Paths in Graphs: Breadth-First Search

Michael Levin

Department of Computer Science and Engineering
University of California, San Diego

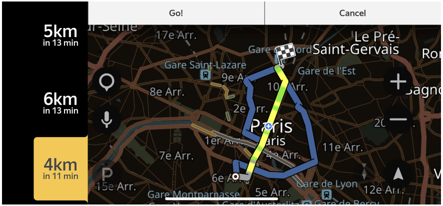
Graph Algorithms
Algorithms and Data Structures

Outline

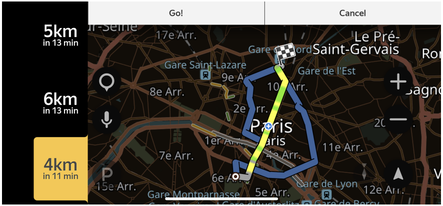
Applications

Applications

Applications



Applications



Minimize Transfers

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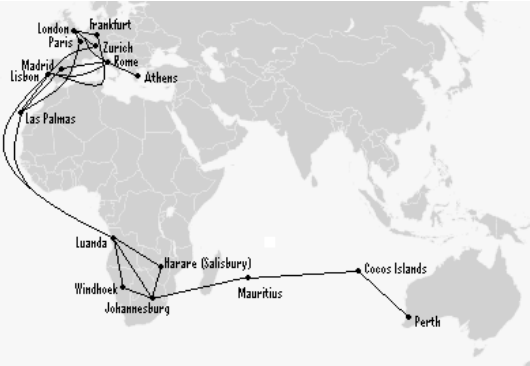

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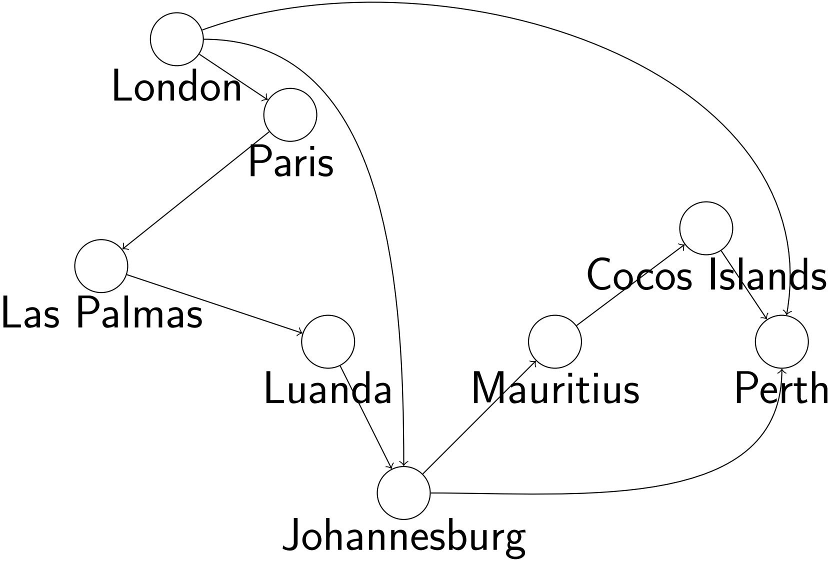
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Luanda
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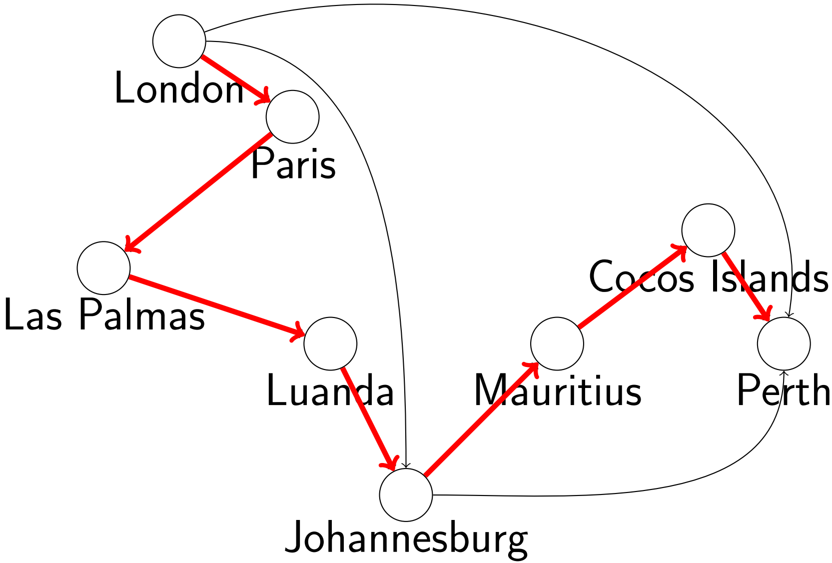
Mauritius
Cocos Islands
Perth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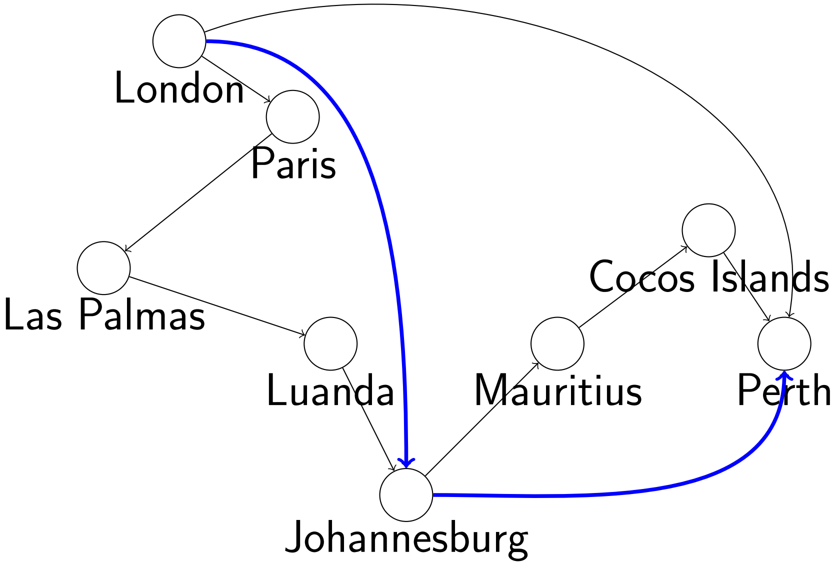
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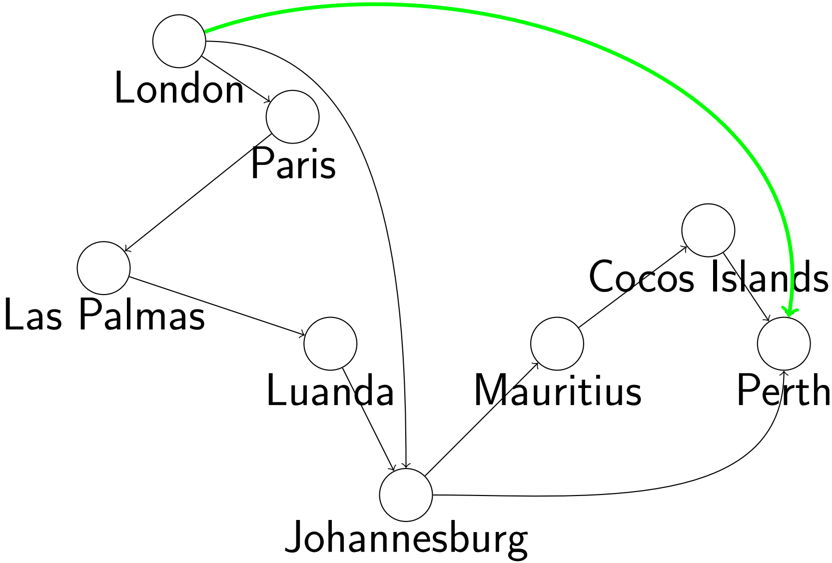
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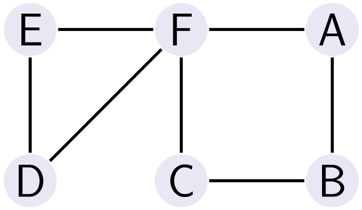
Outline

Path Length

Path length $L(P)$ is the number of edges in the path.



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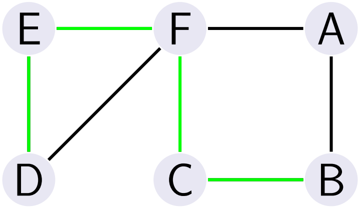

$$L(B - C - F - E - D) = 4$$

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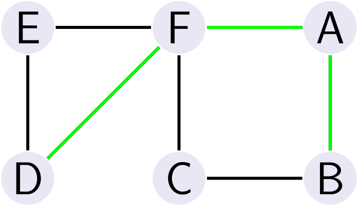
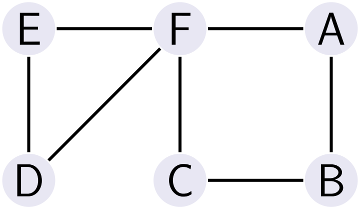


$L(B - C - F - E - D) = 4$
$L(B - A - F - D) = 3$

Distance

The distance between two nodes is the length of the shortest path between them.



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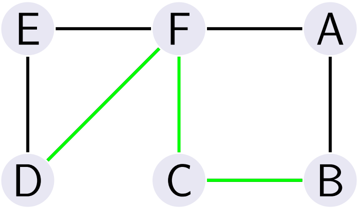
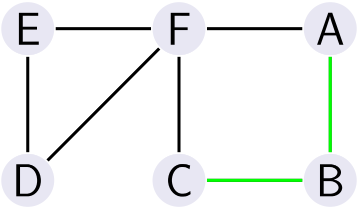


$d(B, D) = 3$

Distance

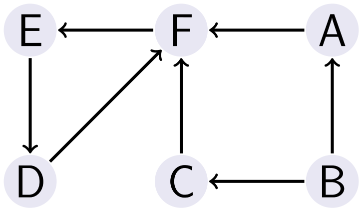
The distance between two nodes is the length of the shortest path between them.



$d(B, D) = 3$
$d(A, C) = 2$

Distance

The distance between two vertices is the length of the shortest path between them.



Distance

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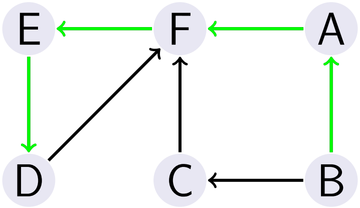

$d(B, D) = 4$

Distance

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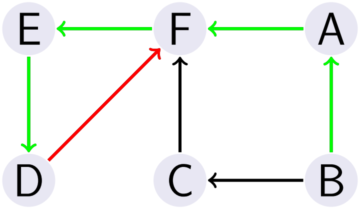

$d(B, D) = 4$

Distance

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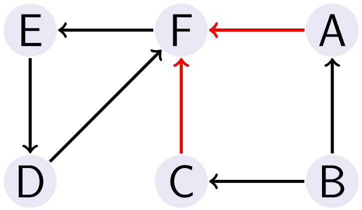

$d(B, D) = 4$
$d(A, C) = \infty$

Distance

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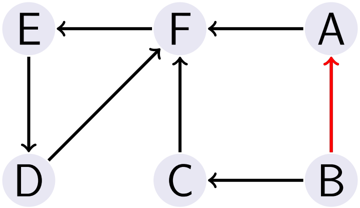

$d(B, D) = 4$
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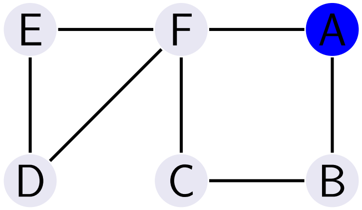
Distance



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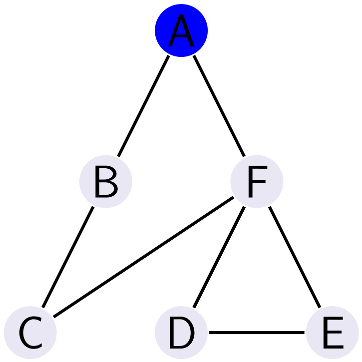

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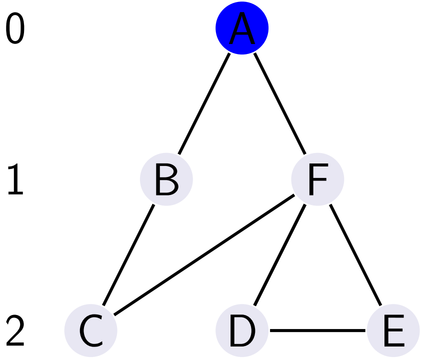

