CS420

NFA -> DFA

Monday, February 7, 2022

UMass Boston CS

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta \colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

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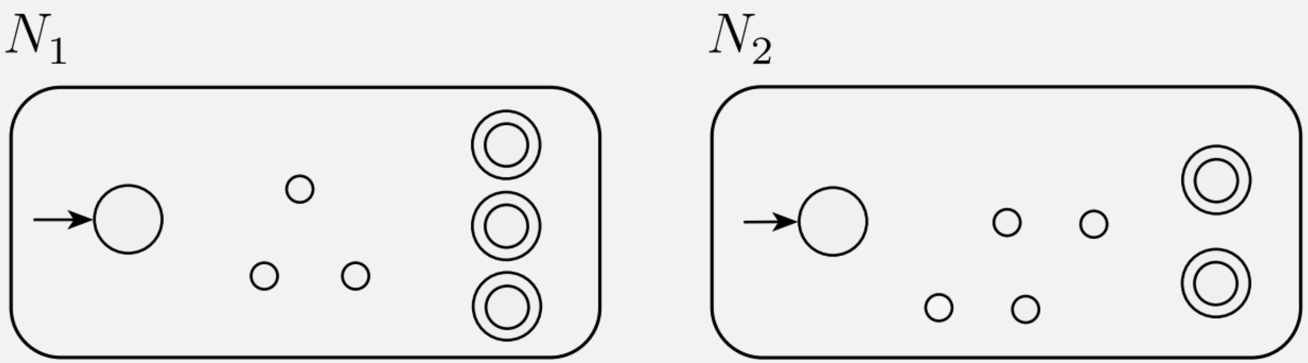
Announcements

- HW 1 in
- HW 2 out
  - Due Sun 2/13 11:59pm EST
- Ask HW questions publicly on Piazza
  - So the entire class can benefit from the discussion
  - Make it anonymous if you want to

