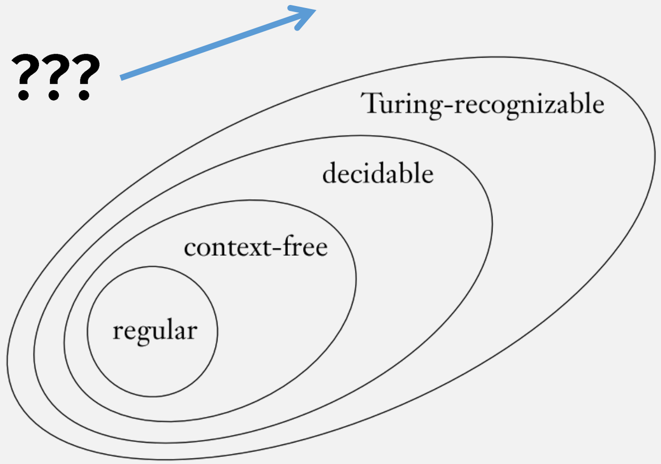
UMB CS 420

Unrecognizability

Wednesday, April 6, 2022



Announcements

- HW 8 in
  - ~~Due Wed 4/6 11:59pm EST~~

- HW 9 out
  - Due Sun 4/17 11:59pm EST

Last Time: Showing Mapping Reducibility

Step 1:
Show there is computable fn f … by creating a TM

Language A is ***mapping reducible*** to language B, written $A \leq_{\mathrm{m}} B$, if there is a $\boxed{\text{computable function } f\colon \Sigma^* \longrightarrow \Sigma^*}$, where for every w,

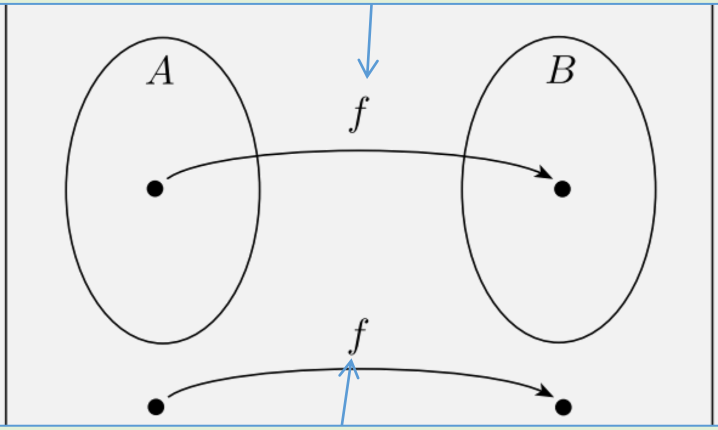
$$\boxed{w \in A \iff f(w) \in B.}$$

"if and only if"

Step 2:
Prove the iff is true for f

The function f is called the ***reduction*** from A to B.

Step 2a: "forward" direction ($\Rightarrow$): if $w \in A$ then $f(w) \in B$



Step 2b: "reverse" direction ($\Leftarrow$): if $f(w) \in B$ then $w \in A$

Step 2b: Equivalent (contrapositive): if $w \notin A$ then $f(w) \notin B$

A function $f\colon \Sigma^* \longrightarrow \Sigma^*$ is a ***computable function*** if some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

Last Time: Using Mapping Reducibility

To prove decidability …

- If $A \leq_m B$ and B is decidable, then A is decidable.

Known

Unknown
(want to prove)

To prove undecidability …

- If $A \leq_m B$ and A is undecidable, then B is undecidable.

Be careful with the **direction of the reduction!**

Flashback: EQ_{TM} is undecidable

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Proof by contradiction:

- Assume EQ_{TM} has *decider* R; use to create E_{TM} *decider*:

$= \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

$S =$ "On input $\langle M\rangle$, where M is a TM:

1. Run R on input $\langle M, M_1\rangle$, where M_1 is a TM that rejects all inputs.
2. If R accepts, *accept*; if R rejects, *reject*."

Alternate Proof: EQ_{TM} is undecidable

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Show **mapping reducibility:** $E_{\mathsf{TM}} \leq_{\mathrm{m}} EQ_{\mathsf{TM}}$

Step 1: create computable fn f, computed by TM S

$S =$ "On input $\langle M\rangle$, where M is a TM:

1. Construct: $\langle M, M_1\rangle$, where M_1 is a TM that rejects all inputs.
2. Output: $\langle M, M_1\rangle$

Step 2: show iff requirements of mapping reducibility

☑ $\Rightarrow$ If $<M> \in E_{\mathsf{TM}}$, then $<M, M_1> \in EQ_{\mathsf{TM}}$

☑ $\Leftarrow$ If $<M> \notin E_{\mathsf{TM}}$, then $<M, M_1> \notin EQ_{\mathsf{TM}}$

And use theorem …

If $A \leq_{\mathrm{m}} B$ and A is undecidable, then B is undecidable.

Flashback: E_{TM} is undecidable

$E_{\mathsf{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

Proof, by contradiction:

- Assume E_{TM} has *decider* R; use to create A_{TM} *decider*:

$S =$ "On input $\langle M, w\rangle$, an encoding of a TM M and a string w:

1. Use the description of M and w to construct the TM M_1

> $M_1 =$ "On input x:
> **1.** If $x \neq w$, *reject*.
> **2.** If $x = w$, run M on input w and *accept* if M does."

2. Run R on input $\langle M_1\rangle$.

3. If R accepts, *reject*; if R rejects, *accept*."

If M accepts w, M_1 not in E_{TM}!

Alternate Proof: E_{TM} is undecidable

$E_{\mathsf{TM}} = \{\langle M \rangle |\ M \text{ is a TM and } L(M) = \emptyset\}$

Show **mapping reducibility??:** $A_{\mathsf{TM}} \leq_{\mathrm{m}} E_{\mathsf{TM}}$

Step 1: create computable fn f: $<M, w> \rightarrow <M_1>$, computed by S

S = "On input $\langle M, w\rangle$, an encoding of a TM M and a string w:

1. Use the description of M and w to construct the TM M_1

   M_1 = "On input x:
   1. If $x \neq w$, *reject*.
   2. If $x = w$, run M on input w and *accept* if M does."

