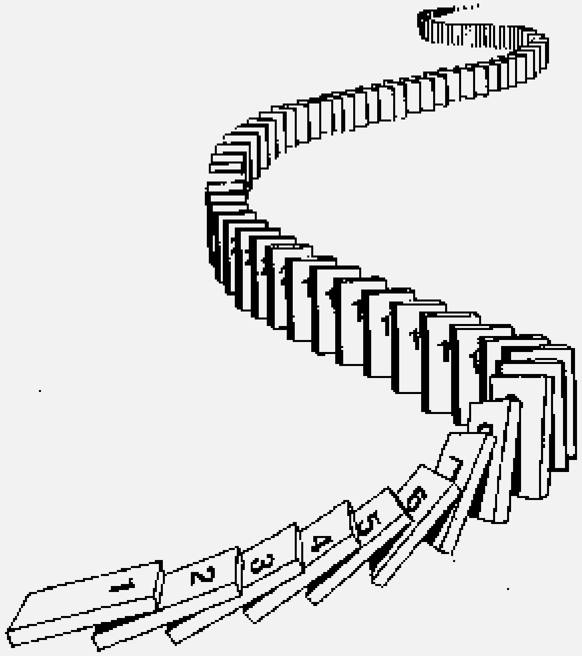
UMB CS 420

Inductive Proofs

Thursday, October 6, 2022



Kinds of Mathematical Proof

- **Deductive proof** (from before)
  - Starting from assumptions and known definitions,
  - Reach conclusion by making logical inferences

- **Inductive proof** (now)
  - …
  - Use this when working with <u>recursive</u> definitions

Proof by Induction

To Prove: a ***Statement*** about a recursively defined "thing" x:

1. Prove: *Statement* for base case of x (usually easy)
2. Prove: *Statement* for recursive case of x:
   - Assume: **induction hypothesis** (IH)

     I.e., *Statement* is true for some x_{smaller}
     - E.g., if x is number, then "smaller" = lesser number

   - Prove: *Statement* for x_{larger} - using IH (and known definitions, theorems …)
     - Usually, must show that going from x_{smaller} to x_{larger} preserves *Statement*

Natural Numbers Are Recursively Defined

A **Natural Number** is:

- $\mathbf{0}$
- Or $\boldsymbol{k+1}$, where k is a Natural Number

Self-reference

But definition is valid because self-reference is “smaller”

So proving things about Natural Numbers requires induction!

Last Time: Proof By Induction Example (Sipser Ch 0)

Prove true: $$P_t = PM^t - Y\left(\frac{M^t - 1}{M - 1}\right)$$

- P_t = loan balance after t months
- t = # months
- P = principal = original amount of loan
- M = interest (multiplier)
- Y = monthly payment

(Details of these variables not too important here)

Last Time: Proof By Induction Example (Sipser Ch 0)

Prove true: $P_t = PM^t - Y\left(\frac{M^t - 1}{M - 1}\right)$

Proof: by **induction** on natural number t

An inductive proof exactly follows the recursive definition (here, natural numbers) that the induction is "on"

Base Case, $t = 0$:

- Goal: **Show** $P_0 = P$
- Proof of Goal:

$$P_0 = PM^0 - Y\left(\frac{M^0 - 1}{M - 1}\right) = P$$

Plug in $t = 0$

Simplify, to get to goal statement

A Natural Number is:

- 0
- Or $k + 1$, where k is a natural number

Last Time: Proof By Induction Example (Sipser Ch 0)

An inductive proof exactly follows the **recursive definition** (here, natural numbers) **that the induction is "on"**

A Natural Number is:

- 0
- $k + 1$, for some nat num k

Prove true: $P_t = PM^t - Y\left(\frac{M^t - 1}{M - 1}\right)$

Inductive Case: $t = k + 1$, for some nat num k

- Inductive Hypothesis (IH), **assume statement true for some** $t =$ (smaller) k

"Connect together" known definitions and statements

$$P_k = PM^k - Y\left(\frac{M^k - 1}{M - 1}\right)$$

- Goal **statement to prove, for** $t = k+1$:

$$P_{k+1} = PM^{k+1} - Y\left(\frac{M^{k+1} - 1}{M - 1}\right)$$

Plug in IH

Simplify, to derive goal statement

- Proof of Goal:

$$P_{k+1} = P_k M - Y = \left[PM^k - Y\left(\frac{M^k - 1}{M - 1}\right)\right] M - Y = PM^{k+1} - Y\left(\frac{M^{k+1} - 1}{M - 1}\right)$$

