Syntax (new2)

$$
\begin{aligned}
e &= x \mid \lambda x.e \mid e\,e && \text{(Expressions)}\\
v &= \lambda x.e && \text{(Values)}\\
a &= A[v] && \text{(Answers)}\\
A &= [\,] \mid A[\lambda x.A]\,e && \text{(Answer Contexts)}\\
\hat{A} &= [\,] \mid A[\hat{A}]\,e && \text{(Partial Answer Contexts – outer)}\\
\check{A} &= [\,] \mid A[\lambda x.\check{A}] && \text{(Partial Answer Contexts – inner)}\\
E &= [\,] \mid E\,e \mid A[E] \mid \hat{A}[A[\lambda x.\check{A}[E[x]]]\,E] && \text{(Evaluation Contexts)}\\
& \qquad \hat{A}[\check{A}] \in A
\end{aligned}
$$

Notions of Reduction (new2)

$$\hat{A}[A_1[\lambda x.\check{A}[E[x]]]\,A_2[v]] \;\;\boldsymbol{\beta}_{\mathbf{need}}\;\; \hat{A}[A_1[A_2[\check{A}[E[x]]\{x := v\}]]]$$
$$\hat{A}[\check{A}] \in A$$

$\rightarrow$: compatible closure of $\boldsymbol{\beta}_{\mathbf{need}}$

$\twoheadrightarrow$: reflexive, transitive closure of $\rightarrow$

$\Rightarrow$: parallel reduction of $\boldsymbol{\beta}_{\mathbf{need}}$ redexes

$\longmapsto$: standard reduction relation

$E[e] \longmapsto E[e']$ if $e\ \boldsymbol{\beta}_{\mathbf{need}}\ e'$

$\mapsto\!\!\!\twoheadrightarrow$: reflexive, transitive closure of $\longmapsto$

Definition 1 ($\Rightarrow$)**.** *Parallel reduction*

$$
\begin{array}{rcll}
e & \Rightarrow & e & (1)\\
\hat{A}[A_1[\lambda x.\check{A}[E[x]]]\,A_2[v]] & \Rightarrow & \hat{A}'[A_1'[A_2'[\check{A}'[E'[x]]\{x := v'\}]]] & (2)\\
& \textit{if} & \hat{A}[\check{A}] \in A,\ \hat{A}'[\check{A}'] \in A, & \\
& & \hat{A} \Rightarrow \hat{A}',\ A_1 \Rightarrow A_1',\ A_2 \Rightarrow A_2',\ \check{A} \Rightarrow \check{A}',\ E \Rightarrow E',\ v \Rightarrow v' & \\
e_1\,e_2 & \Rightarrow & e_1'\,e_2' & (3)\\
& \textit{if} & e_1 \Rightarrow e_1',\ e_2 \Rightarrow e_2' & \\
\lambda x.e & \Rightarrow & \lambda x.e' & (4)\\
& \textit{if} & e \Rightarrow e' &
\end{array}
$$

Definition 2 ($\Rightarrow$ for contexts)**.** *Parallel reduction of contexts.*

If all subterms in a context E parallel reduce, then $E \Rightarrow E'$, where each e in E is replaced with e' in E', and $e \Rightarrow e'$.

$$
\begin{array}{cccl}
[\,] & \Rightarrow & [\,] & \\
A_1[\lambda x.A_2]\,e & \Rightarrow & A_1'[\lambda x.A_2']\,e' & \textit{if } A_1 \Rightarrow A_1',\ A_2 \Rightarrow A_2',\ e \Rightarrow e'\\
A[\hat{A}]\,e & \Rightarrow & A'[\hat{A}']\,e' & \textit{if } A \Rightarrow A',\ \hat{A} \Rightarrow \hat{A}',\ e \Rightarrow e'\\
A[\lambda x.\check{A}] & \Rightarrow & A'[\lambda x.\check{A}'] & \textit{if } A \Rightarrow A',\ \check{A} \Rightarrow \check{A}'\\
E\,e & \Rightarrow & E'\,e' & \textit{if } E \Rightarrow E',\ e \Rightarrow e'\\
A[E] & \Rightarrow & A'[E'] & \textit{if } A \Rightarrow A',\ E \Rightarrow E'\\
\hat{A}[A[\lambda x.\check{A}[E_1[x]]]\,E_2] & \Rightarrow & \hat{A}'[A'[\lambda x.\check{A}'[E_1'[x]]]\,E_2'] & \textit{if } \hat{A}[\check{A}],\ \hat{A} \Rightarrow \hat{A}',\ A \Rightarrow A',\ \check{A} \Rightarrow \check{A}',\ E_1 \Rightarrow E_1',\ E_2 \Rightarrow E_2'
\end{array}
$$