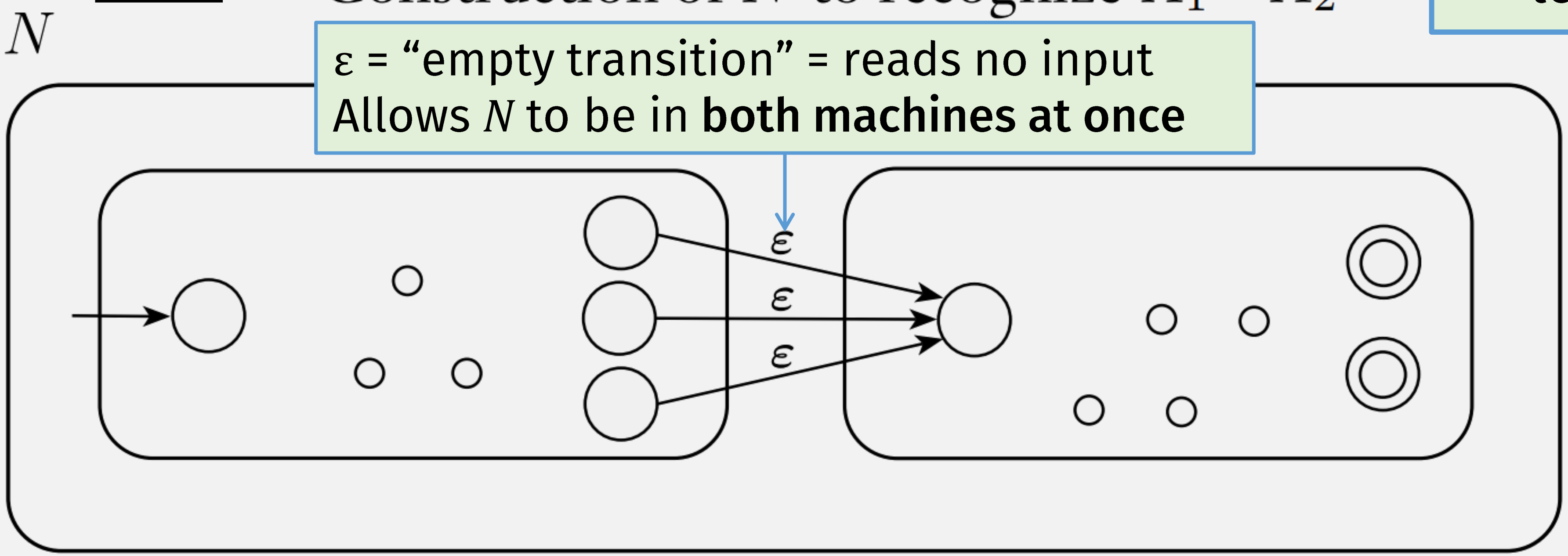
Last Time



Let N_1 recognize A_1, and N_2 recognize A_2.

Want: Construction of N to recognize $A_1 \circ A_2$

ε = "empty transition" = reads no input
Allows *N* to be in **both machines at once**



Does this prove concatentation is closed for regular languages?

Flashback: A DFA's Language

- For DFA $M = (Q, \Sigma, \delta, q_0, F)$

- M ***accepts*** w if $\hat{\delta}(q_0, w) \in F$

- M ***recognizes language*** A if $A = \{w|\ M \text{ accepts } w\}$

- A language is a **regular language** if a DFA recognizes it

An NFA's Language

- For NFA $N = (Q, \Sigma, \delta, q_0, F)$

intersection

accept states

- N ***accepts*** w if $\hat{\delta}(q_0, w) \cap F \neq \emptyset$ ← not empty
  - i.e., if the final states have at least one accept state

- Language of N = $L(N)$ = $\left\{ w \mid \hat{\delta}(q_0, w) \cap F \neq \emptyset \right\}$

Q: How does an NFA's language relate to regular languages?
All we know so far: A language is regular if a **DFA** recognizes it

So is Concatenation Closed for Reg Langs?

- Concatenation of DFAs produces an NFA

- But a language is only regular if a DFA recognizes it

- To finish the proof that concat is closed …
  … we must <u>prove</u> that NFAs *also* recognize regular languages.

Specifically, we must <u>prove</u>:

NFAs ⇔ regular languages

How to Prove a Statement: $X \Leftrightarrow Y$

- $X \Leftrightarrow Y$ = "X if and only if Y" = X iff Y = X <=> Y
- Proof at minimum has 2 required parts:

1. $\Rightarrow$ if X, then Y
   - "forward" direction
   - assume X, then use it to prove Y
2. $\Leftarrow$ if Y, then X
   - "reverse" direction
   - assume Y, then use it to prove X

Proving NFAs Recognize Regular Langs

<u>Theorem:</u>
A language L is regular **if and only if** some NFA N recognizes L.

<u>Proof</u>:
⇒ If L is regular, then some NFA N recognizes it.

- *Easier*
- <u>We know</u>: if L is **regular**, then a **DFA** exists that recognizes it.
- <u>So to prove this part</u>: Convert that DFA to an equivalent NFA! (see HW 2)

⇐ If an NFA N recognizes L, then L is regular.

- *Harder*
- <u>We know:</u> for L to be **regular**, there must be a **DFA** recognizing it
- <u>Proof Idea for this part</u>: Convert given NFA to equivalent DFA

"equivalent" = "recognizes the same language"

How to convert **NFA→DFA**?