2. Output: $\langle M_1 \rangle$.
3. ~~If R accepts, *reject*; if R rejects, *accept*.~~"

Step 2: show iff requirements of mapping reducibility:

? $\Rightarrow$ If $<M, w> \in A_{\mathsf{TM}}$, then $<M_1> \boxed{\notin} E_{\mathsf{TM}}$

? $\Leftarrow$ If $<M, w> \notin A_{\mathsf{TM}}$, then $<M_1> \boxed{\in} E_{\mathsf{TM}}$

- This reduces A_{TM} to $\overline{E_{\mathsf{TM}}}$!!
- It's good enough, if: **undecidable langs are <u>closed</u> under complement**

Undecidable Langs Closed under Complement

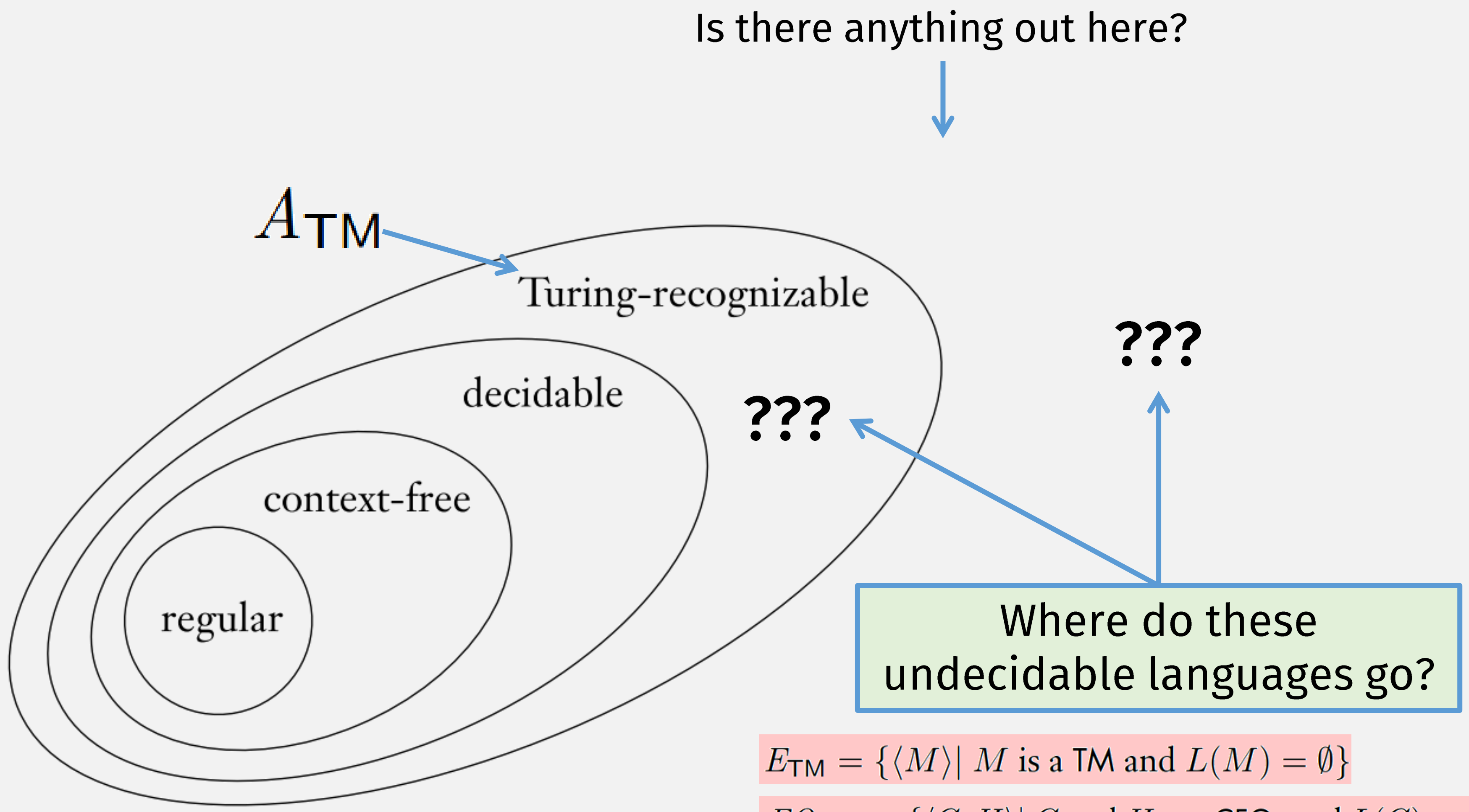
Proof by contradiction

- Assume some lang L is undecidable and $\overline{L}$ is decidable …
  - Then $\overline{L}$ has a decider

Contradiction!

- … then we can create decider for L from decider for $\overline{L}$ …
  - Because **decidable languages are closed under complement** (hw8)**!**

Turing Unrecognizable?



$E_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

$EQ_{\text{CFG}} = \{\langle G, H \rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$

$EQ_{\text{TM}} = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Thm: Some langs are not Turing-recognizable

Proof: requires 2 lemmas

- Lemma 1: The **set of all languages** is *uncountable*
  - Proof: Show there is a bijection with another uncountable set …
    - … The set of all infinite binary sequences

- Lemma 2: The **set of all TMs** is *countable*

- Therefore, **some language is not recognized by a TM**
  (pigeonhole principle)

Mapping a Language to a Binary Sequence

All Possible Strings

Some Language (subset of above)

Its (unique) Binary Sequence

$$\begin{array}{rcccccccccccc}
\Sigma^* & = \{ & \varepsilon, & 0, & 1, & 00, & 01, & 10, & 11, & 000, & 001, & \cdots & \} \\
A & = \{ & & 0, & & 00, & 01, & & & 000, & 001, & \cdots & \} \\
\chi_A & = & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & \cdots &
\end{array}$$