Definition 3 ($\diamond$)**.** *Standard reduction sequences $\mathcal{R}$*

- $x \subset \mathcal{R}$
- *If* $e_1 \diamond \cdots \diamond e_m \in \mathcal{R}$, *then* $\lambda x.e_1 \diamond \cdots \diamond \lambda x.e_m \in \mathcal{R}$
- *If* $e_1 \diamond \cdots \diamond e_m \in \mathcal{R}$ *and* $e'_1 \diamond \cdots \diamond e'_n \in \mathcal{R}$ *then* $(e_1\, e'_1) \diamond \cdots \diamond (e_m\, e'_1) \diamond (e_m\, e'_2) \diamond \cdots \diamond (e_m\, e'_n) \in \mathcal{R}$
- *If* $e_1 \diamond \cdots \diamond e_m \in \mathcal{R}$ *and* $e_0 \longmapsto e_1$, *then* $e_0 \diamond e_1 \diamond \cdots \diamond e_m \in \mathcal{R}$

Definition 4 (size function $|\cdot|$)**.**

$$
\begin{aligned}
|e \Rightarrow e| &= 0 \\
|\hat{A}[A_1[\lambda x.\check{A}[E[x]]]\, A_2[v]] \Rightarrow \hat{A}'[A'_1[A'_2[\check{A}'[E'[x]]\{x := v'\}]]]| &= |\hat{A} \Rightarrow \hat{A}'| + |A_1 \Rightarrow A'_1| + |\check{A}[E[x]] \Rightarrow \check{A}'[E'[x]]| \\
&\quad + |A_2 \Rightarrow A'_2| + \#(x, \check{A}'[E'[x]]) \times |v \Rightarrow v'| + 1 \\
|(e_1\, e_2) \Rightarrow (e'_1\, e'_2)| &= |e_1 \Rightarrow e'_1| + |e_2 \Rightarrow e'_2| \\
|\lambda x.e \Rightarrow \lambda x.e'| &= |e \Rightarrow e'|
\end{aligned}
$$

Definition 5 (size function for contexts)**.** *Size of context is equal to sum of sizes of subcontexts and subterms.*

Theorem 1 (Curry-Feys Standardization)**.** $e \twoheadrightarrow e'$ *iff* $e_1 \diamond \cdots \diamond e_n \in \mathcal{R}$ *s.t.* $e = e_1$ *and* $e' = e_n$.

Proof. By lemma 1 □

Lemma 1. *If* $e \Rightarrow e'$ *and* $e' \diamond e'_2 \diamond \cdots \diamond e'_n \in \mathcal{R}$, *then there exists* $e_1 \diamond e_2 \diamond \cdots \diamond e_p \in \mathcal{R}$ *s.t.* $e_1 = e$ *and* $e_p = e'_n$.

Proof. By triple lexicographic induction on 1) length n of given standard reduction sequence, 2) $|e \Rightarrow e'|$, and 3) structure of e. Proceed by case analysis on last step in derivation of $e \Rightarrow e'$

Case $e = e'$**.** (base)

Then $e_1 = e = e'$ and $e_p = e'_n$

Case $e \Rightarrow e'$ by $\Rightarrow def(2)$**.**

$e = \hat{A}[A_1[\lambda x.\check{A}[E[x]]]\, A_2[v]]$,
$e' = \hat{A}'[A'_1[A'_2[\check{A}'[E'[x]]\{x := v'\}]]]$
Let $e'' = \hat{A}[A_1[A_2[\check{A}[E[x]]\{x := v\}]]]$
Since e is $\boldsymbol{\beta}_{\mathbf{need}}$ redex, $e \longmapsto e'' \Rightarrow e'$
By IH and size lemma(4), there exists $e_2 \diamond \cdots \diamond e_p \in \mathcal{R}$ s.t. $e_2 = e''$ and $e_p = e'_n$

Case $e \Rightarrow e'$ by $\Rightarrow def(3)$**.**

$e = e_3\, e_4$, $e' = e'_3\, e'_4$, $e_3 \Rightarrow e'_3$, $e_4 \Rightarrow e'_4$

Subcase $e' \longmapsto e'_2$**.**

By lemma 2, IH, and reasoning from redex text.

