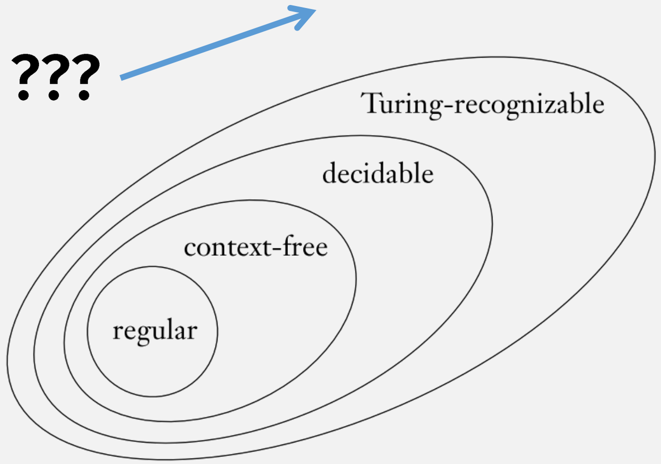
UMB CS 420

Unrecognizability

Monday, April 24, 2023



Announcements

- HW 9 in
  - ~~Due Sun 4/23 11:59pm EST~~

- HW 10 out
  - Due Sun 4/30 11:59pm EST

Quiz Preview

- If a language is undecidable, which of the following statements about its recognizability cannot be true?

Last Time: Showing Mapping Reducibility

Step 1:
Show there is computable fn f … by creating a TM

Language A is ***mapping reducible*** to language B, written $A \leq_m B$, if there is a $\boxed{\text{computable function } f\colon \Sigma^* \longrightarrow \Sigma^*}$, where for every w,

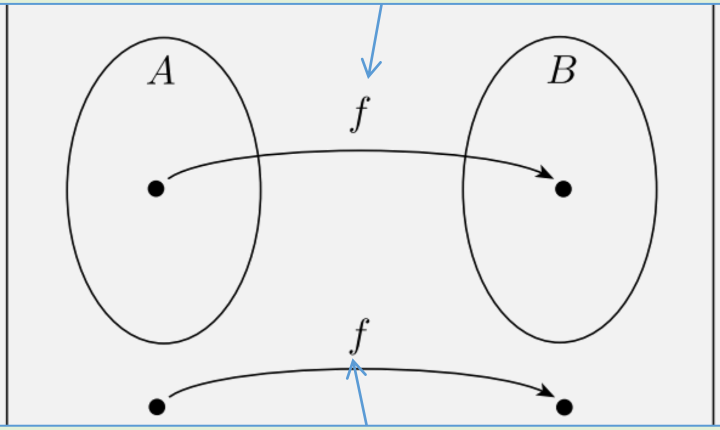
$$\boxed{w \in A \iff f(w) \in B.}$$

"if and only if"

Step 2:
Prove the iff is true for f

The function f is called the ***reduction*** from A to B.

Step 2a: "forward" direction (⇒): if $w \in A$ then $f(w) \in B$



Step 2b: "reverse" direction (⇐): if $f(w) \in B$ then $w \in A$

Step 2b: Equivalent (contrapositive): if $w \notin A$ then $f(w) \notin B$

A function $f\colon \Sigma^* \longrightarrow \Sigma^*$ is a ***computable function*** if some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

Last Time: Using Mapping Reducibility

To prove decidability …

- If $A \leq_{\mathrm{m}} B$ and B is decidable, then A is decidable.

Known

Unknown
(want to prove)

To prove undecidability …

- If $A \leq_{\mathrm{m}} B$ and A is undecidable, then B is undecidable.

Undecidability Proof Technique #4:
Mapping Reducibility
+ this theorem

Be careful with the direction of the **reduction,**
i.e., **what is** known and **what is** unknown!

Flashback: EQ_{TM} is undecidable

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Proof by **contradiction**:

- Assume EQ_{TM} has *decider* R; use to create E_{TM} *decider*:

$= \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

$S =$ "On input $\langle M\rangle$, where M is a TM:

1. Run R on input $\langle M, M_1\rangle$, where M_1 is a TM that rejects all inputs.

2. If R accepts, *accept*; if R rejects, *reject*."

Alternate Proof: EQ_{TM} is undecidable

$EQ_{\text{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Proof by mapping reducibility: $E_{\text{TM}} \leq_{\text{m}} EQ_{\text{TM}}$

Step 1: create computable fn *f*, computed by TM *S*

$S =$ "On input $\langle M\rangle$, where M is a TM:

1. Construct: $\langle M, M_1\rangle$, where M_1 is a TM that rejects all inputs.
2. Output: $\langle M, M_1\rangle$

Step 2: show iff requirements of mapping reducibility

Do for HW 10!

And use theorem …

If $A \leq_{\text{m}} B$ and A is undecidable, then B is undecidable.

Flashback: E_{TM} is undecidable

$E_{\mathsf{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

Proof, by **contradiction**:

- Assume E_{TM} has *decider* R; use to create A_{TM} *decider*:

$S =$ "On input $\langle M, w\rangle$, an encoding of a TM M and a string w:

1. Use the description of M and w to construct the TM M_1

$M_1 =$ "On input x:

1. If $x \neq w$, *reject*.

2. If $x = w$, run M on input w and *accept* if M does."

2. Run R on input $\langle M_1\rangle$.

3. If R accepts, *reject*; if R rejects, *accept*."

If M accepts w, then M_1 not in E_{TM}!
So do the opposite!

M_1 :
- accepts w if M does
- rejects everything else

Alternate Proof: E_{TM} is undecidable

$E_{\mathsf{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

Proof, by mapping reducibility??: $A_{\mathsf{TM}} \leq_{\mathrm{m}} E_{\mathsf{TM}}$

Step 1: create computable fn f: $\langle M, w\rangle \rightarrow \langle M_1\rangle$, computed by S

$S =$ "On input $\langle M, w\rangle$, an encoding of a TM M and a string w:

1. Use the description of M and w to construct the TM M_1

   $M_1 =$ "On input x:
   1. If $x \neq w$, *reject*.
   2. If $x = w$, run M on input w and *accept* if M does."