Each digit represents one possible string:
- 1 if lang has that string,
- 0 otherwise

Thm: Some langs are not Turing-recognizable

This is an "existence" proof, but it's not "constructive", i.e., it doesn't give an example of an unrecognizable language

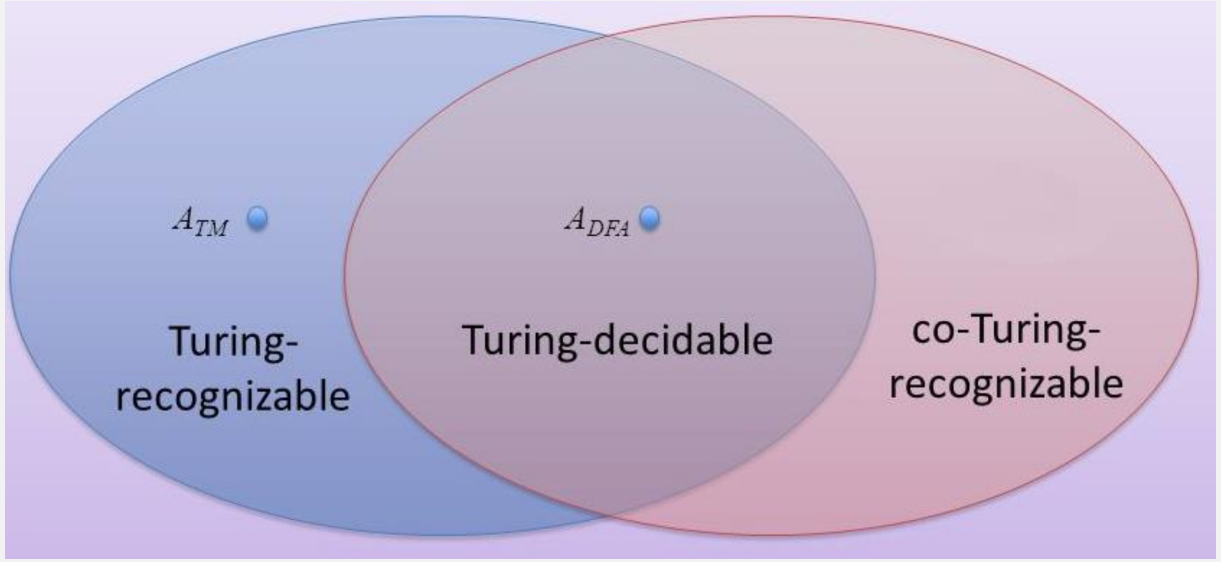
Proof: requires 2 lemmas

- Lemma 1: The **set of all languages** is *uncountable*
  - Proof: Show there is a bijection with another uncountable set …
    - … The set of all infinite binary sequences
    - ➢ Now just prove set of infinite binary sequences is uncountable (exercise)
- Lemma 2: The **set of all TMs** is *countable*
  - Because every TM M can be encoded as a string $<M>$
  - And set of all strings is countable
- Therefore, some language is not recognized by a TM ■

Co-Turing-Recognizability

- A language is **co-Turing-recognizable** if …
- … it is the complement of a Turing-recognizable language.

Thm: Decidable ⇔ Recognizable & co-Recognizable



Thm: Decidable ⇔ Recognizable & co-Recognizable

⇒ If a language is **decidable**, then it is **recognizable** and **co-recognizable**

- Decidable => Recognizable:
  - A decider is a recognizer, bc decidable langs are a subset of recognizable langs
- Decidable => Co-Recognizable:
  - To create co-decider from a decider … switch reject/accept of all inputs
  - A co-decider is a co-recognizer, for same reason as above

⇐ If a language is **recognizable** and **co-recognizable**, then it is **decidable**

Thm: Decidable ⇔ Recognizable & co-Recognizable

⇒ If a language is **decidable**, then it is **recognizable** and **co-recognizable**

- Decidable => Recognizable:
  - A decider is a recognizer, bc decidable langs are a subset of recognizable langs
- Decidable => Co-Recognizable:
  - To create co-decider from a decider … switch reject/accept of all inputs
  - A co-decider is a co-recognizer, for same reason as above

⇐ If a language is **recognizable** and **co-recognizable**, then it is **decidable**

- Let M_1 = recognizer for the language,
- and M_2 = recognizer for its complement
- Decider M:
  - Run 1 step on M_1,
  - Run 1 step on M_2,
  - Repeat, until one machine accepts. If it's M_1, accept. If it's M_2, reject

Termination Arg: Either M_1 or M_2 must accept and halt, so M halts and is a decider

A Turing-unrecognizable language

- We've proved:

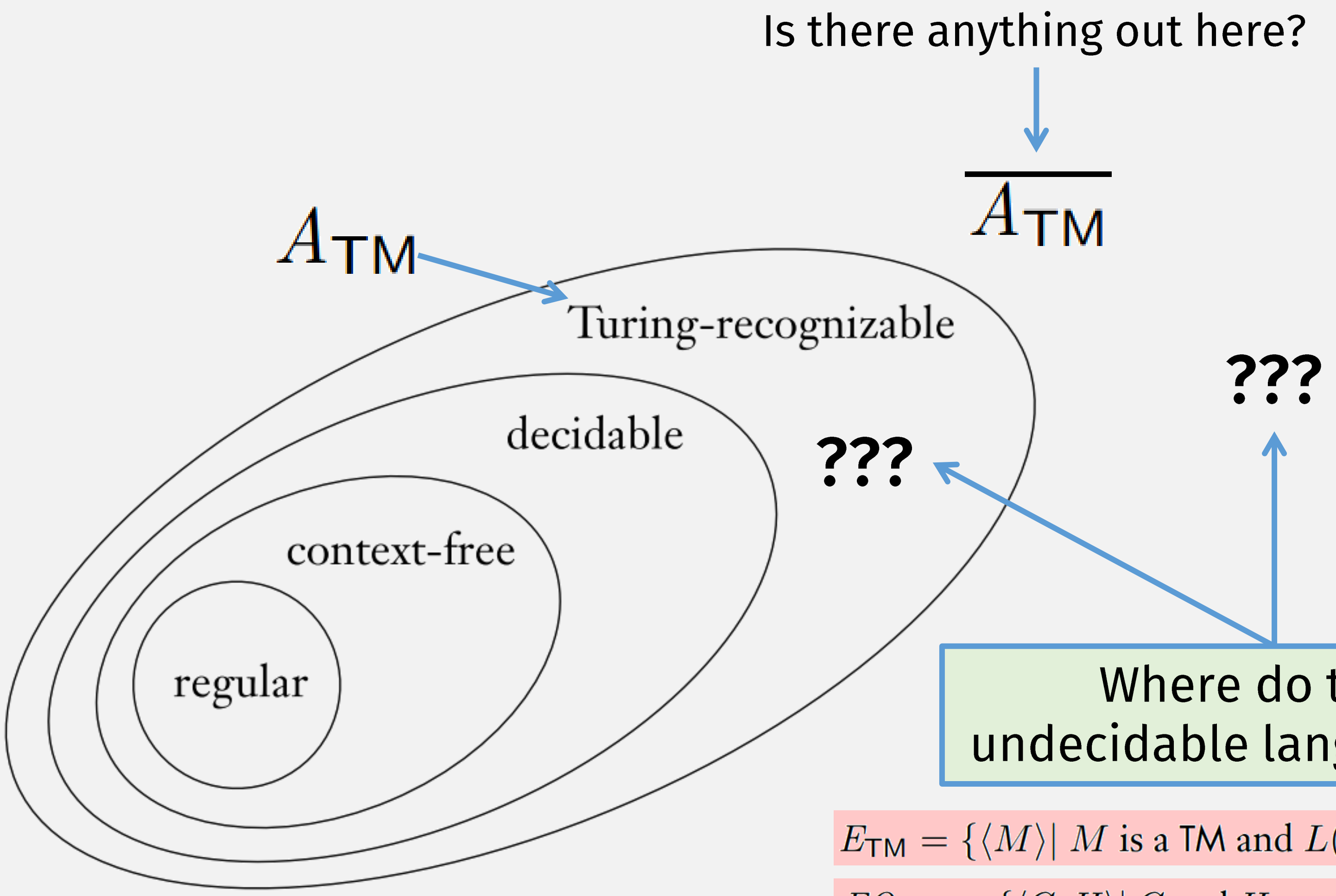
  $\boxed{A_{\mathsf{TM}} \text{ is Turing-recognizable}}$

  A_{TM} is undecidable

- So:

  $\overline{A_{\mathsf{TM}}}$ is not Turing-recognizable

- Because: $\boxed{\text{recognizable \& co-recognizable implies decidable}}$



Where do these undecidable languages go?

$E_{\mathsf{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

$EQ_{\mathsf{CFG}} = \{\langle G, H \rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Using Mapping Reducibility to Prove …

- Decidability
- Undecidability
- Recognizability
- Unrecognizability

More Helpful Theorems

If $A \leq_m B$ and B is $\boxed{\text{Turing-recognizable,}}$ then A is Turing-recognizable.

If $A \leq_m B$ and A is not Turing-recognizable, then B is not Turing-recognizable.

- Same proofs as:

If $A \leq_m B$ and B is decidable, then A is decidable.

If $A \leq_m B$ and A is undecidable, then B is undecidable.

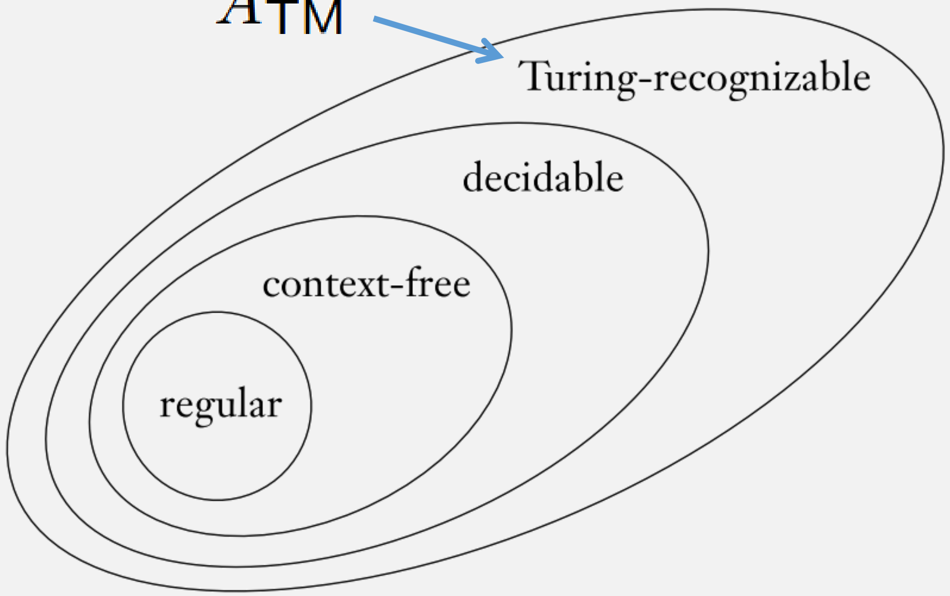
Thm: EQ_{TM} is neither Turing-recognizable nor co-Turing-recognizable.

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1$ and M_2 are TMs and $L(M_1) = L(M_2)\}$

1. EQ_{TM} is not Turing-recognizable

$\overline{A_{\mathsf{TM}}}$

A_{TM}



$\overline{A_{\mathsf{TM}}} \leq_{\mathrm{m}} EQ_{\mathsf{TM}}$ A is not Turing-recognizable, th EQ_{TM} not Turing-recognizable.