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
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3. $\delta \colon Q \times \Sigma \longrightarrow Q$ is the ***transition function***,
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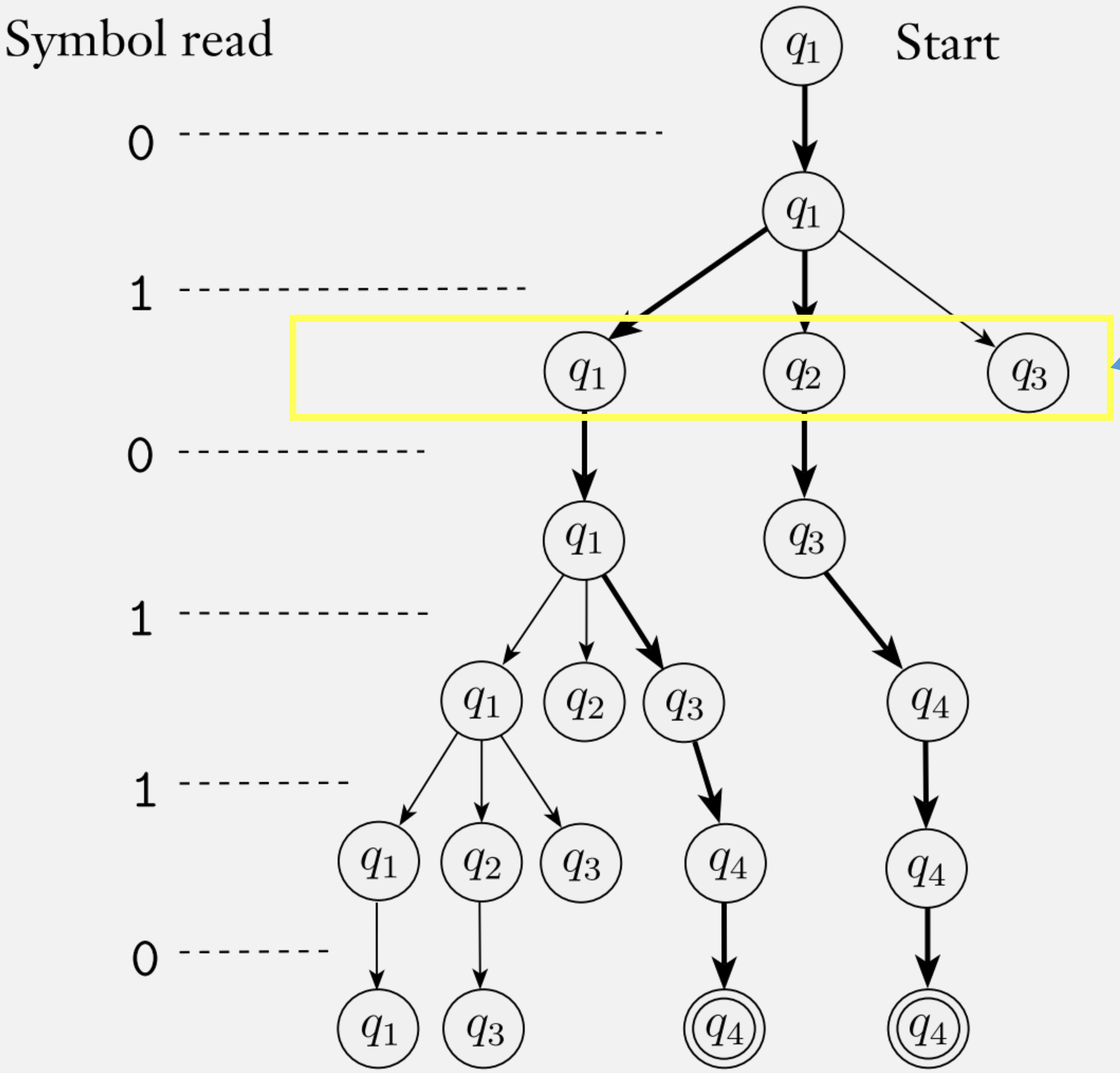
A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta \colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Proof idea:
Let each "state" of the DFA be a set of states in the NFA



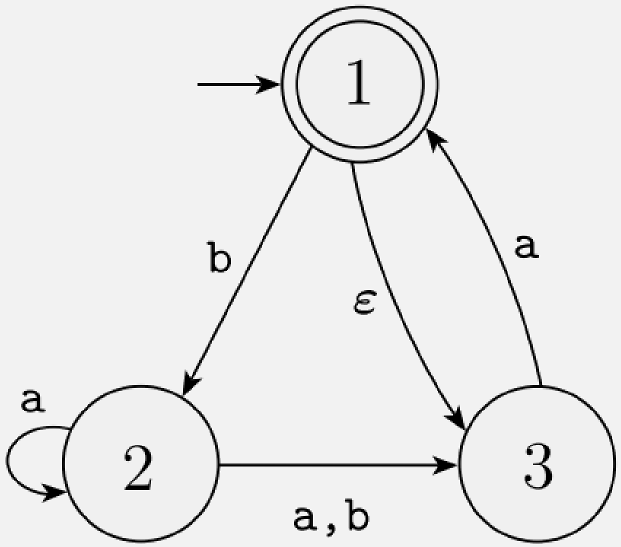
In a DFA, all these states at each step of NFA computation must be only **one** state

