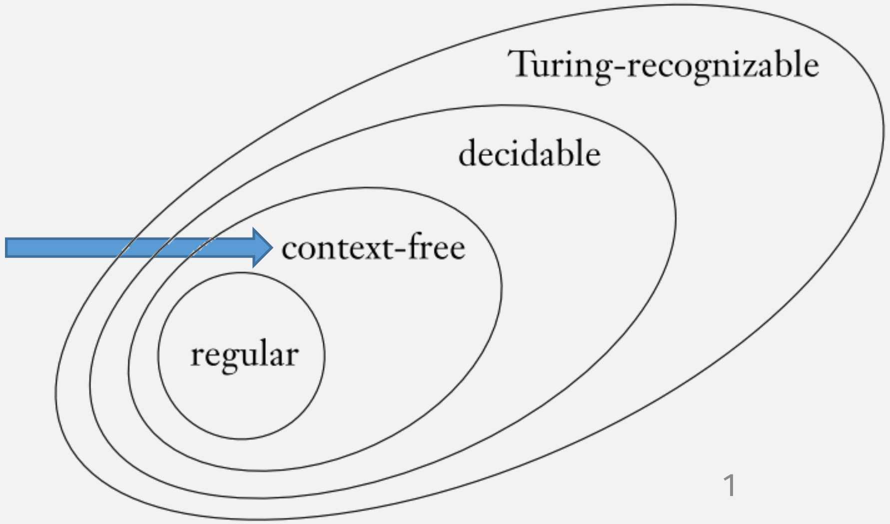
UMB CS 420

Context-Free Languages (CFLs)

Monday, March 6, 2023



Announcements

- HW 4 in
  - ~~due Sun 3/5 11:59pm EST~~
- HW 5 out
  - due Sun 3/19 11:59pm EST
  - (after Spring Break!)

Quiz Preview

- What do we call the class of languages that are generated by a CFG?

Last Time:

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Let B be the language $\{0^n1^n | n \geq 0\}$. We use the pumping lemma to prove that B is not regular. The proof is by contradiction.

- Assume: language B is regular
- So it must satisfy the **Pumping Lemma:**

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contradiction

- <u>Assume:</u> language B is regular
- So it <u>must satisfy</u> the **Pumping Lemma:**
  - All strings $\geq$ length $\boldsymbol{p}$ ...
  - ... can be split into some xyz ... where y <u>is</u> "pumpable"
- Get **contradiction** by finding **counterexample**: a <u>not</u> "pumpable string $\geq$ length $\boldsymbol{p}$: $\boxed{0^p1^p}$
  - Must show string cannot be pumped for <u>all possible splittings</u> into xyz
  - Use pumping lemma condition #3 to eliminate some cases

$$\underbrace{00 \ldots 0}11 \ldots 1$$

y must be in the first $\boldsymbol{p}$ 0s!

- Therefore, B is <u>not</u> regular
  - (This is the **contrapositive** of the Pumping Lemma)
- This is a **contradiction** of the assumption!

Last Time:

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Let B be the language $\{0^n1^n | n \geq 0\}$. We use the pumping lemma to prove that B is not regular. The proof is by contradiction.

If this language is not regular, then what is it???

Maybe? … a **context-free language** (CFL)?

A Context-Free Grammar (CFG)

terminals

Top variable is:
Start variable

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

Variables
(a.k.a., **nonterminals**)
on RHS of rules

Substitution rules
(a.k.a., **productions**)

terminals (analogous to a DFA's alphabet)

A Context-Free Grammar (CFG)

Grammar $G_1 = (V, \Sigma, R, S)$

R is this set of rules (mappings):

terminals

Top variable is:
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$$A \rightarrow 0A1$$
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Variables
(a.k.a., **nonterminals**)

Substitution rules
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terminals (analogous to a DFA's alphabet)

CFG <u>Practical Application</u>:
Used to describe
<u>programming language</u>
<u>syntax</u>!