Subcase *otherwise***.**

Same reasoning as redex text.

Case $e \Rightarrow e'$ by $\Rightarrow def(4)$**.**

Claim holds by IH.

□

Lemma 2. *If $e \Rightarrow e^* \longmapsto e^{**}$, then there exists e^{***} s.t. $e \mapsto\!\!\!\!\twoheadrightarrow e^{***} \Rightarrow e^{**}$.*

Proof. By double lexicographic induction on $|e \Rightarrow e^*|$ and structure of e. Proceed by cases on proof of $e \Rightarrow e^*$.

Case $e \Rightarrow e^*$ by $\Rightarrow def(1)$**.**

$e = e^*$, $e^{***} = e^{**}$

Case $e \Rightarrow e^*$ by $\Rightarrow def(2)$**.**

By the fact that e is a $\boldsymbol{\beta}_{\mathbf{need}}$ redex, size lemma(4), and IH (same reasoning as redex text).

Case $e \Rightarrow e^*$ by $\Rightarrow def(3)$**.**

$e = e_1\, e_2$,
$e^* = e_1'\, e_2'$,
$e_1 \Rightarrow e_1'$, $e_2 \Rightarrow e_2'$
By subcases of $e^* \longmapsto e^{**}$:

Subcase e^* is a $\boldsymbol{\beta}_{\mathbf{need}}$ redex**.**

$e^* = \hat{A}[A_1[\lambda x.\check{A}[E[x]]]\, A_2[v]]$,
$e^{**} = \hat{A}[A_1[A_2[\check{A}[E[x]]\{x := v\}]]]$
(Similar reasoning to Case 3.4 below)

Subsubcase $\hat{A} = [\,]$**.**

$\check{A} = [\,]$, $e_1 \Rightarrow A_1[\lambda x.E[x]]$, $e_2 \Rightarrow A_2[v]$
By lemma 3 and $E[x]$ shape lemma, there exists $A_1'[\lambda x.E'[x]]$ and $A_2'[v']$ s.t.
$\underline{e_1 \mapsto\!\!\!\!\twoheadrightarrow A_1'[\lambda x.E'[x]] \Rightarrow A_1[\lambda x.E[x]]}$ and
$\underline{e_2 \mapsto\!\!\!\!\twoheadrightarrow A_2'[v'] \Rightarrow A_2[v]}$. Thus,

$$\begin{aligned} e = e_1\, e_2 &\mapsto\!\!\!\!\twoheadrightarrow A_1'[\lambda x.E'[x]]\, e_2 \\ &\mapsto\!\!\!\!\twoheadrightarrow A_1'[\lambda x.E'[x]]\, A_2'[v'] \\ &\Rightarrow A_1[\lambda x.E[x]]\, A_2[v] \end{aligned}$$

so $e^{***} = A_1'[\lambda x.E'[x]]\, A_2'[v']$

Subsubcase $\hat{A} = A_3[\hat{A}_1]\, e_2'$**.**

Then $\check{A} = A_4[\lambda y.\check{A}_1]$ s.t. $\hat{A}_1[\check{A}_1] \in A$, so $e_1 \Rightarrow A_3[\hat{A}_1[A_1[\lambda x.A_4[\lambda y.\check{A}_1[E[x]]]]\, A_2[v]]]$, which is an answer, bc there is now one more λ than arguments.

Subsubsubcase $\hat{A}_1 = \check{A}_1 = [\,]$, ($\lambda y.$ is innermost lambda)**.**

