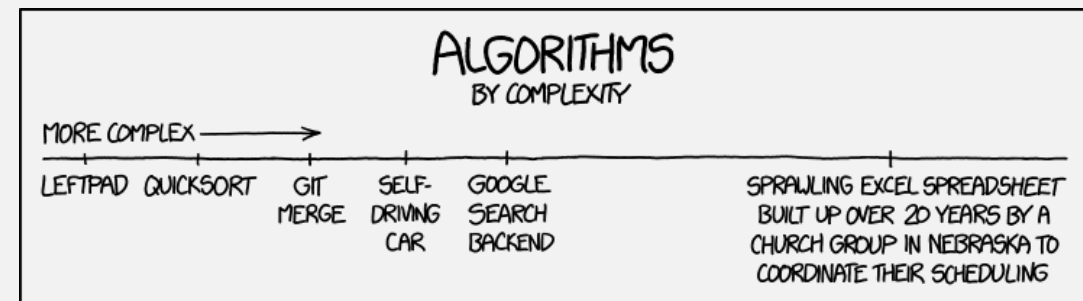# UMB CS 420
Time Complexity

Wednesday, April 26, 2023

Announcements

- HW 10 out
  - Due Sunday April 30, 11:59pm
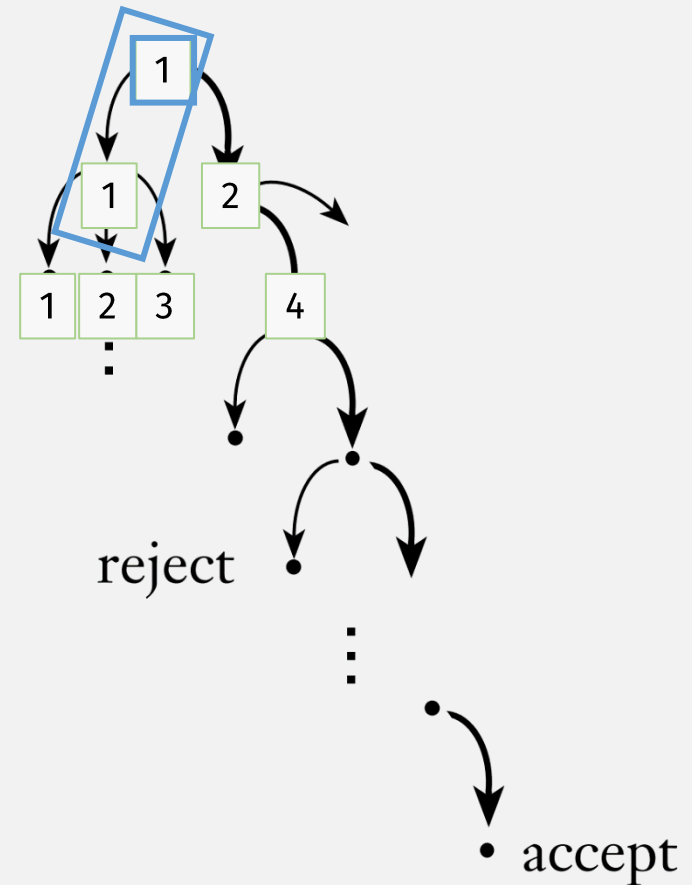
Quiz Preview

- What is the worst case number of steps of a deterministic single-tape Turing machine called?

Flashback: Nondet. TM → Deterministic TM

- ## To <u>simulate</u> NTM with Det. TM:
  - ### Number the nodes at each step
  - ### Deterministically check every tree path, in breadth-first order
    - Root node: 1
    - **1-1**

Nondeterministic computation

Flashback: Nondet. TM → Deterministic TM

- ## To <u>simulate</u> NTM with Det. TM:
  - ### Number the nodes at each step
  - ### Deterministically check every tree path, in breadth-first order
    - Root node: 1
    - 1-1
    - **1-2**

