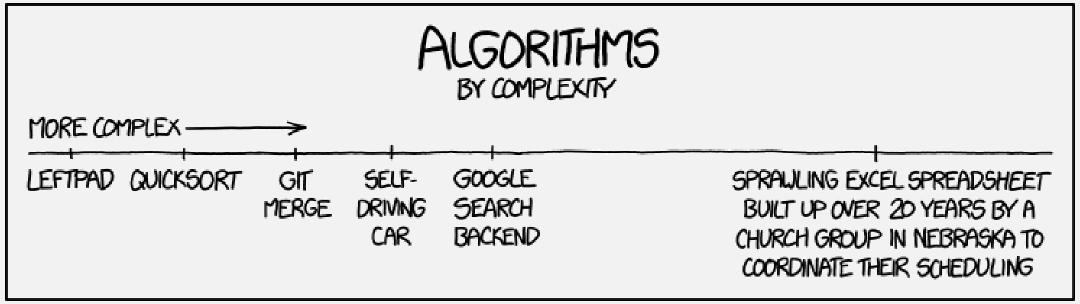
UMB CS 420

Time Complexity

Wednesday, April 13, 2022

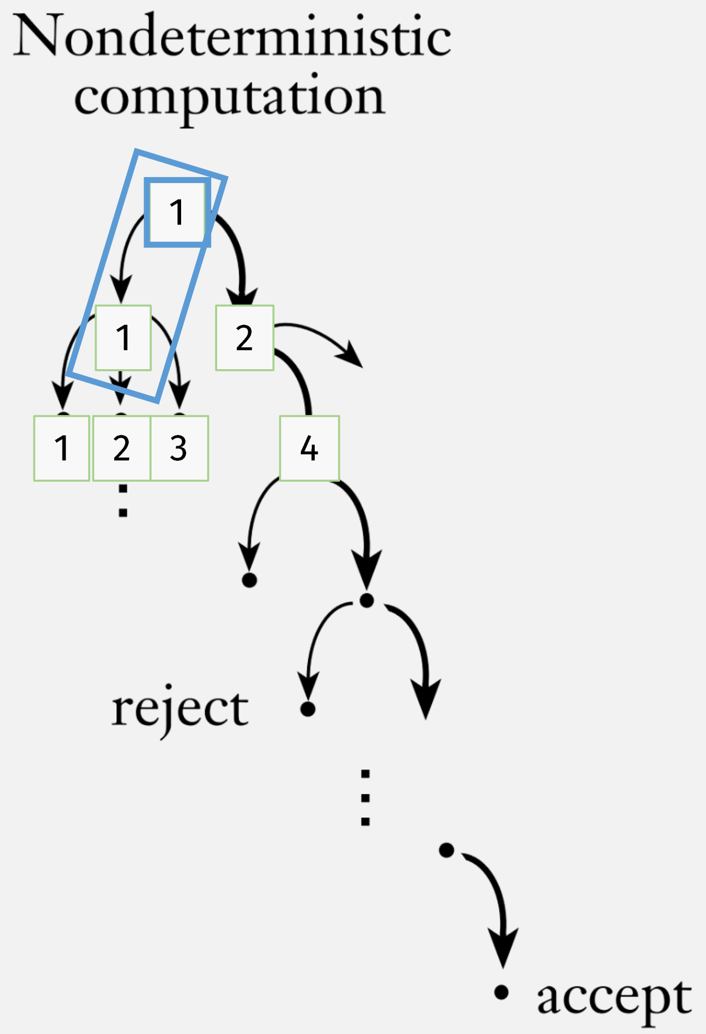


Announcements

- HW 9 out
  - Due Sunday 4/17 11:59pm

Flashback: Nondet. TM → Deterministic TM

- To simulate NTM with Det. TM:
  - Number the nodes at each step
  - Deterministically check every tree path, in breadth-first order
    - Root node: 1
    - **1-1**