A ***context-free grammar*** is a 4-tuple (V, Σ, R, S), where

1. V is a finite set called the ***variables***,
2. Σ is a finite set, disjoint from V, called the ***terminals***,
3. R is a finite set of ***rules***, with each rule being a variable and a string of variables and terminals, and
4. $S \in V$ is the start variable.

$V = \{A, B\}$,

$\Sigma = \{0, 1, \#\}$,

$S = A$,

Java Syntax: Described with CFGs

ORACLE

Java SE > Java SE Specifications > Java Language Specification

Chapter 2

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Chapter 2. Grammars

This chapter describes the context-free grammars used in this specification to define the lexical and syntactic structure of a program

2.1. Context-Free Grammars

A *context-free grammar* consists of a number of *productions*. Each production has an abstract symbol called a *nonterminal* as its *left-hand side*, and a sequence of one or more nonterminal and *terminal* symbols as its *right-hand side*. For each grammar, the terminal symbols are drawn from a specified *alphabet*.

Starting from a sentence consisting of a single distinguished nonterminal, called the *goal symbol*, a given context-free grammar specifies a language, namely, the set of possible sequences of terminal symbols that can result from repeatedly replacing any nonterminal in the sequence with a right-hand side of a production for which the nonterminal is the left-hand side.

(definition of a language)

2.2. The Lexical Grammar

A *lexical grammar* for the Java programming language is given in §3. This grammar has as its terminal symbols the characters of the Unicode character set. It defines a set of productions, starting from the goal symbol *Input* (§3.5), that describe how sequences of Unicode characters (§3.1) are translated into a sequence of input elements (§3.5).

Definition:
A **CFG** describes a **context-free language!**

A **CFG** specifies a language!

```
https://docs.oracle.com/javase/specs/jls/se7/html/jls-2.html
```

Analogies

Regular Language	Context-Free Language (CFL)
Regular Expression	Context-Free Grammar (CFG)
A Reg Expr <u>describes</u> a Regular lang	A CFG <u>describes</u> a CFL

thm

def

(partially)

Python Syntax: Described with a CFG

10. Full Grammar specification

This is the full Python grammar, as it is read by the parser generator and used to parse Python source files:

```
# Grammar for Python

# NOTE WELL: You should also follow all the steps listed at
# https://devguide.python.org/grammar/

# Start symbols for the grammar:
#       single_input is a single interactive statement;
#       file_input is a module or sequence of commands read from an input file;
#       eval_input is the input for the eval() functions.
#       func_type_input is a PEP 484 Python 2 function type comment
# NB: compound_stmt in single_input is followed by extra NEWLINE!
# NB: due to the way TYPE_COMMENT is tokenized it will always be followed by a NEWLINE
single_input: NEWLINE | simple_stmt | compound_stmt NEWLINE
file_input: (NEWLINE | stmt)* ENDMARKER
eval_input: testlist NEWLINE* ENDMARKER
```

(indentation checking probably not describable with a CFG)

https://docs.python.org/3/reference/grammar.html

Many Other Language (partially) ~~Python~~ Syntax: Described with a CFG

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ORACLE

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https://docs.oracle.com/javase/specs/jls/se7/html/jls-2.html

Generating Strings with a CFG

Definition:
A **CFG** describes a **context-free language**!
but what strings are in the language?

$$G_1 =$$

1st rule

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

Strings in CFG's language
= all possible **generated** / **derived** strings

$$\boxed{L(G_1) \text{ is } \{0^n\#1^n \mid n \geq 0\}}$$

"Applying a rule"
= replace LHS variable with RHS

At each step, can choose any variable to replace, and any rule to apply

Stop when string is all terminals

A CFG **generates** a string, by repeatedly applying substitution rules:

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$$

Start variable | After applying 1st rule | 1st rule again | 1st rule again | Use 2nd rule | Use last rule

A ***context-free grammar*** is a 4-tuple (V, Σ, R, S), where

1. V is a finite set called the ***variables***,
2. Σ is a finite set, disjoint from V, called the ***terminals***,
3. R is a finite set of ***rules***, with each rule being a variable and a string of variables and terminals, and
4. $S \in V$ is the start variable.