$e_1 \Rightarrow A_3[A_1[\lambda x.A_4[\lambda y.E[x]]]\, A_2[v]] = A_5[\lambda y.E[x]]$, where $A_5 = A_3[A_1[\lambda x.A_4]\, A_2[v]]$
By lemma 3, and $E[x]$ shape lemma, there exists A_5' and $E'[x]$ s.t.
$\underline{e_1 \mapsto\!\!\!\!\twoheadrightarrow A_5'[\lambda y.E'[x]] \Rightarrow A_5[\lambda y.E[x]]}$.
Let $\underline{A_5' = A_3'[A_1'[\lambda x.A_4']\, e_3]}$ s.t. $e_3 \Rightarrow A_2[v]$,
so by lemma 3, there is some $A_2'[v']$ s.t. $\underline{e_3 \mapsto\!\!\!\!\twoheadrightarrow A_2'[v'] \Rightarrow A_2[v]}$. Thus,

$$\begin{aligned} e = e_1\, e_2 &\mapsto\!\!\!\!\twoheadrightarrow A_5'[\lambda y.E'[x]]\, e_2 \\ &= A_3'[A_1'[\lambda x.A_4'[\lambda y.E'[x]]]\, e_3]\, e_2 \\ &\mapsto\!\!\!\!\twoheadrightarrow A_3'[A_1'[\lambda x.A_4'[\lambda y.E'[x]]]\, A_2'[v']]\, e_2 \\ &\Rightarrow A_3[A_1[\lambda x.A_4[\lambda y.E[x]]]\, A_2[v]]\, e_2' \end{aligned}$$

So $e^{***} = A_3'[A_1'[\lambda x.A_4'[\lambda y.E'[x]]]\, A_2'[v']]\, e_2'$

Subsubsubcase *otherwise***.**

Let $\check{A}_1 = \check{A}_2[\lambda z.[\,]]$ s.t. $\lambda z.$ is the innermost unpaired λ so that

$e_1 \Rightarrow A_3[\hat{A}_1[A_1[\lambda x.A_4[\lambda y.\check{A}_2[\lambda z.E[x]]]]\, A_2[v]]]$

Apply lemma 3 to $e_1 \Rightarrow A_5[\lambda z.E[x]]$, $A_5 = A_3[\hat{A}_1[A_1[\lambda x.A_4[\lambda y.\check{A}_2]]\, A_2[v]]]$ and repeat analysis of above subcase.

(remaining subcases check possible Es)

Subcase $e^* = E[e_3]\, e_2'$, $e^{**} = E[e_3']\, e_2'$, $e_3\, \boldsymbol{\beta}_{\mathbf{need}}\, e_3'$**.**

By IH on $e_1 \Rightarrow E[e_3] \longmapsto E[e_3']$.

Subcase $e^* = A[E[e_3]]$, $e^{**} = A[E[e_3']]$, $e_3\, \boldsymbol{\beta}_{\mathbf{need}}\, e_3'$**.**

If $A = [\,]$, check cases of E.

If $A = A_1[\lambda x.A_2]\, e_2'$, then $e^* = A_1[\lambda x.A_2[E[e_3]]]\, e_2'$, where $e_1 \Rightarrow A_1[\lambda x.A_2[E[e_3]]]$

By lemma 3, there exists $A_1'[\lambda x.e_4]$ s.t. $\underline{e_1 \longmapsto\!\!\!\!\!\rightarrow A_1'[\lambda x.e_4] \Rightarrow A_1[\lambda x.A_2[E[e_3]]]}$,

where $A_1' \Rightarrow A_1$ and $e_4 \Rightarrow A_2[E[e_3]]$

Since $e_4 \Rightarrow A_2[E[e_3]] \longmapsto A_2[E[e_3']]$, by IH, there exists e_5 s.t. $\underline{e_4 \longmapsto\!\!\!\!\!\rightarrow e_5 \Rightarrow A_2[E[e_3']]}$, Thus,

$$\begin{aligned} e = e_1\, e_2 &\longmapsto\!\!\!\!\!\rightarrow A_1'[\lambda x.e_4]\, e_2 \\ &\longmapsto\!\!\!\!\!\rightarrow A_1'[\lambda x.e_5]\, e_2 \\ &\Rightarrow A_1[\lambda x.A_2[E[e_3']]]\, e_2' \end{aligned}$$

So $e^{***} = A_1'[\lambda x.e_5]\, e_2$

Subcase $e^* = \hat{A}[A[\lambda x.\check{A}[E_1[x]]]\, E_2[e_3]]$, $e^{**} = \hat{A}[A[\lambda x.\check{A}[E_1[x]]]\, E_2[e_3']]$, $e_3\, \boldsymbol{\beta}_{\mathbf{need}}\, e_3'$, $\hat{A}[\check{A}] \in A$**.**

(Similar reasoning to case 3.1 above)