So design a state in the DFA to be a **set of NFA states**!

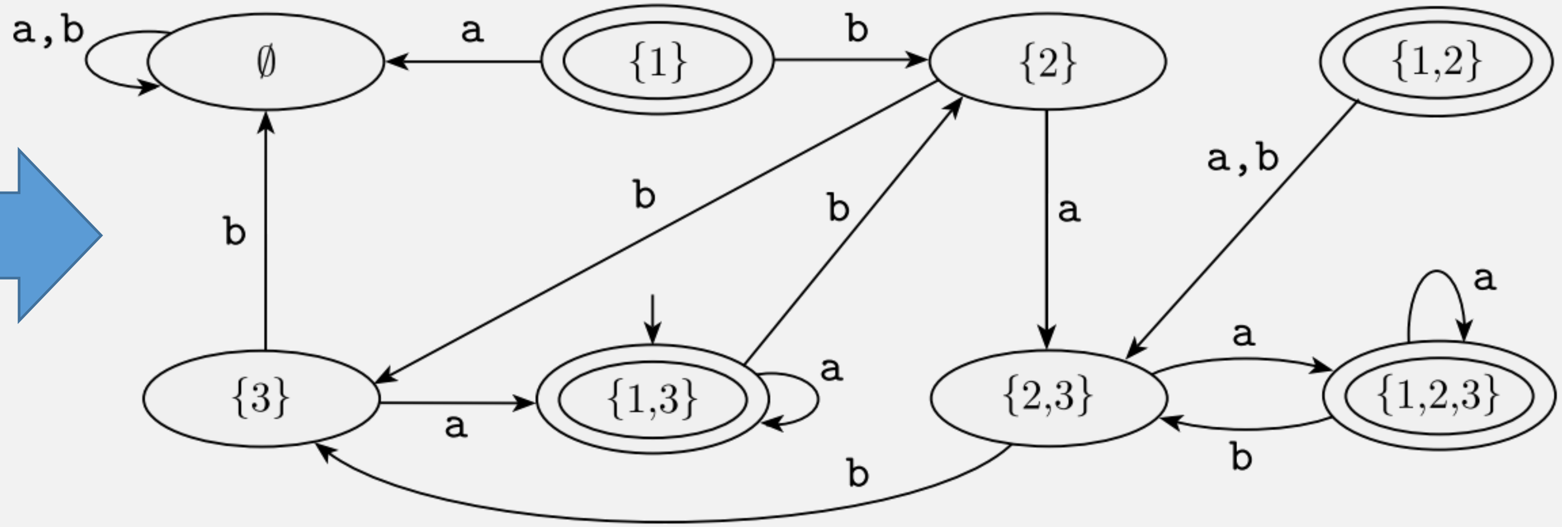
Convert **NFA→DFA**, Formally

- Let NFA N = $(Q, \Sigma, \delta, q_0, F)$
- An equivalent DFA M has states Q' = $\mathcal{P}(Q)$ (power set of Q)

Example:



The NFA N_4



A DFA D that is equivalent to the NFA N_4

No empty transitions

NFA→DFA

Have: NFA $N = (Q, \Sigma, \delta, q_0, F)$

Want: DFA $M = (Q', \Sigma, \delta', {q_0}', F')$

1. $Q' = \mathcal{P}(Q)$

A state for M is a set of states in N

2. For $R \in Q'$ and $a \in \Sigma$,

R = a state in M = a set of states in N

$$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a)$$

Next state for DFA state R =
next states of each NFA state r in R

3. ${q_0}' = \{q_0\}$

4. $F' = \{R \in Q' \mid R \text{ contains an accept state of } N\}$

Flashback: Adding Empty Transitions

- Define the set $\varepsilon\text{-REACHABLE}(q)$
  - … to be all states reachable from q via zero or more empty transitions