Derivations: Formally

Let $G = (V, \Sigma, R, S)$

Single-step

$$\alpha A \beta \underset{G}{\Rightarrow} \alpha \gamma \beta$$

Where:

$\alpha, \beta \in (V \cup \Sigma)^*$ ← Strings of terminals and variables

$A \in V$ ← Variable

$A \to \gamma \in R$ ← Rule

Extended Derivation

Base case: $\alpha \overset{*}{\underset{G}{\Rightarrow}} \alpha$ (0 steps)

Recursive case: (multistep)

- If $\alpha \underset{G}{\Rightarrow} \beta$ and $\beta \overset{*}{\underset{G}{\Rightarrow}} \gamma$

Single step

Recursive call

- Then: $\alpha \overset{*}{\underset{G}{\Rightarrow}} \gamma$

Formal Definition of a CFL

A ***context-free grammar*** is a 4-tuple (V, Σ, R, S), where

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3. R is a finite set of ***rules***, with each rule being a variable and a string of variables and terminals, and
4. $S \in V$ is the start variable.

$$G = (V, \Sigma, R, S)$$

$$L(G) = \left\{ w \in \Sigma^* \mid S \overset{*}{\underset{G}{\Rightarrow}} w \right\}$$

Any language that can be generated by some context-free grammar is called a ***context-free language***

Flashback: $\{0^n 1^n \mid n \geq 0\}$

- Pumping Lemma says it's not a regular language
- It's a context-free language!
  - Proof?
  - Come up with CFG describing it …
  - Hint: It's similar to:

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \cancel{\#}\ \varepsilon$$

$L(G_1)$ is $\{0^n \cancel{\#} 1^n \mid n \geq 0\}$

Statements and Justifications?

Proof: $L = \{0^n 1^n \mid n \geq 0\}$ is a CFL

Statements	**Justifications**
1. If a CFG describes a language, then it is a CFL	1. Definition of CFL
2. CFG G_1 describes L $\begin{array}{l} A \rightarrow 0A1 \\ A \rightarrow B \\ B \rightarrow \varepsilon \end{array}$	2. (Did you come up with examples???)
3. $L = \{0^n 1^n \mid n \geq 0\}$ is a CFL	3. By Statements #1 and #2

Shorthand for multiple rules with same nonterminal LHS

A String Can Have Multiple Derivations

$$\langle \text{EXPR} \rangle \rightarrow \boxed{\langle \text{EXPR} \rangle + \langle \text{TERM} \rangle} \mid \langle \text{TERM} \rangle$$
$$\langle \text{TERM} \rangle \rightarrow \boxed{\langle \text{TERM} \rangle \times \langle \text{FACTOR} \rangle} \mid \langle \text{FACTOR} \rangle$$
$$\langle \text{FACTOR} \rangle \rightarrow (\, \langle \text{EXPR} \rangle \,) \mid \boxed{\mathtt{a}}$$

Want to generate this string: **a + a × a**

- <u>EXPR</u> ⇒
- EXPR + <u>TERM</u> ⇒
- EXPR + TERM × <u>FACTOR</u> ⇒
- EXPR + TERM × **a** ⇒

  ...