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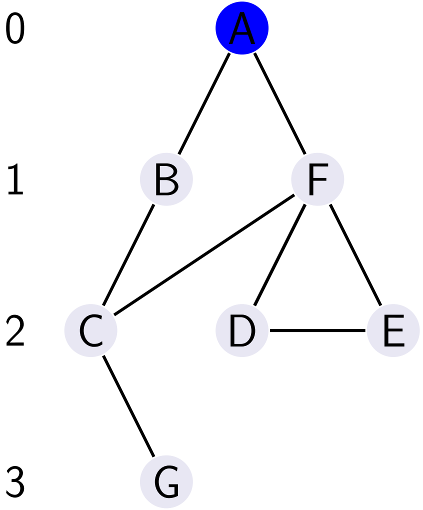

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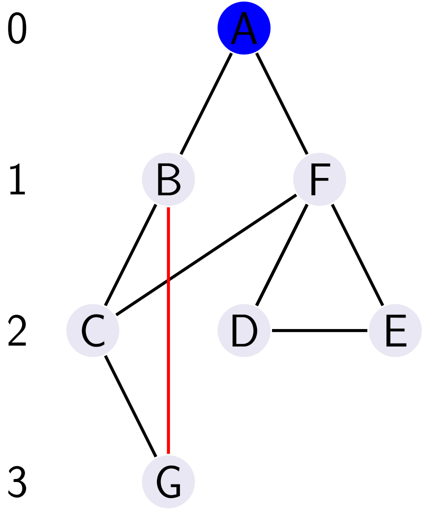

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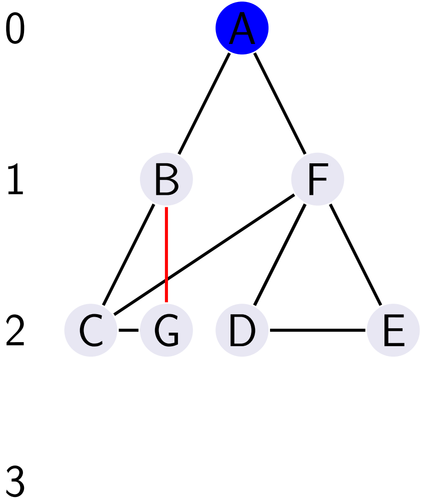

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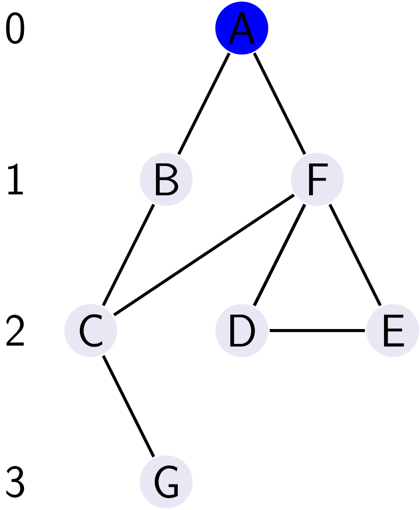

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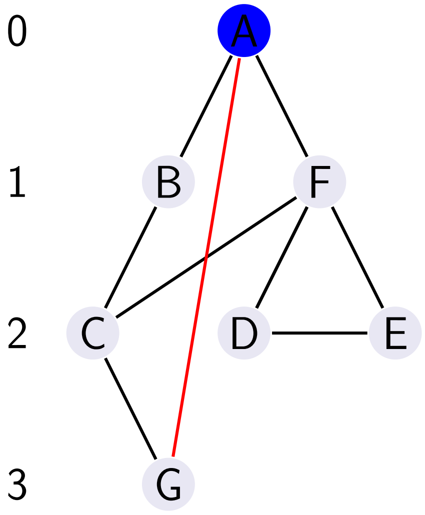

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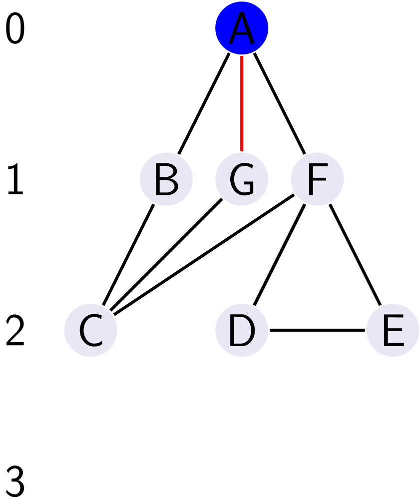

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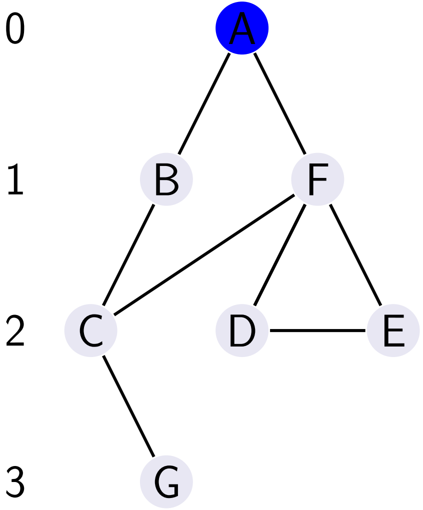

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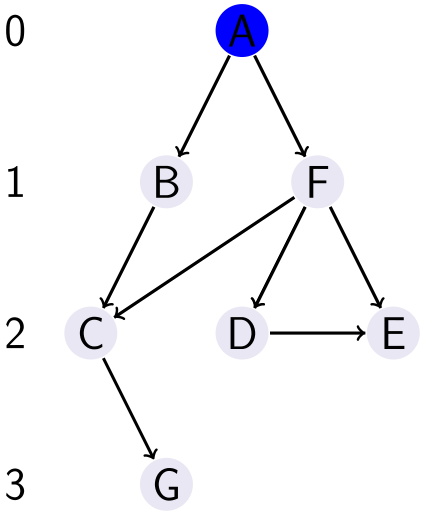

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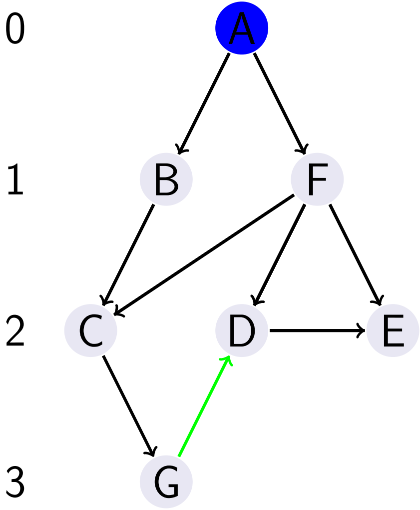

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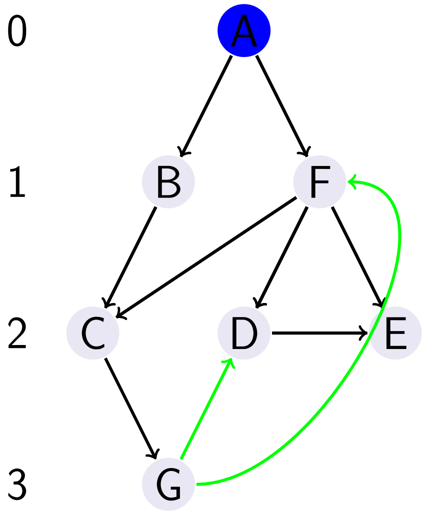

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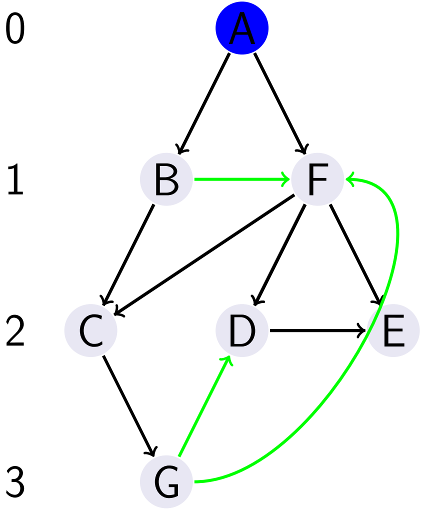

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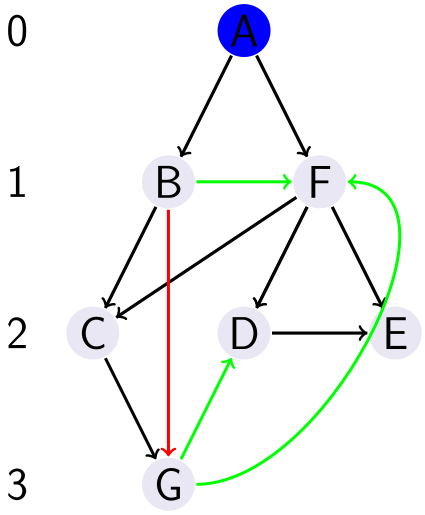
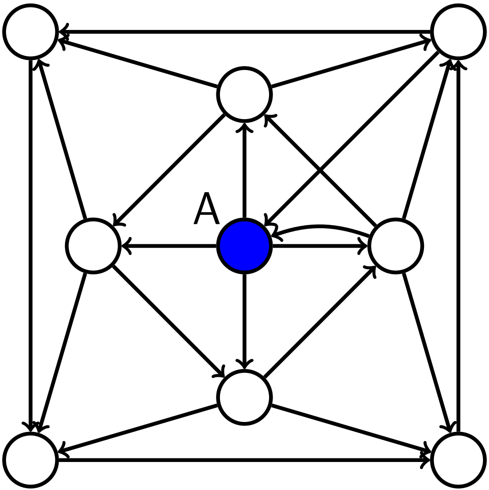


Outline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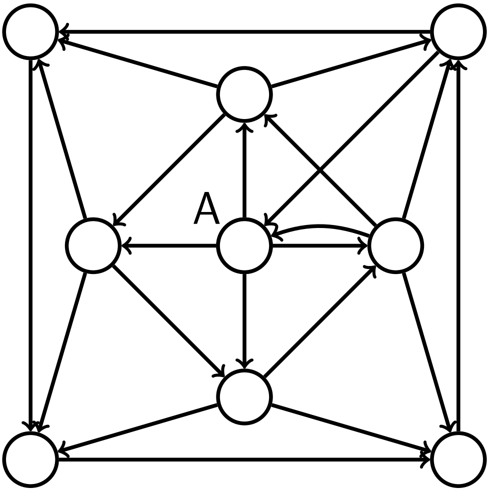
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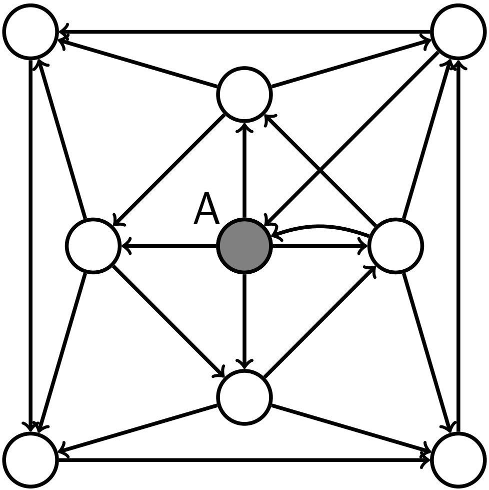
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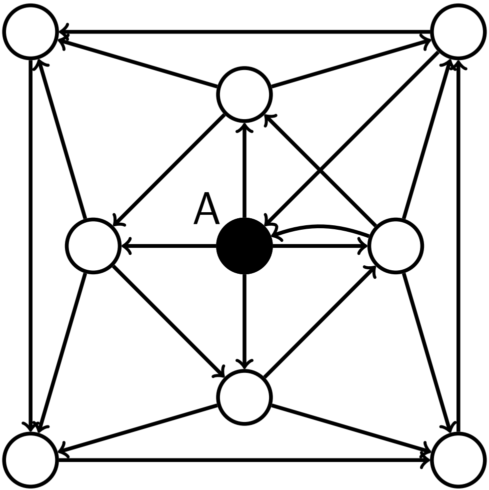
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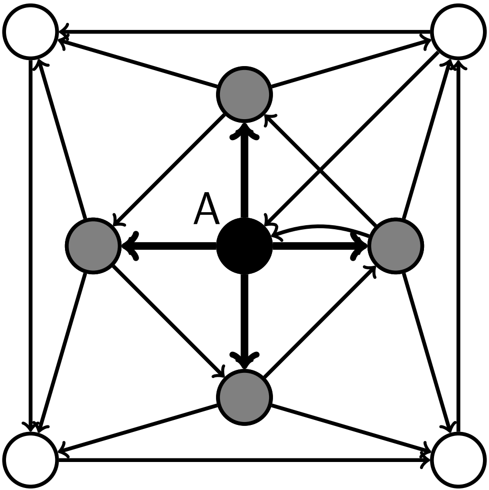
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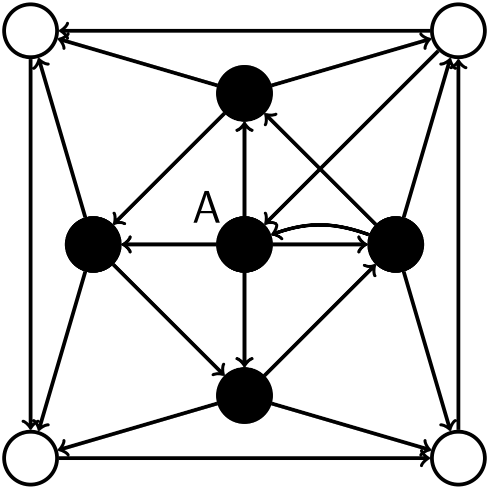
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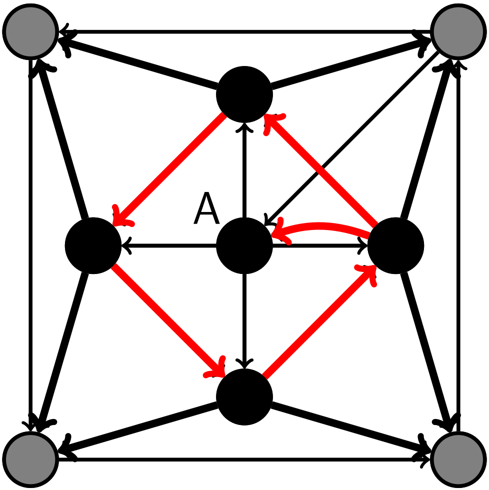