Mapping Reducibility implies Mapping Red. of Complements

Language A is ***mapping reducible*** to language B, written $A \leq_m B$, if there is a computable function $f: \Sigma^* \longrightarrow \Sigma^*$, where for every w,

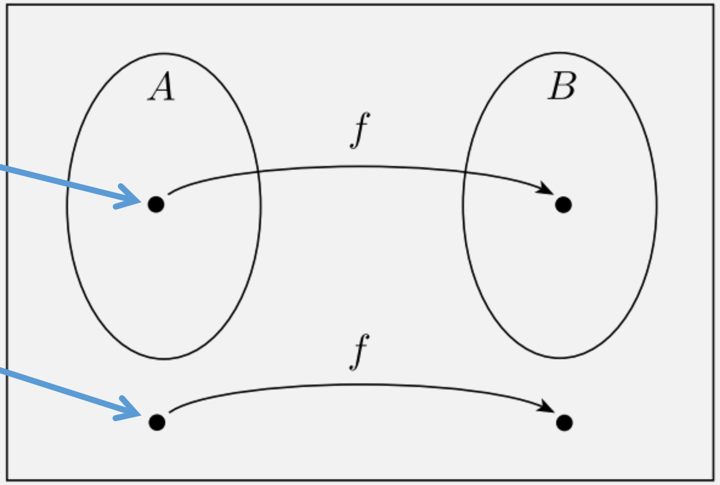
$$\boxed{w \in A \Longleftrightarrow f(w) \in B.}$$

The function f is called the ***reduction*** from A to B.

$\boxed{A \leq_m B}$

implies

$\boxed{\overline{A} \leq_m \overline{B}}$



Thm: EQ_{TM} is neither Turing-recognizable nor co-Turing-recognizable.

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle |\ M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

1. EQ_{TM} is not Turing-recognizable

Two Choices:

- Create Computable fn: $\overline{A_{\mathsf{TM}}} \rightarrow EQ_{\mathsf{TM}}$
- Or Computable fn: $\boxed{A_{\mathsf{TM}} \rightarrow \overline{EQ_{\mathsf{TM}}}}$

Thm: EQ_{TM} is not Turing-recognizable

$EQ_{\text{TM}} = \{\langle M_1, M_2\rangle |\ M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

- Create Computable fn: $A_{\text{TM}} \rightarrow \overline{EQ_{\text{TM}}}$

Step 1

- $\langle M, w\rangle \rightarrow \langle M_1, M_2\rangle$ M_1 and M_2 are TMs and $L(M_1) \neq L(M_2)$

$F =$ "On input $\langle M, w\rangle$, where M is a TM and w a string:

1. Construct the following two machines, M_1 and M_2.

$M_1 =$ "On any input:

1. *Reject.*"

Accepts nothing

$M_2 =$ "On any input:

1. Run M on w. If it accepts, *accept.*"

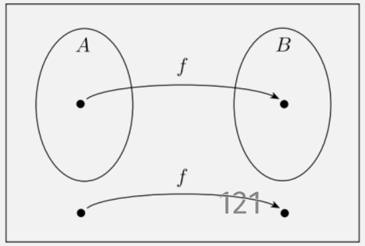
Accepts nothing or everything

2. Output $\langle M_1, M_2\rangle$."

Step 2:

$\Rightarrow$ If M **accepts** w, then $M_1 \neq M_2$

$\Leftarrow$ If M **does not accept** w, then $M_1 = M_2$



Thm: EQ_{TM} is neither Turing-recognizable nor co-Turing-recognizable.

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2 \rangle | \ M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

1. EQ_{TM} is not Turing-recognizable
   - Create Computable fn: $\overline{A_{\mathsf{TM}}} \rightarrow EQ_{\mathsf{TM}}$
   - Or Computable fn: $A_{\mathsf{TM}} \rightarrow \overline{EQ_{\mathsf{TM}}}$
   - **DONE!**

2. $\overline{EQ_{\mathsf{TM}}}$ is not ~~co~~-Turing-recognizable
   - (A lang is co-Turing-recog. if it is complement of Turing-recog. lang)

Previous: EQ_{TM} is not Turing-recognizable

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

- Create Computable fn: $A_{\mathsf{TM}} \rightarrow \overline{EQ_{\mathsf{TM}}}$

Step 1
- $\langle M, w\rangle \rightarrow \langle M_1, M_2\rangle$ M_1 and M_2 are TMs and $L(M_1) \neq L(M_2)$

$F =$ "On input $\langle M, w\rangle$, where M is a TM and w a string:

1. Construct the following two machines, M_1 and M_2.

$M_1 =$ "On any input: ← Accepts nothing

1. *Reject.*"

$M_2 =$ "On any input: ← Accepts nothing or everything

1. Run M on w. If it accepts, *accept*."

2. Output $\langle M_1, M_2\rangle$."

Now: $\overline{EQ}_{\text{TM}}$ is not Turing-recognizable

$EQ_{\text{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

- Create Computable fn: $A_{\text{TM}} \rightarrow \overline{EQ}_{\text{TM}}$

Step 1
- $\langle M, w\rangle \rightarrow \langle M_1, M_2\rangle$ M_1 and M_2 are TMs and $L(M_1) \neq L(M_2)$

F = "On input $\langle M, w\rangle$, where M is a TM and w a string:

1. Construct the following two machines, M_1 and M_2.

M_1 = "On any input: ← Accepts ~~nothing~~ everything

1. *Accept*."

M_2 = "On any input: ← Accepts nothing or everything

1. Run M on w. If it accepts, *accept*."

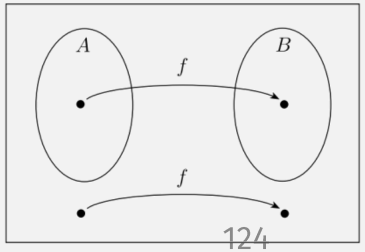
2. Output $\langle M_1, M_2\rangle$."