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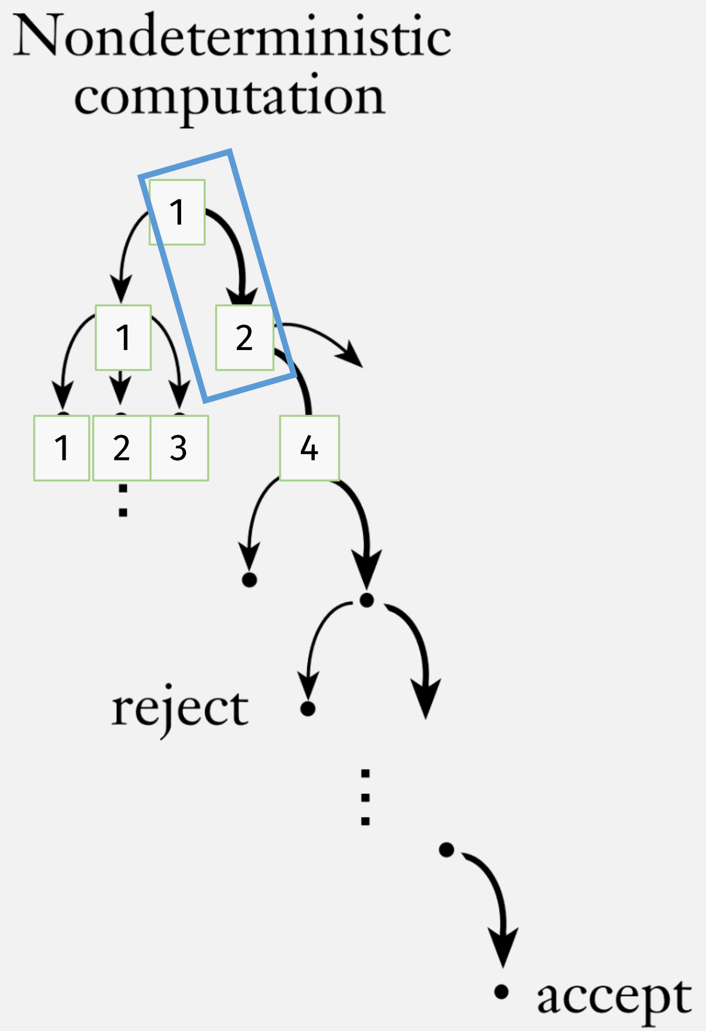

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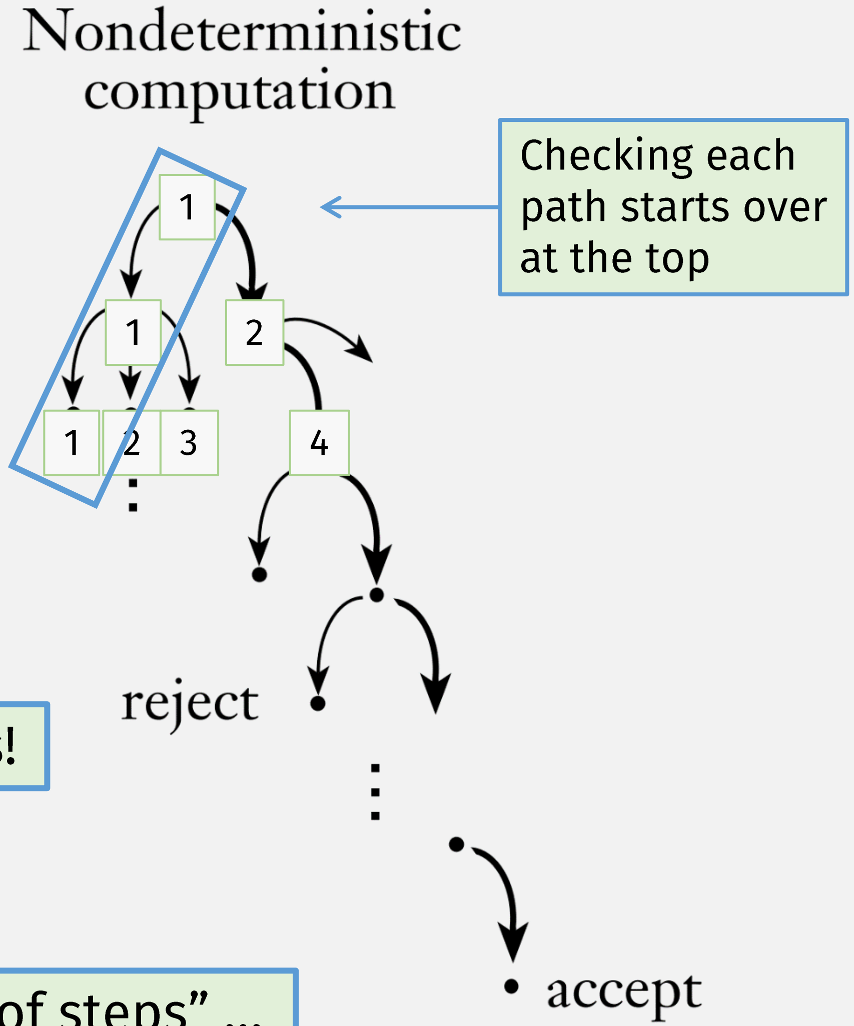