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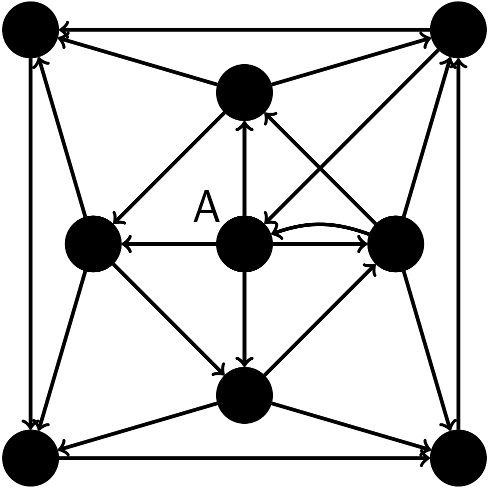
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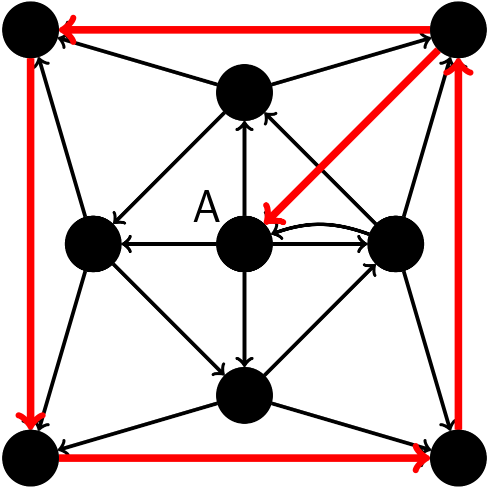
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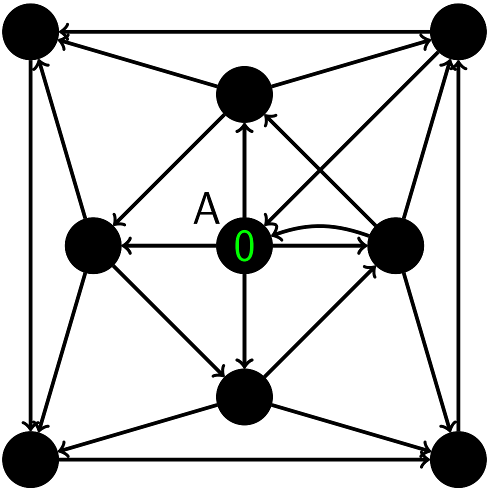
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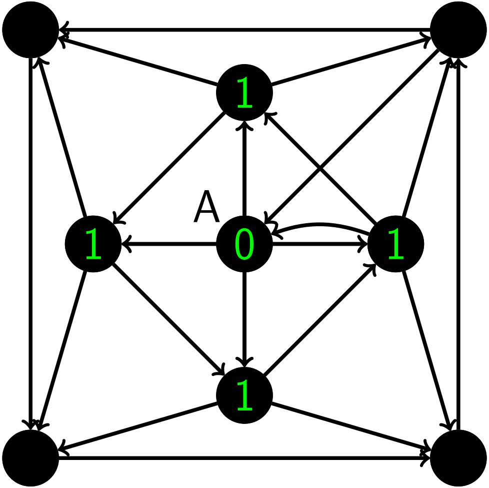
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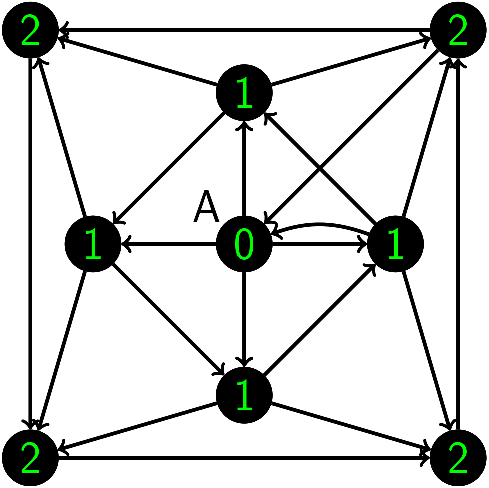
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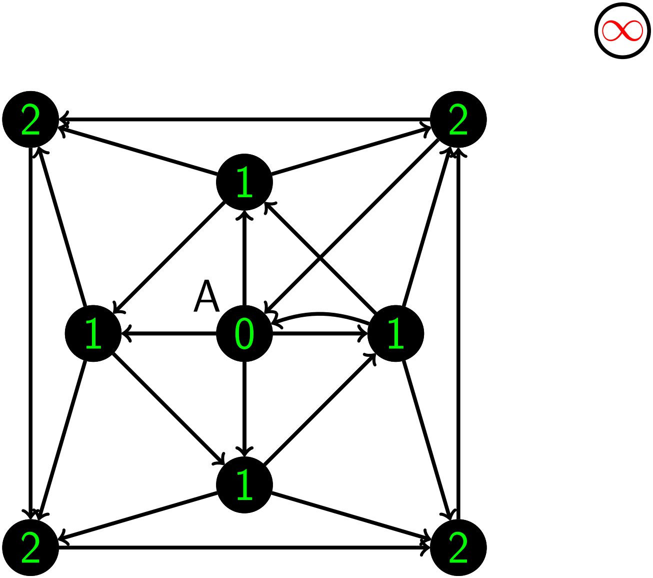