2. Output: $\langle M_1\rangle$.
3. ~~If R accepts, *reject*; if R rejects, *accept*.~~"

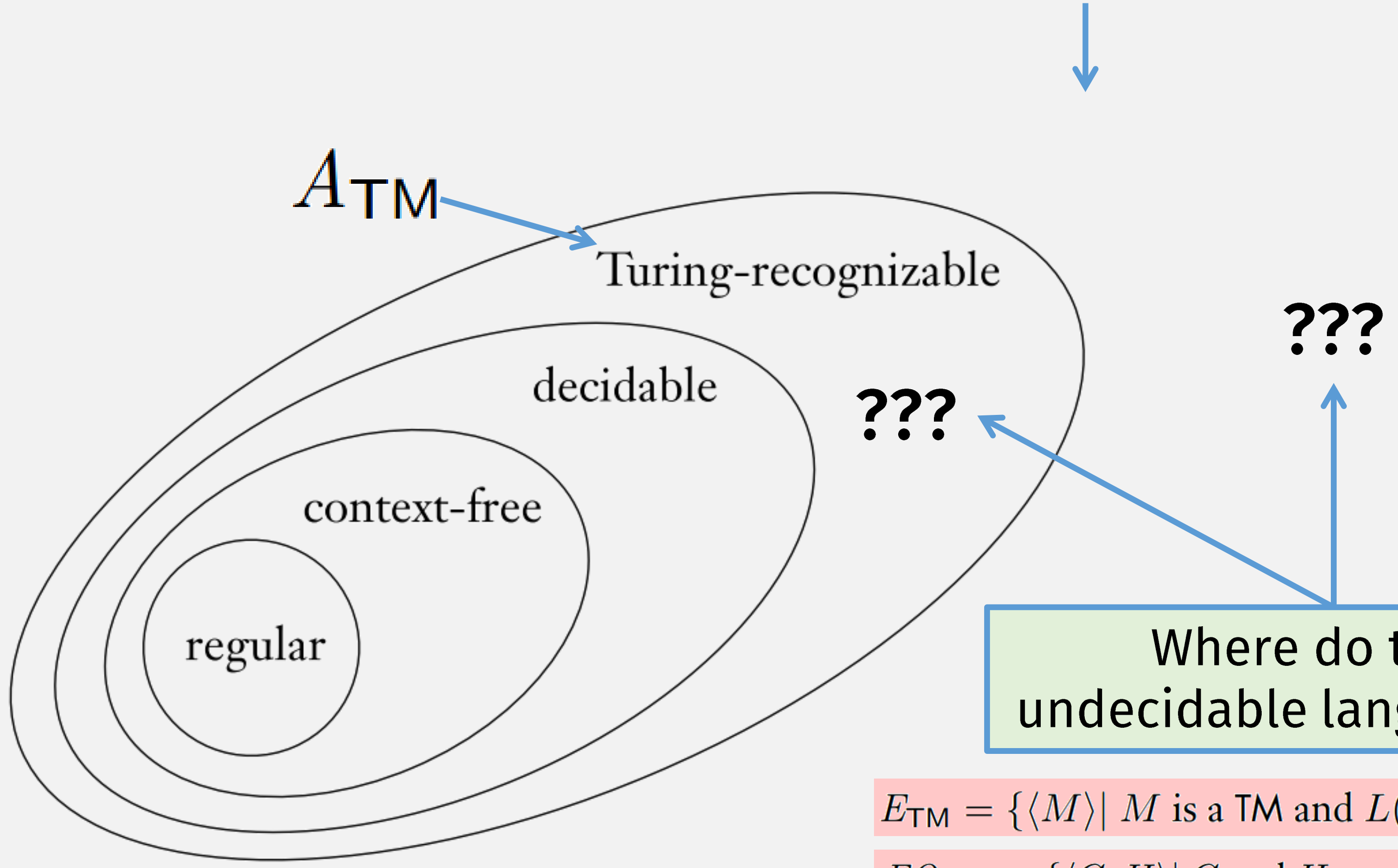
Step 2: show iff requirements of mapping reducibility:

Do for HW 10!

- This reduces A_{TM} to $\overline{E_{\mathsf{TM}}}$!!
- It's good enough, if: undecidable langs are closed under complement

Turing Unrecognizable?

Is there anything out here?



Where do these undecidable languages go?

$E_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

$EQ_{\text{CFG}} = \{\langle G, H \rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$

$EQ_{\text{TM}} = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Thm: Some langs are not Turing-recognizable

Proof: requires 2 lemmas

- Lemma 1: The **set of all languages** is *uncountable* (hw9)

- Lemma 2: The **set of all TMs** is *countable*

- Therefore, some language is not recognized by a TM
  (pigeonhole principle)

Thm: Some langs are not Turing-recognizable

Proof: requires 2 lemmas

- Lemma 1: The **set of all languages** is *uncountable* (hw9)
  - Proof, by contradiction: Assume the set of all languages is countable
    - Then there is a bijection mapping natural numbers to languages (def of countable)

      $1 \rightarrow s_{11}, s_{12}, s_{13}, \ldots$
      $2 \rightarrow s_{21}, s_{22}, s_{23}, \ldots$
      $3 \rightarrow s_{31}, s_{32}, s_{33}, \ldots$
      $\ldots$

      where strings in each language are ordered lexicographically (assumption from problem)
    - But some language is always not mapped to: $s_1, s_2, s_3, \ldots$
      - where $s_1 \neq s_{11}, s_2 \neq s_{22}, s_3 \neq s_{33}, \ldots$ (**diagonalization** technique)
      - and $s_1, s_2, s_3, \ldots$ is alphabetically ordered
    - Thus there is no bijection, which is a contradiction

Thm: Some langs are not Turing-recognizable

Proof: requires 2 lemmas

- Lemma 1: The **set of all languages** is *uncountable*
  - Alternate Proof: Show a bijection with another uncountable set …
    - … The set of all infinite binary sequences (from textbook)

Mapping a Language to a Binary Sequence

All Possible Strings	Σ^* =	$\{\ \varepsilon,\ 0,\ 1,\ 00,\ 01,\ 10,\ 11,\ 000,\ 001,\ \cdots\ \}$
Some Language (subset of above)	A =	$\{\ 0,\ 00,\ 01,\ 000,\ 001,\ \cdots\ \}$
Its (unique) Binary Sequence	χ_A =	$0\ \ 1\ \ 0\ \ 1\ \ 1\ \ 0\ \ 0\ \ 1\ \ 1\ \ \cdots$

Each digit represents one possible string:
- 1 if lang has that string,
- 0 otherwise