Step 2:

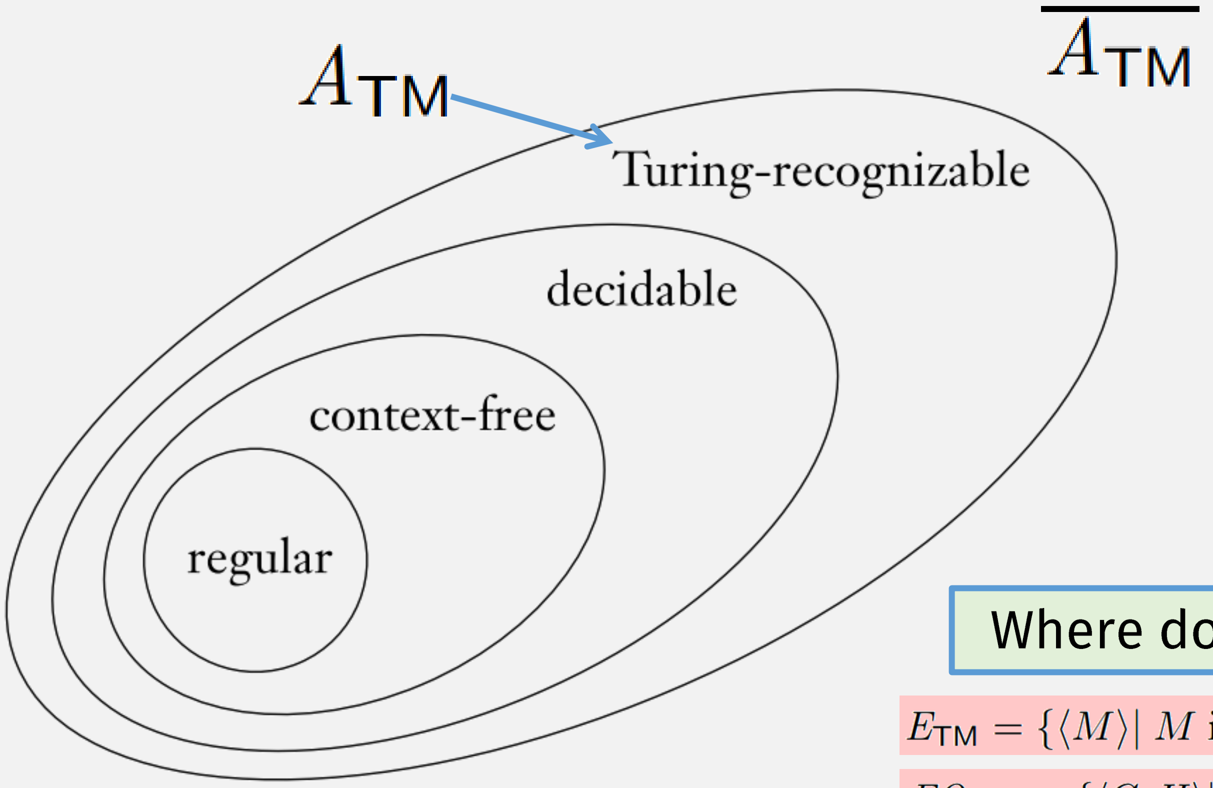
⇒ If M accepts w, then $M_1 \boxed{=} M_2$

⇐ If M does not accept w, then $M_1 \boxed{\neq} M_2$

DONE!



Unrecognizable Languages?



Where do these go?

$E_{\mathsf{TM}} = \{\langle M \rangle |\ M \text{ is a TM and } L(M) = \emptyset\}$

$EQ_{\mathsf{CFG}} = \{\langle G, H \rangle |\ G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2 \rangle |\ M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

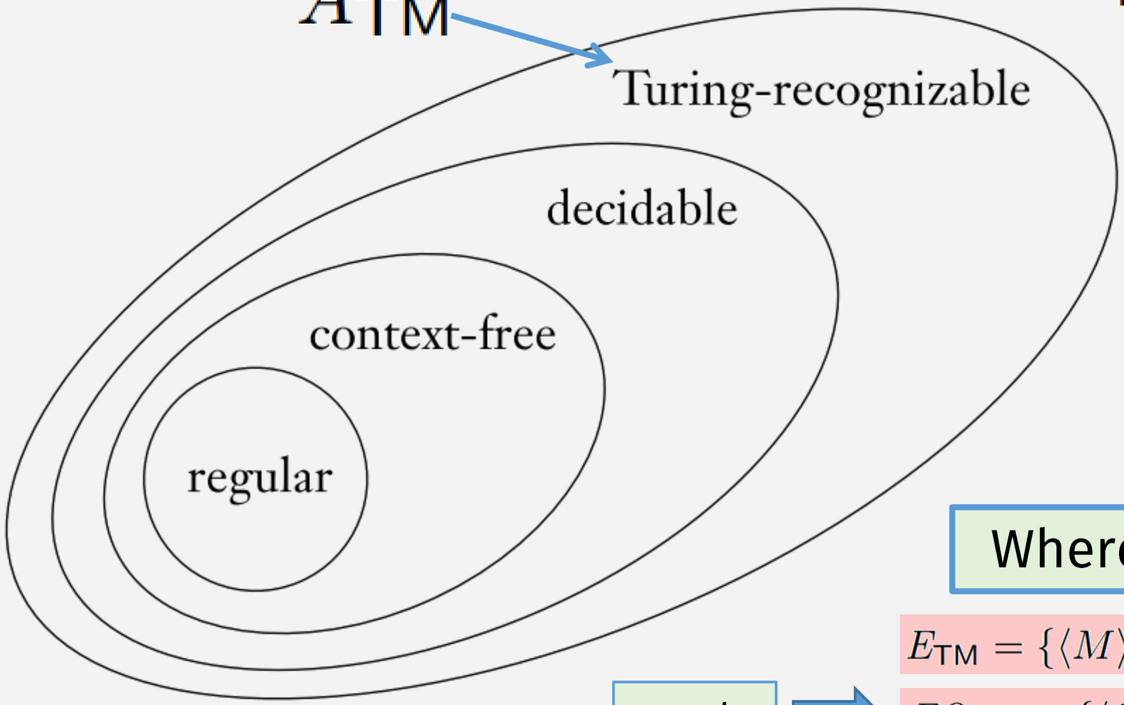
Unrecognizable Languages

A_{TM}

$\overline{A_{\mathsf{TM}}}$

EQ_{TM}

$\overline{EQ_{\mathsf{TM}}}$



Where do these go?