Nondeterministic
computation

Flashback: Nondet. TM → Deterministic TM

- ## To <u>simulate</u> NTM with Det. TM:
  - ## Number the nodes at each step
  - ## Deterministically check every tree path, in breadth-first order
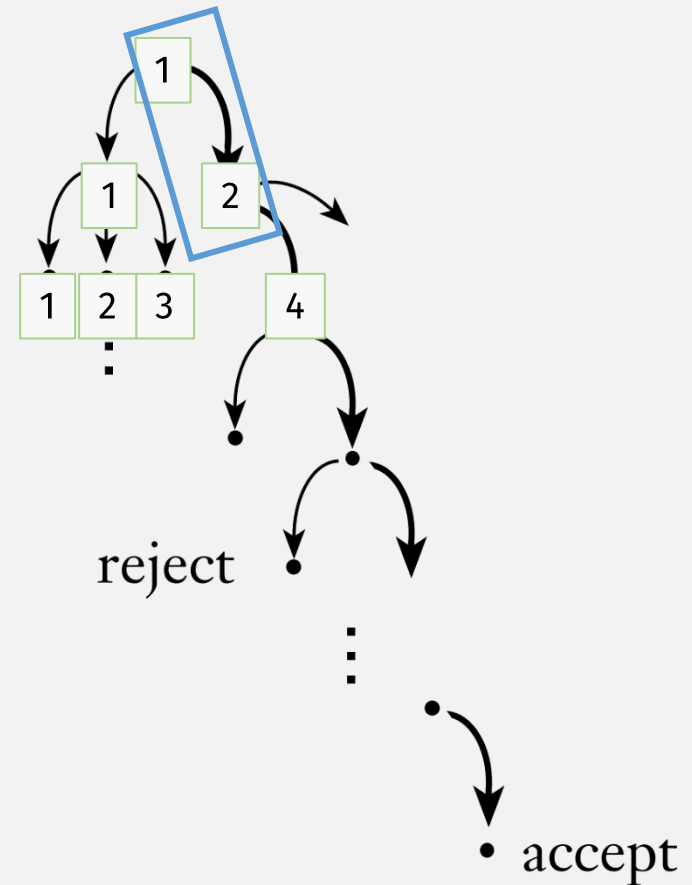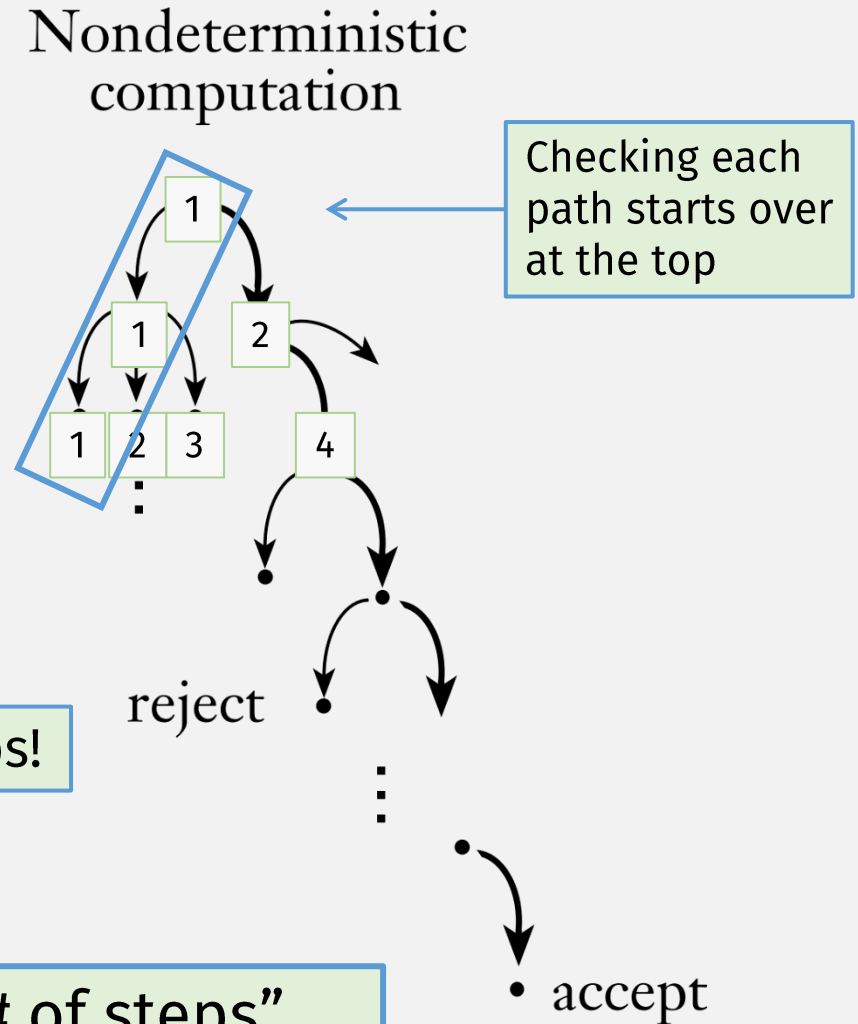   - Root node: 1
    - 1-1
    - 1-2
    - **1-1-1**

Nondeterministic computation

Checking each path starts over at the top

| 1 |
| 1 | 2 |
| 1 | 2 | 3 | 4 |

reject

accept

A **TM** and an **NTM** are "equivalent" …

.. but <u>NOT</u> if we care about the # of steps!

So how inefficient is it?