Thm: Some langs are not Turing-recognizable

This is an "existence" proof,
but it's not "constructive",
i.e., it doesn't give an example of
an unrecognizable language

Proof: requires 2 lemmas

- Lemma 1: The **set of all languages** is *uncountable*
  - Proof: Show there is a bijection with another uncountable set …
    - … The set of all infinite binary sequences
    - ➢ Now just prove set of infinite binary sequences is uncountable (diagonalization)

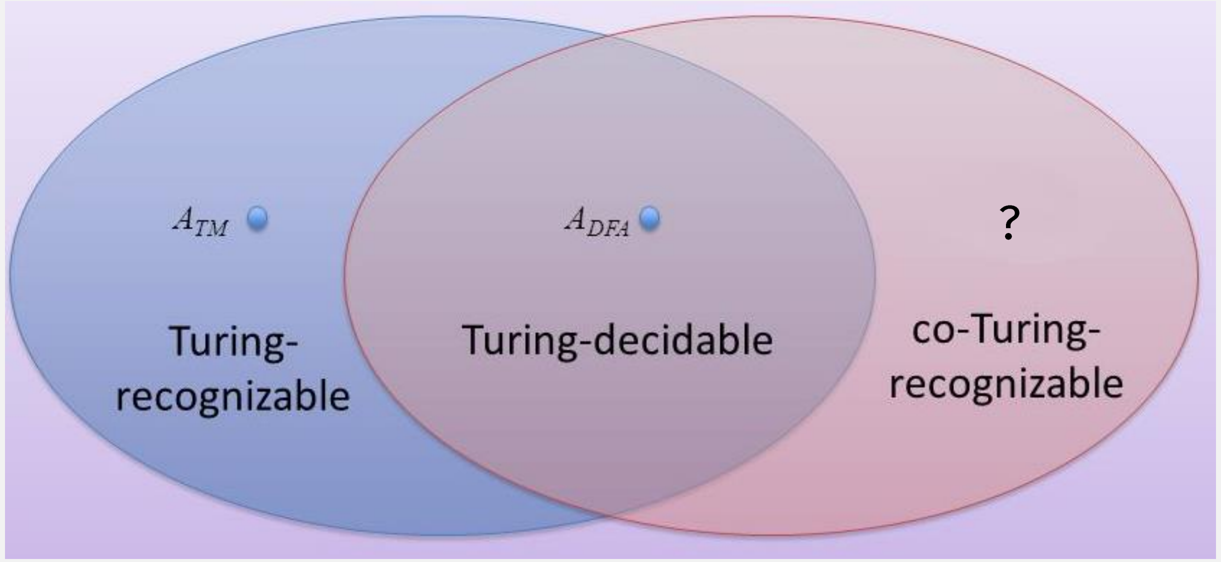
- Lemma 2: The **set of all TMs** is *countable*
  - Because every TM M can be encoded as a string $\langle M \rangle$
  - And set of all strings is countable
  - Order the strings lexicographically (length 0 strings, then length 1 strings, etc)
- Therefore, some language is not recognized by a TM

■

Co-Turing-Recognizability

- A language is **co-Turing-recognizable** if …
- … it is the complement of a Turing-recognizable language.

Thm: Decidable ⇔ Recognizable & co-Recognizable



Thm: Decidable ⇔ Recognizable & co-Recognizable

⇒ If a language is **decidable**, then it is **recognizable** and **co-recognizable**

- Decidable ⇒ Recognizable:
  - A decider is a recognizer (that always halts)
- Decidable ⇒ Co-Recognizable:
  - To create co-decider from a decider … switch reject/accept of all inputs
  - A co-decider is a co-recognizer, for same reason as above

Thm: Decidable ⇔ Recognizable & co-Recognizable

⇒ If a language is **decidable**, then **it is recognizable** and **co-recognizable**

- Decidable ⇒ Recognizable:
  - A decider is a recognizer (that always halts)
- Decidable ⇒ Co-Recognizable:
  - To create co-decider from a decider … switch reject/accept of all inputs
  - A co-decider is a co-recognizer, for same reason as above

⇐ If a language is **recognizable** and **co-recognizable**, then **it is decidable**

- Let M_1 = recognizer for the language,
- and M_2 = recognizer for its complement
- Decider M:
  - Run 1 step on M_1,
  - Run 1 step on M_2,
  - Repeat, until one machine accepts. If it's M_1, accept. If it's M_2, reject

Termination Arg: Either M_1 or M_2 must accept and halt, so M halts and is a decider

A Turing-unrecognizable language

- We've proved:

