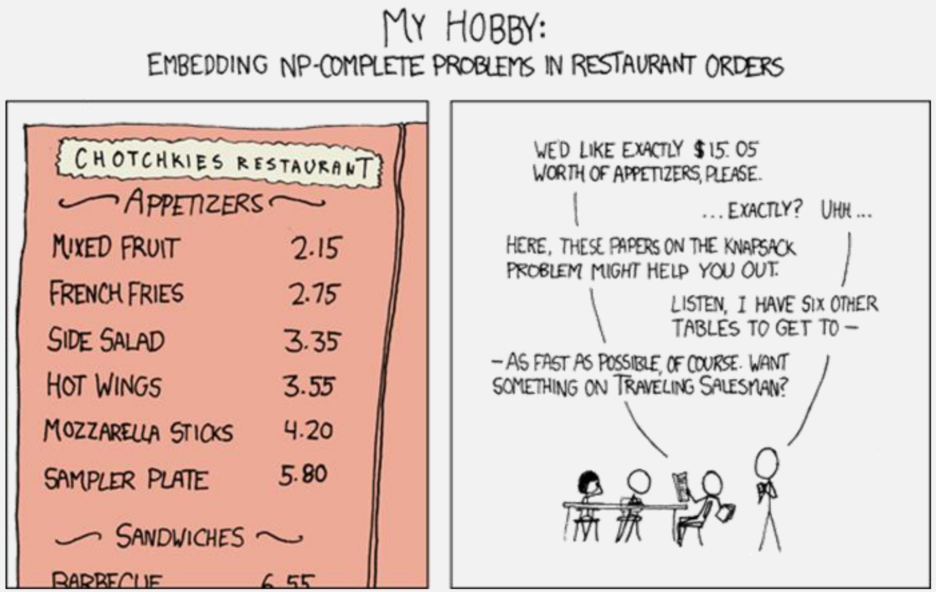
UMB CS 420

NP-Completeness

Monday, May 8, 2023



Announcements

- HW 12 out
  - Due Sunday 5/14 11:59pm

Quiz Preview

Q1 Which of the following are needed to show that a language L is NP-Complete?
1 Point

(select all that apply)

- [ ] it must be in P
- [ ] it must be in NP
- [ ] every language in NP must be poly-time reducible to L
- [ ] L must be poly-time reducible to every other language in NP

Last Time: Verifiers, Formally

$PATH = \{\langle G, s, t\rangle \mid G$ is a directed graph that has a $\boxed{\text{directed path}}$ from s to $t\}$

A possible

Decider …

… with extra argument:
can be any string that helps
to find a result in poly time
(is often just a potential
result itself)

A ***verifier*** for a language A is an algorithm V, where

$$A = \{w \mid V \text{ accepts } \langle w, c\rangle \text{ for some string } c\}.$$

certificate, or ***proof***

We measure the time of a verifier only in terms of the length of w, so a ***polynomial time verifier*** runs in polynomial time in the length of w. A language A is ***polynomially verifiable*** if it has a polynomial time verifier.

- A certificate c has length at most n^k, where n = length of w

Last Time: The class **NP**

DEFINITION

NP is the class of languages that have polynomial time verifiers.

THEOREM

A language is in NP iff it is decided by some nondeterministic polynomial time Turing machine.

2 ways to show that a language is in **NP**

Last Time: **NP** vs **P**

P

The class of languages that have a **deterministic** poly time **decider**

I.e., the class of languages that can be <u>solved</u> "quickly"