Definition of P_{k+1}

Proof by Induction: CS 420 Example

Statement to prove: $\boxed{\text{LANGOF}(G) = \text{LANGOF}(\mathbf{GNFA{\rightarrow}RegExpr}(G))}$

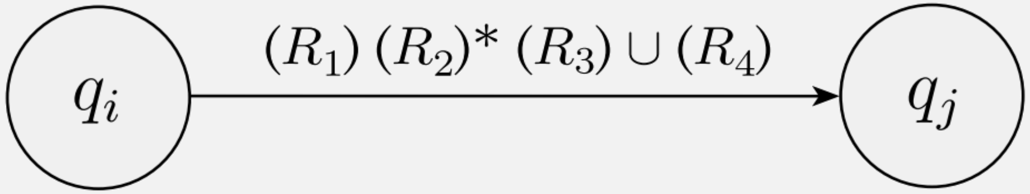
Condition for **GNFA→RegExpr** function to be "correct", i.e., the languages must be equivalent

Last Time: `GNFA→RegExpr` (recursive) function

On GNFA input G:

Base Case

- If G has 2 states, **`return`** the regular expression (from the transition), e.g.:



- Else:

Recursive Case

  - "Rip out" one state
  - "Repair" the machine to get an equivalent GNFA G'
  - Recursively call `GNFA→RegExpr`(G')

Recursive definitions have:
- base case and
- recursive case
(with a "smaller" object)

Proof by Induction: CS 420 Example

Statement to prove: $\boxed{\text{LANGOF}(G) = \text{LANGOF}(\texttt{GNFA→RegExpr}(G))}$

Recursively defined "thing"

Proof: by Induction on # of states in G

☑ 1. Prove *Statement* is true for base case

G has 2 states

$q_i \xrightarrow{R} q_j$

Why is this ok base case?

Statements	Justifications
1. LANGOF ($q_i \xrightarrow{R} q_j$) = LANGOF (R)	1. Definition of GNFA
2. LANGOF (GNFA→RegExpr($q_i \xrightarrow{R} q_j$)) = LANGOF (R)	2. Definition of GNFA→RegExpr
3. LANGOF ($q_i \xrightarrow{R} q_j$) = LANGOF (GNFA→RegExpr($q_i \xrightarrow{R} q_j$))	3. From (1) and (2)

Goal

Don't forget to write out Statements / Justifications !

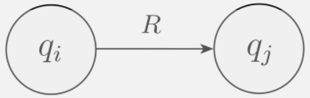
Proof by Induction: CS 420 Example

Statement to prove: LANGOF (G) = LANGOF (**GNFA→RegExpr**(G))

Proof: by Induction on # of states in G

☑ 1. Prove *Statement* is true for base case

G has 2 states

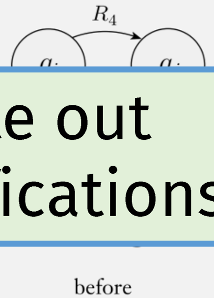


2. Prove *Statement* is true for recursive case:

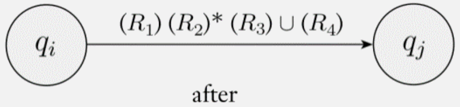
G has > 2 states

- Assume the **induction hypothesis** (IH):
  - *Statement* is true for smaller G'
- Use it to prove *Statement* is true for larger G
  - Show that going from G to G' preserves *Statement*

LANGOF (G')
=
LANGOF (**GNFA→RegExpr**(G'))
(Where G' has less states than G)



Don't forget to write out Statements / Justifications !



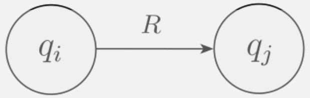
Show that "rip/repair" step converts G to smaller, equivalent G'

Proof by Induction: CS 420 Example

Statement to prove: **LangOf (G) = LangOf (`GNFA→RegExpr`(G))**

Proof: by Induction on # of states in G

☑ 1. Prove *Statement* is true for base case — G has 2 states



☑ 2. Prove *Statement* is true for recursive case: G has > 2 states

- Assume the **induction hypothesis** (IH):
  - *Statement* is true for smaller G'
- Use it to prove *Statement* is true for larger G
  - Show that going from G to G' preserves *Statement*

LangOf (G')
=
LangOf (`GNFA→RegExpr`(G'))
(Where G' has less states than G)

	Statements	**Justifications**
	1. LangOf (G') = LangOf (`GNFA→RegExpr`(G'))	1. IH
	2. LangOf (G) = LangOf ((G'))	2. Correctness of Rip/Repair step (prev)
	3. LangOf (`GNFA→RegExpr`(G)) = LangOf (`GNFA→RegExpr`(G'))	3. Def of `GNFA→RegExpr`
Goal	4. LangOf (G) = LangOf (`GNFA→RegExpr`(G))	4. From (1), (2), and (3)

So Far: How to Prove A Language Is Regular?