First, we need a <u>formal</u> way to count "# of steps" …

A Simpler Example: $A = \{0^k 1^k \mid k \geq 0\}$

M_1 = "On input string w:
1. Scan across the tape and *reject* if a 0 is found to the right of a 1.
2. Repeat if both 0s and 1s remain on the tape:
3.   Scan across the tape, crossing off a single 0 and a single 1.
4. If 0s still remain after all the 1s have been crossed off, or if 1s still remain after all the 0s have been crossed off, *reject*. Otherwise, if neither 0s nor 1s remain on the tape, *accept*."

# of steps (worst case), n = length of w input:

➤ TM Line 1:
- n steps to scan + n steps to return to beginning = 2n steps

A Simpler Example: $A = \{0^k 1^k \mid k \geq 0\}$

M_1 = "On input string w:
1. Scan across the tape and *reject* if a 0 is found to the right of a 1.
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of steps (worst case), n = length of w input:
- ## TM Line 1:
  - n steps to scan + n steps to return to beginning = **2n** steps
- ## Lines 2-3 (loop):
  - steps/iteration (line 3): **$n/2$** steps to find "**1**" + **$n/2$** steps to return = n steps
  - # iterations (line 2): Each scan crosses off 2 chars, so at most **$n/2$** scans
  - Total = steps/iteration * # iterations = n (**$n/2$**) = **$n^2/2$ steps**

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A Simpler Example: $A = \{0^k 1^k \mid k \geq 0\}$
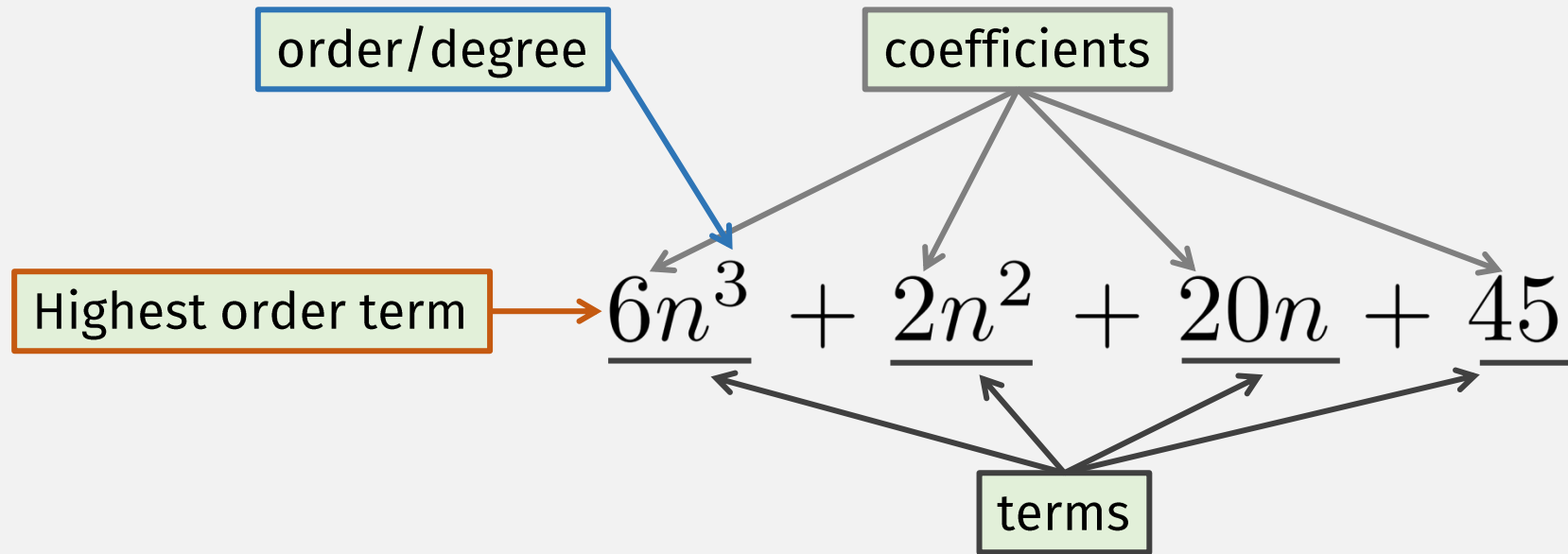
M_1 = "On input string w:
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$n^2/2 + 3n$

of steps (worst case), n = length of w input:

- TM Line 1:
  - n steps to scan + n steps to return to beginning = $2n$ steps
- Lines 2-3 (loop):
  - steps/iteration (line 3): $n/2$ steps to find "1" + $n/2$ steps to return = n steps
  - # iterations (line 2): Each scan crosses off 2 chars, so at most $n/2$ scans
  - Total = steps/iteration * # iterations = n $(n/2)$ = $n^2/2$ steps
- Line 4:
  - n steps to scan input one more time
- Total: $2n + n^2/2 + n$ = $n^2/2 + 3n$ steps