  $\boxed{A_{\mathsf{TM}} \text{ is Turing-recognizable}}$

  A_{TM} is undecidable

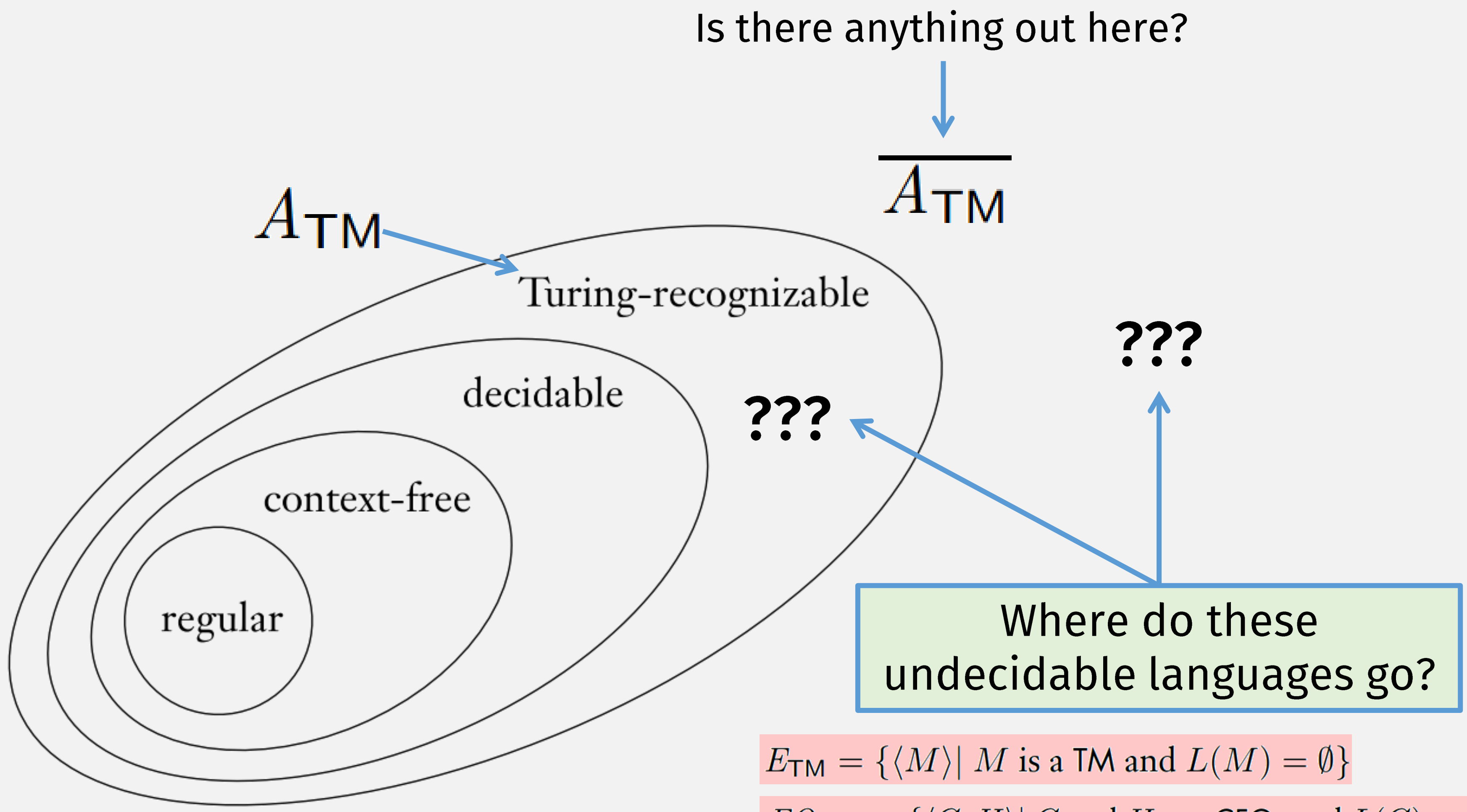
- So:

  $\overline{A_{\mathsf{TM}}}$ is not Turing-recognizable

Unrecognizability
Proof Technique #1

- We know: $\boxed{\textbf{recognizable \& co-recognizable} \Rightarrow \textbf{decidable}}$

  Contrapositive: undecidable ⇒ can't be both recognizable & co-recognizable



$E_{\mathsf{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

$EQ_{\mathsf{CFG}} = \{\langle G, H \rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Mapping Reducibility Can be Used to Prove …

- Decidability
- Undecidability
- Recognizability
- Unrecognizability

More Helpful Theorems

If $A \leq_m B$ and B is $\boxed{\text{Turing-recognizable,}}$ then A is Turing-recognizable.

If $A \leq_m B$ and A is not Turing-recognizable, then B is not Turing-recognizable.

Unrecognizability
Proof Technique #2:
Mapping reducibility
+ this theorem

- Same proofs as:

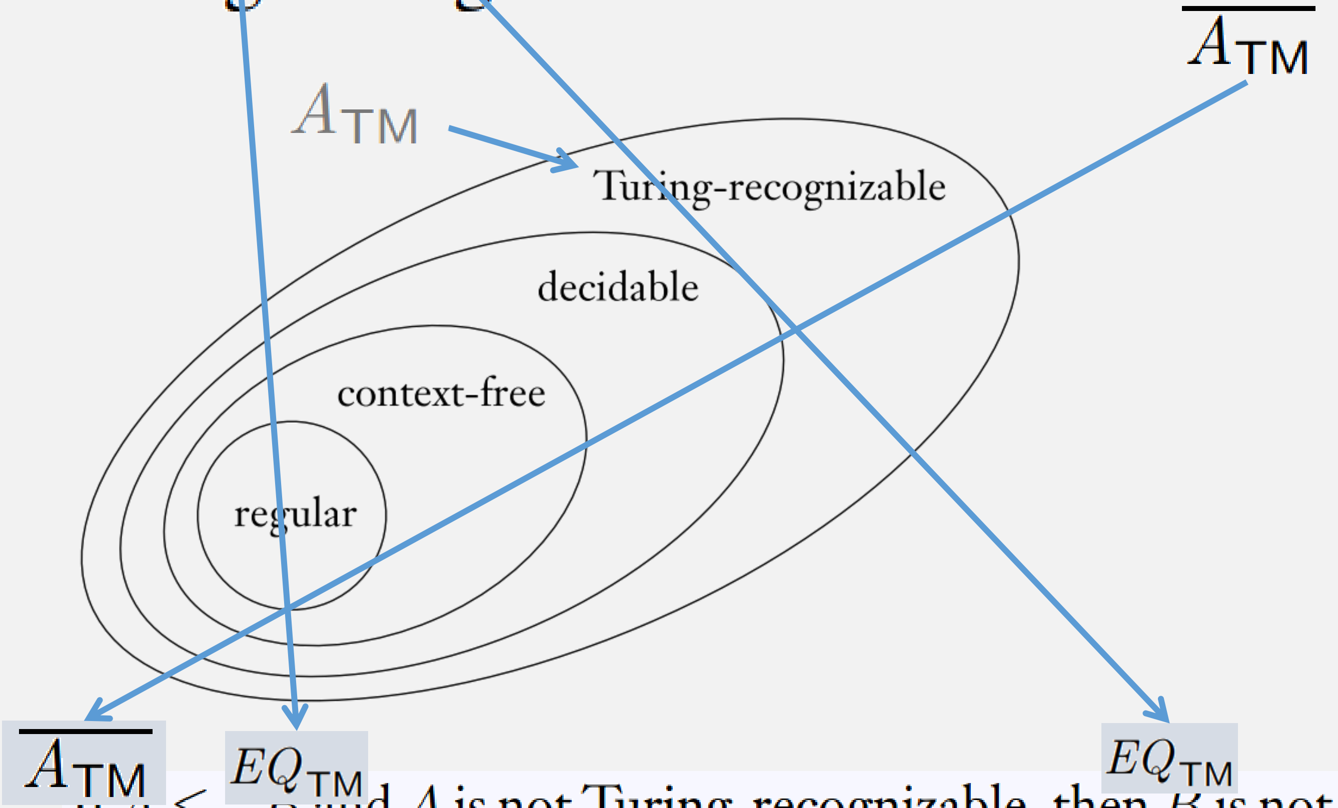
If $A \leq_m B$ and B is $\boxed{\text{decidable,}}$ then A is decidable.

