UMB CS622

# Decidability of Logical Theories

Wednesday, November 3, 2021

# Announcements

- HW6 due tonight
- See piazza announcement about HW problem “plans”

# Hilbert's 23 Open Problems in Math (1900)

1. …

Can't prove "no" unless you first formally define what an **algorithm** is!

10. Is there an algorithm determining whether a polynomial has an integer root?
    Actually:
    "to devise a process according to which it can be determined in a finite number of operations whether the equation is solvable"

23. …

David Hilbert

# A Little Bit of Computation History

<u>1900</u>: Hilbert's 23 Problems

"Computation" = proving things about mathematical statements

<u>1928</u>: Hilbert/Ackermann's "*Entscheidungsproblem*" (decision problem):

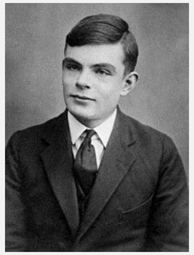
> Is there an <u>algorithm</u> that can determine whether any mathematical statement (about natural numbers) is true or false?

<u>1935</u>: Alonzo Church

- Defined "algorithm" with the λ-calculus
- Proved *Entscheidungsproblem* false by reducing it to …
- … determining whether 2 λ-calculus programs are equivalent
- … and then showed that it is undecidable (analogous to $EQ_{\mathrm{TM}}$)

<u>1936</u>: Alan Turing

- Defined "algorithm" with the Turing Machine
- Proved *Entscheidungsproblem* false by reducing it to … $HALT_{\mathrm{TM}}$
- … and then showed $HALT_{\mathrm{TM}}$ is undecidable



# The Language of Mathematical Statements

$$\mathbf{1.}\ \forall q\, \exists p\, \forall x,y\ \big[p>q \wedge (x,y>1 \rightarrow xy \neq p)\big],$$
$$\mathbf{2.}\ \forall a,b,c,n\ \big[(a,b,c>0 \wedge n>2) \rightarrow a^n + b^n \neq c^n\big], \textbf{ and}$$
$$\mathbf{3.}\ \forall q\, \exists p\, \forall x,y\ \big[p>q \wedge (x,y>1 \rightarrow (xy \neq p \wedge xy \neq p+2))\big]$$

Early theory of "computation" and formal languages research tried to find a "program" to <u>automatically</u> prove these kinds of statements true

1. "Infinitely many prime numbers exist"
   - Euclid proved true 2300 yrs ago
2. Fermat's Last Theorem
   - Wiles proved true in 1994
3. Twin Prime Conjecture: "infinitely many prime pairs exist"
   - Unsolved!

**So there must be some parallel between "proof" and "computation"**

# The Alphabet of Mathematical Statements

- Strings in the language are drawn from the following chars:
  - $\wedge, \vee, \neg$ Boolean operations
  - $(, ), [, ]$ parentheses
  - $\forall, \exists$ quantifiers
  - $x$ variables
  - $R_1, \ldots, R_\mathrm{k}$ Relation symbols

# Formulas

- A mathematical statement is <u>well-formed</u>, i.e., a **formula**, if it's:
  - an atomic formula: $R_i(x_1, \ldots, x_\mathrm{k})$
  - $\phi_1 \wedge \phi_2$, $\phi_1 \vee \phi_2$, or $\neg\phi$
    - where $\phi$, $\phi_1$, and $\phi_2$ are formulas
  - $\forall x\ [\ \phi\ ]$, $\exists x\ [\ \phi\ ]$
    - where $\phi$ is a formula
    - $x$'s "scope" is in the following brackets
    - A free variable is a variable that is outside the scope of a quantifier
  - And all Quantifiers must appear at the front of the formula
    - Prenex normal form

$$R_1(x_1) \wedge R_2(x_1, x_2, x_3)$$
$$\forall x_1\ \left[\ R_1(x_1) \wedge R_2(x_1, x_2, x_3)\right]$$
$$\forall x_1\ \exists x_2\ \exists x_3\ \left[\ R_1(x_1) \wedge R_2(x_1, x_2, x_3)\right]$$

- A **sentence** is a formula with no free variables

# Universes, Models, and Theories

- A **universe** is the set of values that variables can represent
    - E.g., the universe of the natural numbers
- A **model** ($\mathcal{M}$) is:
    - a universe, and
    - an assignment of relations to relation symbols
    - E.g., the model $(\mathcal{N}, \leq)$
- The **language of a model** is the set of all formula that (correctly) use the relations of the model
- A **theory** is the set of all <u>true sentences</u> in a model's language
    - written $\mathrm{Th}(\mathcal{M})$

# Theorem: $\mathrm{Th}(\mathcal{N}, +)$ is decidable

- In the language: $\forall x\, \exists y\, \left[\, x + x = y \,\right]$

- Not in the language: $\exists y \forall x\, \left[\, x + x = y \,\right]$

# A Regular Language About Addition

- Assume an alphabet $\Sigma_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$
  - Columns representing all possible combinations of `0`s and `1`s

- A sequence of these columns is 3 rows of binary numbers

- We show that the following language is regular:

$$B = \{w \in \Sigma_3^* | \text{ the bottom row of } w \text{ is the sum of the top two rows}\}$$