- Want <u>search</u> problems to be in here … but they often are not

NP

The class of languages that have a **deterministic** poly time **verifier**

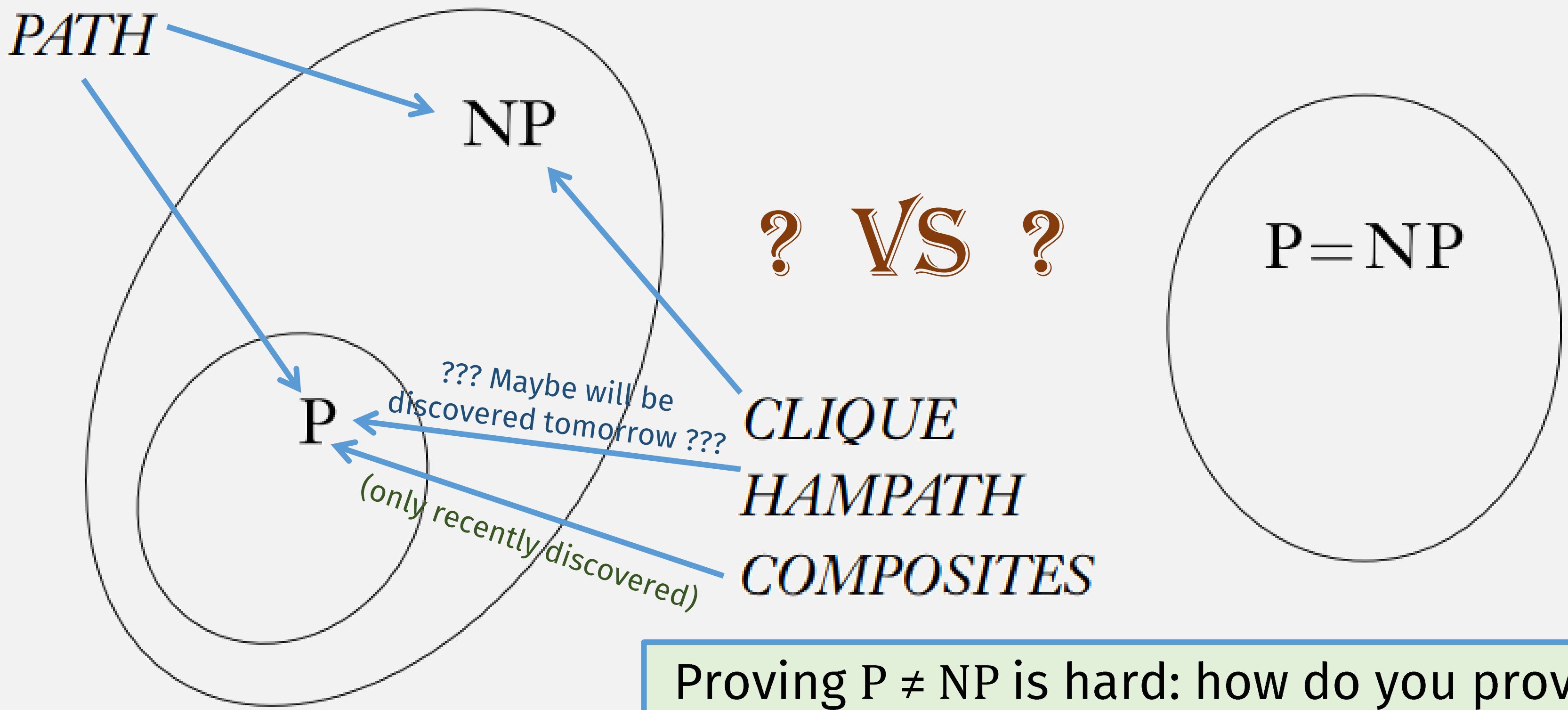
Also, the class of languages that have a **nondeterministic** poly time **decider**

I.e., the class of language that can be <u>verified</u> "quickly"

- Actual <u>search</u> problems (even those not in **P**) are often in here

One of the Greatest unsolved

~~HW~~ Question: Does P = NP?

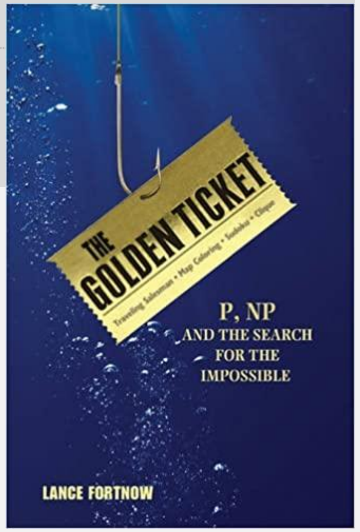


Proving P ≠ NP is hard: how do you prove that an algorithm <u>won't ever</u> have a poly time solution?
(in general, it's hard to prove that something <u>doesn't</u> exist)

Not Much Progress on whether $\mathbf{P} = \mathbf{NP}$?

The Status of the P Versus NP Problem

By Lance Fortnow
Communications of the ACM, September 2009, Vol. 52 No. 9, Pages 78-86
10.1145/1562164.1562186



- One important concept:
  - $\mathbf{NP}$-Completeness

NP-Completeness

DEFINITION

A language B is ***NP-complete*** if it satisfies two conditions:

1. B is in NP, and

2. every A in NP is polynomial time reducible to B.

easy

hard????

Must prove for <u>all</u> langs, not just a single language

What's this?

Flashback: Mapping Reducibility

Language A is ***mapping reducible*** to language B, written $\boxed{A \leq_m B}$, if there is a $\boxed{\text{computable function } f\colon \Sigma^* \longrightarrow \Sigma^*}$, where for every w,

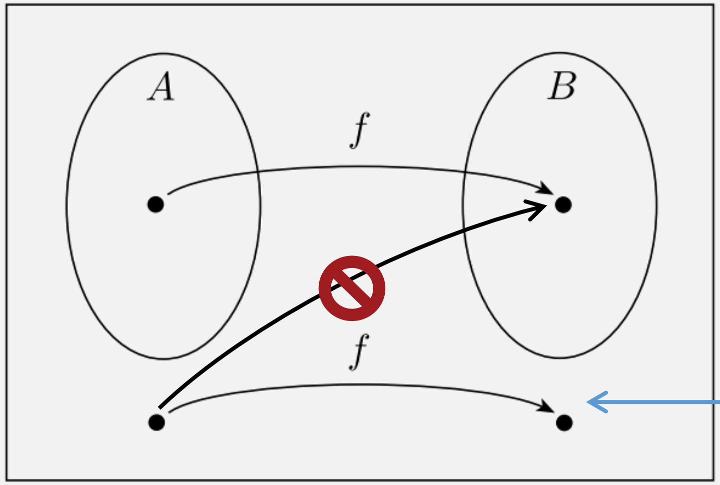
$$\boxed{w \in A \Longleftrightarrow f(w) \in B.}$$

IMPORTANT: "if and only if" …

The function f is called the ***reduction*** from A to B.