If $A \leq_m B$ and A is undecidable, then B is undecidable.

<u>Thm</u>: EQ_{TM} is neither Turing-recognizable nor co-Turing-recognizable.

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle |\ M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

1. EQ_{TM} is not Turing-recognizable



If $A \leq_{\mathrm{m}} B$ and A is not Turing-recognizable, then B is not Turing-recognizable.

Mapping Reducibility implies Mapping Red. of Complements

Language A is ***mapping reducible*** to language B, written $A \leq_{\mathrm{m}} B$, if there is a computable function $f\colon \Sigma^* \longrightarrow \Sigma^*$, where for every w,

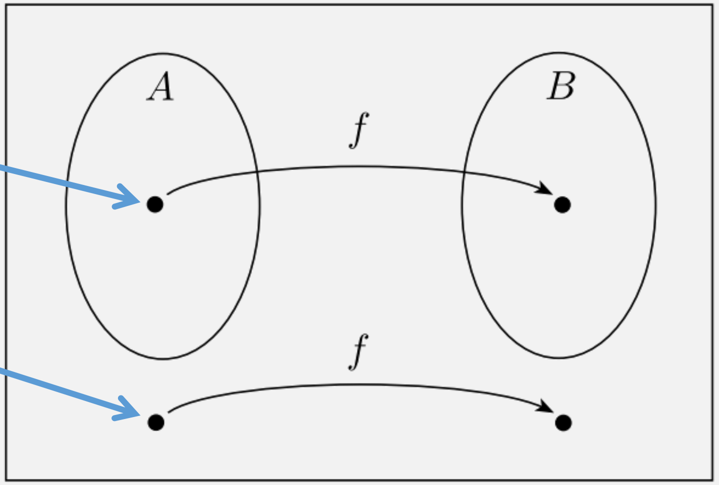
$$\boxed{w \in A \iff f(w) \in B.}$$

The function f is called the ***reduction*** from A to B.

$$\boxed{A \leq_{\mathrm{m}} B}$$

implies

$$\boxed{\overline{A} \leq_{\mathrm{m}} \overline{B}}$$



Thm: EQ_{TM} is neither Turing-recognizable nor co-Turing-recognizable.

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle |\ M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

1. EQ_{TM} is not Turing-recognizable

Two Choices:

- Create Computable fn: $\overline{A_{\mathsf{TM}}} \rightarrow EQ_{\mathsf{TM}}$
- Or Computable fn: $\boxed{A_{\mathsf{TM}} \rightarrow \overline{EQ_{\mathsf{TM}}}}$

And use theorem …

If $A \leq_{\mathrm{m}} B$ and A is not Turing-recognizable, then B is not Turing-recognizable.

Thm: EQ_{TM} is not Turing-recognizable

$EQ_{\text{TM}} = \{\langle M_1, M_2 \rangle |\ M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Step 1
Computable fn

- Create Computable fn: $A_{\text{TM}} \rightarrow \overline{EQ_{\text{TM}}}$
- $\langle M, w \rangle \rightarrow \langle M_1, M_2 \rangle$ M_1 and M_2 are TMs and $L(M_1) \neq L(M_2)$

F = "On input $\langle M, w \rangle$, where M is a TM and w a string:

1. Construct the following two machines, M_1 and M_2.

   M_1 = "On any input: ← Accepts nothing

   1. *Reject*."

   M_2 = "On any input: ← Accepts nothing or everything

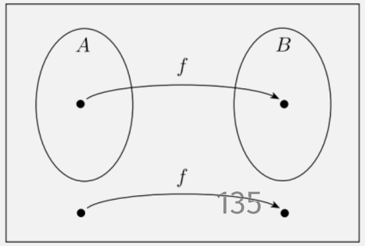
   1. Run M on w. If it accepts, *accept*."

2. Output $\langle M_1, M_2 \rangle$."

Step 2, iff:

⇒ If M accepts w, then $M_1 \neq M_2$

⇐ If M does not accept w, then $M_1 = M_2$



Thm: EQ_{TM} is neither Turing-recognizable nor co-Turing-recognizable.

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle |\ M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

1. EQ_{TM} is not Turing-recognizable
   - Create Computable fn: $\overline{A_{\mathsf{TM}}} \rightarrow EQ_{\mathsf{TM}}$
   - Or Computable fn: $A_{\mathsf{TM}} \rightarrow \overline{EQ_{\mathsf{TM}}}$
   - **DONE!**

And use theorem …

If $A \leq_{\mathrm{m}} B$ and A is not Turing-recognizable, then B is not Turing-recognizable.

(Definition of co-Turing-recognizable)

2. $\overline{EQ_{\mathsf{TM}}}$ is not ~~co~~-Turing-recognizable
   - (A lang is co-Turing-recog. if it is complement of Turing-recog. lang)

Previous: EQ_{TM} is not Turing-recognizable

$EQ_{\text{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

- Create Computable fn: $A_{\text{TM}} \rightarrow \overline{EQ_{\text{TM}}}$

Step 1
- $\langle M, w\rangle \rightarrow \langle M_1, M_2\rangle$ M_1 and M_2 are TMs and $L(M_1) \neq L(M_2)$

F = "On input $\langle M, w\rangle$, where M is a TM and w a string:

1. Construct the following two machines, M_1 and M_2.

M_1 = "On any input: ← Accepts nothing

1. *Reject*."

M_2 = "On any input: ← Accepts nothing or everything

1. Run M on w. If it accepts, *accept*."

2. Output $\langle M_1, M_2\rangle$."

Now: $\overline{EQ}_{\text{TM}}$ is not Turing-recognizable

$EQ_{\text{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

- Create Computable fn: $A_{\text{TM}} \rightarrow \overline{EQ}_{\text{TM}}$

Step 1
- $\langle M, w\rangle \rightarrow \langle M_1, M_2\rangle$ M_1 and M_2 are TMs and $L(M_1) \neq L(M_2)$

$F =$ "On input $\langle M, w\rangle$, where M is a TM and w a string:

1. Construct the following two machines, M_1 and M_2.

   $M_1 =$ "On any input: ← Accepts ~~nothing~~ everything

   1. *Accept*."