A TM and a NTM are "equivalent" ...

.. but **not** if we care about the # of steps!

So **how inefficient is it?**

First**, we need a formal way to count "# of steps" ...**

A Simpler Example: $A = \{0^k 1^k | k \geq 0\}$

M_1 = "On input string w:

1. Scan across the tape and *reject* if a 0 is found to the right of a 1.
2. Repeat if both 0s and 1s remain on the tape:
3. Scan across the tape, crossing off a single 0 and a single 1.
4. If 0s still remain after all the 1s have been crossed off, or if 1s still remain after all the 0s have been crossed off, *reject*. Otherwise, if neither 0s nor 1s remain on the tape, *accept*."

# of steps (worst case), n = length of input:

➢ TM Line 1:

- n steps to scan + n steps to return to beginning = $\boxed{2n \text{ steps}}$

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# of steps (worst case), n = length of input:

- TM Line 1:
  - n steps to scan + n steps to return to beginning = **$2n$ steps**

➢ Lines 2-3 (loop):
  - steps/iteration (line 3): $n/2$ steps to find "1" + $n/2$ steps to return = n steps
  - # iterations (line 2): Each scan crosses off 2 chars, so at most $n/2$ scans
  - Total = steps/iteration * # iterations = $n\ (n/2)$ = $\boxed{n^2/2 \text{ steps}}$

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$\boxed{n^2/2 + 3n}$

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- Lines 2-3 (loop):
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  - Total = steps/iteration * # iterations = $n(n/2)$ = $\boxed{n^2/2 \text{ steps}}$
- Line 4:
  - $\boxed{n \text{ steps}}$ to scan input one more time
- Total: $\boxed{2n + n^2/2 + n}$ = $\boxed{n^2/2 + 3n \text{ steps}}$