$E_{\mathsf{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

next

$EQ_{\mathsf{CFG}} = \{\langle G, H \rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$

Thm: EQ_{CFG} is not Turing-recognizable

Recognizable & co-recognizable implies decidable

- We've proved:
  EQ_{CFG} is undecidable
- We now prove:
  EQ_{CFG} is co-Turing recognizable
- And conclude that:
  - EQ_{CFG} is not Turing recognizable

Thm: EQ_{CFG} is co-Turing-recognizable

$EQ_{\text{CFG}} = \{\langle G, H\rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$

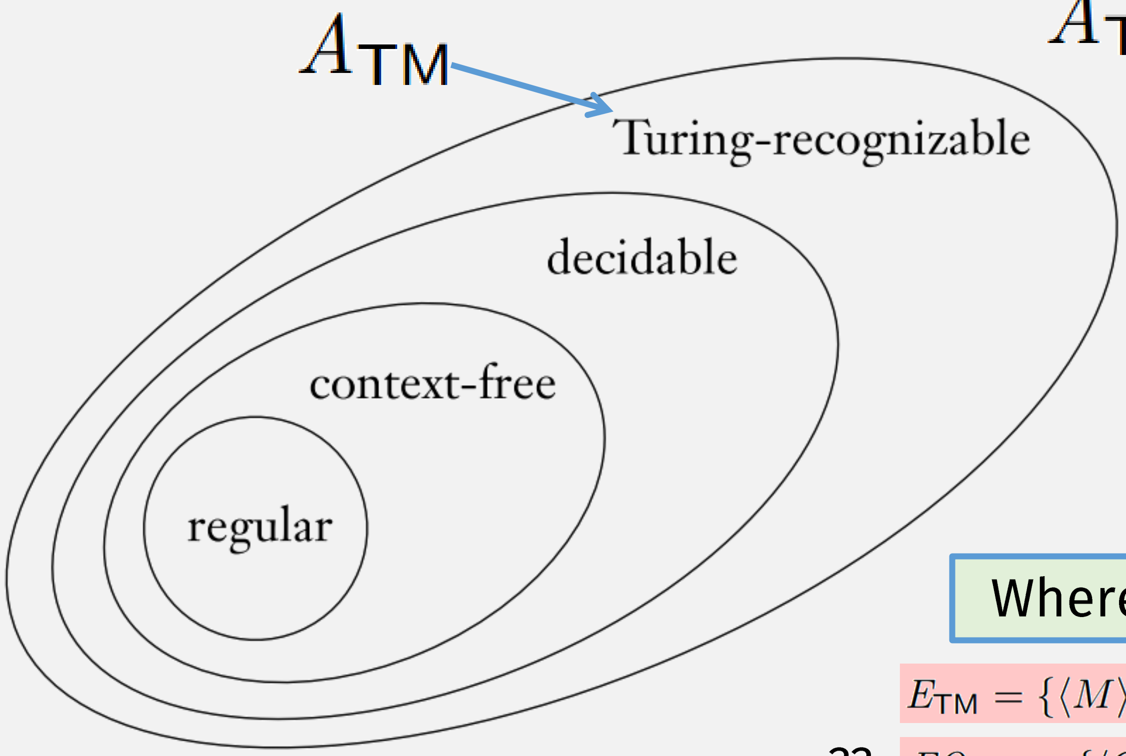
Recognizer for $\overline{EQ}_{\text{CFG}}$:

- On input <G, H>:
  - For every possible string w:
    - Accept if $w \in L(G)$ and $w \notin L(H)$
    - Or accept if $w \in L(H)$ and $w \notin L(G)$
  - Else reject

$A_{\text{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates string } w\}$

This is only a **recognizer** because it loops for ever when $L(G) = L(H)$

Unrecognizable Languages



$\overline{A_{\mathsf{TM}}}$

EQ_{TM}

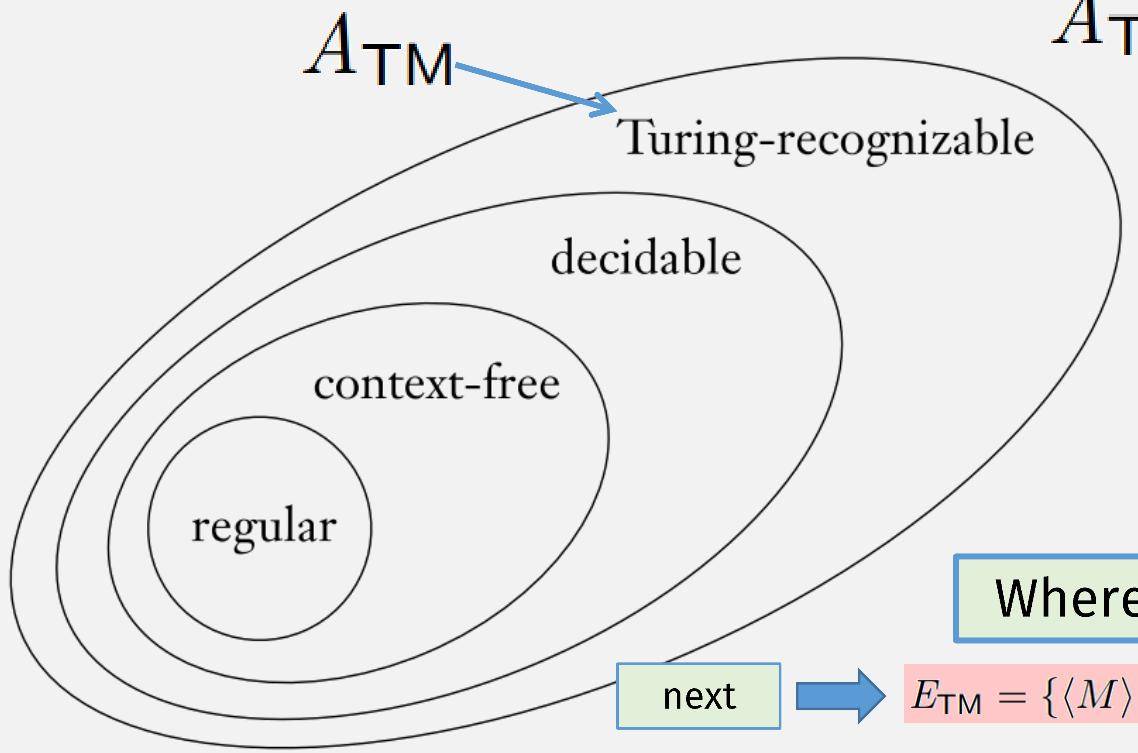
$\overline{EQ_{\mathsf{TM}}}$

Where do these go?