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Interlude: Polynomials

order/degree

coefficients

Highest order term

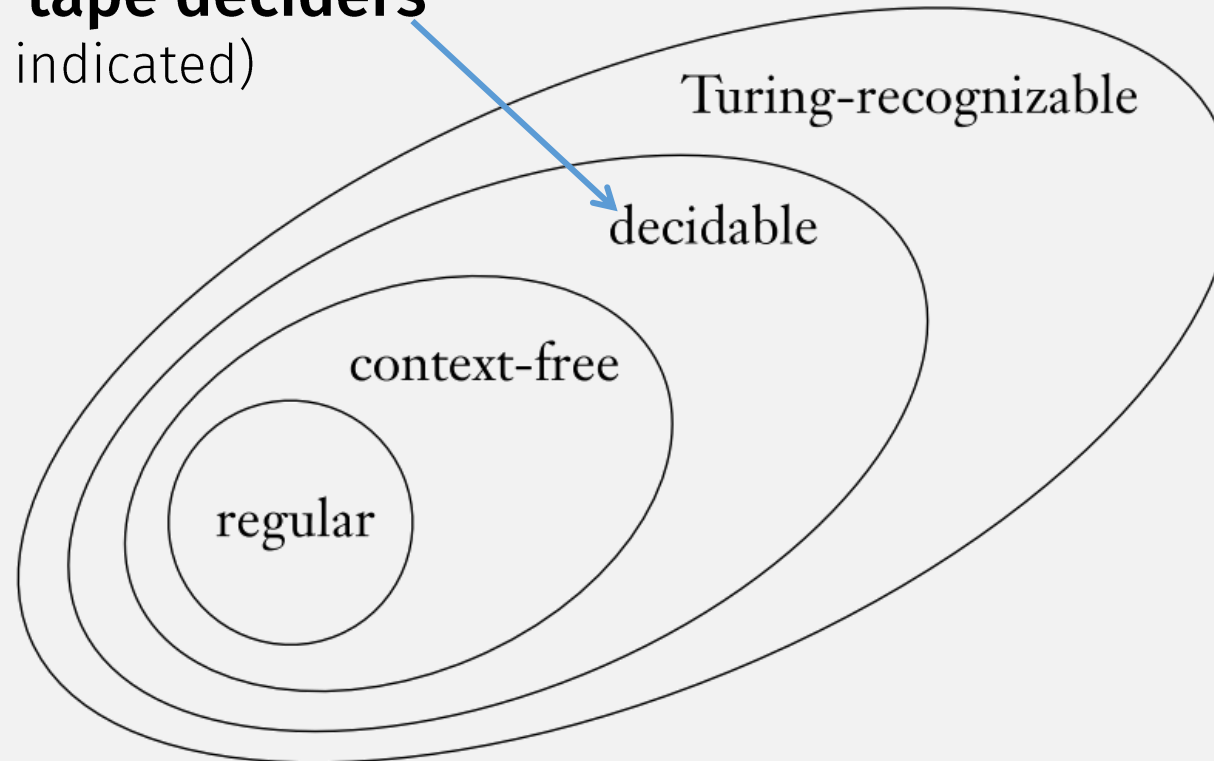$$6n^3 + 2n^2 + 20n + 45$$

terms

Definition: Time Complexity

i.e., a decider (algorithm)

Let M be a deterministic Turing machine that halts on all inputs. The ***running time*** or ***time complexity*** of M is the function $f: \mathcal{N} \longrightarrow \mathcal{N}$, where $f(n)$ is the maximum number of steps that M uses on any input of length n. If $f(n)$ is the running time of M, we say that M runs in time $f(n)$ and that M is an $f(n)$ time Turing machine. Customarily we use n to represent the length of the input.

Running Time or **Time Complexity**
is a property of a (Turing) Machine

Where Are We Now?

We are back in here now:
deterministic, single-tape deciders
(unless otherwise indicated)

Definition: Time Complexity

NOTE: n has no units, it's only *roughly* "length" of the input

n can be:
\# characters,
\# states,
\# nodes, …

We can use any n that is correlated with the input length

Let M be a deterministic Turing machine that halts on all inputs. The ***running time*** or ***time complexity*** of M is the function $f: \mathcal{N} \longrightarrow \mathcal{N}$, where $f(n)$ is the maximum number of steps that M uses on any input of length n. If $f(n)$ is the running time of M, [we] say that M runs in time $f(n)$ and that M is an $f(n)$ time Turing machine. Customarily we use n to represent the length of the [inp]ut.