To show mapping reducibility:

1. create **computable fn**
2. and then show **forward direction**
3. and **reverse direction**
   (or **contrapositive** of **reverse direction**)



… means $\boxed{\overline{A} \leq_m \overline{B}}$

A function $f\colon \Sigma^* \longrightarrow \Sigma^*$ is a ***computable function*** if some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

Polynomial Time Mapping Reducibility

Language A is ***mapping reducible*** to language
if there is a computable function $f: \Sigma^* \longrightarrow \Sigma^*$,

$$w \in A \iff f(w) \in B.$$

The function f is called the ***reduction*** from A t

To show poly time mapping reducibility:

1. create **computable fn**
2. **show computable fn runs in poly time**
3. then show **forward direction**
4. and show **reverse direction**
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Language A is ***polynomial time mapping reducible***, or simply ***polynomial time reducible***, to language B, written $\boxed{A \leq_P B}$, if a polynomial time computable function $f: \Sigma^* \longrightarrow \Sigma^*$ exists, where for every w,

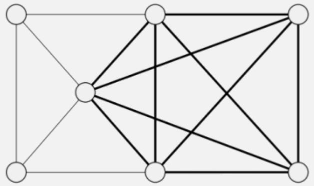
$$w \in A \iff f(w) \in B.$$

The function f is called the ***polynomial time reduction*** of A to B.

A function $f: \Sigma^* \longrightarrow \Sigma^*$ is a **poly time** *computable function* if some **poly time** Turing machine M, on every input w, halts with just $f(w)$ on its tape.

Theorem: *3SAT* is polynomial time reducible to *CLIQUE*.

Last Time: *CLIQUE* is in NP



$CLIQUE = \{\langle G, k\rangle \mid G \text{ is an undirected graph with a } k\text{-clique}\}$

PROOF IDEA The clique is the certificate.

PROOF The following is a verifier V for *CLIQUE*.

V = "On input $\langle\langle G, k\rangle, c\rangle$:

1. Test whether c is a subgraph with k nodes in G.
2. Test whether G contains all edges connecting nodes in c.
3. If both pass, *accept*; otherwise, *reject*."

Theorem: *3SAT* is polynomial time reducible to *CLIQUE*.

??

Boolean Formulas

A Boolean _____	Is …	Example:
Value	TRUE or FALSE (or 1 or 0)	TRUE, FALSE

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Formula ϕ	Combines **vars** and **operations**	$(\overline{x} \wedge y) \vee (x \wedge \overline{z})$

Boolean Satisfiability

- A **Boolean formula** is **satisfiable** if …

- … there is some **assignment** of TRUE or FALSE $(1$ or $0)$ to its variables that makes the entire formula TRUE

- Is $(\overline{x} \wedge y) \vee (x \wedge \overline{z})$ satisfiable?
  - Yes
  - $x = \mathrm{FALSE}$,
    $y = \mathrm{TRUE}$,
    $z = \mathrm{FALSE}$

The Boolean Satisfiability Problem

$SAT = \{\langle\phi\rangle | \phi \text{ is a satisfiable Boolean formula}\}$

Theorem: *SAT* is in **NP**:

- Let $\boldsymbol{n}$ = the number of variables in the formula

Verifier:

On input <ϕ, c>, where c is a possible assignment of variables in ϕ to values:

- Plug values from c into ϕ, **Accept** if result is TRUE

Running Time: $O(\boldsymbol{n})$

Non-deterministic Decider:

On input <ϕ>, where ϕ is a boolean formula:

- Non-deterministically try all possible assignments in parallel
- **Accept** if any satisfy ϕ

Running Time: Checking each assignment takes time $O(\boldsymbol{n})$

Theorem: $3SAT$ is polynomial time reducible to $CLIQUE$.

??

More Boolean Formulas

A Boolean _____	Is …	Example:
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Literal	A var or a negated var	x or $\overline{x}$.

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Clause	**Literals** ORed together	$(x_1 \vee \overline{x_2} \vee \overline{x_3} \vee x_4)$

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Conjunctive Normal Form (**CNF**)	**Clauses** ANDed together	$(x_1 \vee \overline{x_2} \vee \overline{x_3} \vee x_4) \wedge (x_3 \vee \overline{x_5} \vee x_6)$

∧ = AND = "Conjunction"
∨ = OR = "Disjunction"
¬ = NOT = "Negation"

More Boolean Formulas

A Boolean _____	Is …	Example:
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3CNF Formula	Three **literals** in each **clause**	$(x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_3 \vee \overline{x_5} \vee x_6) \wedge (x_3 \vee \overline{x_6} \vee x_4)$

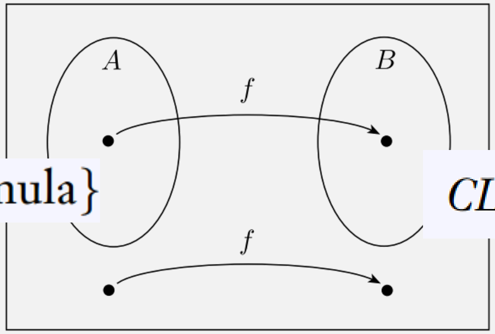
$\wedge$ = AND = “Conjunction”
$\vee$ = OR = “Disjunction”
$\neg$ = NOT = “Negation”

The *3SAT* Problem

$$3SAT = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable 3cnf-formula}\}$$

<u>Theorem</u>: *3SAT* is polynomial time reducible to *CLIQUE*.