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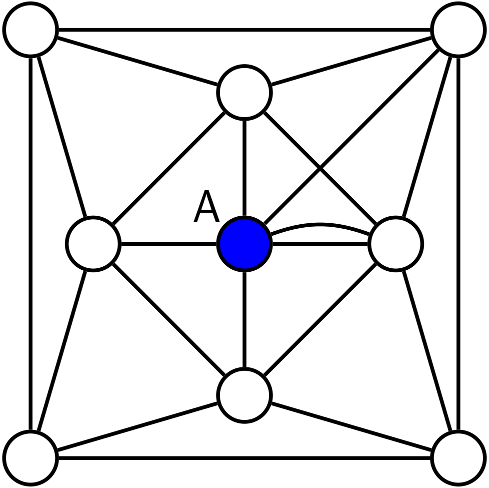
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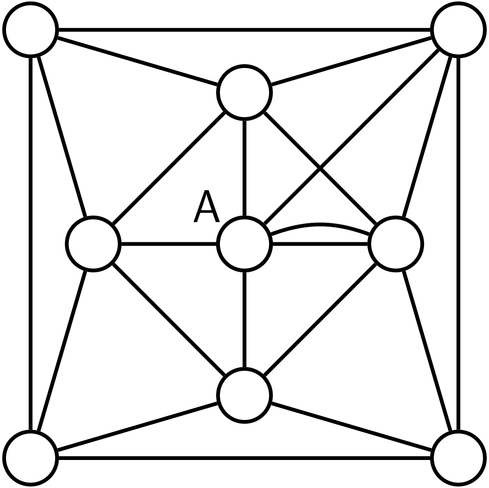
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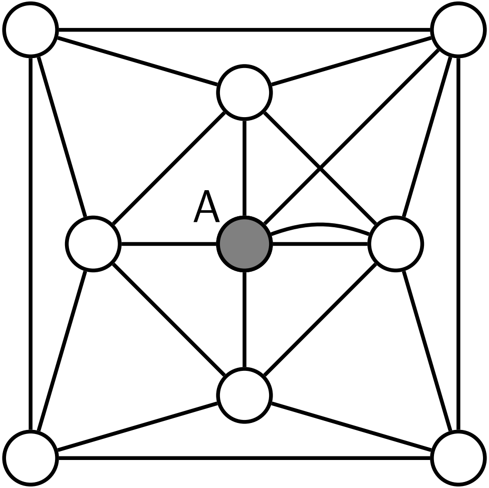
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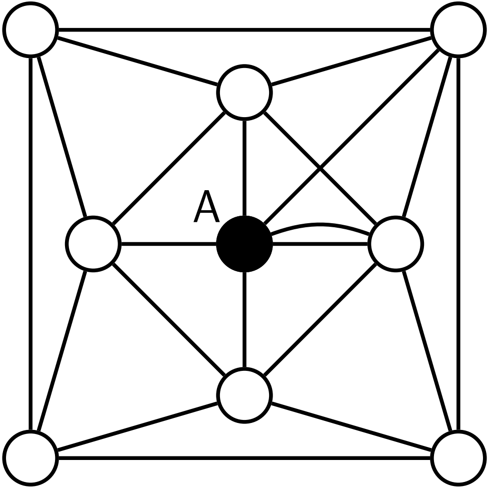
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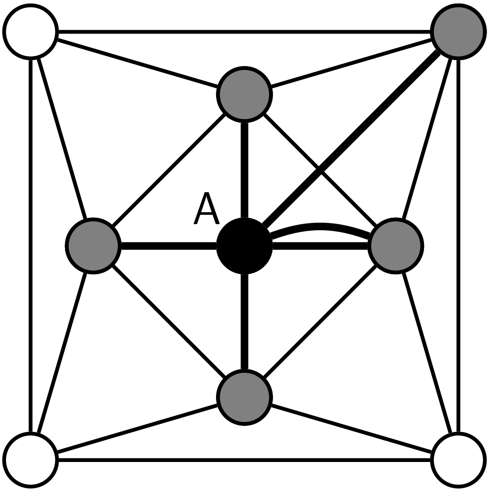
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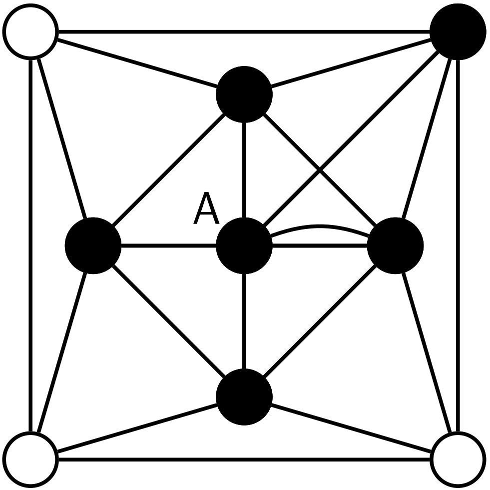
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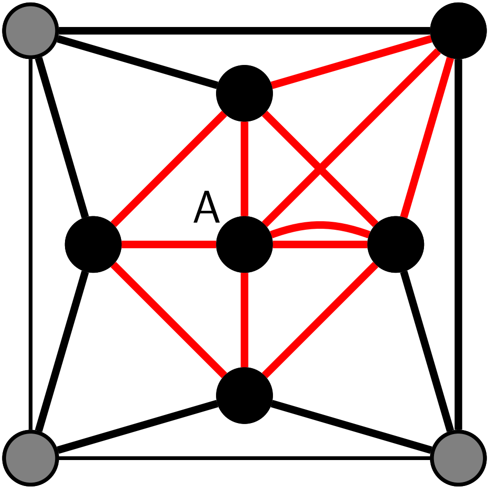
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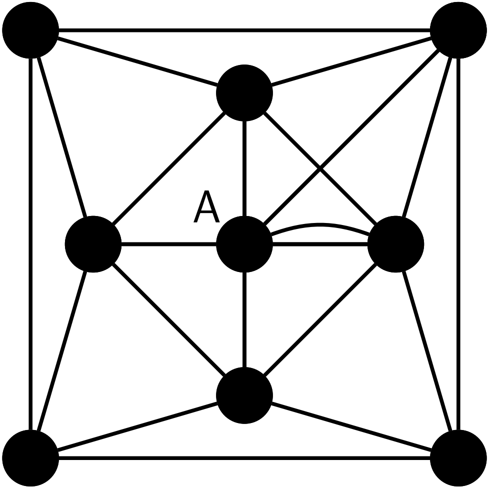
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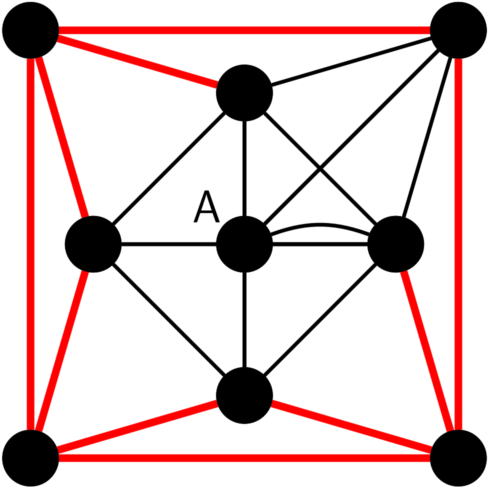
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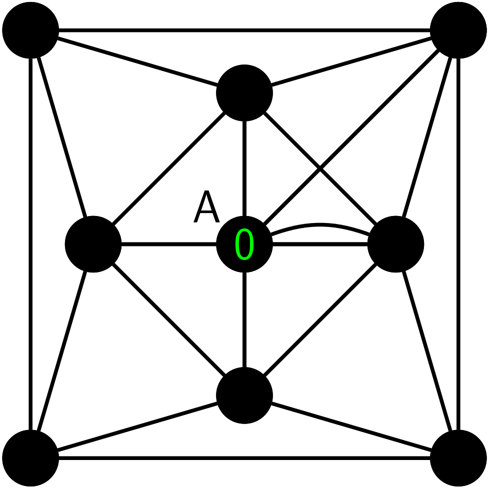
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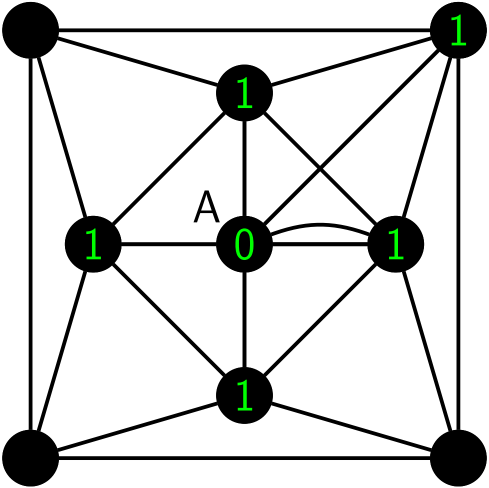
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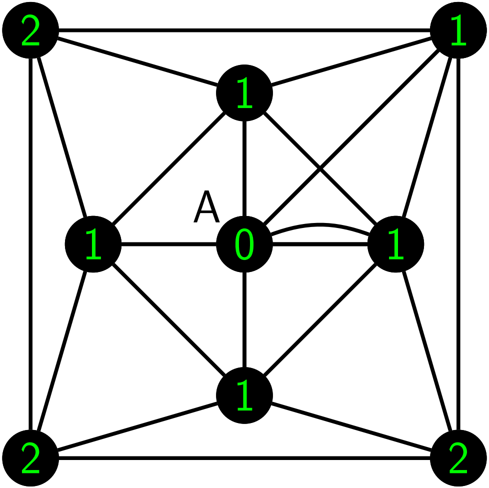
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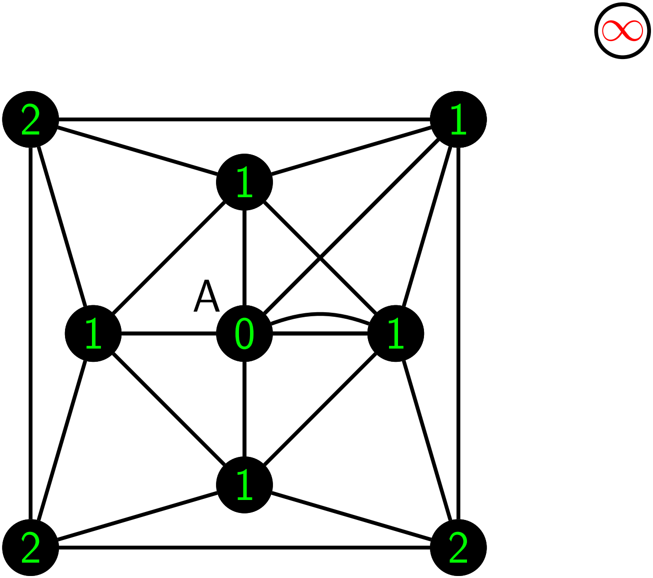
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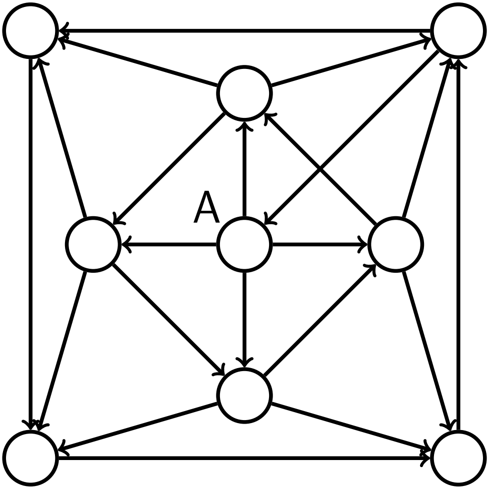
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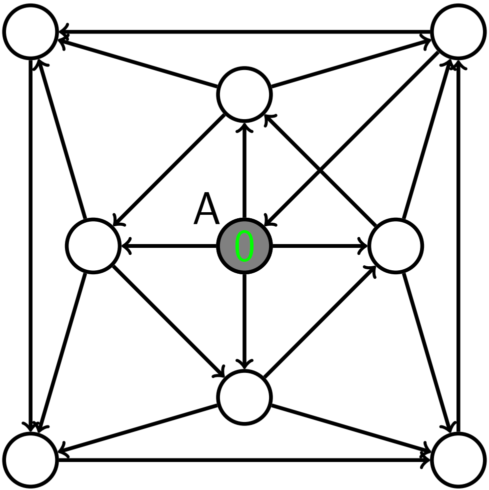
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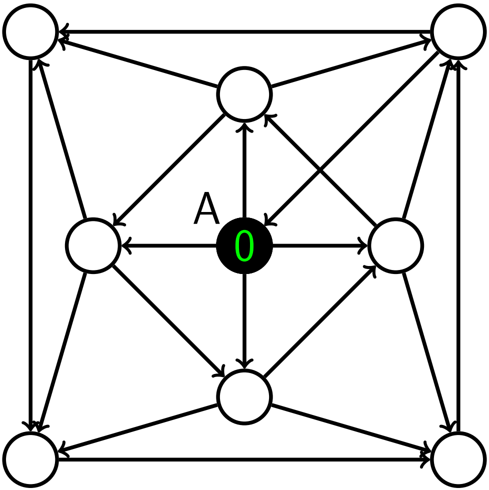
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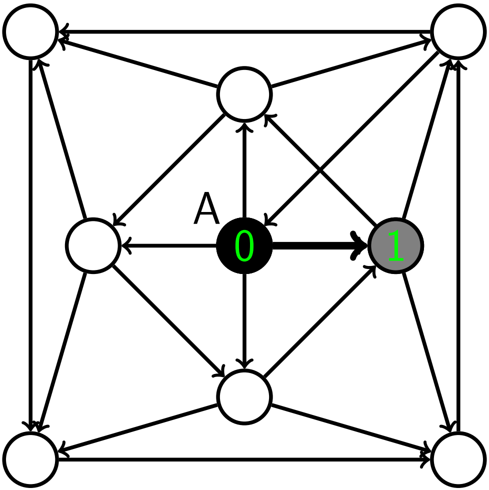
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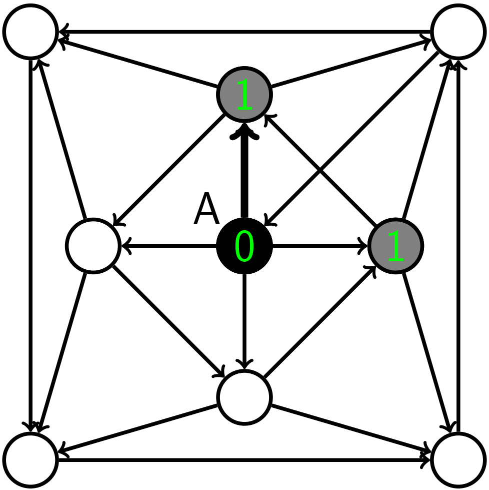

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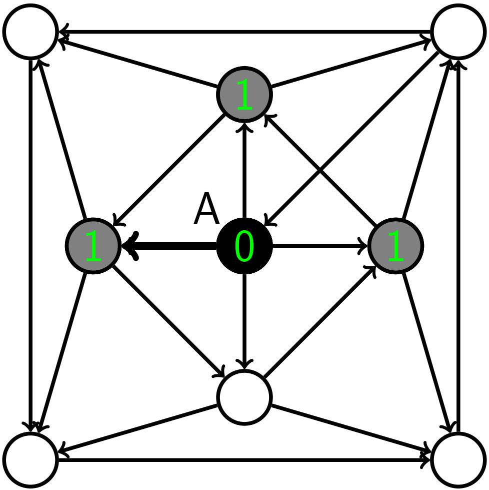
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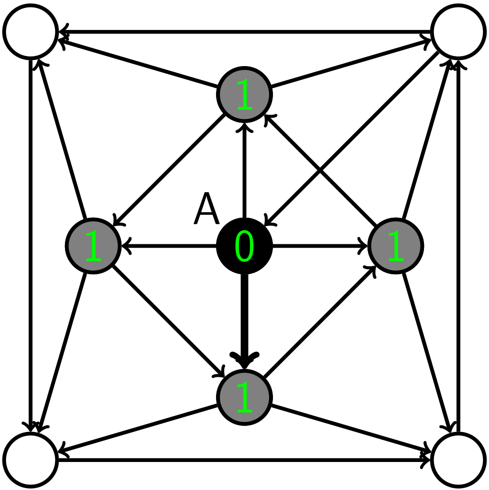
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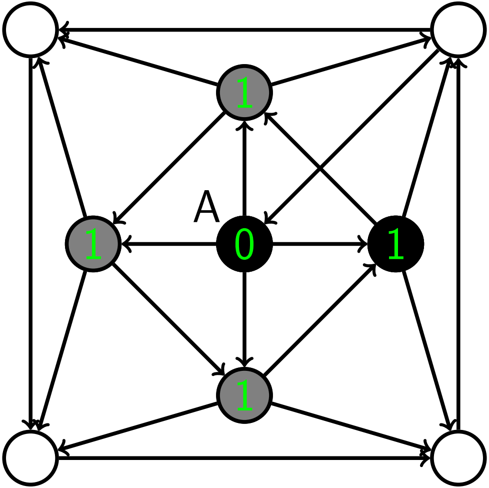

