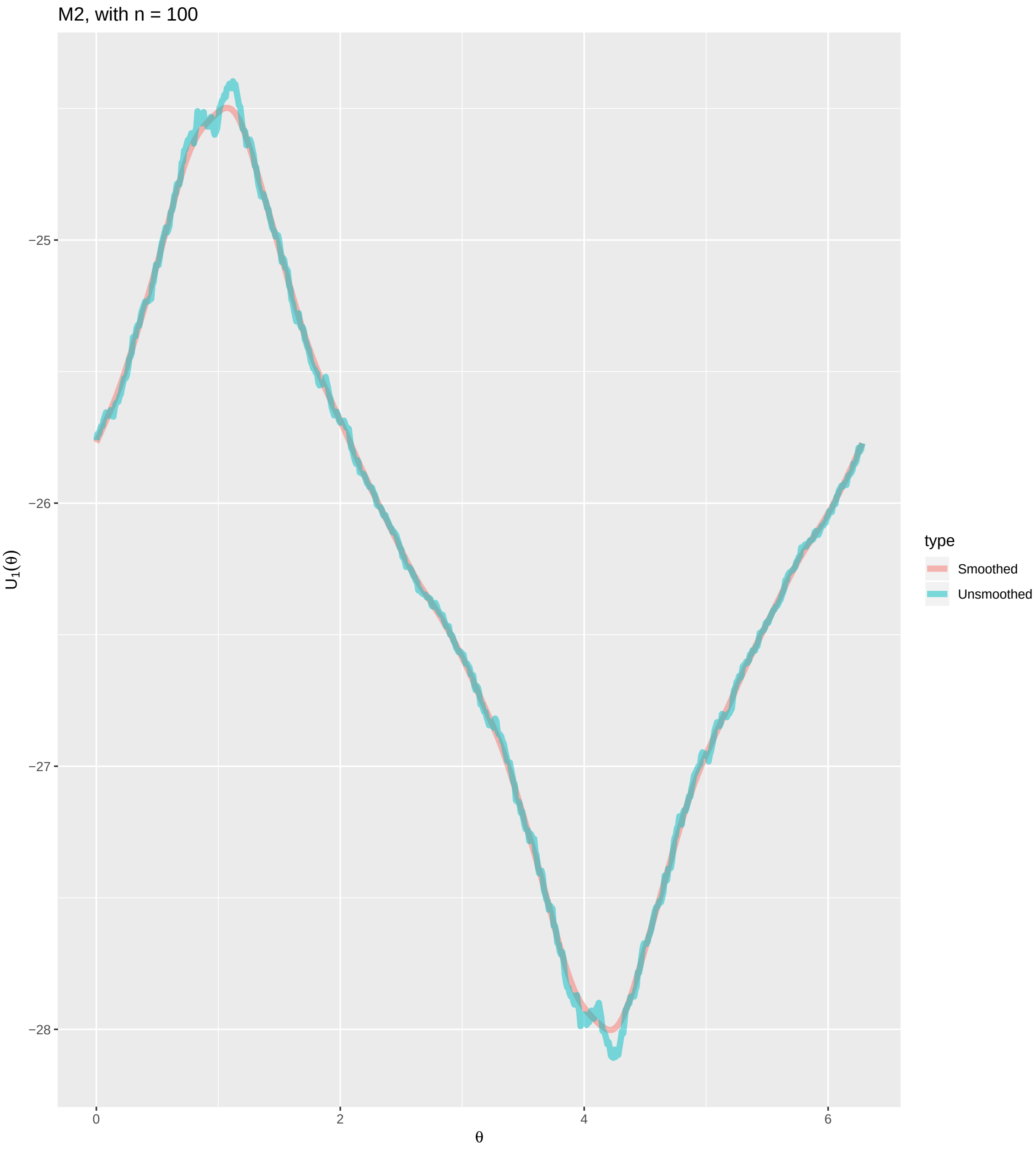

Table 3: Simulation results based on induced smoothing equation (1). With 200 replications. All replications converged to global maximum (verified with an exhaust grid search). Γ is set to be the identity matrix.

	$n = 100$		$n = 200$		$n = 500$	
	PE	ESE	PE	ESE	PE	ESE
			Scenario **M2**			
β_1	-0.582	0.184	-0.599	0.125	-0.593	0.075
β_2	-0.764	0.209	-0.784	0.104	-0.799	0.055
γ_1	0.588	0.118	0.599	0.057	0.593	0.033
γ_2	0.786	0.150	0.797	0.042	0.804	0.024
			Scenario **M3**			
β_1	-0.601	0.161	-0.591	0.096	-0.601	0.052
β_2	-0.774	0.124	-0.797	0.075	-0.797	0.040
γ_1	-0.602	0.120	-0.595	0.066	-0.600	0.044
γ_2	-0.782	0.113	-0.799	0.051	-0.798	0.033