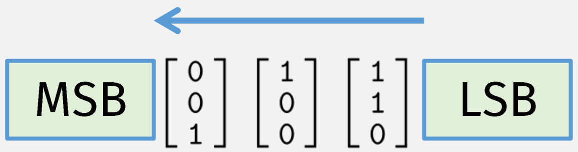
$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in B \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \notin B$$

# Addition: Proof of Regularity

$$B = \{w \in \Sigma_3^* | \text{ the bottom row of } w \text{ is the sum of the top two rows}\}$$

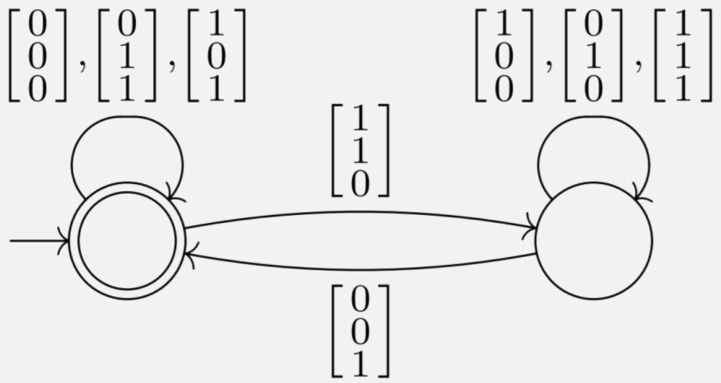
- Create a DFA accepting valid additions

- <u>Key idea</u>: operate on strings in reverse
  - i.e., process least significant bit first
  - This is ok because reverse closed for regular languages




- Reject whenever any column is incorrect

- Use extra state to keep track of "carries"



# Theorem: $\text{Th}(\mathcal{N}, +)$ is decidable (Pressburger Arithmetic)

On input $\phi = Q_1 x_1 Q_2 x_2 \ldots Q_n x_n [\, \psi \,]$:

1. Initially, ignore all the quantifiers $Q_1 \ldots Q_n$ and construct a DFA for $\psi$
   a) For every +, construct a generalized addition DFA over alphabet:

$$\Sigma_i = \left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \right\}$$

   b) Combine those DFAs using (all closed operations for regular languages!):
      - union (for $\vee$),
      - intersection (for $\wedge$),
      - and complement (for $\neg$)

- Call this initial machine $\boldsymbol{A_n}$

# Theorem: $\text{Th}(\mathcal{N}, +)$ is decidable

On input $\phi = Q_1 x_1 Q_2 x_2 \ldots Q_n x_n [\ \psi\ ]$:

- … call this initial machine $\boldsymbol{A_n}$

DFA $\boldsymbol{A_i}$ accepts $i$ rows (numbers) that make formula $Q_{i+1} x_{i+1} \ldots Q_n x_n [\ \psi\ ]$ true

Now handle quantifiers …

2. For every $\exists x_i$, create DFA $\boldsymbol{A_i}$ that is like $\boldsymbol{A_{i+1}}$ but with one less input row
   - Instead, nondeterministically guess the number for the last row

$$\boldsymbol{A_i}\text{'s input} \quad \begin{bmatrix} b_1 \\ \vdots \\ b_{i-1} \\ b_i \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} b_1 \\ \vdots \\ b_{i-1} \\ b_i \\ z \end{bmatrix} \quad \boldsymbol{A_{i+1}}\text{'s input}$$

$$b_j \in \{0,1\}$$

$$z \in \{0,1\}$$

# Theorem: $\mathrm{Th}(\mathcal{N}, +)$ is decidable

On input $\phi$ = $Q_1 x_1 Q_2 x_2$ … $Q_n x_n$ [ $\psi$ ]:

- …

3. For every $\forall x_i$, use equality $\forall x.\phi = \neg\exists x.\neg\phi$ to convert $\forall$ to $\exists$ and then use same construction from the $\exists$ step

After handling all the quantifiers DFA $\boldsymbol{A_o}$ accepts any string when formula $\phi$ is true

Theorem: $\mathrm{Th}(\mathcal{N}, +, \times)$ is undecidable

# *Flashback*: $ALL_{\mathsf{CFG}}$ is undecidable

$ALL_{\mathsf{CFG}} = \{\langle G \rangle |\ G \text{ is a CFG and } L(G) = \Sigma^*\}$

Proof, by contradiction

- Assume $ALL_{\mathsf{CFG}}$ has a decider $R$. Use it to create decider for $A_{\mathsf{TM}}$:

On input <$M$, $w$>:

1. Construct a PDA $P$ that rejects sequences of $M$ configs that accept $w$
2. Convert $P$ to a CFG $G$
3. Give $G$ to $R$:
   - If $R$ accepts, then $M$ has <u>no accepting</u> config sequences for $w$, so reject
   - If $R$ rejects, then $M$ has <u>an accepting</u> config sequence for $w$, so accept