Interlude: Polynomials

order/degree

coefficients

Highest order term

$$\underline{6n^3} + \underline{2n^2} + \underline{20n} + \underline{45}$$

terms

Definition: Time Complexity

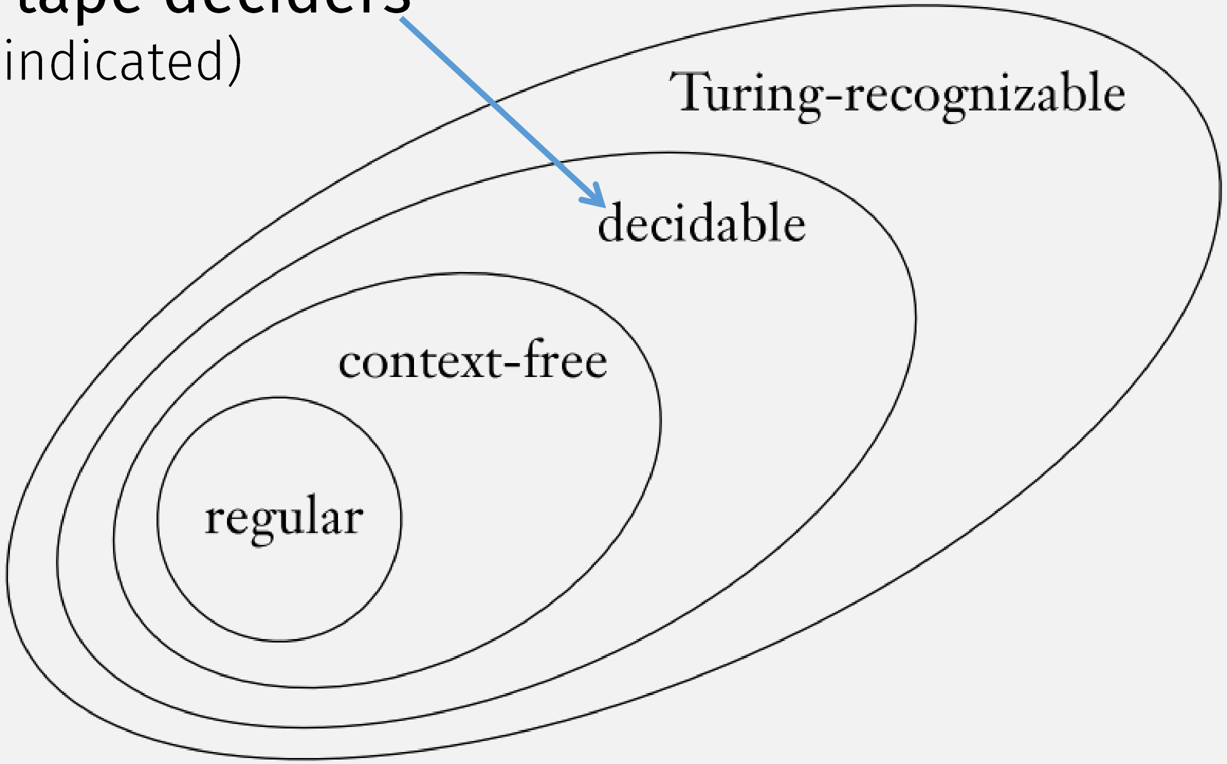
i.e., a decider (algorithm)

Let M be a deterministic Turing machine that halts on all inputs. The ***running time*** or ***time complexity*** of M is the function $f : \mathcal{N} \longrightarrow \mathcal{N}$, where $f(n)$ is the maximum number of steps that M uses on any input of length n. If $f(n)$ is the running time of M, we say that M runs in time $f(n)$ and that M is an $f(n)$ time Turing machine. Customarily we use n to represent the length of the input.

Running Time / **Time Complexity** is a property of a (Turing) Machine

Where Are We Now?

We are back in here now:
deterministic, single-tape deciders
(unless otherwise indicated)



Definition: Time Complexity

NOTE: n has no units, it's only *roughly* "length" of the input

n can be:
\# characters,
\# states,
\# nodes, …

We can use any n that is correlated with the input length

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- Machine M_1 that decides $A = \{0^k 1^k | k \geq 0\}$
  - Running time / Time Complexity: $n^2/2+3n$

M_1 = "On input string w:
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Interlude: Asymptotic Analysis

Total: $\boldsymbol{n^2 + 3n}$

- If $\boldsymbol{n} = 1$
  - $\boldsymbol{n^2} = 1$
  - $\boldsymbol{3n} = 3$
  - Total = 4
- If $\boldsymbol{n} = 10$
  - $\boldsymbol{n^2} = 100$
  - $\boldsymbol{3n} = 30$
  - Total = 130
- If $\boldsymbol{n} = 100$
  - $\boldsymbol{n^2} = 10{,}000$
  - $\boldsymbol{3n} = 300$
  - Total = 10,300
- If $\boldsymbol{n} = 1{,}000$
  - $\boldsymbol{n^2} = 1{,}000{,}000$
  - $\boldsymbol{3n} = 3{,}000$
  - Total = 1,003,000

asymptotic analysis only cares about **large** n

$\boldsymbol{n^2 + 3n} \approx \boldsymbol{n^2}$ as n gets large

Definition: Big-O Notation

Let f and g be functions $f, g\colon \mathcal{N} \longrightarrow \mathcal{R}^{+}$. Say that $\boxed{\boldsymbol{f(n) = O(g(n))}}$ if positive integers c and n_0 exist such that for $\boxed{\text{every integer } n \geq n_0}$,

$$\boxed{f(n) \leq c\, g(n).}$$

"only care about **large** n"

When $f(n) = O(g(n))$, we say that $g(n)$ is an ***upper bound*** for $f(n)$, or more precisely, that $\boxed{g(n) \text{ is an } \textbf{\textit{asymptotic upper bound}}}$ for $f(n)$, to emphasize that we are suppressing constant factors.