$3SAT = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable 3cnf-formula}\}$

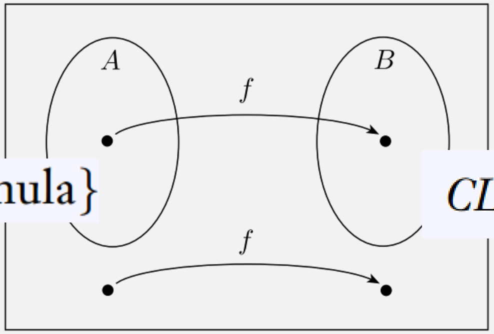


$CLIQUE = \{\langle G, k\rangle \mid G \text{ is an undirected graph with a } k\text{-clique}\}$

To show poly time <u>mapping reducibility</u>:

1. create **computable fn**,
2. show that it **runs in poly time**,
3. then show **forward direction** of mapping red.,
4. and **reverse direction**
   (or **contrapositive** of **reverse direction**)

<u>Theorem</u>: *3SAT* is polynomial time reducible to *CLIQUE*.



$3SAT = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable 3cnf-formula}\}$

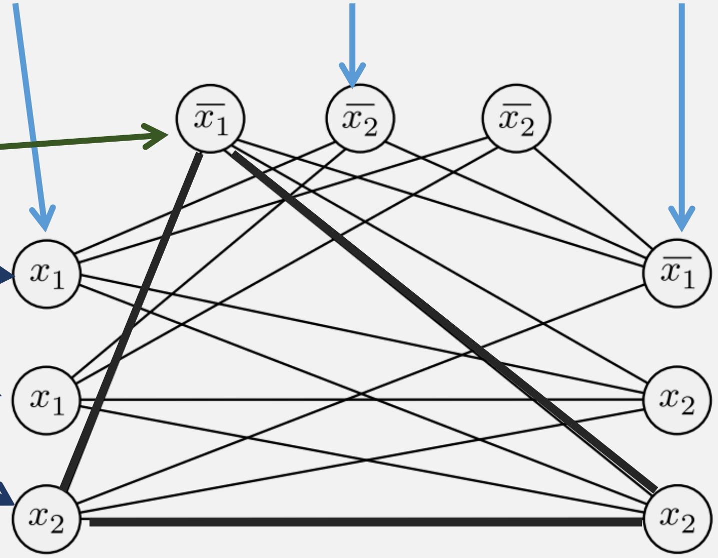
$CLIQUE = \{\langle G, k\rangle \mid G \text{ is an undirected graph with a } k\text{-clique}\}$

<u>Need</u>: poly time <u>computable fn</u> converting a 3cnf-formula …

Example:

$$\phi = (x_1 \vee x_1 \vee \boxed{x_2}) \wedge (\boxed{\overline{x_1}} \vee \overline{x_2} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee \boxed{x_2})$$

- … to a graph containing a clique:
  - Each clause maps to a group of 3 nodes
  - Connect all nodes <u>except</u>:
    - Contradictory nodes
    - Nodes in the same group



Don't forget iff

Runs in **poly time:**
- \# literals = # nodes $O(\boldsymbol{n})$
- \# edges poly in # nodes $O(\boldsymbol{n^2})$

$\Rightarrow$ If $\phi \in 3SAT$
- Then each clause has a TRUE literal
  - Those are <u>nodes in the 3-clique</u>!
  - E.g., $x_1 = 0, x_2 = 1$

$\Leftarrow$ If $\phi \notin 3SAT$
- Then for any assignment, some clause must have a contradiction with another clause
- Then in the graph, some clause's group of nodes won't be connected to another group, <u>preventing the clique</u>

Polynomial Time Mapping Reducibility

Language A is ***polynomial time mapping reducible***, or simply ***polynomial time reducible***, to language B, written $\boxed{A \leq_{\mathrm{P}} B}$, if a polynomial time computable function $f\colon \Sigma^* \longrightarrow \Sigma^*$ exists, where for every w,

$$w \in A \iff f(w) \in B.$$

The function f is called the ***polynomial time reduction*** of A to B.

What is this used for?

poly time poly time

A function $f\colon \Sigma^* \longrightarrow \Sigma^*$ is a ***computable function*** if some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

Flashback: If $A \leq_m B$ and B is decidable, then A is decidable.

Has a decider

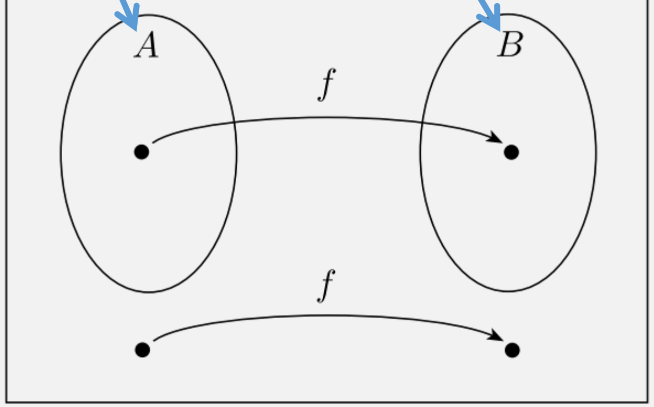
PROOF We let M be the decider for B and f be the reduction from A to B. We describe a decider N for A as follows.

$N =$ "On input w:

1. Compute $f(w)$.
2. Run M on input $f(w)$ and output whatever M outputs."

decides

decides



This proof only works because of the if-and-only-if requirement

Language A is ***mapping reducible*** to language B, written $A \leq_m B$, if there is a computable function $f\colon \Sigma^* \longrightarrow \Sigma^*$, where for every w,

$$\boxed{w \in A \iff f(w) \in B.}$$

The function f is called the ***reduction*** from A to B.