(Defined recursively)

- Base case: $q \in \varepsilon\text{-REACHABLE}(q)$

- Inductive case:

A state is in the reachable set if …

$$\varepsilon\text{-REACHABLE}(q) = \{r \mid p \in \varepsilon\text{-REACHABLE}(q) \text{ and } r \in \delta(p, \varepsilon)\}$$

… there is an empty transition to it from another state in the reachable set

With empty transitions

NFA→DFA

Have: NFA $N = (Q, \Sigma, \delta, q_0, F)$

Want: DFA $M = (Q', \Sigma, \delta', {q_0}', F')$

Almost the same, except …

1. $Q' = \mathcal{P}(Q)$

2. For $R \in Q'$ and $a \in \Sigma$,

$$\delta'(R, a) = \bigcup_{r \in R} \cancel{\delta(r, a)}\ \varepsilon\text{-REACHABLE}(\delta(r, a))$$

Requires extending the fn to sets of states (see HW 2)

3. ${q_0}' = \cancel{\{q_0\}}\ \varepsilon\text{-REACHABLE}(q_0)$

4. $F' = \{R \in Q' \mid R \text{ contains an accept state of } N\}$

Proving NFAs Recognize Regular Langs

Theorem:

A language L is regular **if and only if** some NFA N recognizes L.

Proof:

⇒ If L is regular, then some NFA N recognizes it.

- We know: If L is **regular**, then a **DFA** recognizes it.
- We show: How to convert a DFA to an equivalent NFA

⇐ If an NFA N recognizes L, then L is regular.

- We know: For L to be **regular**, there must be a **DFA** recognizing it
- We show: How to convert NFA N to an equivalent DFA …
- … using the NFA→DFA algorithm we just defined!

∎

Union: $A \cup B = \{x | \, x \in A \text{ or } x \in B\}$

Flashback: Union is Closed For Regular Langs

THEOREM

The class of regular languages is closed under the union operation.

In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

Proof:

- How do we prove that a language is regular?
  - Create a DFA or NFA recognizing it!
- Create machine combining the machines recognizing A_1 and A_2
  - Should we create a DFA or NFA?

Flashback: Union is Closed For Regular Langs

Proof

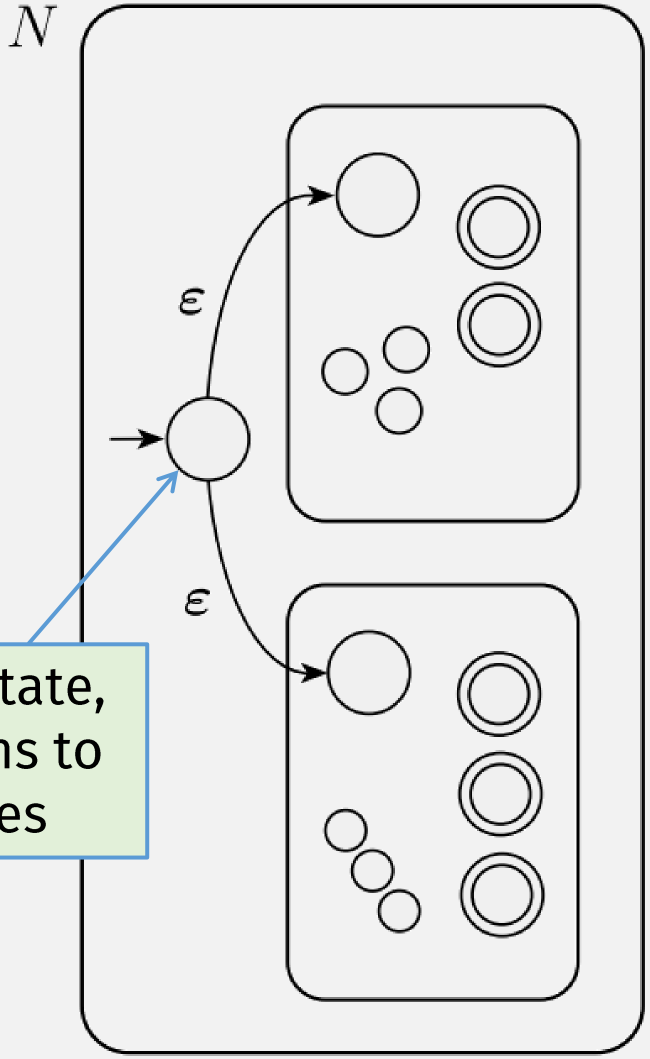
- Given: $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, recognize A_1,
  $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$, recognize A_2,
- Construct: a new machine $M = (Q, \Sigma, \delta, q_0, F)$ using M_1 and M_2
- states of M: $Q = \{(r_1, r_2) | \; r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$ $= Q_1 \times Q_2$
  This set is the ***Cartesian product*** of sets Q_1 and Q_2
- M transition fn: $\delta\big((r_1, r_2), a\big) = \big(\delta_1(r_1, a), \delta_2(r_2, a)\big)$
- M start state: (q_1, q_2)
- M accept states: $F = \{(r_1, r_2) | \; r_1 \in F_1 \text{ or } r_2 \in F_2\}$