In other words: **Keep only highest order term, drop all coefficients**

- **Machine M_1 that decides** $A = \{0^k 1^k \mid k \geq 0\}$
  - is an $\boldsymbol{n^2 + 3n}$ **time Turing machine**
  - is an $O(\boldsymbol{n^2})$ **time Turing machine**
  - has **asymptotic upper bound $O(\boldsymbol{n^2})$**

Definition: Small-o Notation (less used)

Let f and g be functions $f, g\colon \mathcal{N} \longrightarrow \mathcal{R}^+$. Say that $\boldsymbol{f(n) = o(g(n))}$ if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.$$

In other words, $f(n) = o(g(n))$ means that for any real number $c > 0$, a number n_0 exists, where $\boxed{f(n) < c\,g(n)}$ for all $n \geq n_0$.

Analogy: Big-O : ≤ :: small-o : <

Let f and g be functions $f, g\colon \mathcal{N} \longrightarrow \mathcal{R}^+$. Say that $\boldsymbol{f(n) = O(g(n))}$ if positive integers c and n_0 exist such that for every integer $n \geq n_0$,

$$\boxed{f(n) \leq c\,g(n).}$$

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Big-O arithmetic

- $O(\boldsymbol{n^2}) + O(\boldsymbol{n^2})$
  $= O(\boldsymbol{n^2})$

- $O(\boldsymbol{n^2}) + O(\boldsymbol{n})$
  $= O(\boldsymbol{n^2})$

- $\boldsymbol{2n} = O(\boldsymbol{n})$?
  - TRUE

- $\boldsymbol{2n} = O(\boldsymbol{n^2})$?
  - TRUE

- $\boldsymbol{1} = O(\boldsymbol{n^2})$?
  - TRUE

- $\boldsymbol{2^n} = O(\boldsymbol{n^2})$?
  - FALSE

NOTE: **Other courses** might use Big-Θ notation, which is a tighter bound where some of these equalities won't be true, e.g., $\boldsymbol{2n} \neq \Theta(\boldsymbol{n^2})$

NOTE: **In this course,** we use Big-O only, <u>not</u> Big-Θ (so do not confuse the two)

Definition: Time Complexity Classes

Let $t: \mathcal{N} \longrightarrow \mathcal{R}^+$ be a function. Define the ***time complexity class***, $\mathbf{TIME}(t(n))$, to be the collection of all $\boxed{\text{languages}}$ that are decidable by an $O(t(n))$ time Turing machine.

Remember: **TMs** have a time complexity (i.e., a running time), **languages** are in a time complexity class

The complexity class of a **language** is determined by the time complexity (running time) of its deciding **TM**

A **language** could be in more than one **time complexity class**

- Machine M_1 decides language $A = \{0^k 1^k | k \geq 0\}$
  - M_1 has time complexity (running time) of $O(\boldsymbol{n^2})$
  - A is in time complexity class $\mathbf{TIME}(\boldsymbol{n^2})$

A Faster Machine? $A = \{0^k 1^k \mid k \geq 0\}$

$M_2 =$ "On input string w:

1. Scan across the tape and *reject* if a 0 is found to the right of a 1.
2. Repeat as long as some 0s and some 1s remain on the tape:
3. Scan across the tape, checking whether the total number of 0s and 1s remaining is even or odd. If it is odd, *reject*.
4. Scan again across the tape, crossing off every other 0 starting with the first 0, and then crossing off every other 1 starting with the first 1.
5. If no 0s and no 1s remain on the tape, *accept*. Otherwise, *reject*."