- Machine M_1 that decides $A = \{0^k 1^k \,|\, k \geq 0\}$
  - Running time or Time Complexity: $n^2/2 + 3n$

M_1 = "On input string w:
  1. Scan across the tape and *reject* if a 0 is found to the right of a 1.
  2. Repeat if both 0s and 1s remain on the tape:
  3. Scan across the tape, crossing off a single 0 and a single 1.
  4. If 0s still remain after all the 1s have been crossed off, or if 1s still remain after all the 0s have been crossed off, *reject*. Otherwise, if neither 0s nor 1s remain on the tape, *accept*."

Interlude: Asymptotic Analysis

Total: $n^2 + 3n$

- If $n = 1$
  - $n^2 = 1$
  - $3n = 3$
  - Total = 4

- If $n = 10$
  - $n^2 = 100$
  - $3n = 30$
  - Total = 130

- If $n = 100$
  - $n^2 = 10,000$
  - $3n = 300$
  - Total = 10,300

- If $n = 1,000$
  - $n^2 = 1,000,000$
  - $3n = 3,000$
  - Total = 1,003,000

$n^2 + 3n \approx n^2$ as n gets large

asymptotic analysis only cares about <u>large</u> n

Definition: Big-O Notation

Let f and g be functions $f, g: \mathcal{N} \longrightarrow \mathcal{R}^+$. Say that $\boldsymbol{f(n) = O(g(n))}$ if positive integers c and n_0 exist such that for every integer $n \geq n_0$,

$$f(n) \leq c\, g(n).$$

When $f(n) = O(g(n))$, we say that $g(n)$ is an **upper bound** for $f(n)$, or more precisely, that $g(n)$ is an **asymptotic upper bound** for $f(n)$, to emphasize that we are suppressing constant factors.

notation

"only care about <u>large</u> n"

In other words: Keep only highest order term, drop all coefficients

- Machine M_1 that decides $A = \{0^k 1^k \mid k \geq 0\}$
  - is an $\boldsymbol{n^2 + 3n}$ time Turing machine
  - is an $O(\boldsymbol{n^2})$ time Turing machine, i.e., $\boldsymbol{n^2 + 3n} = O(\boldsymbol{n^2})$
  - has asymptotic upper bound $O(\boldsymbol{n^2})$

Definition: Small-o Notation (less used)

Let f and g be functions $f, g\colon \mathcal{N}\longrightarrow\mathcal{R}^{+}$. Say that $\boldsymbol{f(n) = o(g(n))}$ if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

In other words, $f(n) = o(g(n))$ means that for any real number $c > 0$, a number n_0 exists, where $\boxed{f(n) < c\, g(n)}$ for all $n \geq n_0$.

Analogy: **Big-O** : ≤ :: small-o : <

Let f and g be functions $f, g\colon \mathcal{N}\longrightarrow\mathcal{R}^{+}$. Say that $\boldsymbol{f(n) = O(g(n))}$ if positive integers c and n_0 exist such that for every integer $n \geq n_0$,

$$\boxed{f(n) \leq c\, g(n).}$$

When $f(n) = O(g(n))$, we say that $g(n)$ is an **upper bound** for $f(n)$, or more precisely, that $g(n)$ is an **asymptotic upper bound** for $f(n)$, to emphasize that we are suppressing constant factors.

Other "Oh"s (not used in this course)

- "Big Theta" Θ

- "Small Omega" ω

- "Big Omega" Ω

Don't use these by mistake!
Pay attention to our <u>exact</u> definitions!

Big-O arithmetic

- $O(n^2) + O(n^2)$
  $= O(n^2)$

- $O(n^2) + O(n)$
  $= O(n^2)$

- **2n** $= O(n)$?
  - TRUE

- **2n** $= O(n^2)$?
  - TRUE

- **1** $= O(n^2)$?
  - TRUE

- **2^n** $= O(n^2)$?
  - FALSE

NOTE: <u>Other courses</u> might use Big-Θ notation (which is a tighter bound) where some of these equalities won't be true, e.g., **2n** $\neq \Theta(n^2)$

NOTE: <u>In this course</u>, we use Big-O only, <u>not</u> Big-Θ (so do not confuse the two)

Definition: Time Complexity Classes

Let $t: \mathcal{N} \longrightarrow \mathcal{R}^+$ be a function. Define the ***time complexity class***, $\mathbf{TIME}(t(n))$, to be the collection of all languages that are decidable by an $O(t(n))$ time Turing machine.