RIGHTMOST DERIVATION

- <u>EXPR</u> ⇒
- <u>EXPR</u> + TERM ⇒
- <u>TERM</u> + TERM ⇒
- <u>FACTOR</u> + TERM ⇒
- **a** + TERM

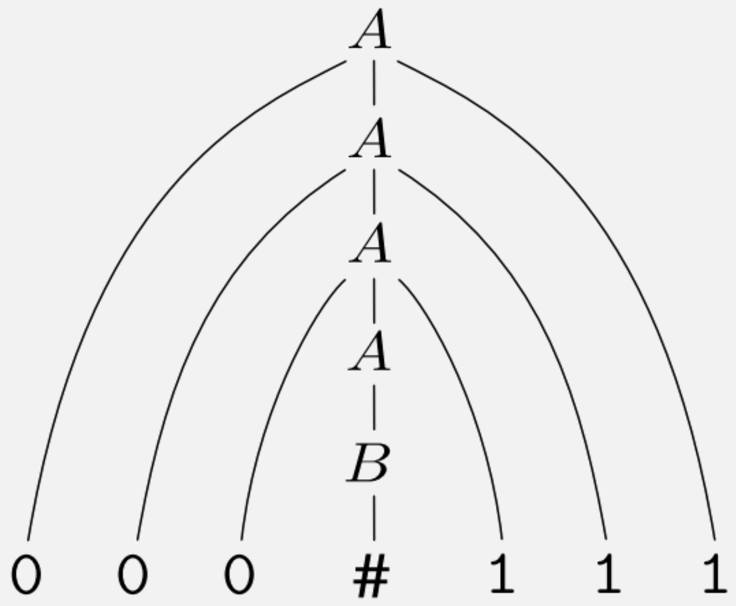
  ...

LEFTMOST DERIVATION

Derivations and Parse Trees

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$$

A derivation may also be represented as a **parse tree**



Multiple Derivations, Single Parse Tree

Leftmost deriviation

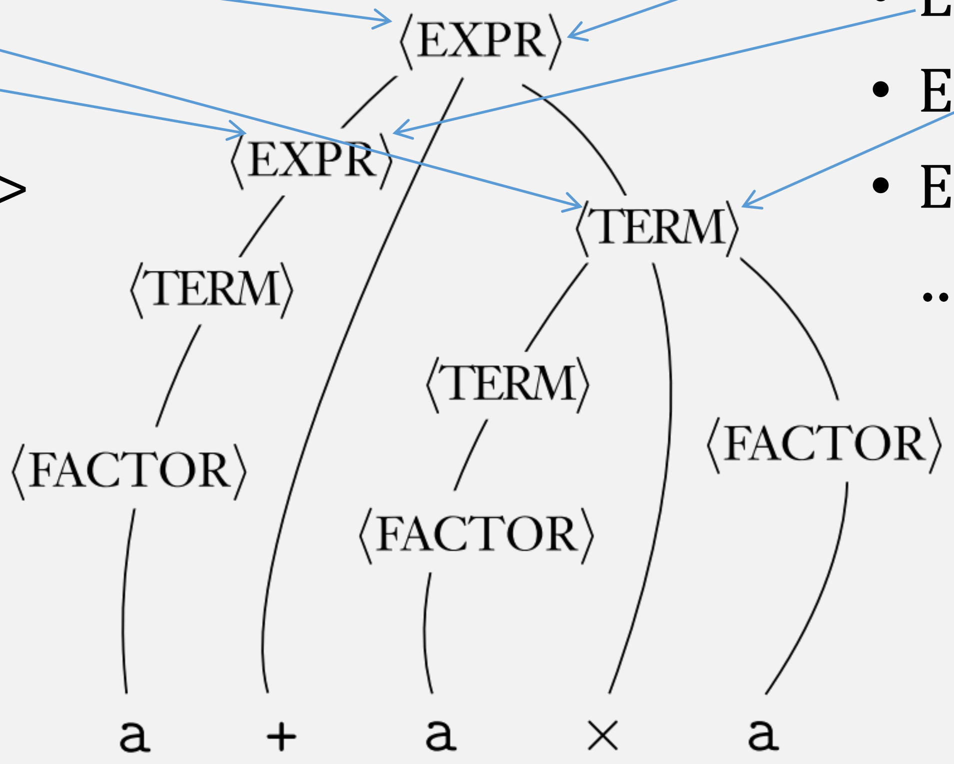
- EXPR =>
- EXPR + TERM =>
- TERM + TERM =>
- FACTOR + TERM =>
- a + TERM