Previously:

$M_1 =$ "On input string w:

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Number of steps (worst case)**,** n = **length of input:**

➢ Line 1:

- $\boldsymbol{n}$ steps to scan + $\boldsymbol{n}$ steps to return to beginning = $\boxed{O(\boldsymbol{n}) \text{ steps}}$

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Number of steps (worst case), n = **length of input:**

- Line 1:
  - $\boldsymbol{n}$ steps to scan + $\boldsymbol{n}$ steps to return to beginning = $O(\boldsymbol{n})$ steps
- Lines 2-4 (loop):
  - steps/iteration (lines 3-4): a scan takes $O(\boldsymbol{n})$ steps
  - # iters (line 2): Each iter crosses off *half* the chars, so at most $O(\mathbf{log}\ \boldsymbol{n})$ scans
  - Total: $O(\boldsymbol{n}) * O(\mathbf{log}\ \boldsymbol{n})$ = $\boxed{O(\boldsymbol{n}\ \mathbf{log}\ \boldsymbol{n}) \text{ steps}}$

Interlude: Logarithms (dual to exponentiation)

- If $2^x = y$ ...
- ... then $\log_2 y = x$
- $\log_2 n = O(\mathbf{log}\ \boldsymbol{n})$
  - "divide and conquer" algorithms = $O(\mathbf{log}\ \boldsymbol{n})$
  - E.g., binary search
- (In computer science, **base-2 is the only base!**)

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Number of steps (worst case), n = **length of input:**

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- Lines 2-4 (loop):
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  - Total: $O(\mathbf{n})$ * $O(\mathbf{log\ n})$ = $O(\mathbf{n\ log\ n})$ steps

➢ Line 5:
  - $O(\mathbf{n})$ steps to scan input one more time

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$\boxed{O(n \log n)}$

Prev: $n^2/2 + 3n = O(n^2)$

Number of steps (worst case), n = **length of input:**

- Line 1:
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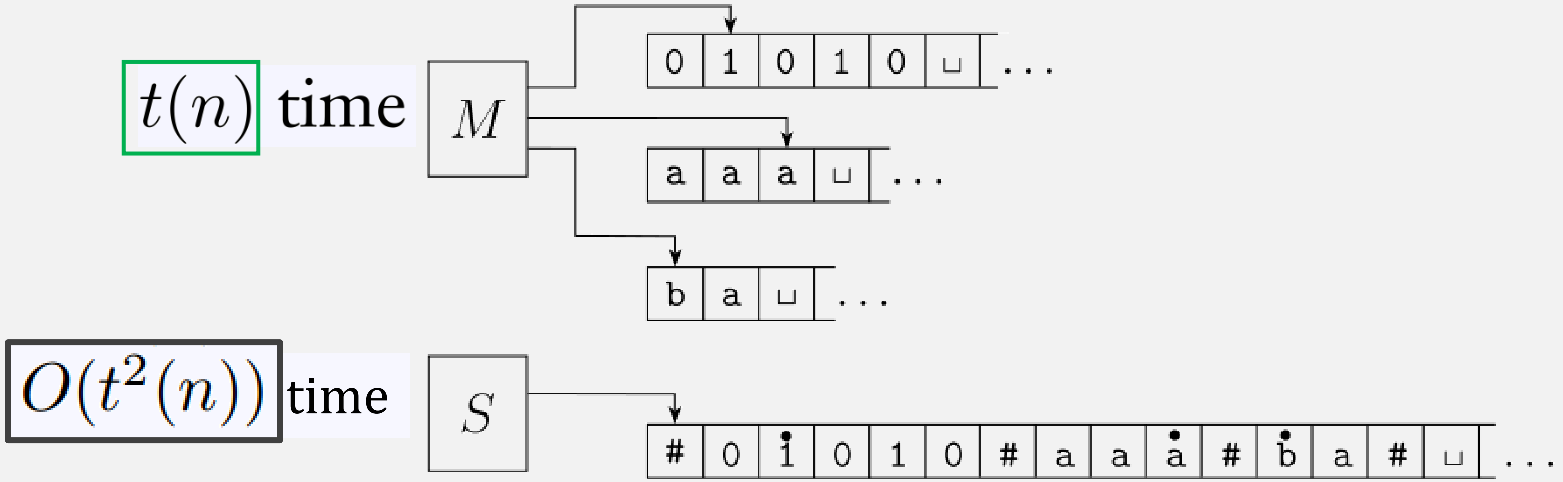
  - # iters (line 2): Each iter crosses off *half* the chars, so at most $O(\log n)$ scans
  - Total: $O(n) * O(\log n)$ = $\boxed{O(n \log n)}$ steps
- Line 5:
  - $\boxed{O(n) \text{ steps}}$ to scan input one more time
- Total: $\boxed{O(n) + O(n \log n) + O(n)}$ =