   $M_2 =$ "On any input: ← Accepts nothing or everything

   1. Run M on w. If it accepts, *accept*."

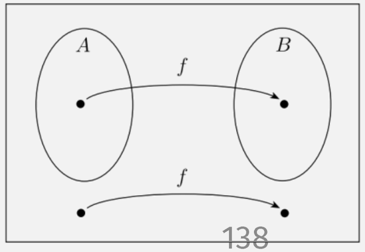
2. Output $\langle M_1, M_2\rangle$."

Step 2, iff:

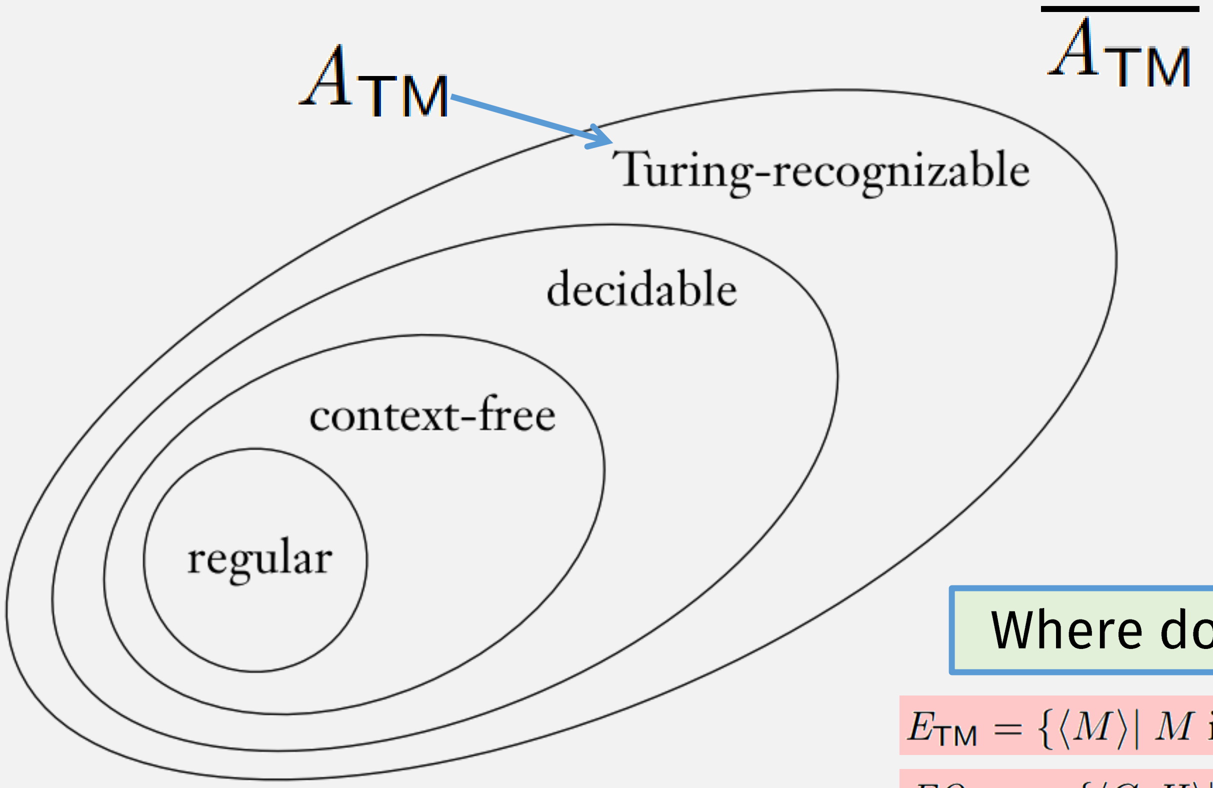
⇒ If M accepts w, then $M_1 \boxed{=} M_2$

⇐ If M does not accept w, then $M_1 \boxed{\neq} M_2$

DONE!



Unrecognizable Languages?



Where do these go?

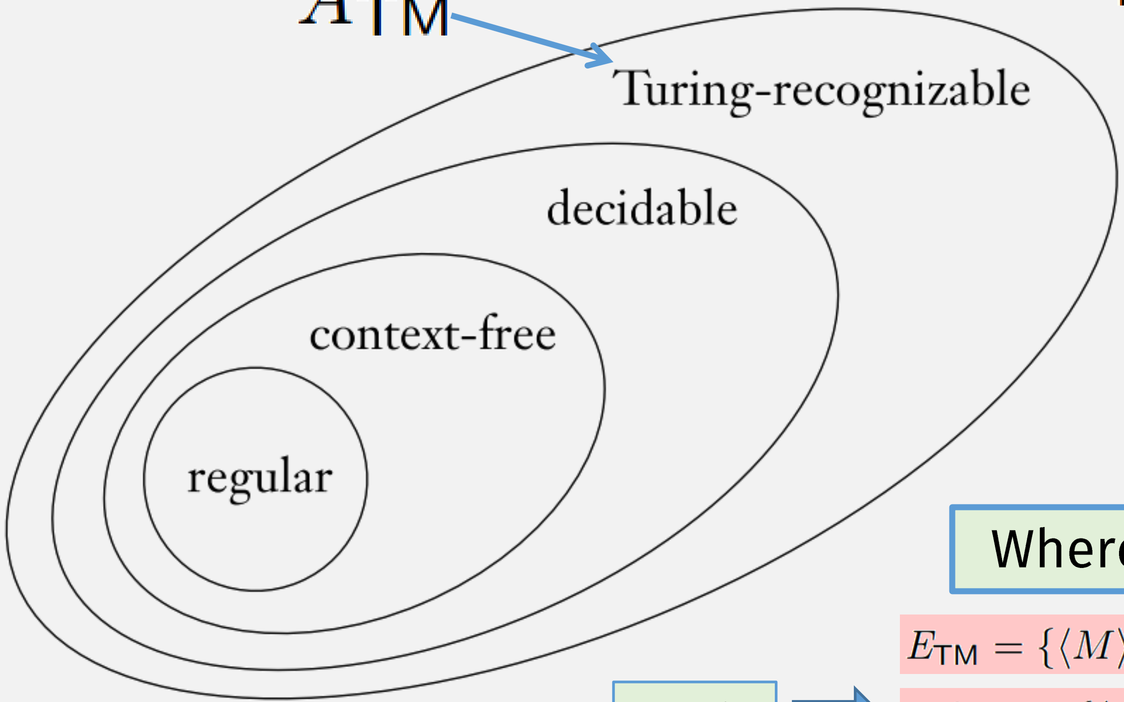
$E_{\mathsf{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

$EQ_{\mathsf{CFG}} = \{\langle G, H\rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Unrecognizable Languages

A_{TM}



$\overline{A_{\mathsf{TM}}}$

EQ_{TM}

$\overline{EQ_{\mathsf{TM}}}$

Where do these go?

