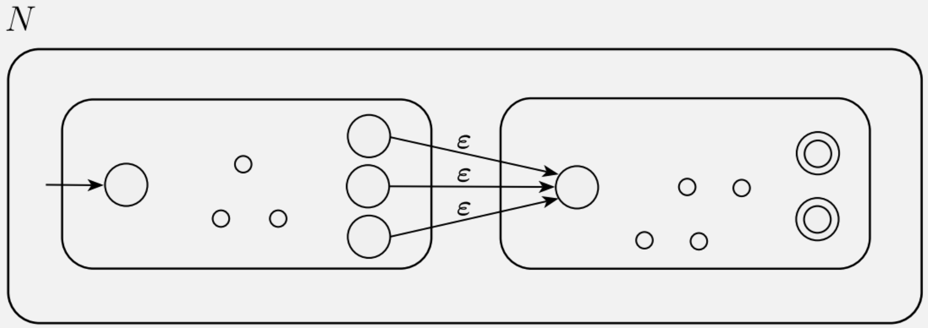
CS420

Combining Automata & Closed Operations

Thursday, September 20, 2022

UMass Boston Computer Science



Announcements

- HW 1
  - Due Sun 9/25 11:59pm EST

Last Time: Is Union Closed For Regular Langs?

In this course, **we are interested in closed operations for a** <u>**set of languages**</u> (here the **set of regular languages**)

(*In general, a* **set** is **closed** under an operation if applying the **operation** to **members of the set** produces a **result in the same set**)

Want to prove this statement

The class of regular languages is closed under the union operation.

In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

A **member of the set of regular languages** is …

… **a regular language,** which itself is a set (of strings) …

… so the **operations** we're interested in are **set operations**

Union: $A \cup B = \{x | \, x \in A \text{ or } x \in B\}$

Last Time: Union of Languages

Let the alphabet Σ be the standard 26 letters $\{\mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}\}$.

If $A = \{\mathtt{good}, \mathtt{bad}\}$ and $B = \{\mathtt{boy}, \mathtt{girl}\}$, then

$$A \cup B = \{\mathtt{good}, \mathtt{bad}, \mathtt{boy}, \mathtt{girl}\}$$

Last Time: Is Union Closed For Regular Langs?

THEOREM

The class of regular languages is closed under the union operation.

In other words, if A_1 and A_2 are regular languages, so is $\boxed{A_1 \cup A_2}$.

Want to prove this statement

Need to show this language is regular (where A_1 and A_2 are regular)

Given

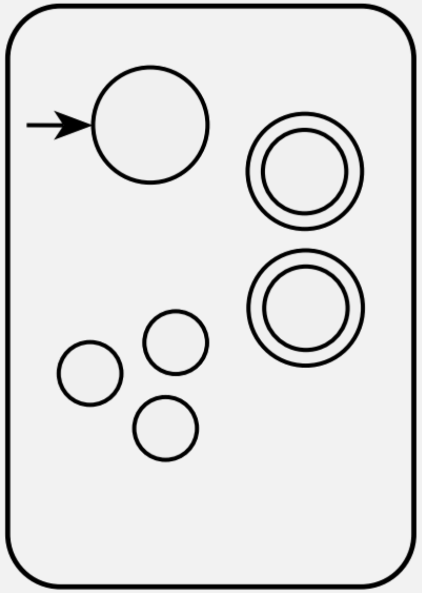
A language is called a ***regular language*** if some finite automaton recognizes it.

- How do we prove that a language is regular?
  - Create a DFA recognizing it!
- So to <u>prove</u> this theorem … <u>create</u> a DFA that recognizes $A_1 \cup A_2$
  - But! We <u>don't know</u> what A_1 and A_2 are!
  - What <u>do</u> we know about A_1 and A_2???