State in M = M_1 state + M_2 state

M step = a step in M_1 + a step in M_2

Accept if either M_1 or M_2 accept

Union is Closed for Regular Languages



Add new start state, and ε-transitions to old start states

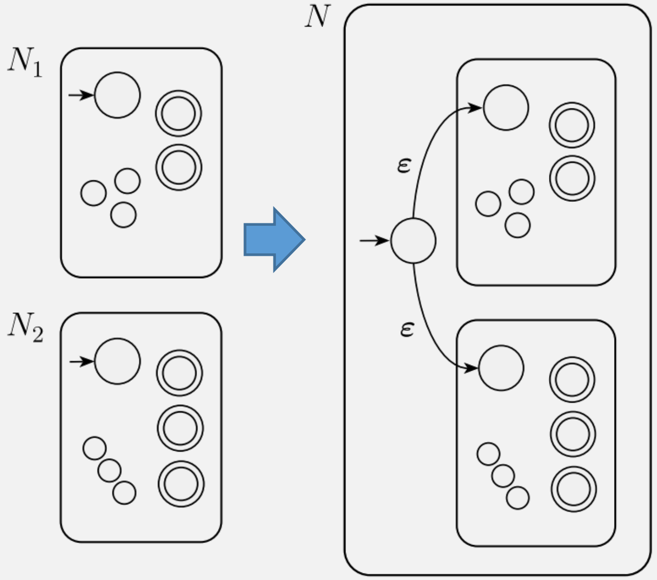
Union is Closed for Regular Languages

PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$.

1. $Q = \{q_0\} \cup Q_1 \cup Q_2$.
2. The state q_0 is the start state of N.
3. The set of accept states $F = F_1 \cup F_2$.



Union is Closed for Regular Languages

PROOF

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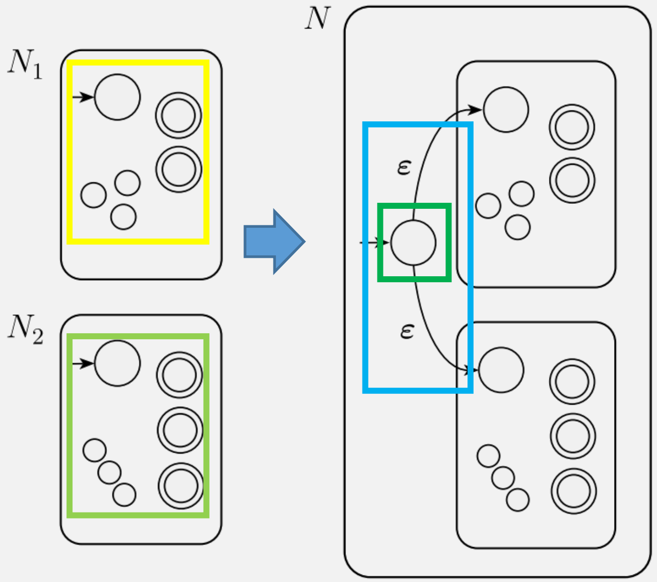
Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$.