$E_{\mathsf{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

next

$EQ_{\mathsf{CFG}} = \{\langle G, H \rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$

Thm: EQ_{CFG} is not Turing-recognizable

Unrecognizability
Proof Technique #1

Recognizable & co-recognizable ⇒ decidable

Contrapositive: undecidable ⇒ can't be both recognizable & co-recognizable

- We didn't prove this yet (but it is true):
  EQ_{CFG} is undecidable
- We now prove:
  EQ_{CFG} is co-Turing recognizable
- And conclude that:
  - EQ_{CFG} is not Turing recognizable

Thm: EQ_{CFG} is co-Turing-recognizable

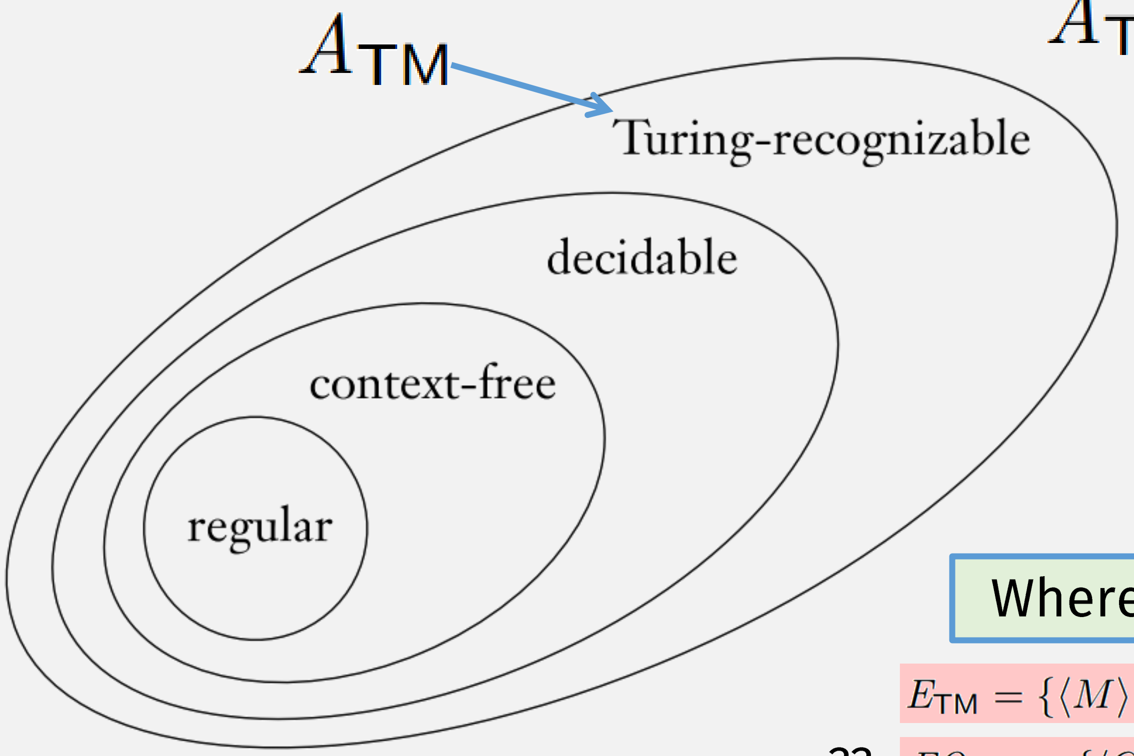
$EQ_{\mathsf{CFG}} = \{\langle G, H\rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$

Recognizer for $\overline{EQ}_{\mathsf{CFG}}$:

M = On input $\langle G, H\rangle$, where G and H are CFGs:
- For every possible string w:
  Accept if
  - $w \in L(G)$ and $w \notin L(H)$, or
  - $w \notin L(G)$ and $w \in L(G)$ → Use decider for: $A_{\mathsf{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates string } w\}$
- Else reject

This is only a **recognizer** because it loops for ever when $L(G) = L(H)$

Unrecognizable Languages



$\overline{A_{\mathsf{TM}}}$

EQ_{TM}

$\overline{EQ_{\mathsf{TM}}}$

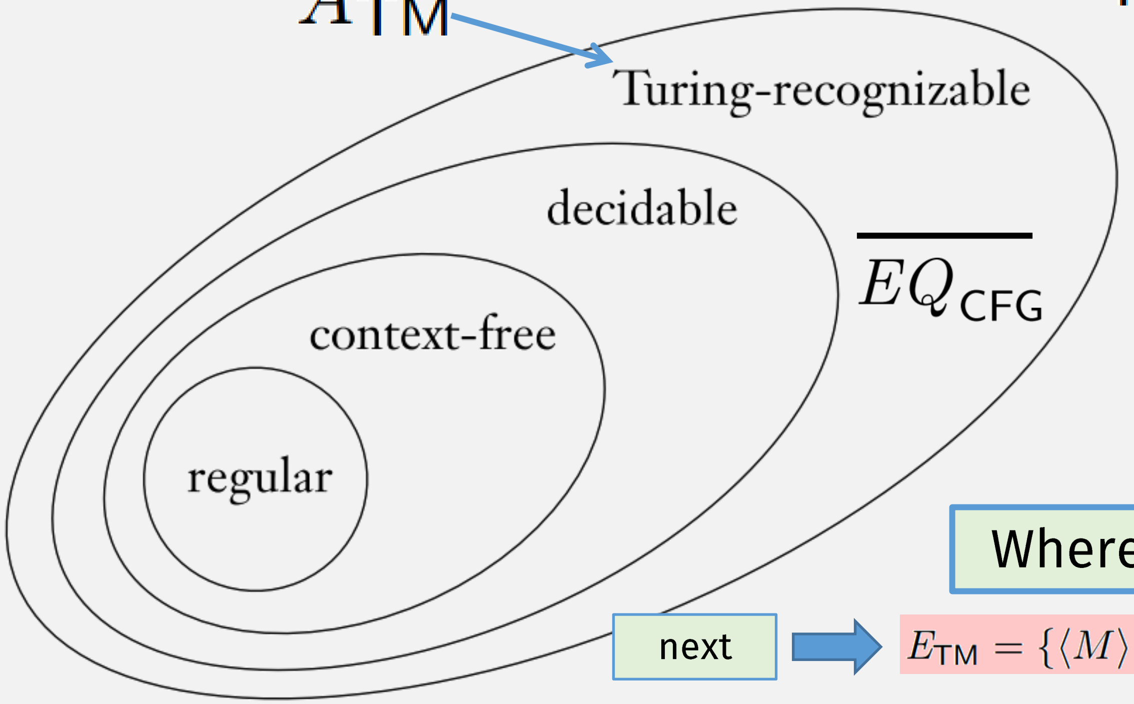
Where do these go?