Union

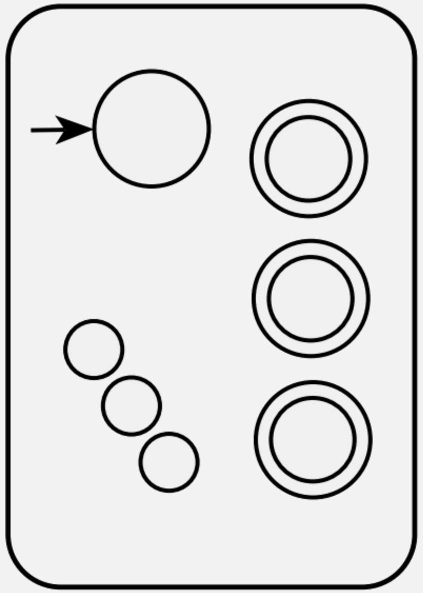
M_1

recognizes A_1



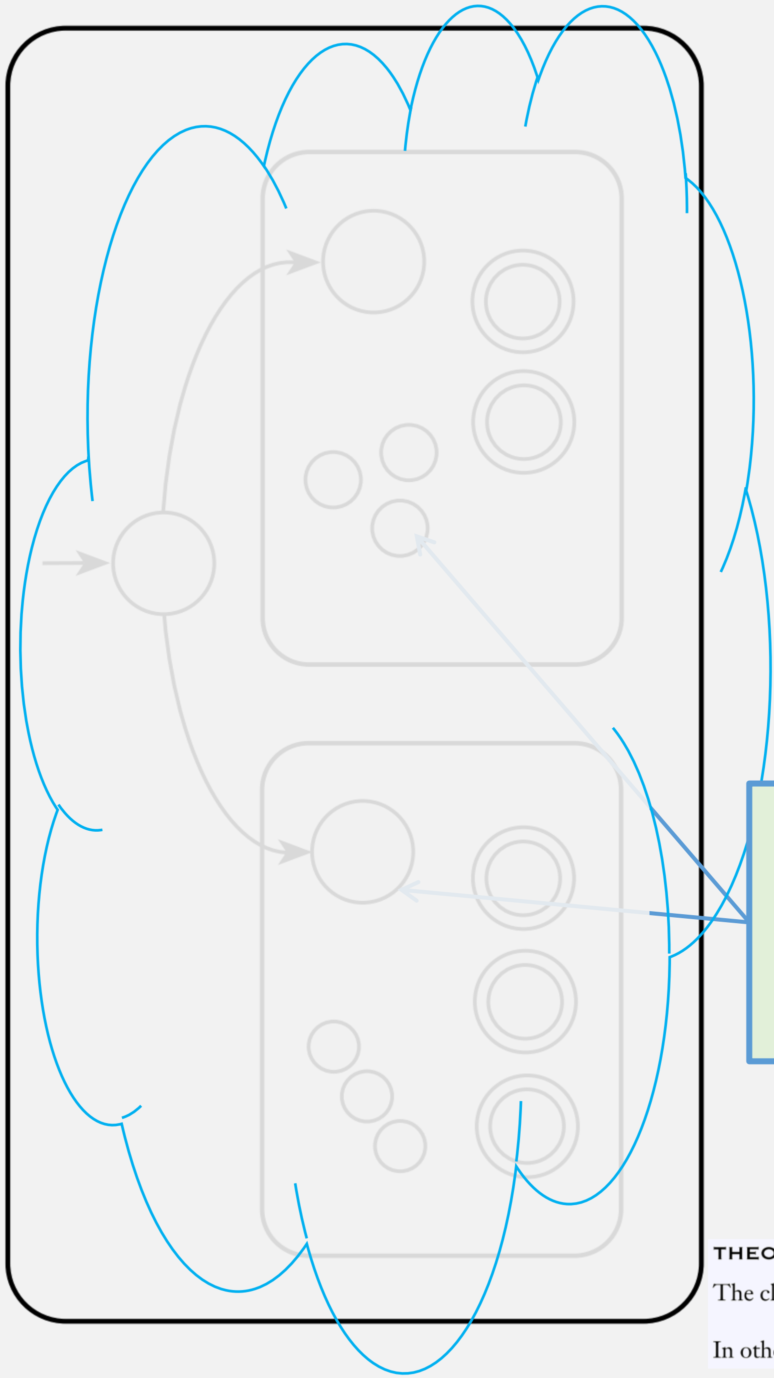
M_2

recognizes A_2



Want: M

Recognizes $A_1 \cup A_2$



Rough sketch Idea:
M is a combination of M_1 and M_2 that checks whether its input is accepted by either M_1 and M_2

But, a DFA can only read its input once!

Need to somehow simulate "being in" both an M_1 and M_2 state simultaneously

THEOREM

The class of regular languages is closed under the union operation.

In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

Union is Closed For Regular Languages

Proof

- Given: $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, recognize A_1,
  $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$, recognize A_2,

Want: M that can simultaneously be in both an M_1 and M_2 state

- Construct: $M = (Q, \Sigma, \delta, q_0, F)$, using M_1 and M_2, that recognizes $A_1 \cup A_2$
- states of M: $Q = \{(r_1, r_2) \mid r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$ $= Q_1 \times Q_2$
  This set is the ***Cartesian product*** of sets Q_1 and Q_2

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
2. Σ is a finite set called the ***alphabet***,
3. $\delta: Q \times \Sigma \longrightarrow Q$ is the ***transition function***,[1]
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.