Terminology: Categories of Bounds

- Exponential time
  - $O(2^{n\text{^}c})$, for $c > 0$, or $2^{O(n)}$ (always base 2)
- Polynomial time
  - $O(n^c)$, for $c > 0$
- Quadratic time (special case of polynomial time)
  - $O(n^2)$
- Linear time (special case of polynomial time)
  - $O(n)$
- Log time
  - $O(\log n)$

Multi-tape vs Single-tape TMs: # of Steps



- For single-tape TM to <u>simulate 1 step</u> of multi-tape:
  1. Scan to find all "heads" = O(length of all M's tapes)
  2. "Execute" transition at all the heads = O(length of all M's tapes)
- # single-tape steps to simulate 1 multitape step (worst case)
  - = O(length of all M's tapes)
  - = $O(t(n))$, If M spends all its steps expanding its tapes
- <u>Total steps (single tape)</u>: $O(t(n))$ per step × $t(n)$ steps =

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  - Deterministically check every tree path, in breadth-first order
    - 1
    - 1-1
    - 1-2
    - **1-1-1**

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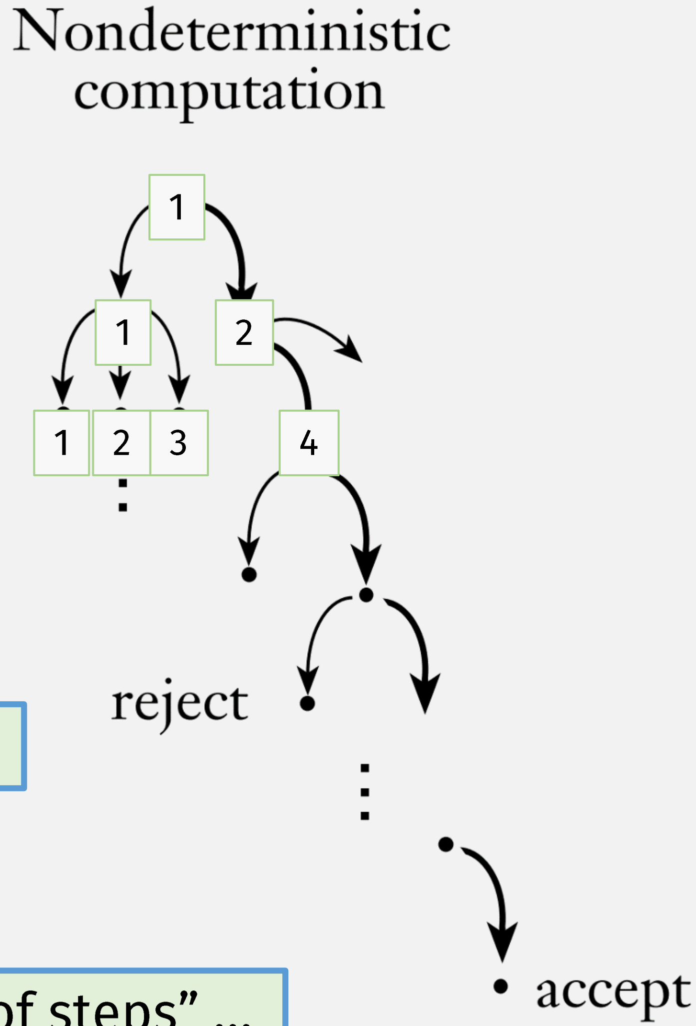
.. but not if we care about the # of steps

How inefficient is it?

First, we need a formal way to count "# of steps" ...