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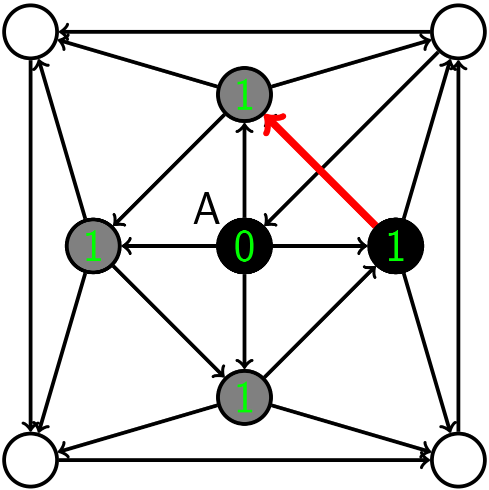

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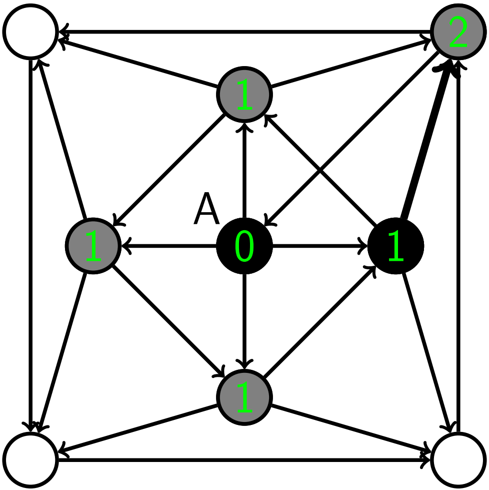

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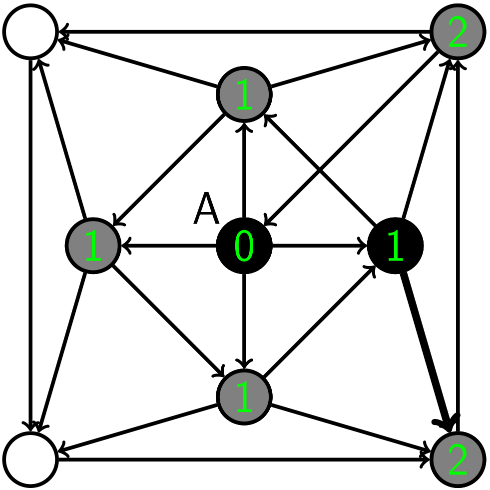

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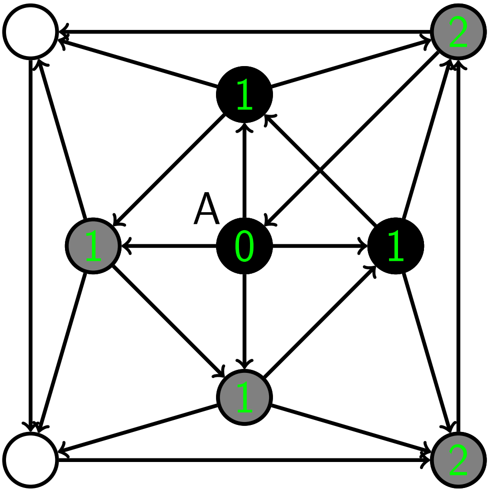
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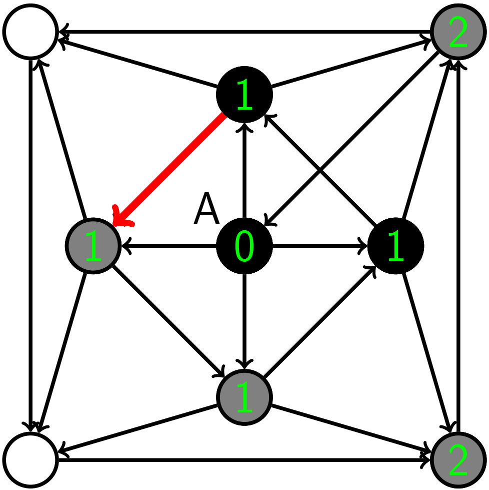

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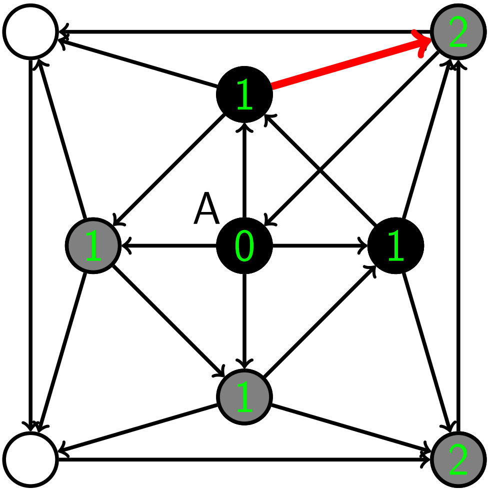
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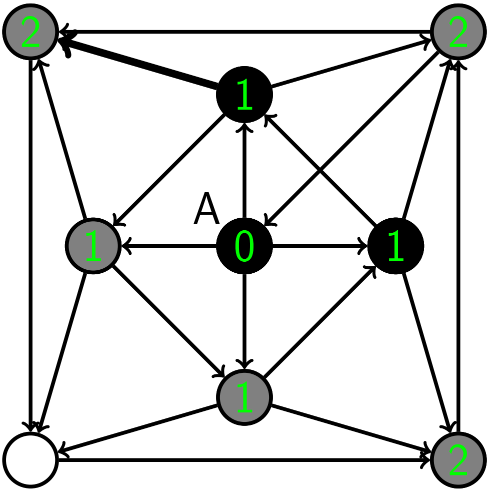
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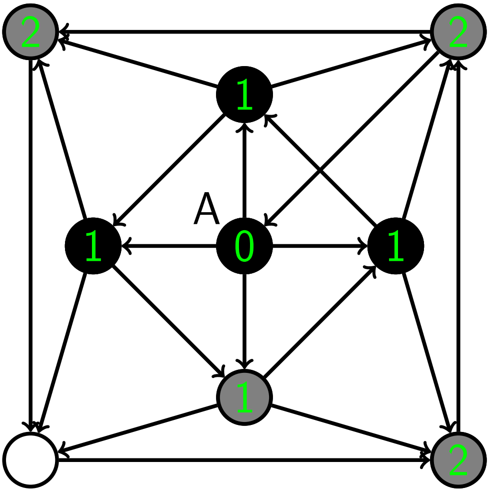
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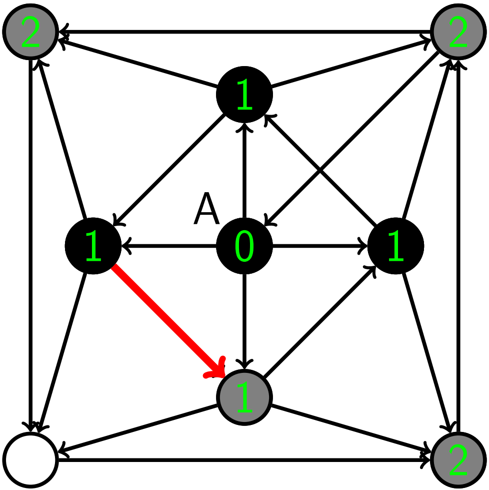
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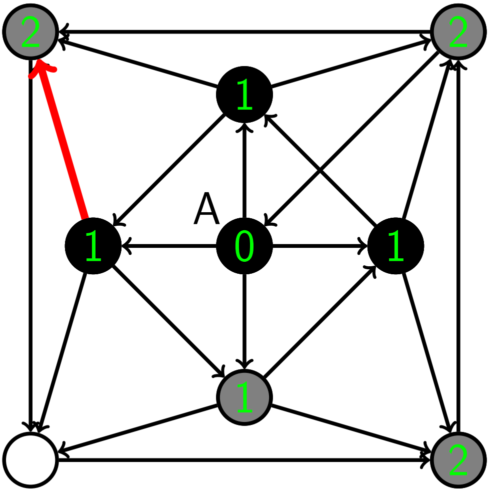

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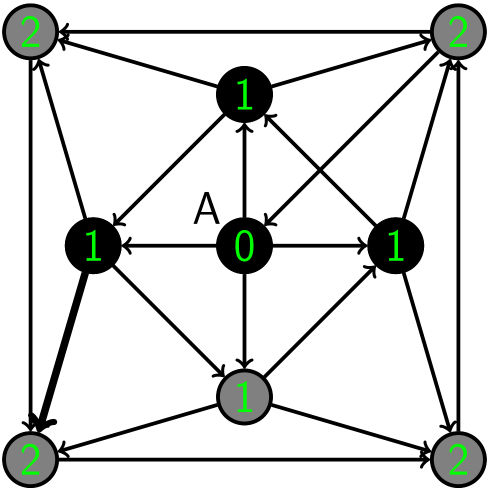
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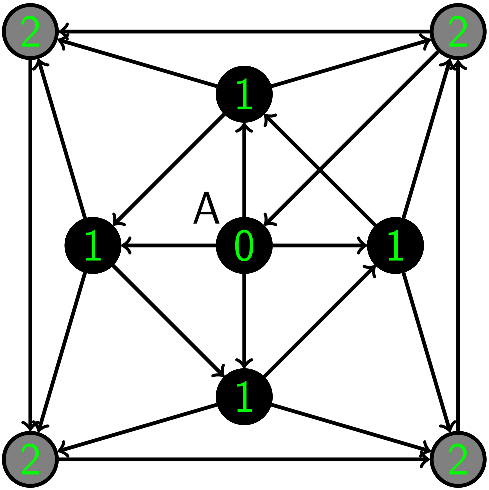
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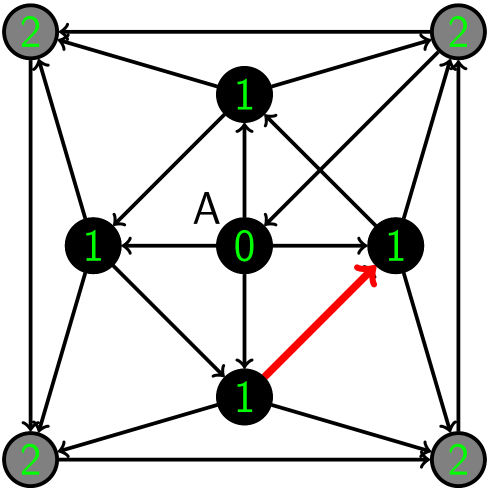
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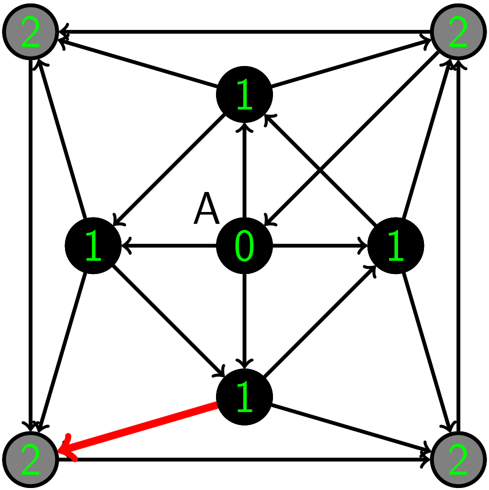
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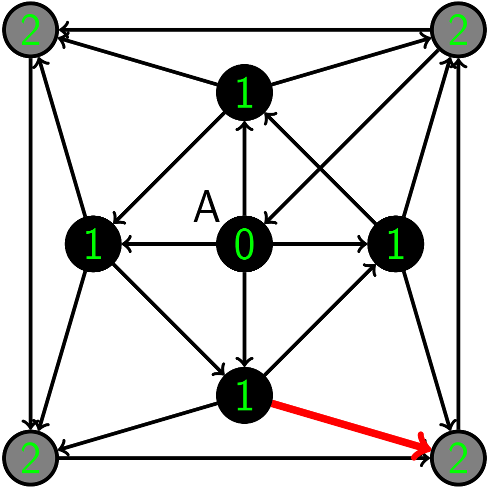