$E_{\mathsf{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

?? $EQ_{\mathsf{CFG}} = \{\langle G, H \rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$

Unrecognizable Languages

A_{TM}



$\overline{A_{\mathsf{TM}}}$

EQ_{TM}

$\overline{EQ_{\mathsf{TM}}}$

EQ_{CFG}

Where do these go?

next

$E_{\mathsf{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

Thm: E_{TM} is not Turing-recognizable

Unrecognizability
Proof Technique #1

Recognizable & co-recognizable ⇒ decidable

Contrapositive: undecidable ⇒ can't be both recognizable & co-recognizable

- We've proved:
  - E_{TM} is undecidable
- We now prove:
  E_{TM} is co-Turing recognizable
- And then conclude that:
  - E_{TM} is not Turing recognizable

Thm: E_{TM} is co-Turing-recognizable

$E_{\mathsf{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

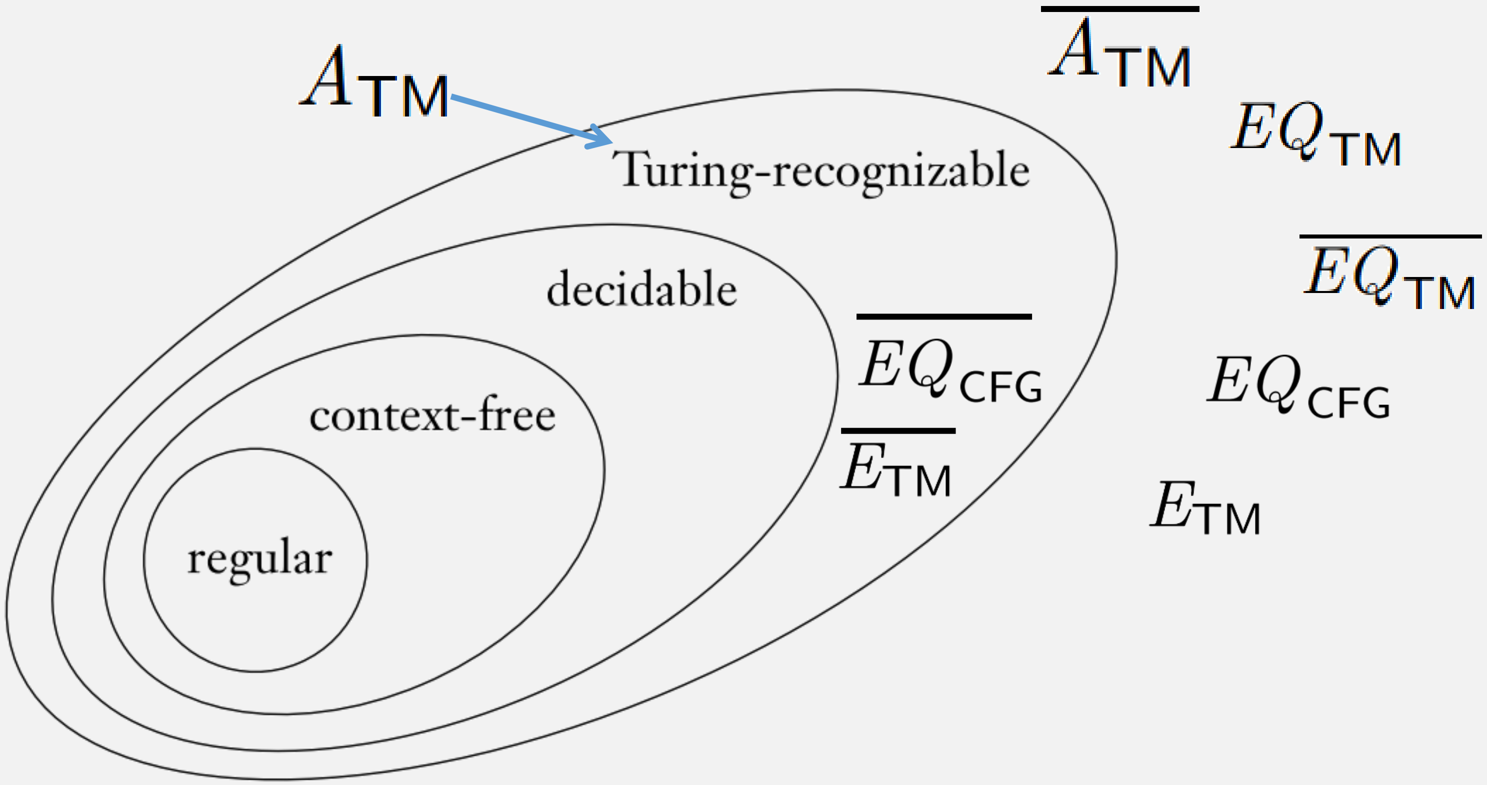
Recognizer for $\overline{E_{\mathsf{TM}}}$: Let $s_1, s_2, \ldots$ be a list of all strings in Σ^*

"On input $\langle M\rangle$, where M is a TM:

1. Repeat the following for $\boxed{i} = 1, 2, 3, \ldots$.
2. Run M for $\boxed{i}$ steps on each input, $s_1, s_2, \ldots, \boxed{s_i}$.
3. If M has accepted any of these, *accept*. Otherwise, continue."

This is only a **recognizer** because it loops for ever when $L(M)$ is empty

Unrecognizable Languages



Check-in Quiz 4/24

On gradescope