Remember: **TMs** have a **time complexity** (i.e., a running time), **languages** are in a **time complexity class**

The **complexity class** of a **language** is determined by the **time complexity** (running time) of its deciding **TM**

A **language** could be in more than one **time complexity class**

- Machine M_1 decides language $A = \{0^k 1^k \mid k \geq 0\}$
  - M_1 has time complexity (running time) of $O(\boldsymbol{n^2})$
  - A is in time complexity class $\mathbf{TIME}(\boldsymbol{n^2})$

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A Faster Machine? $A = \{0^k 1^k \mid k \geq 0\}$

M_2 = "On input string w:
1. Scan across the tape and *reject* if a 0 is found to the right of a 1.
2. Repeat as long as some 0s and some 1s remain on the tape:
3.   Scan across the tape, checking whether the total number of 0s and 1s remaining is even or odd. If it is odd, *reject*.
4.   Scan again across the tape, crossing off every other 0 starting with the first 0, and then crossing off every other 1 starting with the first 1.
5. If no 0s and no 1s remain on the tape, *accept*. Otherwise, *reject*."

M_1 = "On input string w:
1. Scan across the tape and *reject* if a 0 is found to the right of a 1.
2. Repeat if both 0s and 1s remain on the tape:
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Number of steps (worst case), n = length of input:
➢Line 1:
- n steps to scan + n steps to return to beginning = $O(n)$ steps

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Number of steps (worst case), n = length of input:

- Line 1:
  - n steps to scan + n steps to return to beginning = $O(n)$ steps
- Lines 2-4 (loop):
  - steps/iteration (lines 3-4): a scan takes $O(n)$ steps
  - # iters (line 2): Each iter crosses off *half* the chars, so at most $O(\log n)$ scans
  - Total: $O(n) * O(\log n) = O(n \log n)$ steps

Interlude: Logarithms (dual to exponentiation)

- If $2^x = y$ …

- … then $\log_2 y = x$

- $\log_2 n = O(\mathbf{\log\,n})$
  - "divide and conquer" algorithms $= O(\mathbf{\log\,n})$
  - E.g., binary search

- (In computer science, **base-2 is the only base!** So 2 is dropped)

A Faster Machine? $A = \{0^k 1^k | k \geq 0\}$

M_2 = "On input string w:
1. Scan across the tape and *reject* if a 0 is found to the right of a 1.
2. Repeat as long as some 0s and some 1s remain on the tape:
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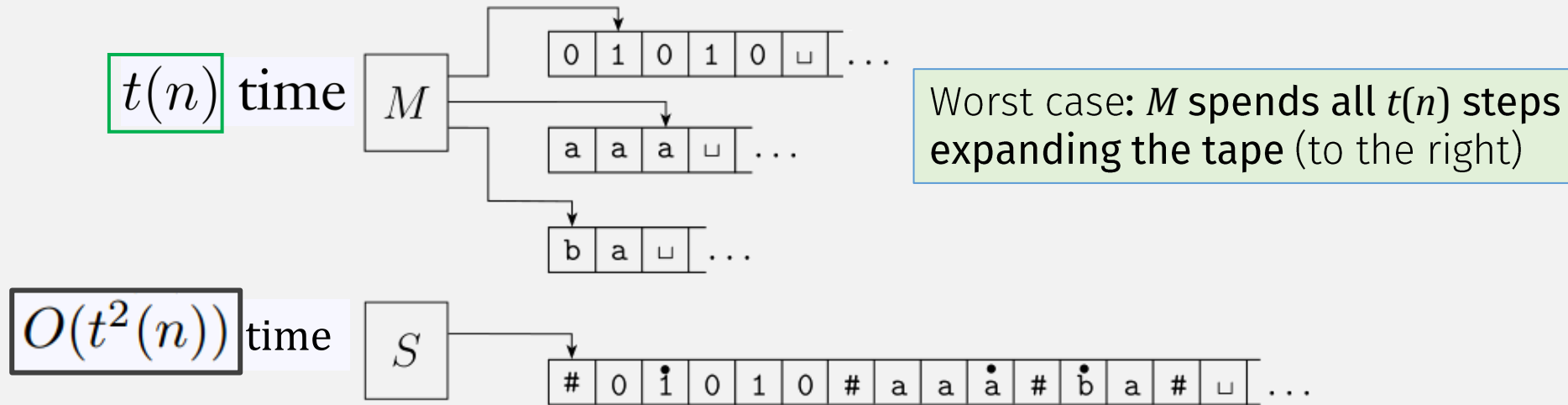**Number of steps** (worst case)**, n = length of input:**

- Line 1:
  - n steps to scan + n steps to return to beginning = $O(n)$ steps
- Lines 2-4 (loop):
  - steps/iteration (lines 3-4): a scan takes $O(n)$ steps
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  - Total: $O(n) * O(\log n) = O(n \log n)$ steps
- Line 5:
  - $O(n)$ steps to scan input one more time

A Faster Machine? $A = \{0^k 1^k \mid k \geq 0\}$

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$O(\boldsymbol{n} \log \boldsymbol{n})$