Subsubcase $\hat{A} = [\,]$**.**

If $\hat{A} = [\,]$, then $\check{A} = [\,]$, so $e_1 \Rightarrow A[\lambda x.E_1[x]]$ and $e_2 \Rightarrow E_2[e_3]$

Since $e_1 \Rightarrow A[\lambda x.E_1[x]]$, by lemma 3, there exists A' and e_5 s.t. $\underline{e_1 \longmapsto\!\!\!\!\!\rightarrow A'[\lambda x.e_5] \Rightarrow A[\lambda x.E_1[x]]}$,

where $A' \Rightarrow A$ and $e_5 \Rightarrow E_1[x]$. By the $E[x]$ shape lemma, $\underline{e_5 = E_1'[x]}$, where $E_1' \Rightarrow E_1$.

Since $e_2 \Rightarrow E_2[e_3] \longmapsto E_2[e_3']$, by IH, there exists e_4 s.t. $\underline{e_2 \longmapsto\!\!\!\!\!\rightarrow e_4 \Rightarrow E_2[e_3']}$. Thus,

$$\begin{aligned} e = e_1\, e_2 &\longmapsto\!\!\!\!\!\rightarrow A'[\lambda x.E_1'[x]]\, e_2 \\ &\longmapsto\!\!\!\!\!\rightarrow A'[\lambda x.E_1'[x]]\, e_4 \\ &\Rightarrow A[\lambda x.E_1[x]]\, E_2[e_3'] \end{aligned}$$

so $e^{***} = A'[\lambda x.E_1'[x]]\, e_4$.

Subsubcase $\hat{A} = A_1[\hat{A}_1]\, e_2'$**.**

Then $\check{A} = A_2[\lambda y.\check{A}_1]$ s.t. $\hat{A}_1[\check{A}_1] \in A$, so $e_1 \Rightarrow A_1[\hat{A}_1[A[\lambda x.A_2[\lambda y.\check{A}_1[E_1[x]]]]\, E_2[e_3]]]$, which is an answer, bc there is now one more λ than arguments.

Subsubsubcase $\hat{A}_1 = \check{A}_1 = [\,]$, ($\lambda y.$ is innermost lambda)**.**

$e_1 \Rightarrow A_1[A[\lambda x.A_2[\lambda y.E_1[x]]]\, E_2[e_3]] = A_3[\lambda y.E_1[x]]$, where $A_3 = A_1[A[\lambda x.A_2]\, E_2[e_3]]$

By lemma 3, and $E[x]$ shape lemma, there exists A_3' and $E_1'[x]$ s.t.

$\underline{e_1 \longmapsto\!\!\!\!\!\rightarrow A_3'[\lambda y.E_1'[x]] \Rightarrow A_3[\lambda y.E_1[x]]}$

Let $\underline{A_3' = A_1'[A'[\lambda x.A_2']\, e_4]}$ s.t. $e_4 \Rightarrow E[e_3] \longmapsto E_2[e_3']$,

so by IH, there is some e_5 s.t. $\underline{e_4 \longmapsto\!\!\!\!\!\rightarrow e_5 \Rightarrow E_2[e_3']}$. Thus,

$$\begin{aligned} e = e_1\, e_2 &\longmapsto\!\!\!\!\!\rightarrow A_3'[\lambda y.E_1'[x]]\, e_2 \\ &= A_1'[A'[\lambda x.A_2'[\lambda y.E_1'[x]]]\, e_4]\, e_2 \\ &\longmapsto\!\!\!\!\!\rightarrow A_1'[A'[\lambda x.A_2'[\lambda y.E_1'[x]]]\, e_5]\, e_2 \\ &\Rightarrow A_1[A[\lambda x.A_2[\lambda y.E_1[x]]]\, E_2[e_3']]\, e_2' \end{aligned}$$

So $e^{***} = A_1'[A'[\lambda x.A_2'[\lambda y.E_1'[x]]]\, e_5]\, e_2$

Subsubsubcase *otherwise***.**

Let $\check{A}_1 = \check{A}_2[\lambda z.[\,]]$ s.t. $\lambda z.$ is the innermost unpaired λ so that
$e_1 \Rightarrow A_1[\hat{A}_1[A[\lambda x.A_2[\lambda y.\check{A}_2[\lambda z.E_1[x]]]]\, E_2[e_3]]]$
Apply lemma 3 to $e_1 \Rightarrow A_3[\lambda z.E_1[x]]$ and repeat analysis of above subcase.