Flashback: Nondet. TM → Deterministic TM

$t(n)$ time → $\boxed{2^{O(t(n))}}$ time

- Simulate NTM with Det. TM:
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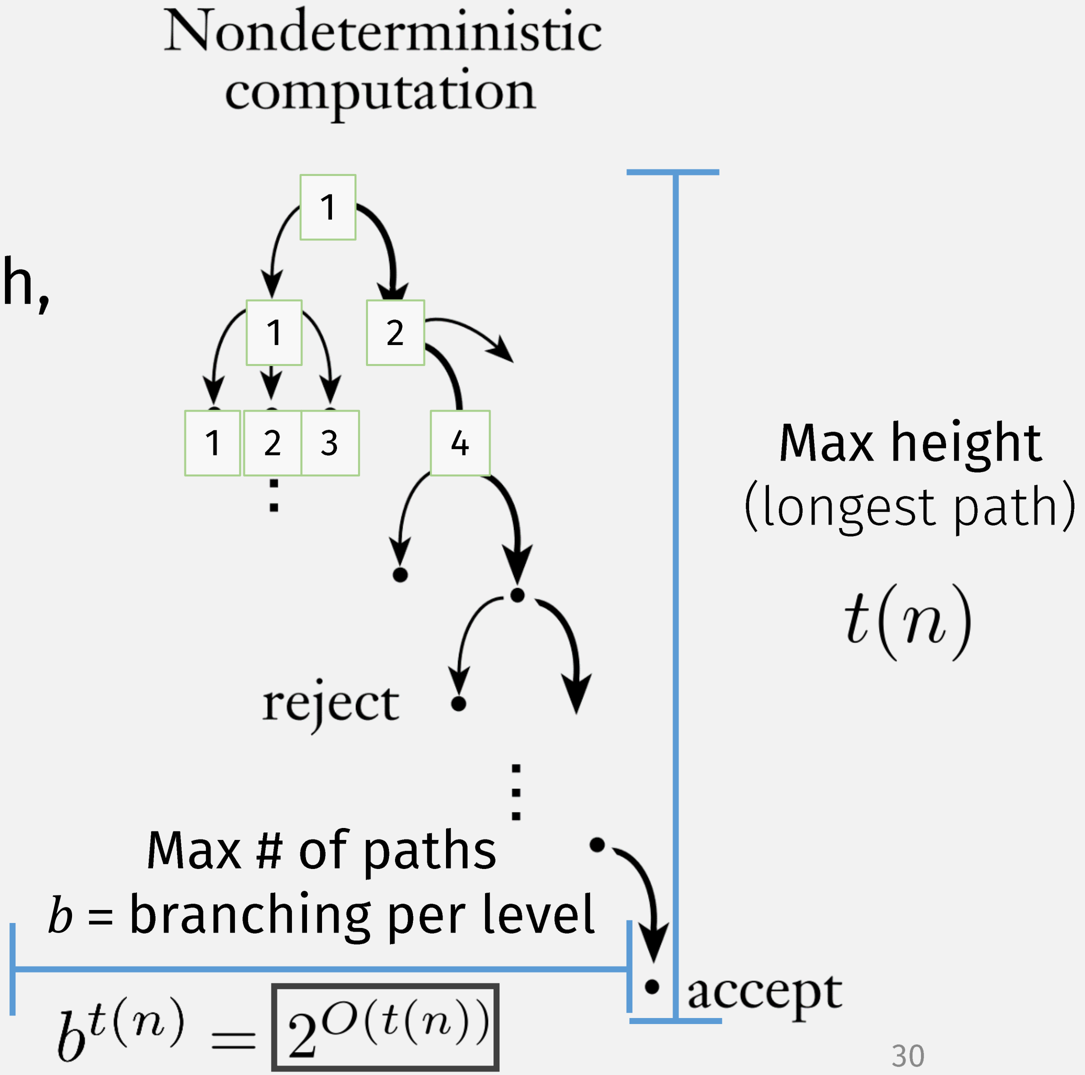

Summary: TM Variations

- If **multi-tape TM:** $t(n)$ time
- Then **equivalent single-tape TM:** $O(t^2(n))$
    - **Quadratically** slower

- If **non-deterministic TM:** $t(n)$ time
- Then **equivalent single-tape TM:** $2^{O(t(n))}$
    - **Exponentially** slower

Check-in Quiz 4/13

On gradescope