Thm: If $A \leq_{\cancel{\mathrm{m}}\,\mathbf{P}} B$ and B ~~is decidable,~~ $\in \mathrm{P}$ then A ~~is decidable.~~ $\in \mathrm{P}$

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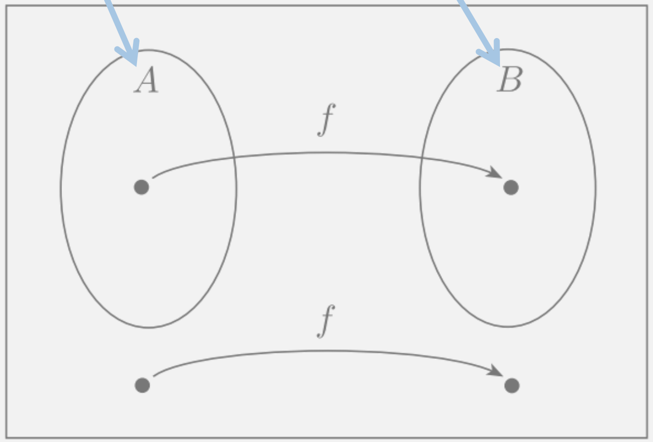

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PROOF We let M be the poly time decider for B and f be the poly time reduction from A to B. We describe a poly time decider N for A as follows.

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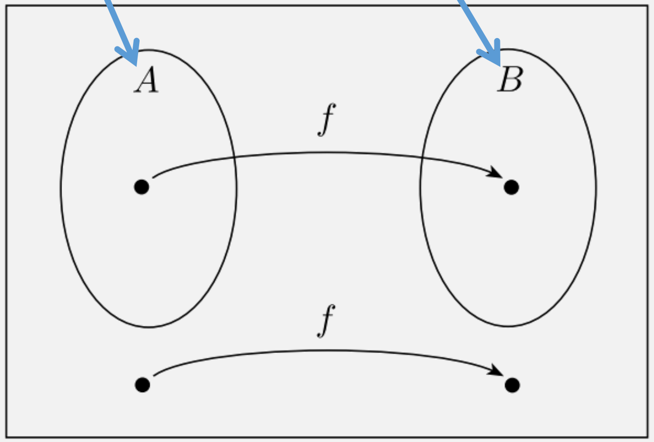

Language A is poly time ***mapping reducible*** to language B, written $A \le_{\mathrm{m}} B$, if there is a computable function $f\colon \Sigma^* \longrightarrow \Sigma^*$, where for every w,

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The function f is called the ***reduction*** from A to B.

NP-Completeness

DEFINITION

A language B is ***NP-complete*** if it satisfies two conditions:

1. B is in NP, and
2. every A in NP is polynomial time reducible to B.

- How does this help the **P** = **NP** problem?

THEOREM

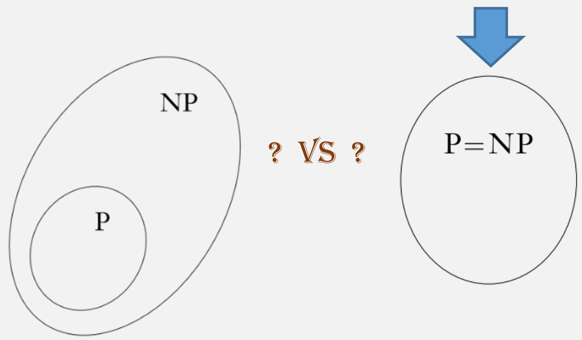
If B is NP-complete and $B \in \mathrm{P}$, then $\mathrm{P} = \mathrm{NP}$.

assume

THEOREM

Proof:

If B is NP-complete and $B \in$ P, then P $=$ NP.



DEFINITION

A language B is ***NP-complete*** if it satisfies two conditions:

1. B is in NP, and $A \leq_\mathrm{P} B$
2. every A in NP is polynomial time reducible to B.

P

r A → verifier for A that ignores its certificate

2. If **a language** $A \in$ **NP**, then $A \in$ **P**
   - Given **a language** $A \in$ **NP** …
   - **… can poly time mapping reduce** A **to** B --- why?
     - because B **is NP-Complete** (assumption)
   - Then A **also** $\in$ **P** …
     - Because $A \leq_\mathrm{P} B$ and $B \in$ P, then $A \in$ P

(prev slide)

So to prove P = NP, we only need to **find a poly-time algorithm** for <u>one</u> NP-Complete problem!

Thus, **if a language** B **is NP**-**complete** and **in P**, then **P = NP**

NP-Completeness

DEFINITION

A language B is ***NP-complete*** if it satisfies two conditions:

1. B is in NP, and
2. every A in NP is polynomial time reducible to B.

- How does this help the **P** = **NP** problem?

THEOREM

If B is NP-complete and $B \in \mathrm{P}$, then $\mathrm{P} = \mathrm{NP}$.

But we still don't know any **NP**-Complete problems!

So to prove **P** = **NP**, we only need to find a poly-time algorithm for <u>one</u> **NP**-Complete problem!

Figuring out the first one is hard!
(just like figuring out the first undecidable problem was hard!)