  …

Rightmost deriviation

- EXPR =>
- EXPR + TERM =>
- EXPR + TERM x FACTOR =>
- EXPR + TERM x a=>

  …



Same parse tree

A **parse tree** represents
a CFG computation … like
a **sequence of states** represents
a DFA computation

A **Parse Tree**
gives
"meaning"
to a string

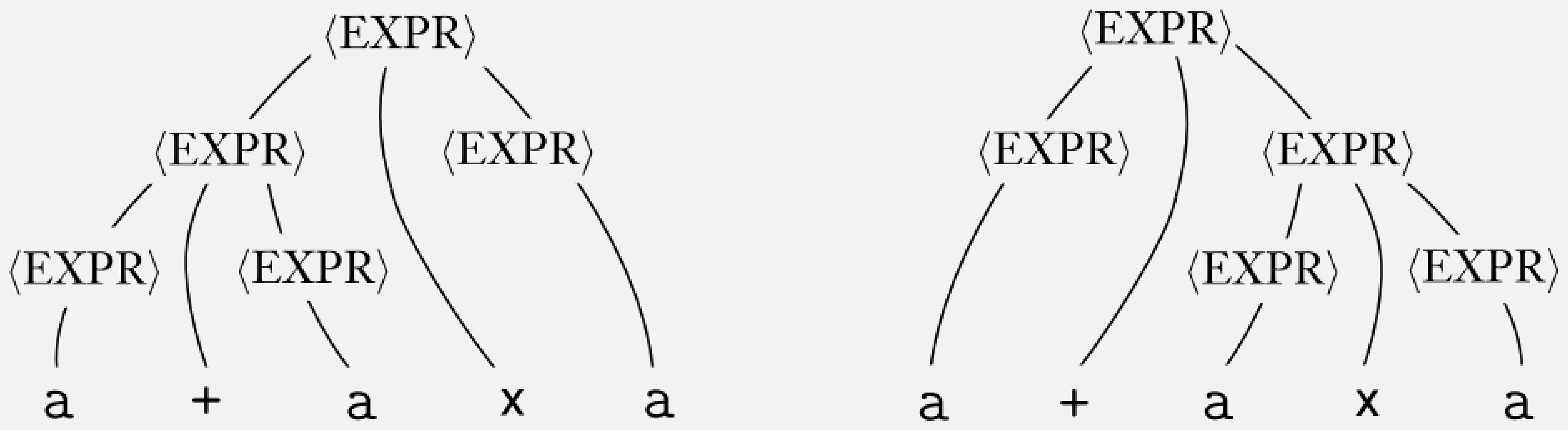
Ambiguity

grammar G_5:

$$\langle \text{EXPR} \rangle \rightarrow \langle \text{EXPR} \rangle \texttt{+} \langle \text{EXPR} \rangle \mid \langle \text{EXPR} \rangle \texttt{x} \langle \text{EXPR} \rangle \mid \texttt{(}\, \langle \text{EXPR} \rangle \,\texttt{)} \mid \texttt{a}$$

Same **string,**
different **derivation,**
and different **parse tree!**

So this string has
two meanings!



Ambiguity

A string w is derived ***ambiguously*** in context-free grammar G if it has two or more different leftmost derivations. Grammar G is ***ambiguous*** if it generates some string ambiguously.

An ambiguous grammar can give a string <u>multiple meanings</u>, ie represent <u>two different computations</u>!
(why is this <u>bad</u>?)

Real-life Ambiguity ("Dangling" `else`)

- What is the result of this `C` program?

```
if (1) if (0) printf("a"); else printf("2");
```

```
if (1)
  if (0)
    printf("a");
  else
    printf("2");
```

VS

```
if (1)
  if (0)
    printf("a");
else
  printf("2");
```

This string has 2 parsings, and thus 2 meanings!