A state of M is a pair:
- the first part is a state of M_1 and
- the second part is a state of M_2

So the states of M is all possible combinations of the states of M_1 and M_2

Union is Closed For Regular Languages

<u>*Proof*</u>

- Given: $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, recognize A_1,
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$a) = (\delta_1(r_1, a), \delta_2(r_2, a))$

A step in M is both
a step in M_1, and a step in M_2

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Union is Closed For Regular Languages

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  This set is the ***Cartesian product*** of sets Q_1 and Q_2
- M transition fn: $\delta\big((r_1, r_2), a\big) = \big(\delta_1(r_1, a), \delta_2(r_2, a)\big)$
- M start state: (q_1, q_2)

Start state of M is both
start states of M_1 and M_2

Union is Closed For Regular Languages

__*Proof*__

- Given: $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, recognize A_1,
  $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$, recognize A_2,
- Construct: $M = (Q, \Sigma, \delta, q_0, F)$, using M_1 and M_2, that recognizes $A_1 \cup A_2$
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- M start state: (q_1, q_2)

Accept if <u>either</u> M_1 or M_2 accept

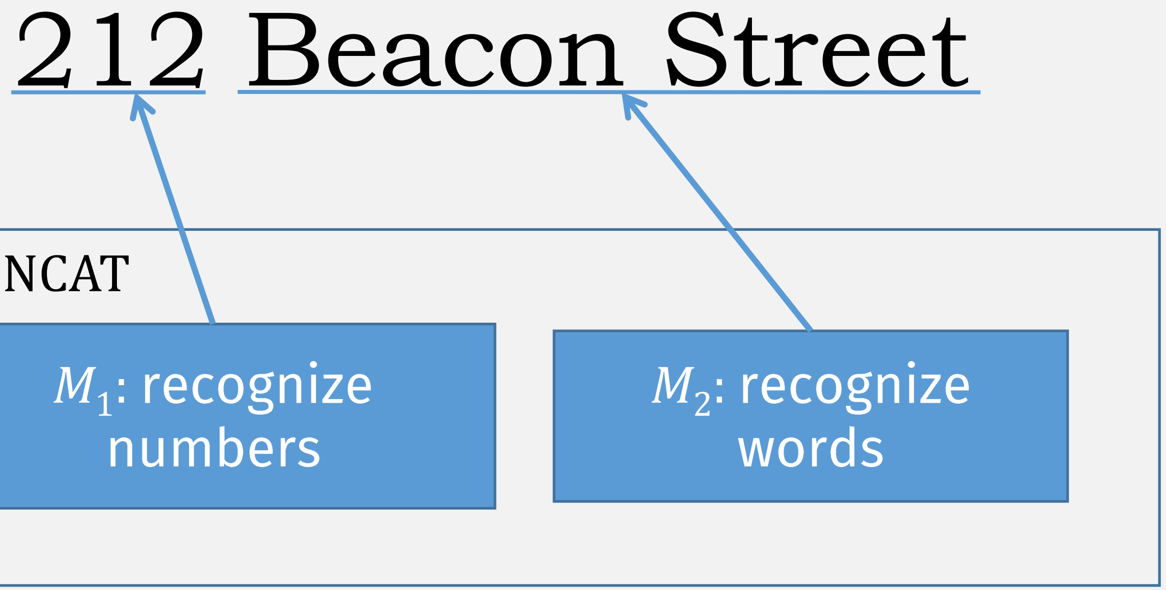
Remember:
Accept states must be subset of Q

- M accept states: $F = \{(r_1, r_2) \mid r_1 \in F_1 \text{ or } r_2 \in F_2\}$

(Q.E.D.) ■

Another operation: Concatenation

Example: **Recognizing street addresses**



Concatenation: $A \circ B = \{xy | \ x \in A \text{ and } y \in B\}$

Concatenation of Languages

Let the alphabet Σ be the standard 26 letters $\{\mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}\}$.

If $A = \{\mathtt{good}, \mathtt{bad}\}$ and $B = \{\mathtt{boy}, \mathtt{girl}\}$, then

$$A \circ B = \{\mathtt{goodboy}, \mathtt{goodgirl}, \mathtt{badboy}, \mathtt{badgirl}\}$$

Is Concatenation Closed?