- Construct DFA
- Construct NFA
- Create Regular Expression

Slightly different because of recursive definition

R is a ***regular expression*** if R is

1. a for some a in the alphabet Σ,
2. ε,
3. $\emptyset$,
4. $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,
5. $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or
6. (R_1^*), where R_1 is a regular expression.

Proof by Induction

To Prove: a ***Statement*** about a recursively defined "thing" x:

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   - Assume: **induction hypothesis** (IH)

     I.e., *Statement* is true for some x_{smaller}
     - E.g., if x is number, then "smaller" = lesser number
     - E.g., if x is regular expression, then "smaller" = …
   - Prove: *Statement* for x_{larger} - using IH (and known definitions, theorems …)
     - Usually, must show that going from x_{smaller} to x_{larger} preserves *Statement*

1. a for some a in the alphabet Σ,
2. ε,
3. $\emptyset$,
4. $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,
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6. (R_1^*), where R_1 is a regular expression.

Whole reg expr

"smaller"

Thm: Reverse is Closed for Regular Langs

For any string $w = w_1 w_2 \cdots w_n$, the ***reverse*** of w, written $w^{\mathcal{R}}$, is the string w in reverse order, $w_n \cdots w_2 w_1$.

For any language A, let $A^{\mathcal{R}} = \{w^{\mathcal{R}} | \, w \in A\}$

Theorem: if A is regular, so is $A^{\mathcal{R}}$

Proof: by induction on the regular expression of A

Thm: Reverse is Closed for Regular Langs

if A is regular, so is $A^{\mathcal{R}}$

Proof: by Induction on regular expression of A: (6 cases)

Base cases

1. a for some a in the alphabet Σ, same reg. expr. represents $A^{\mathcal{R}}$ so it is regular
2. ε, same reg. expr. represents $A^{\mathcal{R}}$ so it is regular
3. $\emptyset$, same reg. expr. represents $A^{\mathcal{R}}$ so it is regular

Inductive cases

4. $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,
5. $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or
6. (R_1^*), where R_1 is a regular expression.

Need to Prove: if A is a regular language, described by reg expr $R_1 \cup R_2$, then $A^{\mathcal{R}}$ is regular

"smaller"

IH1: if A_1 is a regular language, described by reg expr R_1, then $A_1{}^{\mathcal{R}}$ is regular

IH1: if A_2 is a regular language, described by reg expr R_2, then $A_2{}^{\mathcal{R}}$ is regular

Thm: Reverse is Closed for Regular Langs

if A is regular, so is $A^{\mathcal{R}}$

Proof: by Induction on regular expression of A: (Case # 4)

Statements	Justifications
1. Language A is regular, with reg expr $R_1 \cup R_2$	1. Given
2. R_1 and R_2 are regular expressions	2. Def of Regular Expression
3. R_1 and R_2 describe regular langs A_1 and A_2	3. Reg Expr ⇔ Reg Lang (Prev Thm)
4. If A_1 is a regular language, then $A_1{}^{\mathcal{R}}$ is regular	4. IH
5. If A_2 is a regular language, then $A_2{}^{\mathcal{R}}$ is regular	5. IH
6. $A_1{}^{\mathcal{R}}$ and $A_2{}^{\mathcal{R}}$ are regular	6. By (3), (4), and (5)
7. $A_1{}^{\mathcal{R}} \cup A_2{}^{\mathcal{R}}$ is regular	7. Union Closed for Reg Langs
8. $A_1{}^{\mathcal{R}} \cup A_2{}^{\mathcal{R}} = (A_1 \cup A_2)^{\mathcal{R}}$	8. Reverse and Union Ops Commute
9. $A = A_1 \cup A_2$	9. By (1), (2), and (3)
Goal 10. $A^{\mathcal{R}}$ is regular	10. By (7), (8), (9)

Thm: Reverse is Closed for Regular Langs

if A is regular, so is $A^{\mathcal{R}}$

Proof: by Induction on regular expression of A: (6 cases)

Base cases

☑ **1.** a for some a in the alphabet Σ,

☑ **2.** ε,

☑ **3.** $\emptyset$,

Inductive cases

☑ **4.** $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,

5. $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or

6. (R_1^*), where R_1 is a regular expression.

Remaining cases will use similar reasoning

In-Class quiz 10/6

See gradescope