The Cook-Levin Theorem

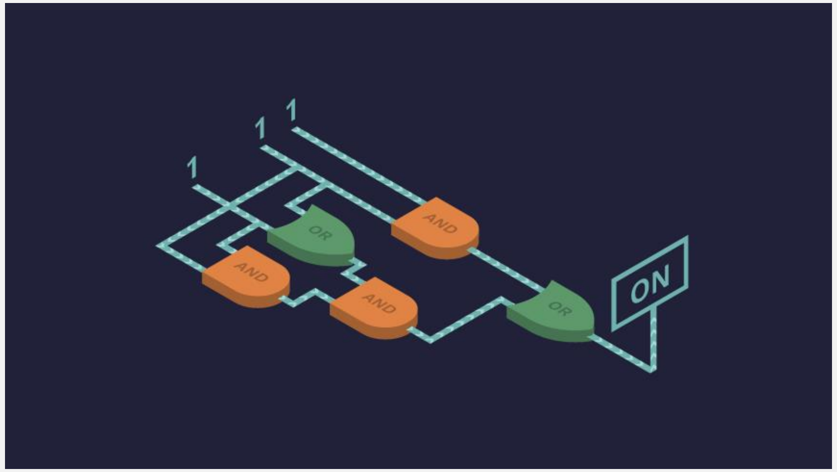
The first **NP**-Complete problem

THEOREM

SAT is NP-complete.

(havent proven yet)

It sort of makes sense that every problem can be reduced to it …



After this, it'll be much easier to find other **NP**-Complete problems!

THEOREM

If B is NP-complete and $B \leq_P C$ for C in NP, then C is NP-complete.

known

unknown

THEOREM

Key Thm: If B is NP-complete and $B \leq_P C$ for C in NP, then C is NP-complete.

To use this theorem, C must be in **NP**

DEFINITION

A language B is ***NP-complete*** if it satisfies two conditions:

1. B is in NP, and
2. every A in NP is polynomial time reducible to B.

Proof:

- Need to show: C is **NP**-complete:
  - it's in **NP** (given), and
  - every lang A in **NP** reduces to C in poly time (must show)
- For every language A in **NP**, reduce $A \rightarrow C$ by:
  - First reduce $A \rightarrow B$ in poly time
    - Can do this because B is **NP**-Complete
  - Then reduce $B \rightarrow C$ in poly time
    - This is given
- Total run time: Poly time + poly time = poly time

THEOREM

Using: If B is NP-complete and $B \leq_P C$ for C in NP, then C is NP-complete.

3 steps to prove a language C is **NP**-complete:

1. Show C is in **NP**
2. Choose B, the **NP**-complete problem to reduce from
3. Show a poly time mapping reduction from B to C

To show poly time mapping reducibility:
1. create **computable fn**,
2. show that it **runs in poly time**,
3. then show **forward direction** of mapping red.,
4. and **reverse direction**
(or **contrapositive** of **reverse direction**)

THEOREM

Using: If B is NP-complete and $B \leq_P C$ for $\boxed{C \text{ in NP}}$, then $\boxed{C \text{ is NP-complete.}}$

3 steps to prove a language C is **NP**-complete:

1. Show C is in **NP**
2. Choose B, the **NP**-complete problem to reduce from
3. Show a poly time mapping reduction from B to C

Example:

Let $C = 3SAT$, to prove $\boxed{3SAT \text{ is } \mathbf{NP}\text{-Complete:}}$

1. Show $3SAT$ is in **NP**

Flashback: 3SAT is in **NP**

3$SAT = \{\langle\phi\rangle \mid \phi$ is a satisfiable Boolean formula$\}$

Let $\boldsymbol{n}$ = the number of variables in the formula

Verifier:

On input <ϕ, c>, where c is a possible assignment of variables in ϕ to values:

- Accept if c satisfies ϕ

Running Time: $O(\boldsymbol{n})$

Non-deterministic Decider:

On input <ϕ>, where ϕ is a boolean formula:

- Non-deterministically try all possible assignments in parallel
- Accept if any satisfy ϕ

Running Time: Checking each assignment takes time $O(\boldsymbol{n})$

THEOREM

Using: If $\boxed{B \text{ is NP-complete}}$ and $\boxed{B \leq_P C}$ for C in NP, then C is NP-complete.

3 steps to prove a language is **NP**-complete:

1. Show C is in **NP**
2. Choose B, the **NP**-complete problem to reduce from
3. Show a poly time mapping reduction from B to C

Example:

Let $C = 3SAT$, to prove $3SAT$ is **NP**-Complete:

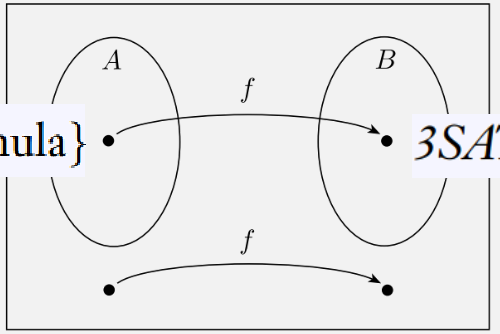
☑ 1. Show $3SAT$ is in **NP**

☑ $\boxed{2.}$ Choose B, the **NP**-complete problem to reduce from: SAT

$\boxed{3.}$ Show a poly time mapping reduction from SAT to $3SAT$

<u>Theorem</u>: *SAT* is Poly Time Reducible to *3SAT*

$SAT = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable Boolean formula}\}$



$3SAT = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable 3cnf-formula}\}$