THEOREM

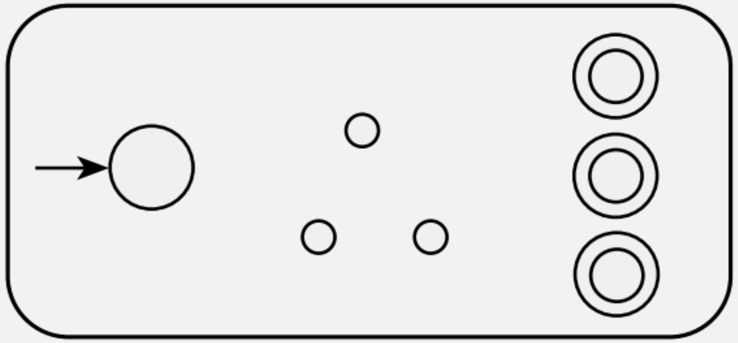
The class of regular languages is closed under the concatenation operation.

In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

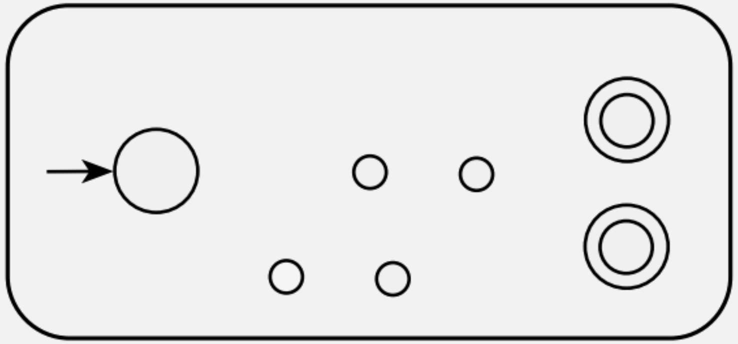
- **Construct a new machine M recognizing $A_1 \circ A_2$?** (like union)
  - Using **DFA M_1** (which recognizes A_1),
  - and **DFA M_2** (which recognizes A_2)

Concatentation

M_1



M_2

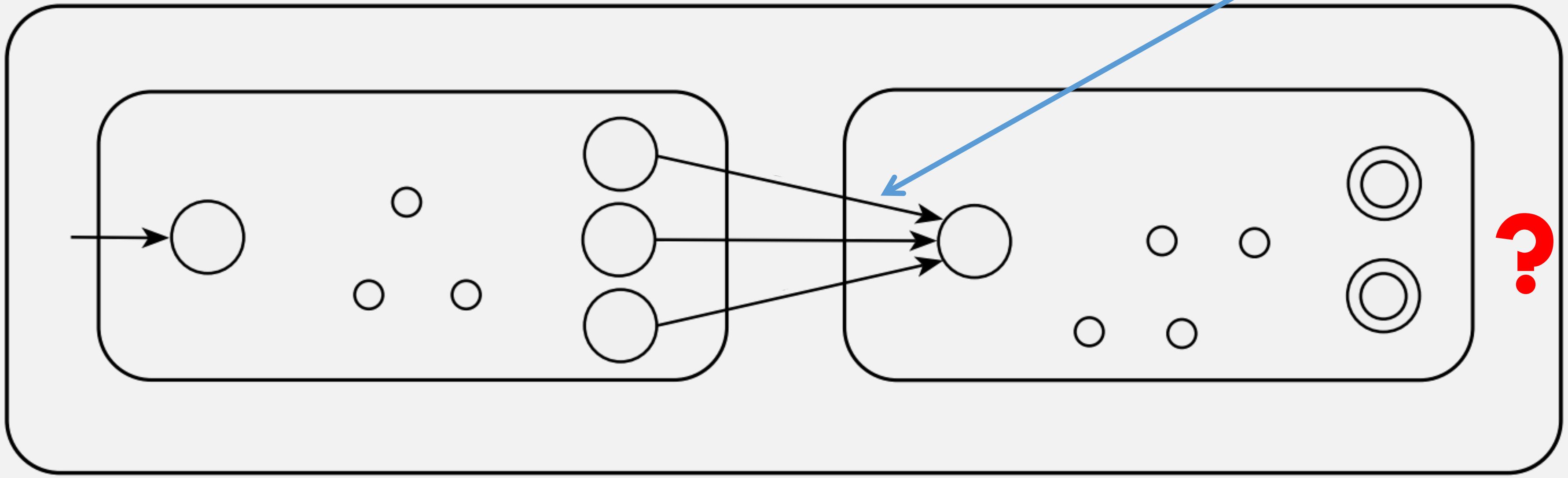


PROBLEM: Can only read input once, can't backtrack

Let M_1 recognize A_1, and M_2 recognize A_2.

Want: Construction of M to recognize $A_1 \circ A_2$

Need to switch machines at some point, but when?



???