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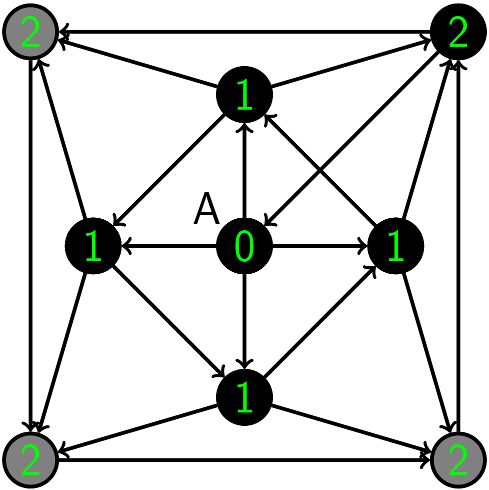
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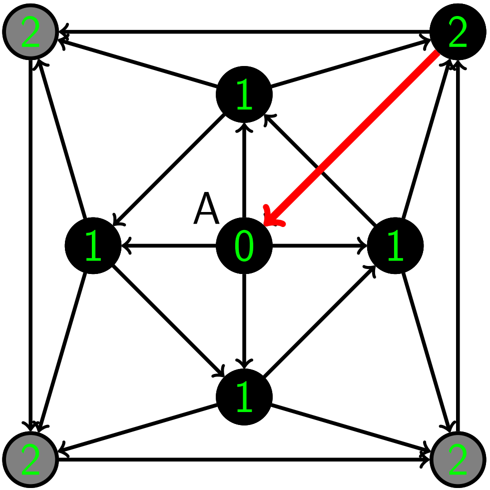

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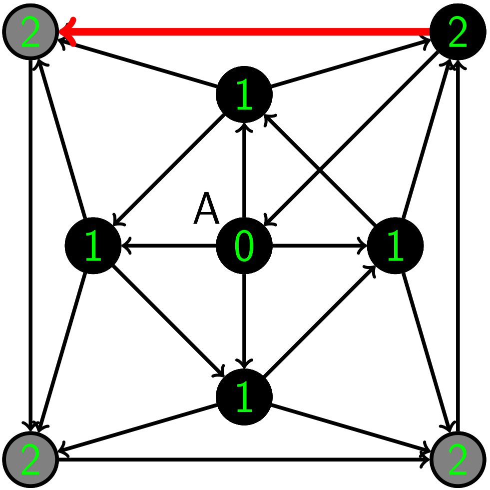

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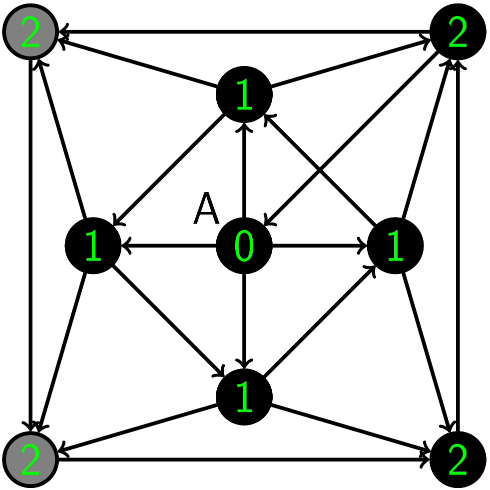

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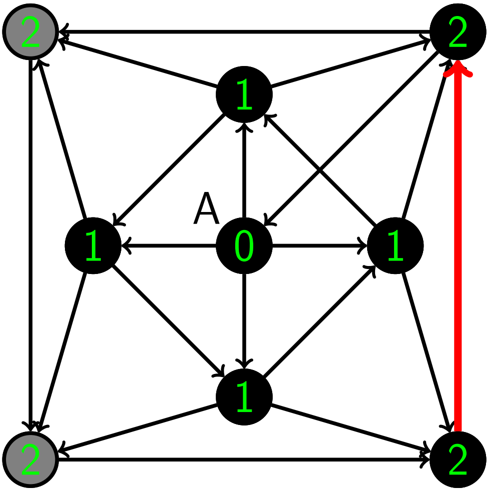
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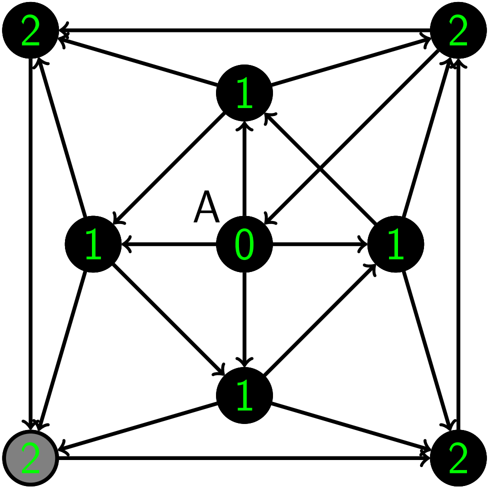
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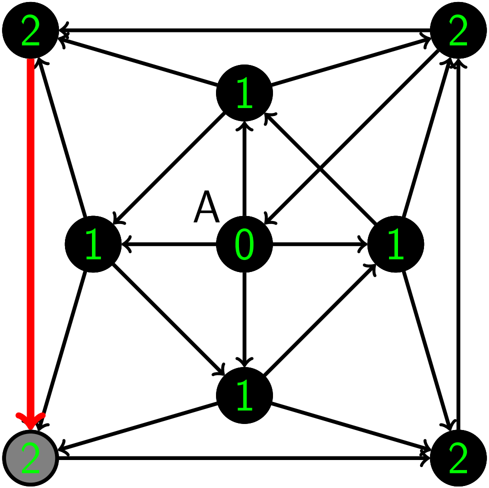

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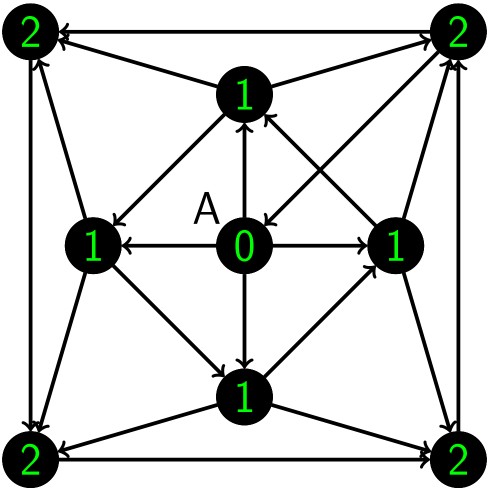
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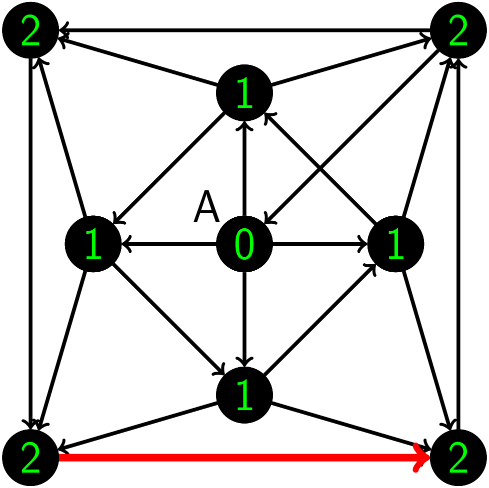
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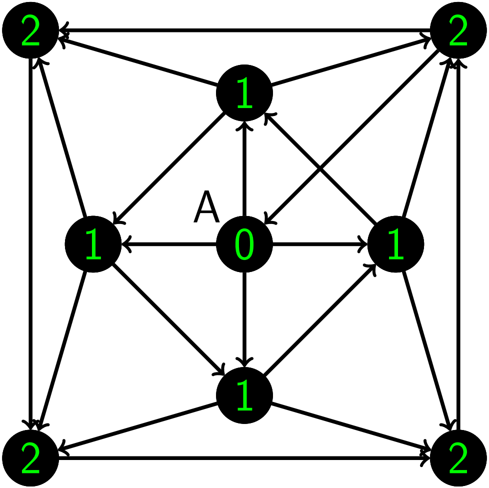
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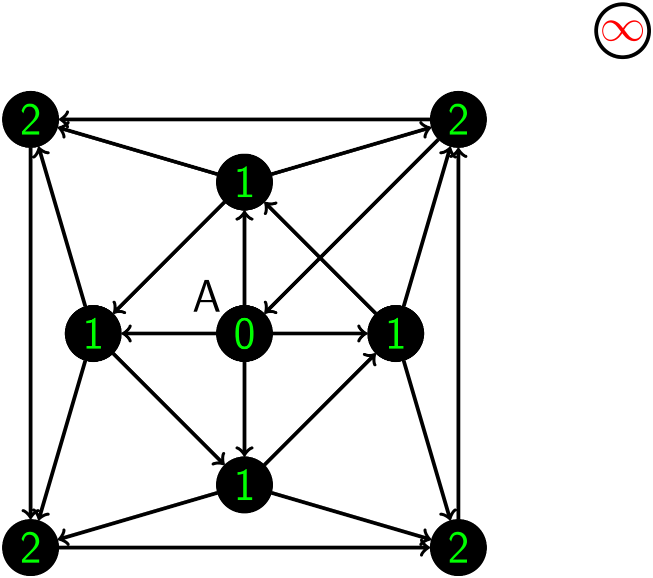

∞
2
2
1
A
1
0
1
1
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2

Outline

Breadth-first search

BFS(G, A)

```
for all u ∈ V:
  dist[u] ← ∞
dist[A] ← 0
Q ← {A} {queue containing just A}
while Q is not empty:
  u ← Dequeue(Q)
  for all (u, v) ∈ E:
    if dist[v] = ∞:
      Enqueue(Q, v)
      dist[v] ← dist[u] + 1
```

Breadth-first search

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Breadth-first search

$\texttt{BFS}(G, A)$

$\texttt{for all } u \in V\texttt{:}$
  $\texttt{dist}[u] \leftarrow \infty$
$\texttt{dist}[A] \leftarrow 0$
$Q \leftarrow \{A\}$ $\{\texttt{queue containing just } A\}$
$\texttt{while } Q \texttt{ is not empty:}$
  $u \leftarrow \texttt{Dequeue}(Q)$
  $\texttt{for all } (u, v) \in E\texttt{:}$
    $\texttt{if dist}[v] = \infty\texttt{:}$
      $\texttt{Enqueue}(Q, v)$
      $\texttt{dist}[v] \leftarrow \texttt{dist}[u] + 1$

Breadth-first search

BFS(G, A)

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Breadth-first search

BFS(G, A)

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```

Breadth-first search

BFS(G, A)

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Breadth-first search

BFS(G, A)

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Breadth-first search

BFS(G, A)