Prev: $\boldsymbol{n^2/2 + 3n = O(n^2)}$

Number of steps (worst case), n = length of input:
- Line 1:
  - $\boldsymbol{n}$ steps to scan + $\boldsymbol{n}$ steps to return to beginning = $O(\boldsymbol{n})$ steps
- Lines 2-4 (loop):
  - steps/iteration (lines 3-4): a scan takes $O(\boldsymbol{n})$ steps
  - # iters (line 2): Each iter crosses off *half* the chars, so at most $O(\boldsymbol{\log n})$ scans
  - Total: $O(\boldsymbol{n}) * O(\boldsymbol{\log n}) = O(\boldsymbol{n \log n})$ steps
- Line 5:
  - $O(\boldsymbol{n})$ steps to scan input one more time
- Total: $O(\boldsymbol{n}) + O(\boldsymbol{n \log n}) + O(\boldsymbol{n}) =$

Terminology: Categories of Bounds

- Exponential time
  - $O(2^{n^c})$, for $c > 0$, or $2^{O(n)}$ (always base 2)
- Polynomial time
  - $O(n^c)$, for $c > 0$
- Quadratic time (special case of polynomial time)
  - $O(n^2)$
- Linear time (special case of polynomial time)
  - $O(n)$
- Log time
  - $O(\log n)$

Multi-tape vs Single-tape TMs: # of Steps

$t(n)$ time M

| 0 | 1 | 0 | 1 | 0 | ␣ | ... |

| a | a | a | ␣ | ... |

| b | a | ␣ | ... |

Worst case: M spends all $t(n)$ steps expanding the tape (to the right)

$O(t^2(n))$ time S

| # | 0 | 1̇ | 0 | 1 | 0 | # | a | a | ȧ | # | ḃ | a | # | ␣ | ... |

- For single-tape TM to <u>simulate 1 step</u> of multi-tape: $t(n)$
  1. Scan to find all "heads" = O(length of <u>all</u> M's tapes)
  2. "Execute" transition at all the heads = O(length of <u>all</u> M's tapes)
- # single-tape steps to simulate 1 multitape step (worst case)
  - = O(length of all M's tapes)
  - = $O(t(n))$, If M spends all its steps expanding its tapes
- <u>Total steps (single tape)</u>: $O(t(n))$ per step × $t(n)$ steps =

Flashback: Nondet. TM → Deterministic TM

Nondeterministic computation

- **Simulate NTM with Det. TM:**
  - Number the nodes at each step
  - Deterministically check every tree path, in breadth-first order
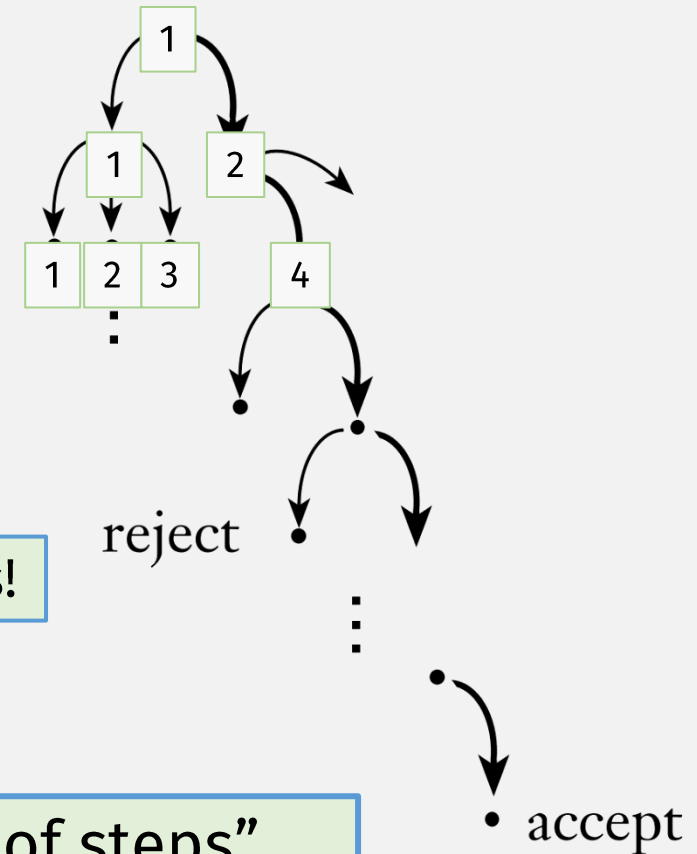   - 1
    - 1-1
    - 1-2
    - **1-1-1**

A **TM** and an **NTM** are "equivalent" …

.. but <u>NOT</u> if we care about the # of steps!