$\textbf{Concatenation: } A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$

Overlapping Concatenation Example

- Let M_1 recognize language A = {**ab**, **abc**}
- and M_2 recognize language B = {**cde**}
- Want: Construct M to recognize $A \circ B$ = {$\boxed{\textbf{ab}}$**cde**, $\boxed{\textbf{ab}}$**ccde**}

- If M sees **ab** …
- M must decide to <u>either</u>:

Concatenation: $A \circ B = \{xy | \ x \in A \text{ and } y \in B\}$

Overlapping Concatenation Example

- Let M_1 **recognize language** A = {**ab**, **abc**}
- and M_2 **recognize language** B = {**cde**}
- Want: **Construct** M **to recognize** $A \circ B$ = {**abcde**, **abccde**}

- If M **sees ab** ...
- M **must decide to** <u>**either**</u>**:**
  - **stay in** M_1 (correct, if full input is **abccde**)

Concatenation: $A \circ B = \{xy | \, x \in A \text{ and } y \in B\}$

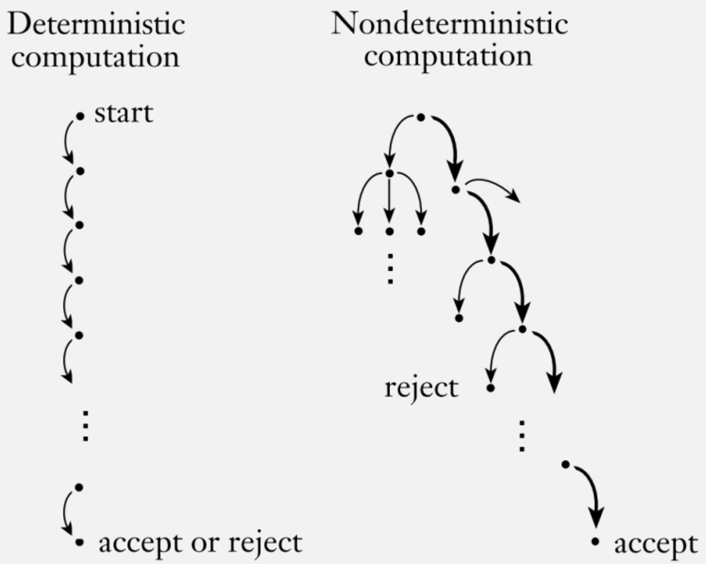
Overlapping Concatenation Example

- Let M_1 **recognize language** A = {**ab**, **abc**}
- and M_2 **recognize language** B = {**cde**}
- Want: **Construct** M **to recognize** $A \circ B$ = {**abcde**, **abccde**}

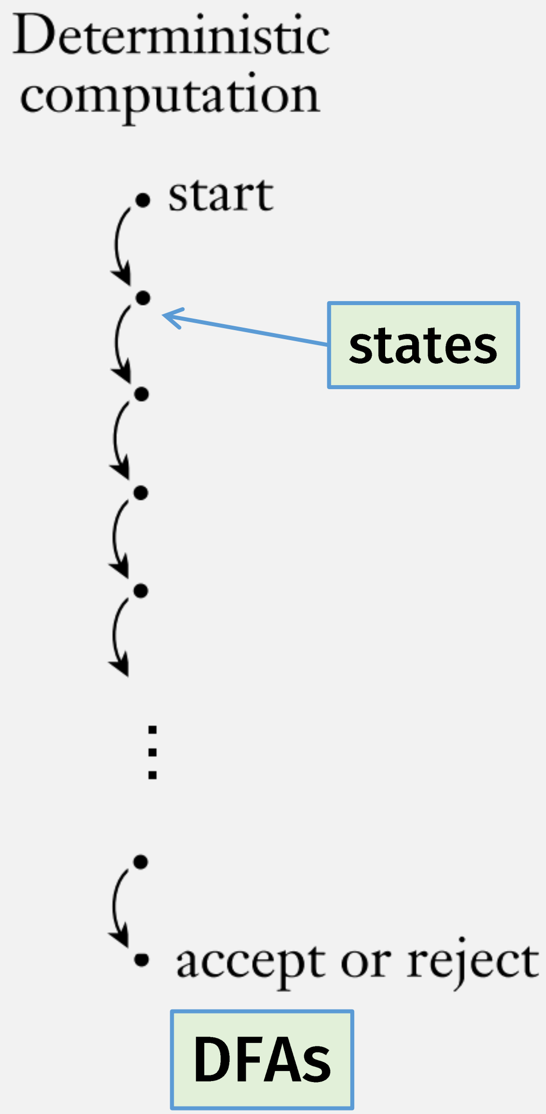
- If M **sees ab** …
- M **must decide to** __either__:
  - **stay in** M_1 (correct, if full input is **abccde**)
  - or **switch to** M_2 (correct, if full input is **abcde**)
- __But__ **to recognize** $A \circ B$, **it needs to handle** __both cases__**!!**
  - **Without backtracking**