Ambiguous grammars are confusing. A computation on a string should ideally have only one result.

Thus in practice, we typically focus on the **unambiguous** subset of CFGs (CFLs) (more on this later)

Problem is, there's no easy way to create an **unambiguous** grammar (it's up to language designers to "be careful")

Designing Grammars : Basics

1. Think about what you want to "link" together

- E.g., $0^n 1^n$
    - $A \rightarrow 0A1$
    - # 0s and # 1s are "linked"

- E.g., XML
    - $\text{ELEMENT} \rightarrow \text{<TAG>CONTENT</TAG>}$
    - Start and end tags are "linked"

2. Start with small grammars and then combine (just like FSMs)

Designing Grammars: Building Up

- Start with small grammars and then combine (just like FSMs)
  - To create a grammar for the language $\{0^n1^n | n \geq 0\} \cup \{1^n0^n | n \geq 0\}$
  - First create grammar for lang $\{0^n1^n | n \geq 0\}$:

$$S_1 \to 0S_11 \mid \varepsilon$$

  - Then create grammar for lang $\{1^n0^n | n \geq 0\}$:

$$S_2 \to 1S_20 \mid \varepsilon$$

  - Then combine:

$$S \to S_1 \mid S_2$$
$$S_1 \to 0S_11 \mid \varepsilon$$
$$S_2 \to 1S_20 \mid \varepsilon$$

New start variable and rule combines two smaller grammars

"|" = "or" = union (combines 2 rules with same left side)

(Closed) Operations on CFLs?

- Start with small grammars and then combine (just like FSMs)
- "Or": $S \rightarrow S_1 \mid S_2$
- "Concatenate": $S \longrightarrow S_1 S_2$
- "Repetition": $S' \longrightarrow S' S_1 \mid \varepsilon$

Could you write out the full proof?

In-class Example: Designing grammars

alphabet Σ is $\{0,1\}$

$\{w \mid w \text{ starts and ends with the same symbol}\}$

- $S \rightarrow \mathtt{0}C'\mathtt{0} \mid \mathtt{1}C'\mathtt{1} \mid \varepsilon$ — "string starts/ends with same symbol, middle can be anything"
- $C' \rightarrow C'C \mid \varepsilon$ — "middle: all possible terminals, repeated (ie, all possible strings)"
- $C \rightarrow \mathtt{0} \mid \mathtt{1}$ — "all possible terminals"

Next Time:

Regular Languages	Context-Free Languages (CFLs)
Regular Expression	Context-Free Grammar (CFG)
A Reg Expr describes a Regular Lang	A CFG describes a CFL
Finite Automaton (FSM)	???
An FSM recognizes a Regular Lang	A ??? recognizes a CFL

Next Time:

Regular Languages	Context-Free Languages (CFLs)
Regular Expression	Context-Free Grammar (CFG)
A Reg Expr <u>describes</u> a Regular Lang	A CFG <u>describes</u> a CFL
Finite Automaton (FSM)	**Push-down Automaton** (PDA)
An FSM <u>recognizes</u> a Regular Lang	A **PDA** <u>recognizes</u> a CFL

Next Time:

	Regular Languages	Context-Free Languages (CFLs)	
thm	Regular Expression	Context-Free Grammar (CFG)	def
	A Reg Expr describes a Regular Lang	A CFG describes a CFL	
def	Finite Automaton (FSM)	**Push-down Automaton** (PDA)	thm
	An FSM recognizes a Regular Lang	A **PDA** recognizes a CFL	
	DIFFERENCE:	DIFFERENCE:	
	A Regular Lang is defined with a FSM	A CFL is defined with a CFG	
	Proved: Reg Expr ⇔ Reg Lang	*Must prove*: **PDA** ⇔ CFL	

Check-in Quiz 3/6

On gradescope