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Breadth-first search

$\mathtt{BFS}(G, A)$

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  for all (u, v) ∈ E:
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      dist[v] ← dist[u] + 1
```

Running time

Lemma

The running time of breadth-first search is $O(|E| + |V|)$.

Proof

Running time

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Proof

- Each vertex is enqueued at most once

Running time

Lemma

The running time of breadth-first search is $O(|E| + |V|)$.

Proof

- Each vertex is enqueued at most once
- Each edge is examined either once (for directed graphs) or twice (for undirected graphs) $\square$

Outline

Reachability

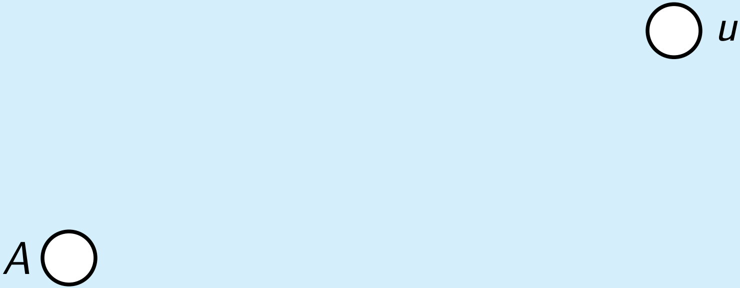
Definition

Node u is reachable from node A if there is a path from A to u

Lemma

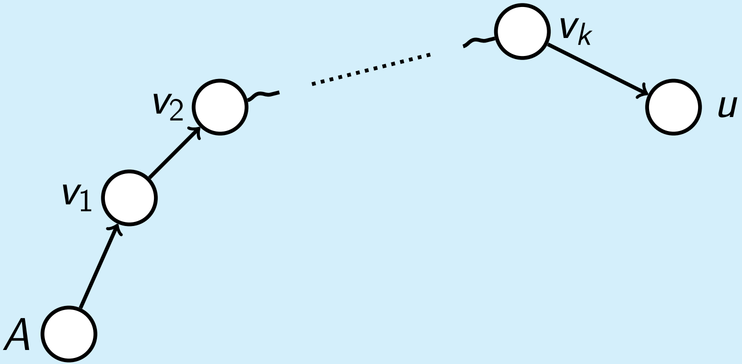
Reachable nodes are discovered at some point, so they get a finite distance estimate from the source. Unreachable nodes are not discovered at any point, and the distance to them stays infinite.

Proof



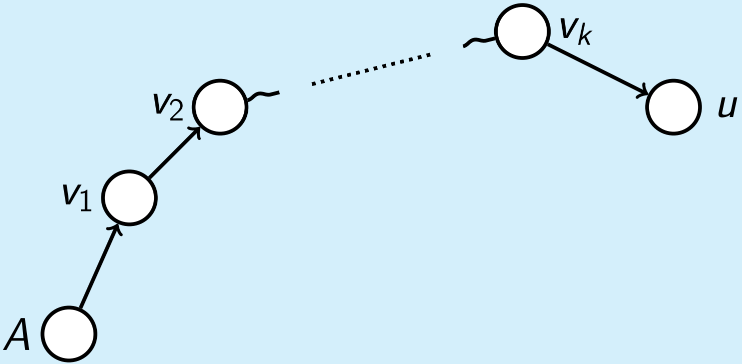
- u — reachable undiscovered closest to A

Proof



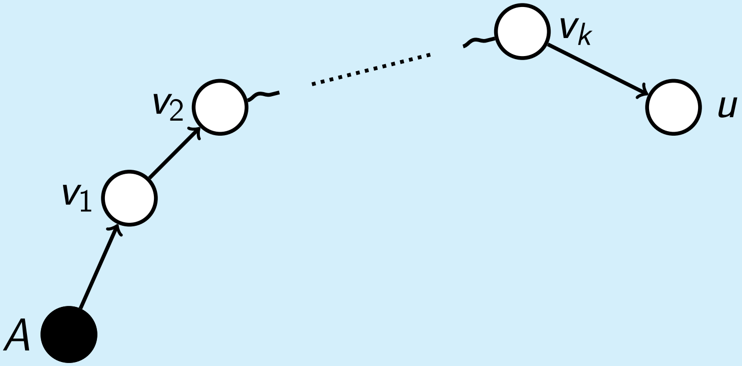
- u — reachable undiscovered closest to A
- $A - v_1 - \cdots - v_k - u$ — shortest path

Proof



- u — reachable undiscovered closest to A
- $A - v_1 - \cdots - v_k - u$ — shortest path
- u is discovered while processing v_k

Proof



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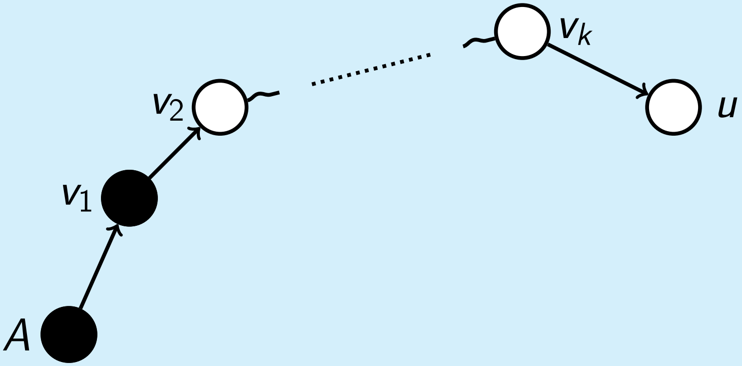
Proof



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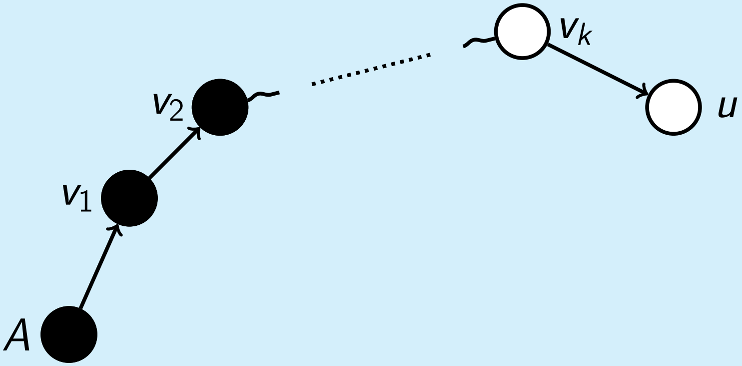
Proof



- u — reachable undiscovered closest to A
- $A - v_1 - \cdots - v_k - u$ — shortest path
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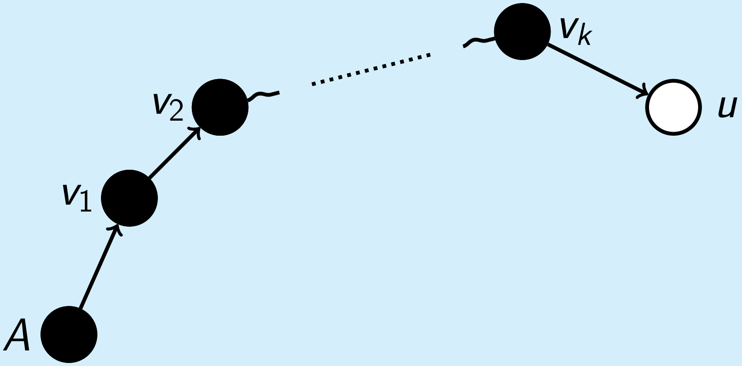
Proof



- u — reachable undiscovered closest to A
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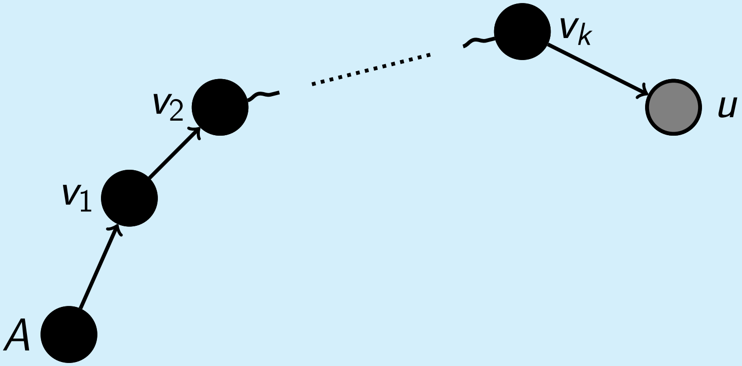
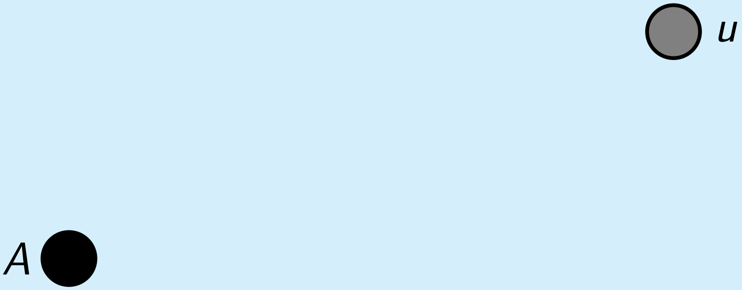
Proof



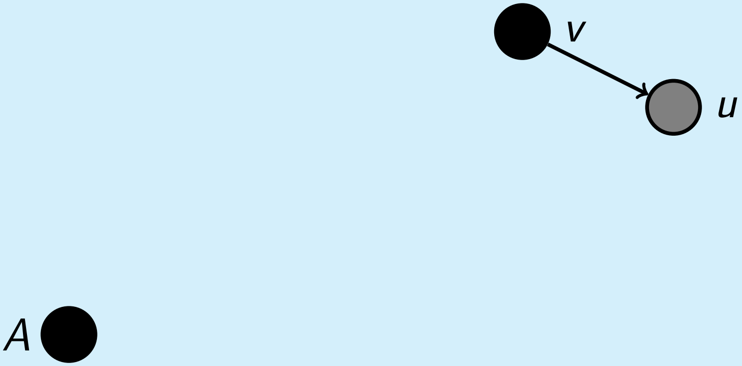
- u — reachable undiscovered closest to A
- $A - v_1 - \cdots - v_k - u$ — shortest path
- u is discovered while processing v_k

Proof



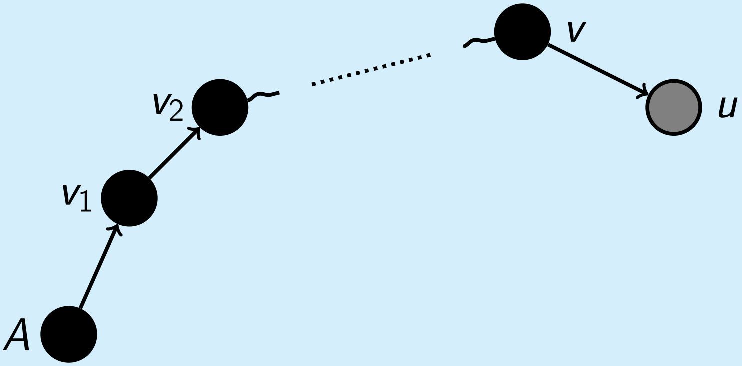
- u — first unreachable discovered

Proof



- u — first unreachable discovered
- u was discovered while processing v

Proof



- u — first unreachable discovered
- u was discovered while processing v
- u is reachable through v $\square$

Order Lemma

Lemma

By the time node u at distance d from A is dequeued, all the nodes at distance at most d have already been discovered (enqueued).

Order Lemma Proof

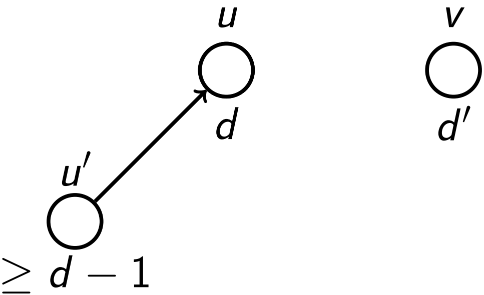
Consider the first time the order was broken

Order Lemma Proof

Consider the first time the order was broken

$d' \leq d$

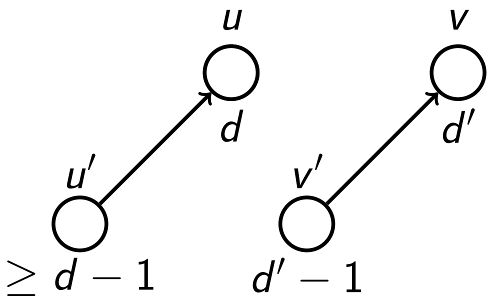
Order Lemma Proof



Consider the first time the order was broken

$d' \leq d$

Order Lemma Proof



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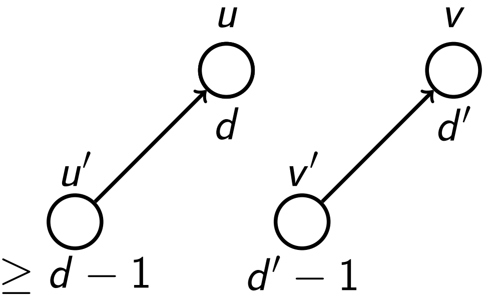
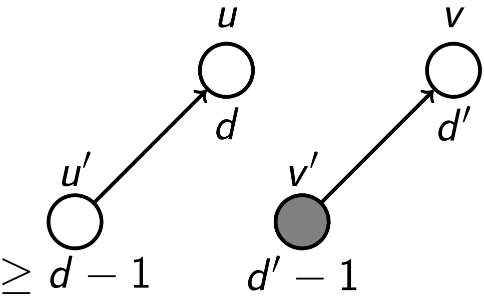
Order Lemma Proof



Consider the first time the order was broken

$d' \leq d \quad \Rightarrow d' - 1 \leq d - 1$, so v' was discovered before u' was dequeued

Order Lemma Proof



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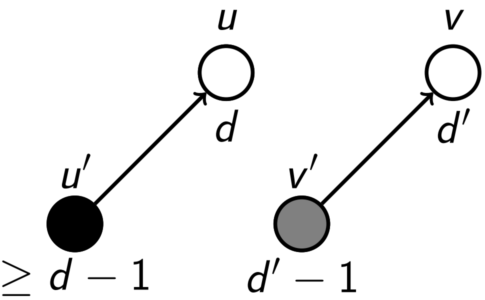

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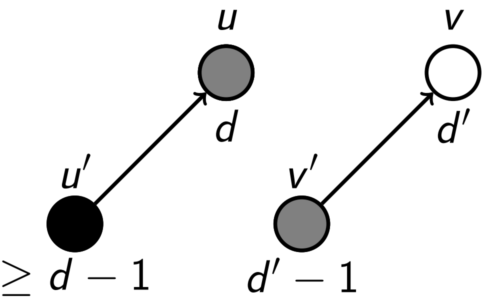
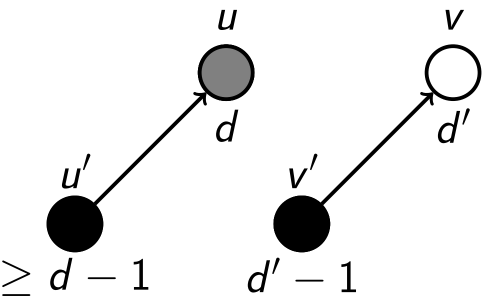
Order Lemma Proof



Consider the first time the order was broken