A DFA can't do this!
(We need a new kind of machine)

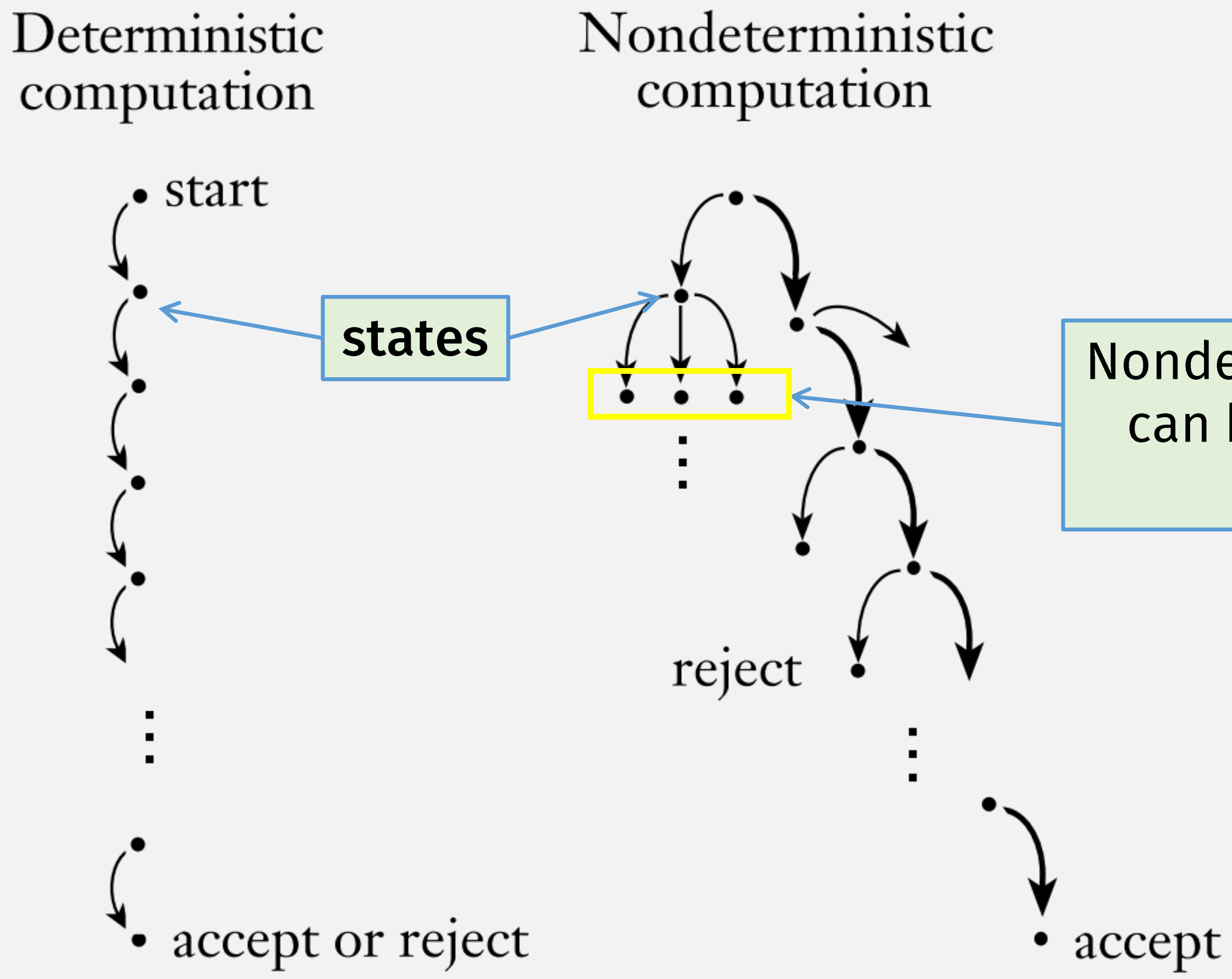
Nondeterminism



Deterministic vs Nondeterministic



Deterministic vs Nondeterministic



Nondeterministic computation can be in multiple states at the same time

DFAs

New FA

Finite Automata: The Formal Definition

DEFINITION

deterministic

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
2. Σ is a finite set called the ***alphabet***,
3. $\delta\colon Q \times \Sigma \longrightarrow Q$ is the ***transition function***,
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.