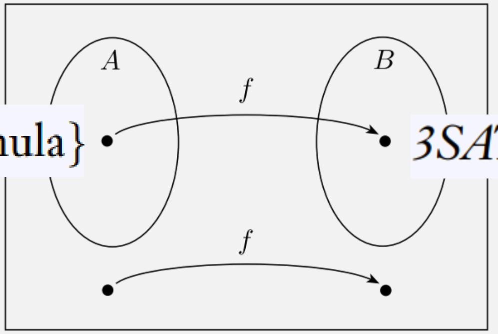
To show poly time <u>mapping reducibility</u>:

1. create **computable fn** *f*,
2. show that it **runs in poly time**,
3. then show **forward direction** of mapping red.,
   - ⇒ if $\phi \in SAT$, then $f(\phi) \in 3SAT$
4. and **reverse direction**
   - ⇐ if $f(\phi) \in 3SAT$, then $\phi \in SAT$

   (or **contrapositive** of **reverse direction**)
   - ⇐ (alternative) if $\phi \notin SAT$, then $f(\phi) \notin 3SAT$

Theorem: *SAT* is Poly Time Reducible to *3SAT*

$SAT = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable Boolean formula}\}$



$3SAT = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable 3cnf-formula}\}$

Want: poly time computable fn converting a Boolean formula ϕ to 3CNF:

1. Convert ϕ to CNF (an AND of OR clauses)
   a) Use DeMorgan's Law to push negations onto literals

      $\neg(P \lor Q) \boxed{\iff} (\neg P) \land (\neg Q)$ $\quad$ $\neg(P \land Q) \boxed{\iff} (\neg P) \lor (\neg Q)$ $\boxed{O(\boldsymbol{n})}$

   b) Distribute ORs to get ANDs outside of parens

      $(P \lor (Q \land R)) \boxed{\Leftrightarrow} ((P \lor Q) \land (P \lor R))$ $\boxed{O(\boldsymbol{n})}$

2. Convert to 3CNF by adding new variables

   $(a_1 \lor a_2 \lor a_3 \lor a_4) \boxed{\Leftrightarrow} (a_1 \lor a_2 \lor z) \land (\overline{z} \lor a_3 \lor a_4)$ $\boxed{O(\boldsymbol{n})}$

Remaining step: show iff relation holds …

… this thm is a special case, don't need to separate forward/reverse dir bc each step is already a known "law"

THEOREM

Using: If B is NP-complete and $B \leq_P C$ for C in NP, then C is NP-complete.

3 steps to prove a language is **NP**-complete:

1. Show C is in **NP**
2. Choose B, the **NP**-complete problem to reduce from
3. Show a poly time mapping reduction from B to C

Example:

Let $C = 3SAT$, to prove $3SAT$ is **NP**-Complete:

☑ 1. Show $3SAT$ is in **NP**

☑ 2. Choose B, the **NP**-complete problem to reduce from: SAT

☑ 3. Show a poly time mapping reduction from SAT to $3SAT$

Each **NP**-complete problem we prove makes it easier to prove the next one!

THEOREM

Using: If B is NP-complete and $B \leq_P C$ for C in NP, then C is NP-complete.

3 steps to prove a language is **NP**-complete:

1. Show C is in **NP**
2. Choose B, the **NP**-complete problem to reduce from
3. Show a poly time mapping reduction from B to C

Example:

Let C = ~~*3SAT*~~ ***CLIQUE***, to prove ~~*3SAT*~~ ***CLIQUE*** is **NP**-Complete:

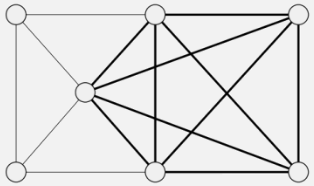
?1. Show ~~*3SAT*~~ ***CLIQUE*** is in **NP**

?2. Choose B, the **NP**-complete problem to reduce from: ~~*SAT*~~ ***3SAT***

?3. Show a poly time mapping reduction from ***3SAT*** to ~~*3SAT*~~ ***CLIQUE***

Flashback:

$CLIQUE$ is in NP



$CLIQUE = \{\langle G, k\rangle |\ G \text{ is an undirected graph with a } k\text{-clique}\}$

PROOF IDEA The clique is the certificate.

Let $\boldsymbol{n}$ = # nodes in G

c is at most $\boldsymbol{n}$

PROOF The following is a $\boxed{\text{verifier } V}$ for $CLIQUE$.

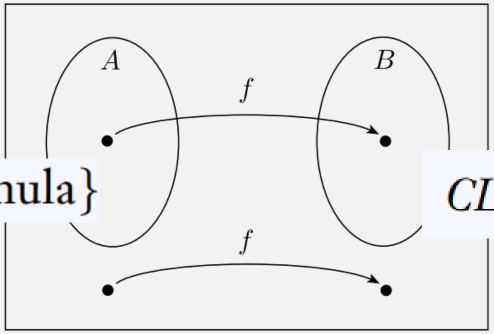
V = "On input $\langle\langle G, k\rangle, c\rangle$:

1. Test whether c is a subgraph with k nodes in G.
2. Test whether G contains all edges connecting nodes in c.
3. If both pass, *accept*; otherwise, *reject*."

For each node in c, check whether it's in G: $O(\boldsymbol{n})$

For each pair of nodes in c, check whether there's an edge in G: $O(\boldsymbol{n}^2)$

Flashback: $\mathit{3SAT}$ is polynomial time reducible to CLIQUE.



