# CS 7480 Spring 2013 Control Flow Analysis

notes by: Stephen Chang

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## 1 Summary

General monotone flow analysis framework.

1. Monotonic, continuous functions
2. Lattices
   - represents a program property, ie, a "fact" lattice
   - ordering is information ordering
   - modeling programs as functions on lattices

## 2 Monotonic, continuous functions

**Definition 1.** *A* least upper bound (lub) *on set $S$:*

1. $\forall x \in S : x \leq \operatorname{lub} S$,
   *ie,* $\operatorname{lub} S$ *is an upper bound of* $S$
2. $\forall x \in S : x \leq y \implies \operatorname{lub} S \leq y$,
   *ie,* $\operatorname{lub} S$ *is less than or equal to all upper bounds of* $S$.

If ordering is information ordering, then $\operatorname{lub} S$ is the "most precise" upper bound.

**Definition 2.** *In a* complete partial order (cpo)*, every chain has a lub.*

**Definition 3.** *A* pointed set *has a bottom ($\perp$) element.*

**Definition 4.** *A function $f$ is* monotone *if* $\forall a, b : a \leq b \implies f(a) \leq f(b)$.

In other words, more information gives better conclusions. But we don't need to worry much about monotonicity because it's pretty much always true.

**Definition 5.** *A function $f$ is* continuous *if* $\forall$ *chains* $C : f(\operatorname{lub} C) = \operatorname{lub} f(C)$.

In other words, the function and lub operations commute. So you "can sneak up on an answer."

**Theorem 1.** *continuous $\implies$ monotone*

*Proof.* .

1. $x \leq y \implies \text{lub}\,\{x, y\} = y$
2. From the definition of continuous:

$$f(\text{lub}\,\{x, y\}) = \text{lub}\, f(\{x, y\})$$

Using step 1:

$$f(y) = \text{lub}\, f(\{x, y\}) = \text{lub}\,\{f(x), f(y)\}$$

so

$$f(x) \leq f(y)$$

meaning $f$ is monotone.

$\square$

**Theorem 2.** *monotonic $\implies$ continuous, when cpo chains are finite*

*Proof.* Consider (finite) chain $C = x_1 \leq \cdots \leq x_n$. Want to show:

$$f(\text{lub}\, C) = \text{lub}\, f(C)$$

1. $\text{lub}\, C = x_n$, so $f(\text{lub}\, C) = f(x_n)$
2. By monotonicity:

$$f(x_1) \leq \cdots \leq f(x_n)$$

so

$$\text{lub}\, f(C) = f(x_n)$$

$\square$

**Theorem 3.** *With an infinite chain, monotone $\not\Longrightarrow$ continuous.*

*Proof.* Counterexample:
$f : \mathbb{N}_\infty \to \mathbb{N}_\infty$, $f(i) = 0$, for $i \in \mathbb{N}$, $f(\infty) = 1$
Consider the chain $C = \mathbb{N}$. $f(\text{lub}\, C) = f(\infty) = 1$, but $\text{lub}\, f(C) = 0$. $\square$

**Theorem 4** (Least Fixed-point)**.** *If $D$ is a pointed cpo and $f : D \to D$ is continuous, then $f$ has a* least fixed-point (lfp)*,* $\text{fix}\, f = \text{lub}\,\{f^n\bot \mid n \geq 0\}$.

Recall that a recursive function is a search for a fixed point. If there is a least fixed-point, then there is a “best answer”.

*Proof.* Summary:

1. Show that $\{f^n\bot \mid n \geq 0\}$ has a lub.

2. Show that the lub is a fixed-point.

3. Show that the lub is a least fixed-point.

Proof:

1.

2.

3.

$\square$