Also called a **Deterministic Finite Automata (DFA)**

Precise Terminology is Important

- A **finite automata** or **finite state machine** (**FSM**) defines …
  … computation with a finite number of states

- There are many kinds of FSMs

- We've learned one kind, the **Deterministic Finite Automata (DFA)**
  - (So currently, the terms DFA and FSM refer to the same definition)

- We will learn other kinds, e.g., **Nondeterministic Finite Automata (NFA)**

- Be careful with terminology!

Nondeterministic Finite Automata (NFA)

DEFINITION

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta \colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Compare with DFA:

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

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Difference

Power set, i.e. a transition results in set of states

Power Sets

- A power set is the set of all subsets of a set

- Example: S = {**a**, **b**, **c**}

- Power set of S =
  - {{ }, {**a**}, {**b**}, {**c**}, {**a**, **b**}, {**a**, **c**}, {**b**, **c**}, {**a**, **b**, **c**}}
  - Note: includes the empty set!

Nondeterministic Finite Automata (NFA)

DEFINITION

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

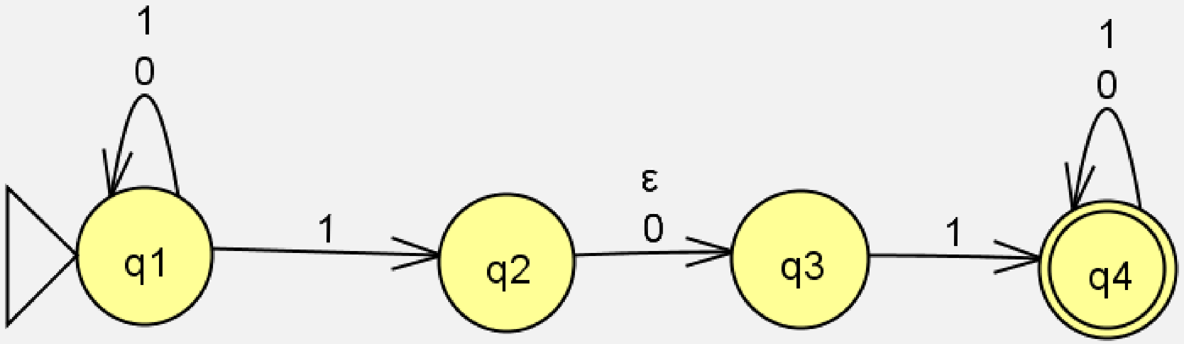
1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta \colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Transition label can be "empty", i.e., machine can transition without reading input

$$\boxed{\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}}$$

NFA Example

- Come up with a formal description of the following NFA:



DEFINITION

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4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

The formal description of N_1 is $(Q, \Sigma, \delta, q_1, F)$, where

$\delta : Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$

1. $Q = \{q_1, q_2, q_3, q_4\}$,
2. $\Sigma = \{0,1\}$,
3. δ is given as

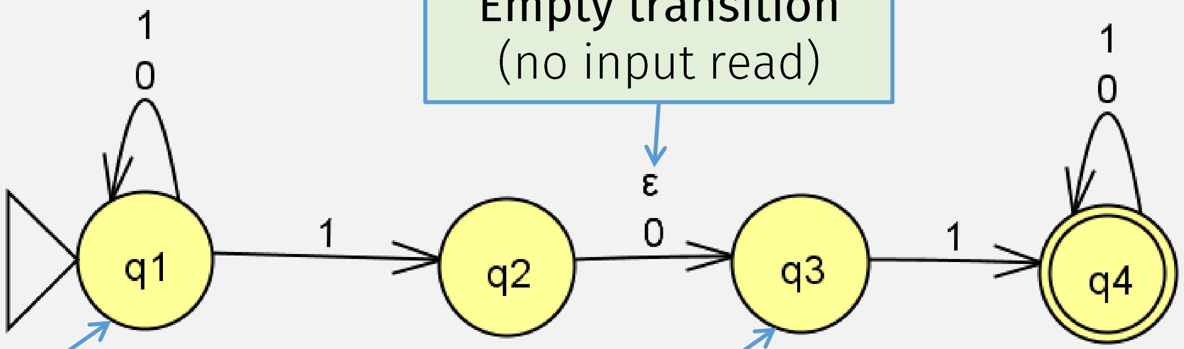
Empty transition (no input read)

	0	1	ε
q_1	$\{q_1\}$	$\{q_1, q_2\}$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\{q_3\}$
q_3	$\emptyset$	$\{q_4\}$	$\emptyset$
q_4	$\{q_4\}$	$\{q_4\}$	$\emptyset$,

Result of transition is a set

4. q_1 is the start state, and
5. $F = \{q_4\}$.