Case $e \Rightarrow e'$ by $\Rightarrow def(4)$**.**

Impossible because not a $\boldsymbol{\beta}_{\mathbf{need}}$ redex.

$\square$

Lemma 3.

1. *If* $e \Rightarrow A[\lambda x.e']$, *there exists* A' *and* e'' *s.t.* $e \longmapsto\!\!\!\!\rightarrow A'[\lambda x.e''] \Rightarrow A[\lambda x.e']$, *where* $A' \Rightarrow A$ *and* $e'' \Rightarrow e'$.

2. *If* $e \Rightarrow x$, *then* $e \longmapsto\!\!\!\!\rightarrow x$.

Proof. Same reasoning as redex text. $\square$

Lemma 4 (Size)**.** *If* $s_e = |e \Rightarrow e'|$ *and* $s_v = |v \Rightarrow v'|$ *then:*

1. $e\{x := v\} \Rightarrow e'\{x := v'\}$
2. $|e\{x := v\} \Rightarrow e'\{x := v'\}| \leq |e \Rightarrow e'| + \#(x, e') \times |v \Rightarrow v'|$

Proof. By induction on e. Case analysis on last step in $e \Rightarrow e'$.

Case $e \Rightarrow e'$ by $\Rightarrow def(1)$**.**

$e = e'$. Proof by subcases on e. Same reasoning as redex book.

Case $e \Rightarrow e'$ by $\Rightarrow def(2)$**.**

$e = \hat{A}[A_1[\lambda y.\check{A}[E[y]]]\, A_2[v_1]]$,
$e' = \hat{A}'[A_1'[A_2'[\check{A}'[E'[y]]\{y := v_1'\}]]]$
$|e \Rightarrow e'| = |\hat{A} \Rightarrow \hat{A}'| + |A_1 \Rightarrow A_1'| + |A_2 \Rightarrow A_2'| + |\check{A}[E[y]] \Rightarrow \check{A}'[E'[y]]| + \#(y, \check{A}'[E'[y]]) \times |v_1 \Rightarrow v_1'| + 1$
By IH:
$|\hat{A}\{x := v\} \Rightarrow \hat{A}'\{x := v'\}| \leq |\hat{A} \Rightarrow \hat{A}'| + \#(x, \hat{A}') \times |v \Rightarrow v'|$
$|A_1\{x := v\} \Rightarrow A_1'\{x := v'\}| \leq |A_1 \Rightarrow A_1'| + \#(x, A_1') \times |v \Rightarrow v'|$
$|\check{A}[E[y]]\{x := v\} \Rightarrow \check{A}'[E'[y]]\{x := v'\}| \leq |\check{A}[E[y]] \Rightarrow \check{A}'[E'[y]]| + \#(x, \check{A}'[E'[y]]) \times |v \Rightarrow v'|$
$|A_2\{x := v\} \Rightarrow A_2'\{x := v'\}| \leq |A_2 \Rightarrow A_2'| + \#(x, A_2') \times |v \Rightarrow v'|$
$|v_1\{x := v\} \Rightarrow v_1'\{x := v'\}| \leq |v_1 \Rightarrow v_1'| + \#(x, v_1') \times |v \Rightarrow v'|$