$d' \leq d \quad \Rightarrow d' - 1 \leq d - 1$, so v' was discovered before u' was dequeued

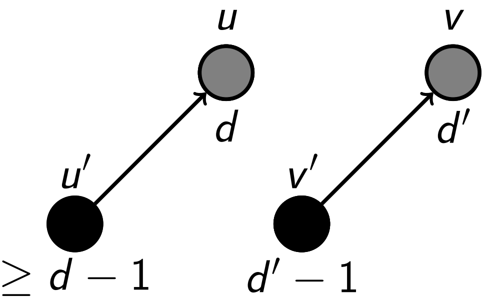
Order Lemma Proof



Consider the first time the order was broken

$d' \leq d \quad \Rightarrow d' - 1 \leq d - 1$, so v' was discovered before u' was dequeued

Order Lemma Proof



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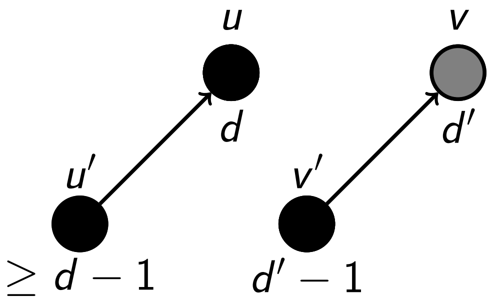

Consider the first time the order was broken
$d' \leq d \quad \Rightarrow d' - 1 \leq d - 1$, so v' was discovered before u' was dequeued

Queue property

Queue:	d	d	d	$\ldots$	d	d	$d+1$	$d+1$	$\ldots$	$d+1$

Lemma

At any moment, if the first node in the queue is at distance d from A, then all the nodes in the queue are either at distance d from A or at distance $d+1$ from A. All the nodes in the queue at distance d go before (if any) all the nodes at distance $d+1$.

Queue property

Proof

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- Nodes at distance $> d + 1$ will be discovered when all d are gone $\square$

Outline

Correct distances

Lemma

When node u is discovered (enqueued), `dist[u]` is assigned exactly $d(A, u)$.

Correct distances

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- Inductive step: suppose proved for all nodes at distance $\leq k$ from $A \rightarrow$ prove for nodes at distance $k + 1$

Correct distances

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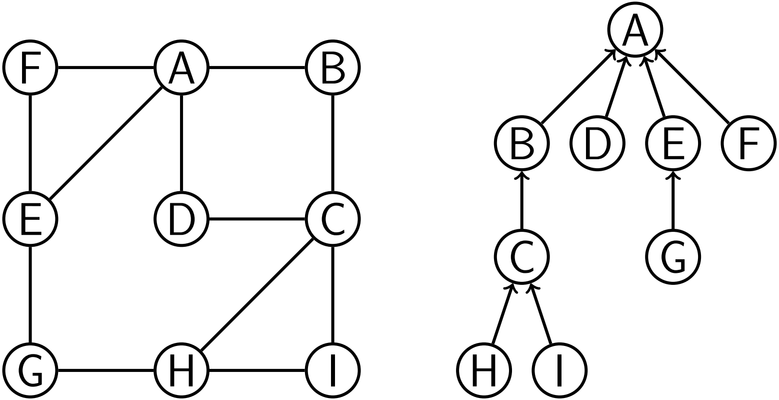
- So $d(A,u) = k$, and $\text{dist}[v] \leftarrow \text{dist}[u] + 1 = k+1$ $\square$

Outline

Shortest-path tree



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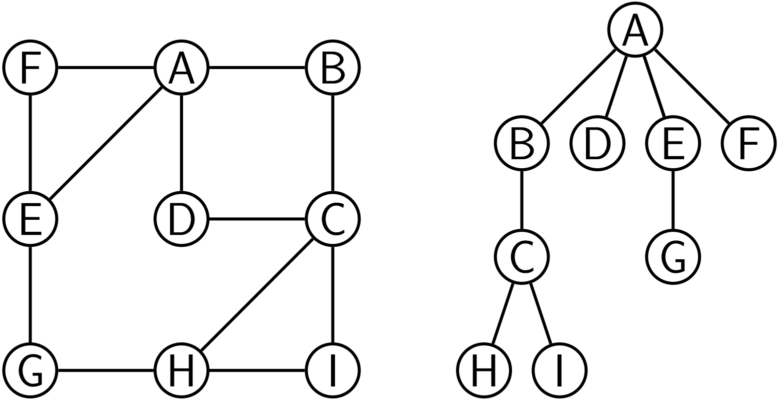
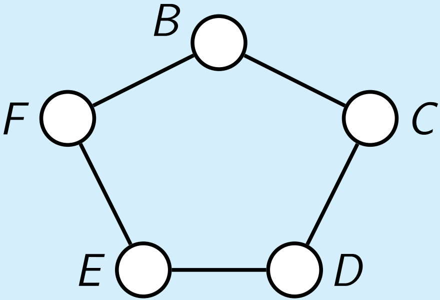


Lemma

Shortest-path tree is indeed a tree, i.e. it doesn't contain cycles (it is a connected component by construction).

Proof



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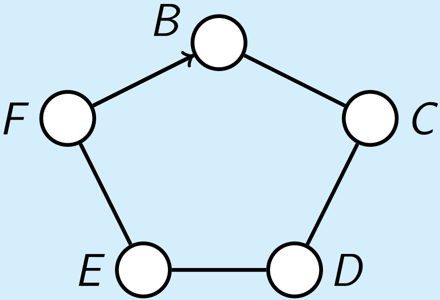

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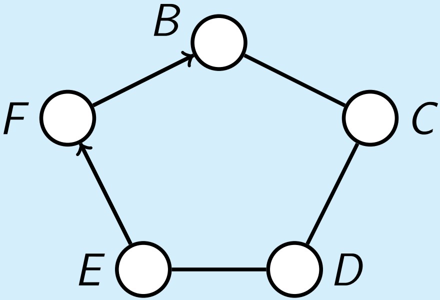

- Only one outgoing edge from each node

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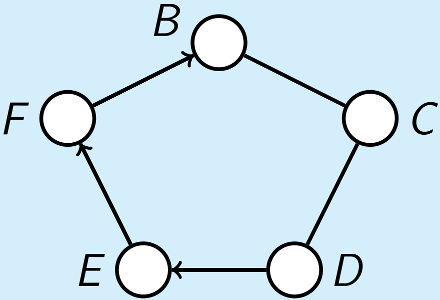

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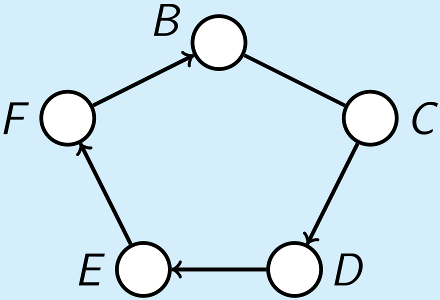

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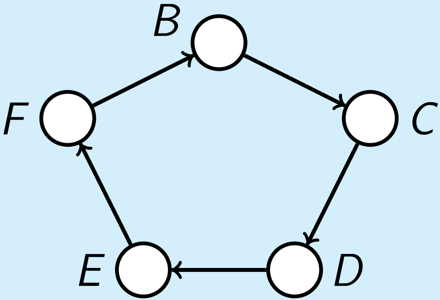

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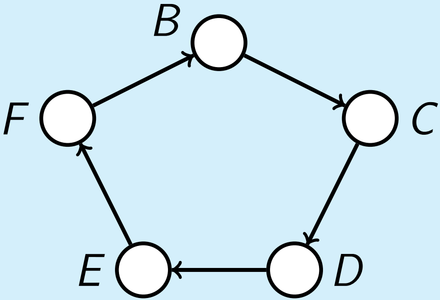

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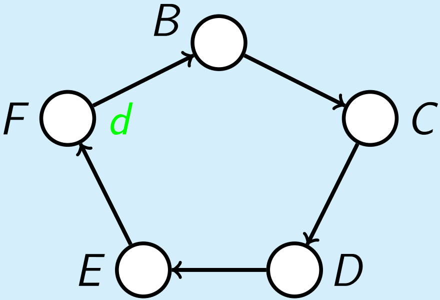

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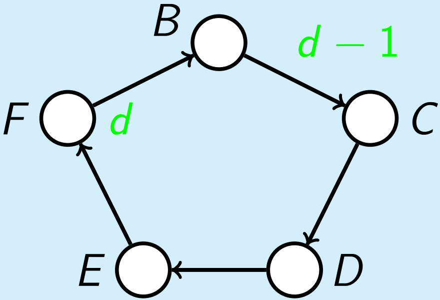

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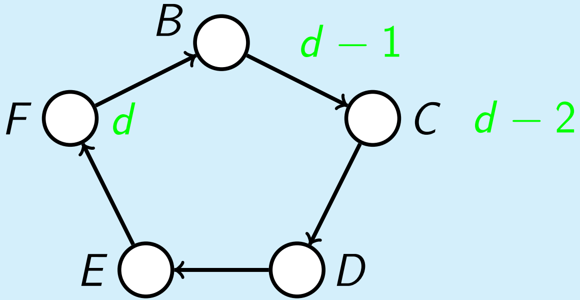

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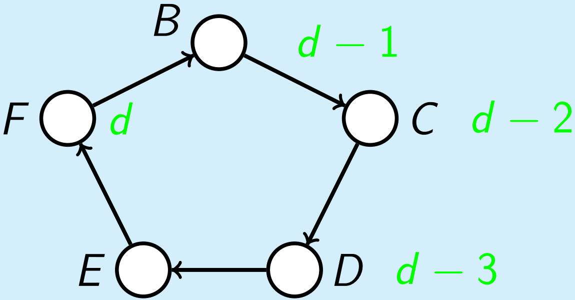

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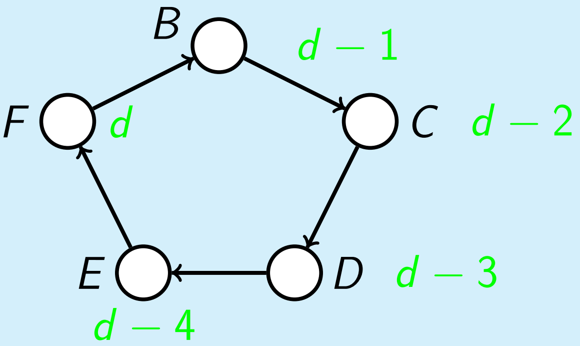

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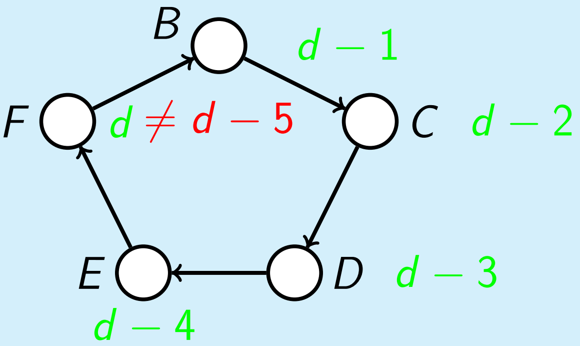

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Constructing shortest-path tree

BFS(G, A)

```
for all u ∈ V:
  dist[u] ← ∞, prev[u] ← nil
dist[A] ← 0
Q ← {A} {queue containing just A}
while Q is not empty:
  u ← Dequeue(Q)
  for all (u, v) ∈ E:
    if dist[v] = ∞:
      Enqueue(Q, v)
      dist[v] ← dist[u] + 1, prev[v] ← u
```

Reconstructing Shortest Path

```
ReconstructPath(A, u, prev)

result ← empty
while u ≠ A:
    result.append(u)
    u ← prev[u]
return Reverse(result)
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- Works in $O(|E| + |V|)$