Empty transition (no input read)

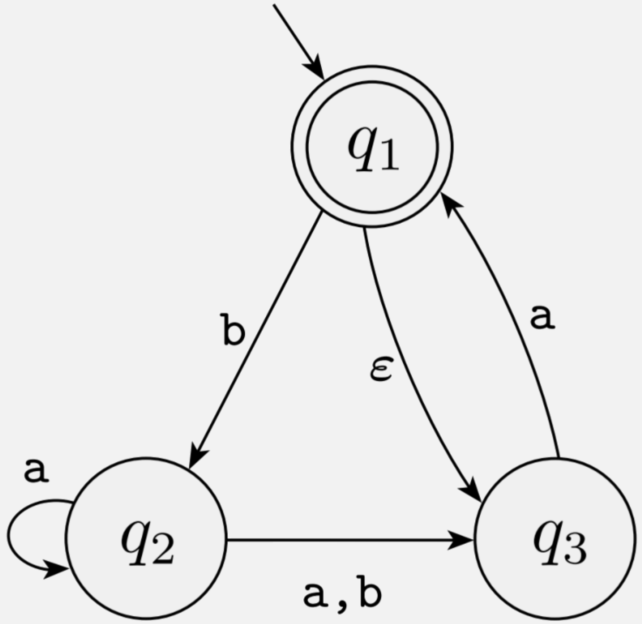


Multiple 1 transitions

No 0 transition

In-class Exercise

- Come up with a formal description for the following NFA
  - $\Sigma = \{\texttt{a}, \texttt{b}\}$



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5. $F \subseteq Q$ is the set of accept states.

In-class Exercise Solution

Let $N = (Q, \Sigma, \delta, q_0, F)$

- $Q = \{ q_1, q_2, q_3 \}$
- $\Sigma = \{ \texttt{a}, \texttt{b} \}$
- δ ...
- $q_0 = q_1$
- $F = \{ q_1 \}$

$\delta(q_1, \texttt{a}) = \{ \}$
$\delta(q_1, \texttt{b}) = \{ q_2 \}$
$\delta(q_1, \varepsilon) = \{ q_3 \}$
$\delta(q_2, \texttt{a}) = \{ q_2, q_3 \}$
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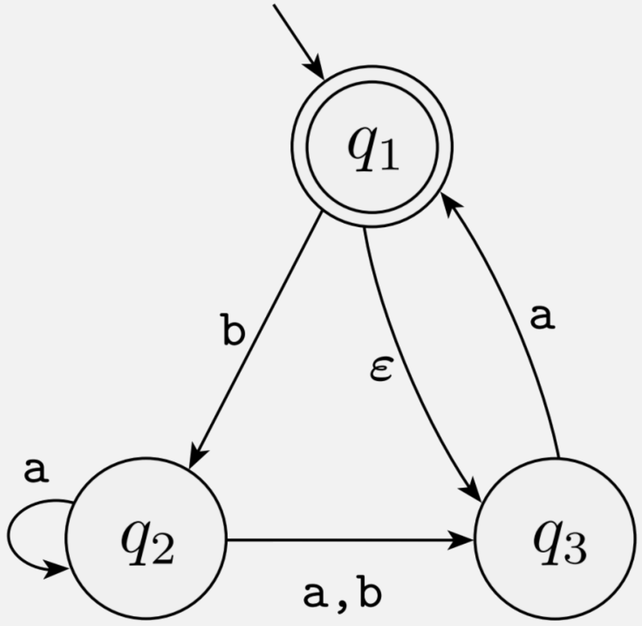
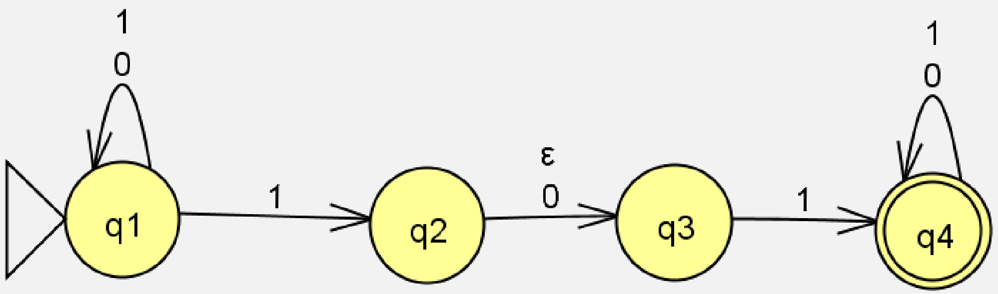


Next Time: Running Programs, NFAs (JFLAP demo): `010110`



Check-in Quiz 9/20

On gradescope