$E_{\mathsf{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

?? $EQ_{\mathsf{CFG}} = \{\langle G, H \rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$

Unrecognizable Languages



$\overline{A_{\mathsf{TM}}}$

EQ_{TM}

$\overline{EQ_{\mathsf{TM}}}$

EQ_{CFG}

Where do these go?

next

$E_{\mathsf{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

Thm: E_{TM} is not Turing-recognizable

Recognizable & co-recognizable implies decidable

- We've proved:
  - E_{TM} is undecidable
- We now prove:
  E_{TM} is co-Turing recognizable
- And then conclude that:
  - E_{TM} is not Turing recognizable

<u>Thm</u>: E_{TM} is co-Turing-recognizable

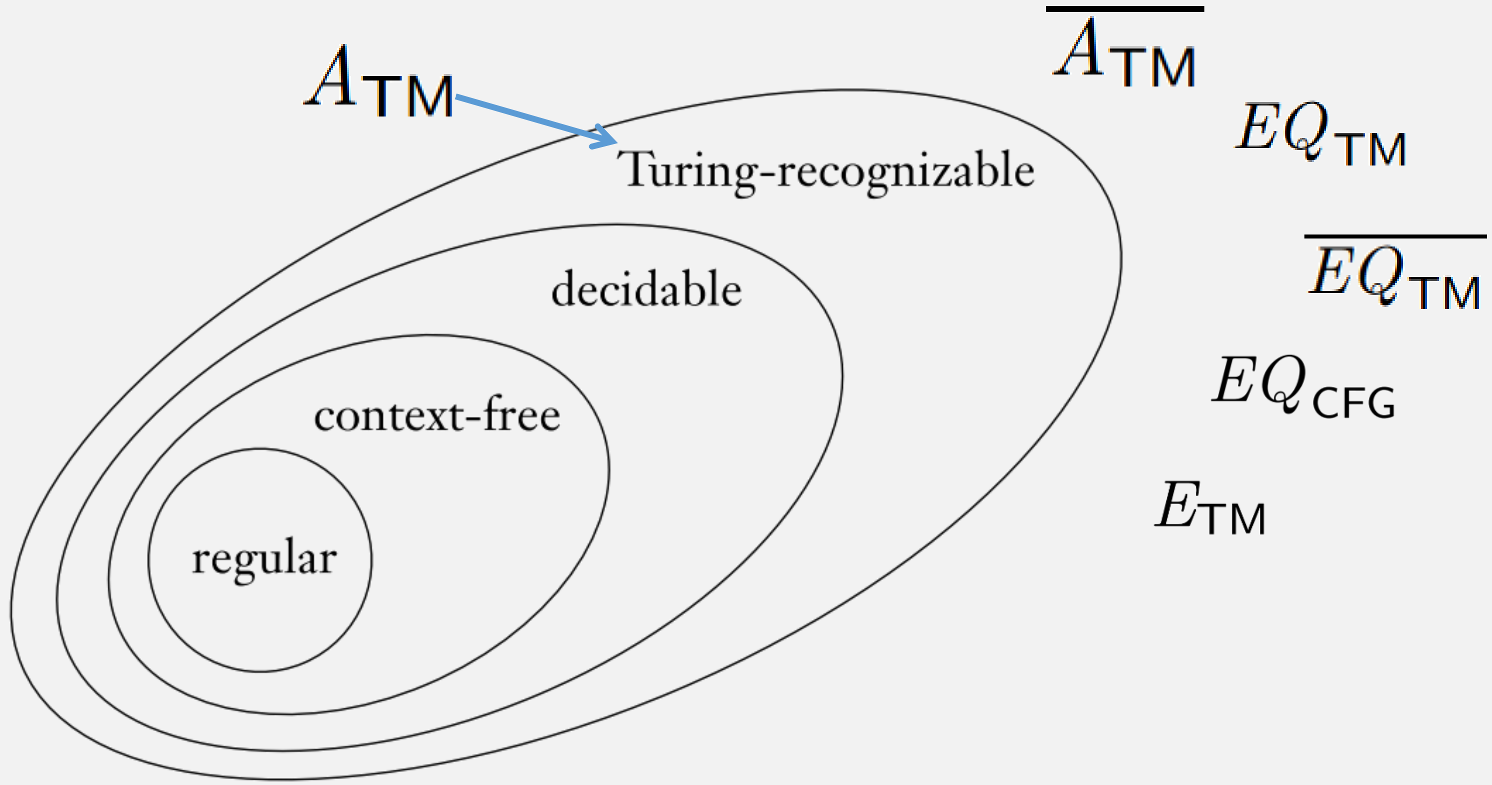
$E_{\mathsf{TM}} = \{\langle M \rangle |\ M \text{ is a TM and } L(M) = \emptyset\}$

Recognizer for $\overline{E_{\mathsf{TM}}}$: Let $s_1, s_2, \ldots$ be a list of all strings in Σ^*

"On input $\langle M \rangle$, where M is a TM:
1. Repeat the following for $i = 1, 2, 3, \ldots$.
2. Run M for i steps on each input, $s_1, s_2, \ldots, s_i$.
3. If M has accepted any of these, *accept*. Otherwise, continue."

This is only a **recognizer** because it loops for ever when $L(M)$ is empty

Unrecognizable Languages



Check-in Quiz 4/6

On gradescope