**Insight:** Any machine that can validate accepting TM config sequences must represent an undecidable language!

# Theorem: $\text{Th}(\mathcal{N}, +, \times)$ is undecidable

Proof sketch, by contradiction

- Assume $\text{Th}(\mathcal{N}, +, \times)$ has a decider $R$. Use it to create decider for $A_{\text{TM}}$:

On input <$M$, $w$>:

This "validates" accepting config sequences, using + and ×

1. Construct a formula $\exists x.\phi_{M,w}$ that is true iff M accepts $w$
2. Give the formula to $R$ and accept if it accepts

**Insight**: A TM configuration represents a number!

# *Flashback:* LBA Configurations

- How many possible configurations does an LBA have?
  - $q$ states
  - $g$ tape alphabet chars
  - tape of length $n$

- Possible Configurations = $qng^n$
  - $g^n$ = number of possible tape configurations
  - $qn$ = all the possible head positions

# Proof Sketch $\mathrm{Th}(\mathcal{N}, +, \times)$ is undecidable

- A sequence of TM configurations is just a large number
  - In Base-$g$ ($g$ = number of tape alphabet chars)

- So in formula $\exists x. \phi_{M,w}$
  - $x$ is a number representing a sequence of configs
  - $\phi_{M,w}$ ”checks”, using plus and times, that it is a valid seq that accepts $w$

# “Checking” a TM Sequence with + and ×

$w$

# Analogy: Checking Digits in a Number

Example:

- Check that a given number has:
  - First digit: $\mathbf{5}$
  - Second digit: $\mathbf{4}$
  - Third digit: $\mathbf{3}$

- Equivalent to checking that the number is $543$
  - $\mathbf{5} \times 10 \times 10 + \mathbf{4} \times 10 + \mathbf{3} = 543$

Note the required operations:
$+$ and $\times$!

TM Configurations

Configuration Sequence

# ~~Analogy:~~ Checking ~~Digits~~ in a ~~Number~~

Example:

- Check that a given number has:
  - First digit: $\mathbf{5}$
  - Second digit: $\mathbf{4}$
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TM Configurations

Configuration Sequence

# ~~Analogy:~~ Checking ~~Digits~~ in a ~~Number~~

Example:

- Check that a given number has:
  - First digit: ~~**5**~~ $\boldsymbol{C_1}$
  - Second digit: ~~**4**~~ $\boldsymbol{C_2}$
  - Third digit: ~~**3**~~ $\boldsymbol{C_3}$
- Equivalent to checking that the number is 543
  - $\mathbf{5} \times 10 \times 10 + \mathbf{4} \times 10 + \mathbf{3} = 543$

TM Configurations

Configuration Sequence

# ~~Analogy:~~ Checking ~~Digits~~ in a ~~Number~~

Example:

- Check that a given number has:
  - First digit: ~~**5**~~ $c_1$
  - Second digit: ~~**4**~~ $c_2$
  - Third digit: ~~3~~ $c_3$

Configuration Sequence $c_1 c_2 c_3$

- Equivalent to checking that the ~~number~~ is ~~543~~
  - $\cancel{\mathbf{5} \times 10 \times 10 + \mathbf{4} \times 10 + \mathbf{3} = 543}$

$c_1$ $g$ $g$ $c_2$ $g$ $c_3$ $c_1 c_2 c_3$

You can't do check TM config sequences without both + and ×!

× by itself is insufficient (it's decidable)

# Gödel's (1st) Incompleteness Theorem

# Completeness

- A theory is **complete** if …
- … every sentence (i.e., true statement) in the language is provable
- For now, we just assume that a proof is some string representing a sequence of steps
  - Analogy: You can think of a sequence of configurations as a kind of "proof" that a machine accepts some string
- <u>Key</u>: A proof can be validated by a decider

# Godel's ($1^{st}$) Incompleteness Theorem

- Any theory that satisfies the following must be **incomplete**:
  - Recognizable
  - Undecidable
  - Has the ability to "prove" true statements

- Proof is by contradiction:
  - If such a theory were complete, then we could create a decider

# Thm: provable statements in $\mathrm{Th}(\mathcal{N}, +, \times)$ is Turing-recognizable

- Recognizer $P$ = On input $\phi$:
  - Check all possible strings …
  - For each, try to validate whether it’s a proof of $\phi$
  - Accept if we find a proof

# Thm: Some true statement in $\mathrm{Th}(\mathcal{N}, +, \times)$ is not provable

- Proof by contradiction: Assume all true statements provable
- Create decider for $\mathrm{Th}(\mathcal{N}, +, \times)$

On input $\phi$:

- Run recognizer $P$ on <u>both</u> $\phi$ *and* $\neg\phi$
- One must be true so $P$ will halt and accept one of them
  - If $P$ halts and accepts $\phi$, then accept
  - If $P$ halts and accepts $\neg\phi$, then reject

# Godel's (1st) Incompleteness Theorem

- (Very Roughly)
  - Any theory that is <u>undecidable but recognizable</u> is incomplete.

- Compare with our previous theorem about recognizability:
  - Decidable ⇔ Turing-recognizable and co-Turing-recognizable
  - So any language that is <u>undecidable but recognizable</u> must be co-Turing-recognizable

# Check-in Quiz 11/3

On gradescope