$$\begin{aligned}
& |e\{x := v\} \Rightarrow e'\{x := v'\}| \\
= {} & |\hat{A}\{x := v\} \Rightarrow \hat{A}'\{x := v'\}| + |A_1\{x := v\} \Rightarrow A_1'\{x := v'\}| + |A_2\{x := v\} \Rightarrow A_2'\{x := v'\}| + \\
& |\check{A}[E[y]]\{x := v\} \Rightarrow \check{A}'[E'[y]]\{x := v'\}| + \\
& \#(y, \check{A}'[E'[y]]) \times |v_1\{x := v\} \Rightarrow v_1'\{x := v'\}| + 1 \\
\leq {} & |\hat{A} \Rightarrow \hat{A}'| + \#(x, \hat{A}') \times |v \Rightarrow v'| + \\
& |A_1 \Rightarrow A_1'| + \#(x, A_1') \times |v \Rightarrow v'| + \\
& |A_2 \Rightarrow A_2'| + \#(x, A_2') \times |v \Rightarrow v'| + \\
& |\check{A}[E[y]] \Rightarrow \check{A}'[E'[y]]| + \#(x, \check{A}'[E'[y]]) \times |v \Rightarrow v'| + \\
& \#(y, \check{A}'[E'[y]]) \times (|v_1 \Rightarrow v_1'| + \#(x, v_1') \times |v \Rightarrow v'|) + 1 \\
= {} & |\hat{A} \Rightarrow \hat{A}'| + \#(x, \hat{A}') \times |v \Rightarrow v'| + \\
& |A_1 \Rightarrow A_1'| + \#(x, A_1') \times |v \Rightarrow v'| + \\
& |A_2 \Rightarrow A_2'| + \#(x, A_2') \times |v \Rightarrow v'| + \\
& |\check{A}[E[y]] \Rightarrow \check{A}'[E'[y]]| + \#(x, \check{A}'[E'[y]]) \times |v \Rightarrow v'| + \\
& \#(y, \check{A}'[E'[y]]) \times |v_1 \Rightarrow v_1'| + \#(y, \check{A}'[E'[y]]) \times \#(x, v_1') \times |v \Rightarrow v'| + 1 \\
= {} & |\hat{A} \Rightarrow \hat{A}'| + |A_1 \Rightarrow A_1'| + |A_2 \Rightarrow A_2'| + |\check{A}[E[y]] \Rightarrow \check{A}'[E'[y]]| + \#(y, \check{A}'[E'[y]]) \times |v_1 \Rightarrow v_1'| + 1 + \\
& (\#(x, \hat{A}') + \#(x, A_1') + \#(x, A_2') + \#(x, \check{A}'[E'[y]]) + \#(y, \check{A}'[E'[y]]) \times \#(x, v_1')) \times |v \Rightarrow v'| \\
= {} & |e \Rightarrow e'| + \#(x, e') \times |v \Rightarrow v'|
\end{aligned}$$

Case $e = e_1\, e_2$**.**

Same reasoning as redex book.

Case $e = \lambda x.e_1$**.**

Same reasoning as redex book.

$\square$

Lemma 5 (Unique Decomposition)**.** *For all terms e and variables x, if $e \neq E'[x]$, x not bound by E', then e is either an answer $A[v]$, or there exists a unique evaluation context E and $\boldsymbol{\beta}_{\mathbf{need}}$ redex e' such that $e = E[e']$.*

Proof. By structural induction on e.

Case $e = x$**.**

Impossible because $e = E[x]$, $E = [\,]$

Case $e = \lambda x.e_1$**.**

Claim holds with e an answer, $A = [\,]$, $v = e$.

Case $e = e_1\, e_2$**.**

$e_1 \neq E''[x]$, otherwise, $e = E'[x]$, where $E' = E''\, e_2$. By IH, e_1 is either an answer or uniquely decomposes, so proceed by subcases on e_1.

Subcase $e_1 = E_1[e_3]$**.**

By IH, e_3 is a $\boldsymbol{\beta}_{\mathbf{need}}$ redex. Then the claim holds with $E = E_1\, e_2$ and $e' = e_3$.

Subcase $e_1 = A_1[v_1]$**.**

Let $v_1 = \lambda x.e_3$. Proceed by subsubcases on e_3.

Subsubcase $e_3 = E_1[y]$**.**

where y is a variable bound by the answer context $A_1[\lambda x.[\,]]\, e_2$. Proceed by subsubsubcases on e_2

Subsubsubcase $e_2 = E_2[e_4]$**.**

e_4 is a $\boldsymbol{\beta}_{\mathbf{need}}$ redex. Then the claim holds with $E = A_1[\lambda x.E_1[y]]\, E_2$ and $e' = e_4$.

Subsubsubcase $e_2 = A_2[v_2]$**.**

Then e is a $\boldsymbol{\beta}_{\mathbf{need}}$ redex.

Subsubcase $e_3 \neq E_1[y] = E_2[e_4]$**.**

e_4 is a $\boldsymbol{\beta}_{\mathbf{need}}$ redex. Claim holds with $E = A_1[\lambda x.E_2[\ ]]\, e_2$ and $e' = e_4$.

Subsubcase $e_3 \neq E_1[y] = A_2[v_2]$**.**

Claim holds where e is an answer with $A = A_1[\lambda x.A_2[\ ]]\, e_2$ and $v = v_2$

$\square$