So how inefficient is it?
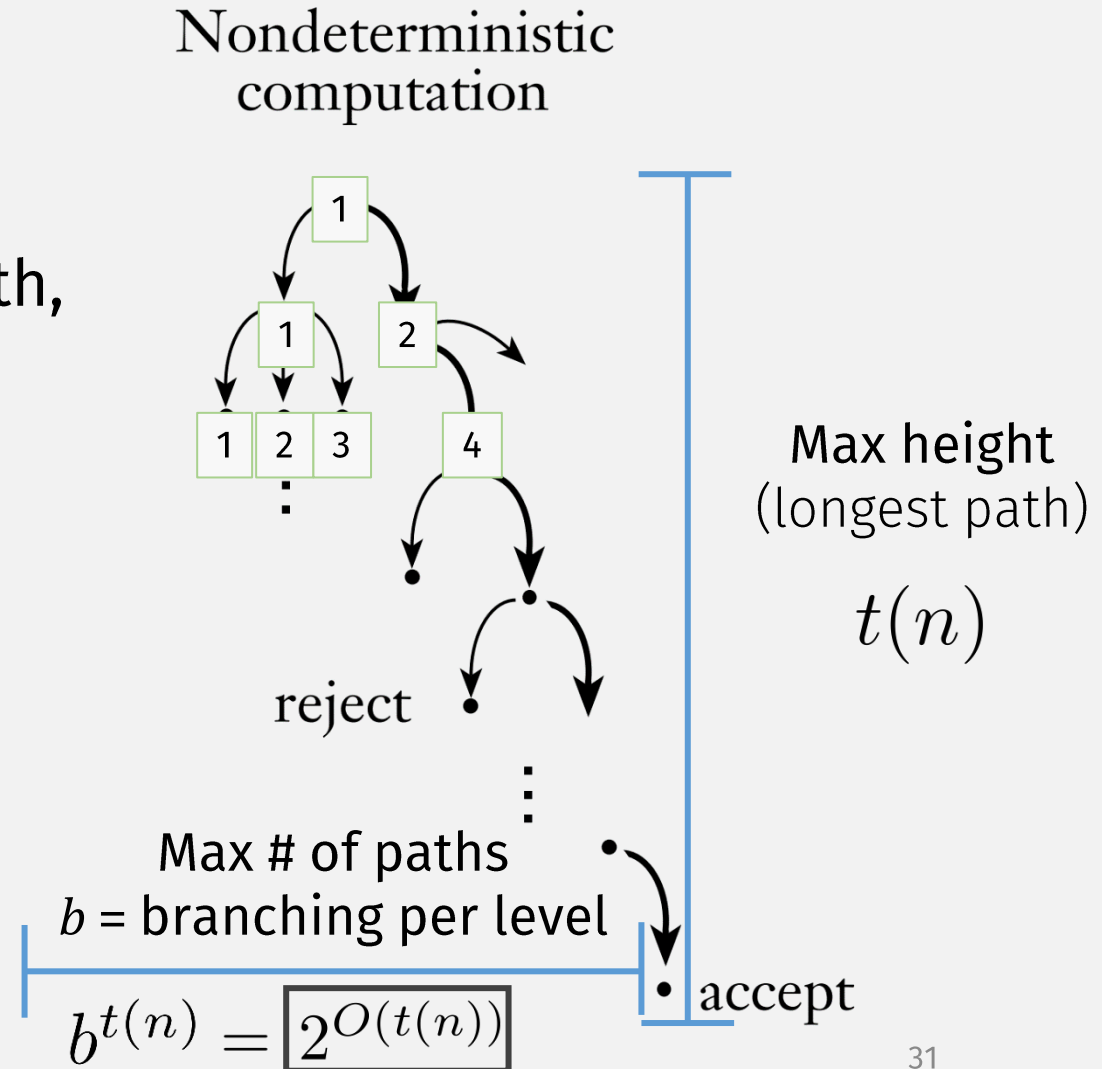
First, we need a <u>formal</u> way to count "# of steps" …

1

1   2

1 2 3   4

reject

accept

Flashback: Nondet. TM → Deterministic TM

$t(n)$ time $2^{O(t(n))}$ time

- **Simulate NTM with Det. TM:**
  - Number the nodes at each step
  - Deterministically check every tree path, in breadth-first order
    - 1
    - 1-1
    - 1-2
    - **1-1-1**

Nondeterministic computation

Max height
(longest path)

$t(n)$

reject

Max # of paths
b = branching per level

$b^{t(n)} = 2^{O(t(n))}$

accept

Summary: TM Variations

- If **multi-tape TM:** $t(n)$ time
- Then **equivalent single-tape TM:** $O(t^2(n))$
  - **Quadratically** slower

- If **non-deterministic TM:** $t(n)$ time
- Then **equivalent single-tape TM:** $2^{O(t(n))}$
  - **Exponentially** slower

Check-in Quiz 4/26

On gradescope