$\mathit{3SAT} = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable 3cnf-formula}\}$

$\mathit{CLIQUE} = \{\langle G, k\rangle \mid G \text{ is an undirected graph with a } k\text{-clique}\}$

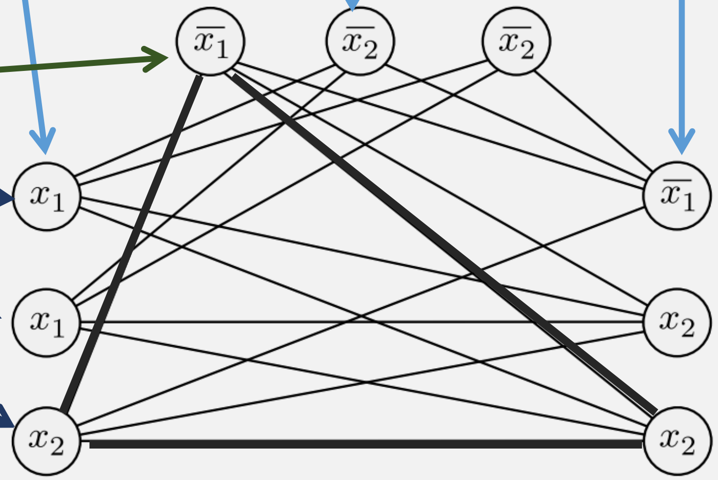
<u>Need</u>: poly time <u>computable fn</u> converting a 3cnf-formula …

Example:

$$\phi = (x_1 \vee x_1 \vee \boxed{x_2}) \wedge (\boxed{\overline{x_1}} \vee \overline{x_2} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee \boxed{x_2})$$

- … to a graph containing a clique:
  - Each clause maps to a group of 3 nodes
  - Connect all nodes <u>except</u>:
    - Contradictory nodes
    - Nodes in the same group

Don't forget iff



Runs in **poly time:**
- \# literals = # nodes $O(\boldsymbol{n})$
- \# edges poly in # nodes $O(\boldsymbol{n^2})$

$\Rightarrow$ If $\phi \in \mathit{3SAT}$

- Then each clause has a TRUE literal
  - Those are nodes in the clique!
  - E.g., $x_1 = 0$, $x_2 = 1$

$\Leftarrow$ If $\phi \notin \mathit{3SAT}$

- For any assignment, some clause must have a contradiction with another clause
- Then in the graph, some clause's group of nodes won't be connected to another group, preventing the clique

THEOREM

Using: If B is NP-complete and $B \leq_P C$ for C in NP, then C is NP-complete.

3 steps to prove a language is **NP**-complete:

1. Show C is in **NP**
2. Choose B, the **NP**-complete problem to reduce from
3. Show a poly time mapping reduction from B to C

Example:

Let C = ~~*3SAT*~~ ***CLIQUE***, to prove ~~*3SAT*~~ ***CLIQUE*** is **NP**-Complete:

☑1. Show ~~*3SAT*~~ ***CLIQUE*** is in **NP**

☑2. Choose B, the **NP**-complete problem to reduce from: ~~*SAT*~~ ***3SAT***

☑3. Show a poly time mapping reduction from ***3SAT*** to ~~*3SAT*~~ ***CLIQUE***

NP-Complete problems, so far

- $SAT = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable Boolean formula}\}$ (havent proven yet)
- $3SAT = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable 3cnf-formula}\}$ (reduced *SAT* to *3SAT*)
- $CLIQUE = \{\langle G, k\rangle \mid G \text{ is an undirected graph with a } k\text{-clique}\}$ (reduced *3SAT* to *CLIQUE*)

Each **NP**-complete problem we prove makes it easier to prove the next one!

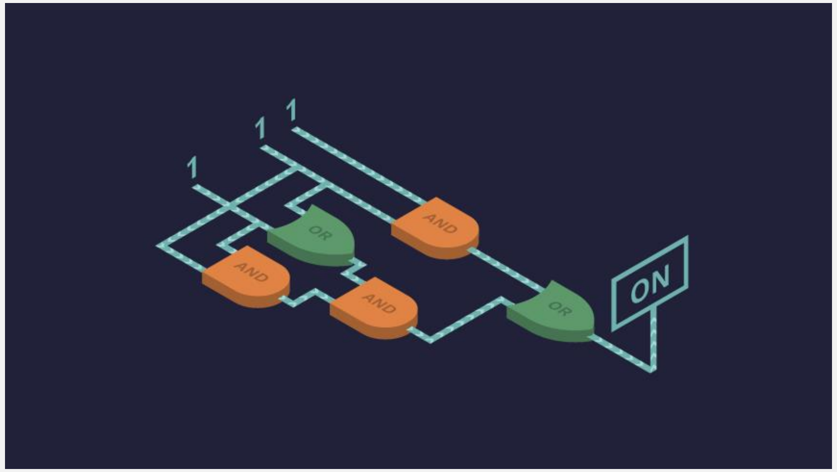
Next Time: The Cook-Levin Theorem

The first **NP**-Complete problem

THEOREM

SAT is NP-complete.

It sort of makes sense that every problem can be reduced to it …



After this, it'll be much easier to find other **NP**-Complete problems!

THEOREM

If B is NP-complete and $B \leq_{\mathrm{P}} C$ for C in NP, then C is NP-complete.

Quiz 5/8

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