Definitions
A set Σ of wffs is *satisfiable* iff there is a truth assignment that satisfies every member of Σ.

Exercise 1.7.1
Assume every finite subset of Σ is satisfiable. Show that the same is true of at least one of the sets $\Sigma; \alpha$ and $\Sigma; \neg\alpha$.

Proof:
Proof by contradiction. Assume that neither every finite subset of $\Sigma; \alpha$ nor every finite subset of $\Sigma; \neg\alpha$ is satisfiable. That means there is some finite $\Sigma_1 \subseteq \Sigma$ such that $\Sigma_1; \alpha$ is unsatisfiable and that there is some finite $\Sigma_2 \subseteq \Sigma$ such that $\Sigma_2; \neg\alpha$ is unsatisfiable. However, $\Sigma_1 \cup \Sigma_2$ is a finite subset of Σ, so it must be satisfiable, from the stated assumption. This means that there exists some truth assignment v such that $\bar{v}(\varphi) = T$, for $\varphi \in (\Sigma_1 \cup \Sigma_2)$. However, either $\bar{v}(\alpha) = T$ or $\bar{v}(\alpha) = F$, which means that either $\Sigma_1; \alpha$ or $\Sigma_2; \neg\alpha$ is satisfiable. This is a contradiction from what was previously stated.

Exercise 1.7.2
Let Δ be a set of wffs such that (i) every finite subset of Δ is satisfiable, and (ii) for every wff α, either $\alpha \in \Delta$ or $(\neg\alpha) \in \Delta$. Define the truth assignment v:

$$v(A) = \begin{cases} T & \text{iff } A \in \Delta, \\ F & \text{iff } A \notin \Delta \end{cases}$$

for each sentence symbol. Show that for every wff φ, $\bar{v}(\varphi) = T$ iff $\varphi \in \Delta$.

Proof:
$\Rightarrow$ We need to show that, for every wff φ, if $\bar{v}(\varphi) = T$, then $\varphi \in \Delta$.

If $\bar{v}(\varphi) = T$, but $\varphi \notin \Delta$, then by (ii), $(\neg\varphi) \in \Delta$. However, $\bar{v}(\neg\varphi) = F$, which means that there is a finite subset of Δ that is unsatisfiable. However, this cannot be because according to (i), every finite subset of Δ is satisfiable. Therefore, if $\bar{v}(\varphi) = T$, then $\varphi \in \Delta$.

$\Leftarrow$ We need to show that, for every wff φ, if $\varphi \in \Delta$, then $\bar{v}(\varphi) = T$.

We will prove this by structural induction on φ.

Case $\varphi = A$
Since $A \in \Delta$, then $v(A) = T$, by definition of v.

Case $\varphi = \alpha \wedge \beta$
We first show that if $\varphi \in \Delta$, then both α and β are in Δ. If $\varphi \in \Delta$ and $\alpha \notin \Delta$, then by (ii), $\neg\alpha \in \Delta$. However, this means that $\{\varphi, \neg\alpha\}$ is a finite subset of Δ. However, $\{\varphi, \neg\alpha\}$ is unsatisfiable because φ is true when $\neg\alpha$ is false and vice versa. Therefore, $\{\varphi, \neg\alpha\}$ cannot be a finite subset of Δ because according to (i), every finite subset of Δ is satisfiable. Therefore, if $\varphi \in \Delta$, then $\alpha \in \Delta$. We can also make a similar argument for β and therefore, if $\varphi \in \Delta$, then $\alpha, \beta \in \Delta$.

If $\alpha \in \Delta$, then by the induction hypothesis, $\bar{v}(\alpha) = T$. Similarly, $\bar{v}(\beta) = T$. Since $\varphi = \alpha \wedge \beta$, then $\bar{v}(\varphi) = T$. Therefore, if $\varphi \in \Delta$, then $\bar{v}(\varphi) = T$.

Case $\varphi = \alpha \vee \beta$
Case $\varphi = \alpha \rightarrow \beta$
Case $\varphi = \alpha \leftrightarrow \beta$
Case $\varphi = \neg\alpha$

A proof strategy that is similar to the one used for the conjunction case can also be used for these cases.

Therefore, we have shown that if $\varphi \in \Delta$, then $\bar{v}(\varphi) = T$, for all cases.

Since we have proven both directions, we can conclude that for every wff φ, if $\varphi \in \Delta$, then $\bar{v}(\varphi) = T$.

Compactness Theorem
A set of wffs is satisfiable iff every finite subset is satisfiable.

Proof:
$\Rightarrow$ We need to show that if a set of wffs is satisfiable, then every finite subset is satisfiable.

If a set of wffs is satisfiable, then every finite subset of the set is automatically satisfiable.

$\Leftarrow$ We need to show that if every finite subset of a set of wffs is satisfiable, then the set itself is satisfiable.

Say the set in question is called Σ.

1. Enumerate every wff $\alpha_1, \alpha_2, \ldots$
2. Let $\Delta_0 = \Sigma$
3. Let $\Delta_{n+1} = \begin{cases} \Delta_n; \alpha n + 1 & \text{if this is finitely satisfiable,} \\ \Delta_n; \neg\alpha_{n+1} & \text{otherwise.} \end{cases}$ From the result of Exercise 1.7.1, we know that every Δ_n is satisfiable.
4. Let $\Delta = \bigcup_n \Delta_n$
5. We know that every finite subset of Δ is satisfiable because every finite subset of Δ is also a subset of some Δ_n, which is finitely satisfiable. We also know that for any wff α, either $\alpha \in \Delta$ or $\neg\alpha \in \Delta$. Therefore, if we have a truth assignment v such that $v(A) = T$ iff $A \in \Delta$, from Exercise 1.7.2, we know that if $\varphi \in \Delta$, then $\bar{v}(\varphi) = T$. Since every wff in Δ is satisfied by v, then Δ is satisfiable.
6. Since $\Sigma \subseteq \Delta$, v must also safisfy Σ, so therefore Σ is satisfiable.

Corollary
If $\Sigma \models \tau$, then there is a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \tau$.

Exercise 1.7.3 - Proof of Compactness Theorem Using Corollary
$\Rightarrow$ We must show that if a set of wffs is satisfiable, then every finite subset is satisfiable.

If a set of wffs is satisfiable, then every finite subset is automatically satisfiable.

$\Leftarrow$ We must show that if every finite subset of a set of wffs is satisfiable, then the set itself is satisfiable.

Proof by contradiction. Assume that we have a set of wffs Σ such that every finite subset of Σ is satisfiable, but that Σ itself is unsatisfiable. This means that $\Sigma \models \tau$, for any wff τ. According to the corollary, there is a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \tau$. Let $\tau = \alpha \wedge \neg\alpha$, which is unsatisfiable. This means that there is no $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \tau$. We have a contradiction, so therefore, if every finite subset of Σ is satisfiable, Σ must be satisfiable.