1. $Q = \{q_0\} \cup Q_1 \cup Q_2$.
2. The state q_0 is the start state of N.
3. The set of accept states $F = F_1 \cup F_2$.
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q, a) = \begin{cases} \delta_1(?, a) & \boxed{q \in Q_1} \\ \delta_2(?, a) & \boxed{q \in Q_2} \\ \{q_1 ? q_2\} & \boxed{q = q_0 \text{ and } a = \varepsilon} \\ \emptyset \quad ? & \boxed{q = q_0 \text{ and } a \neq \varepsilon} \end{cases}$$



Concatenation is Closed for Regular Langs

PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

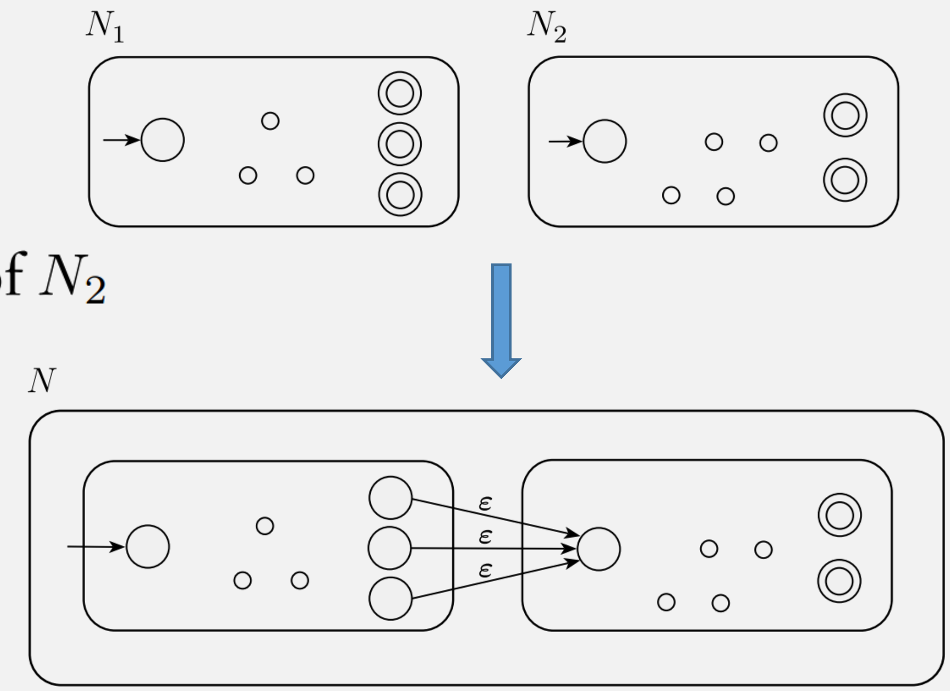
1. $Q = Q_1 \cup Q_2$

2. The state q_1 is the same as the start state of N_1

3. The accept states F_2 are the same as the accept states of N_2

4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \text{ and } a \neq \boldsymbol{\varepsilon} \\ \delta_1(q,a) \cup \{q_2\} & q \in F_1 \text{ and } a = \boldsymbol{\varepsilon} \\ \delta_2(q,a) & q \in Q_2. \end{cases}$$



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List of Closed Ops for Reg Langs (so far)

- ☑ Union
- ☑ Concatentation
- Kleene Star (repetition)

Star: $A^* = \{x_1x_2\ldots x_k |\, k \geq 0 \text{ and each } x_i \in A\}$

Kleene Star Example

Let the alphabet Σ be the standard 26 letters $\{\mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}\}$.

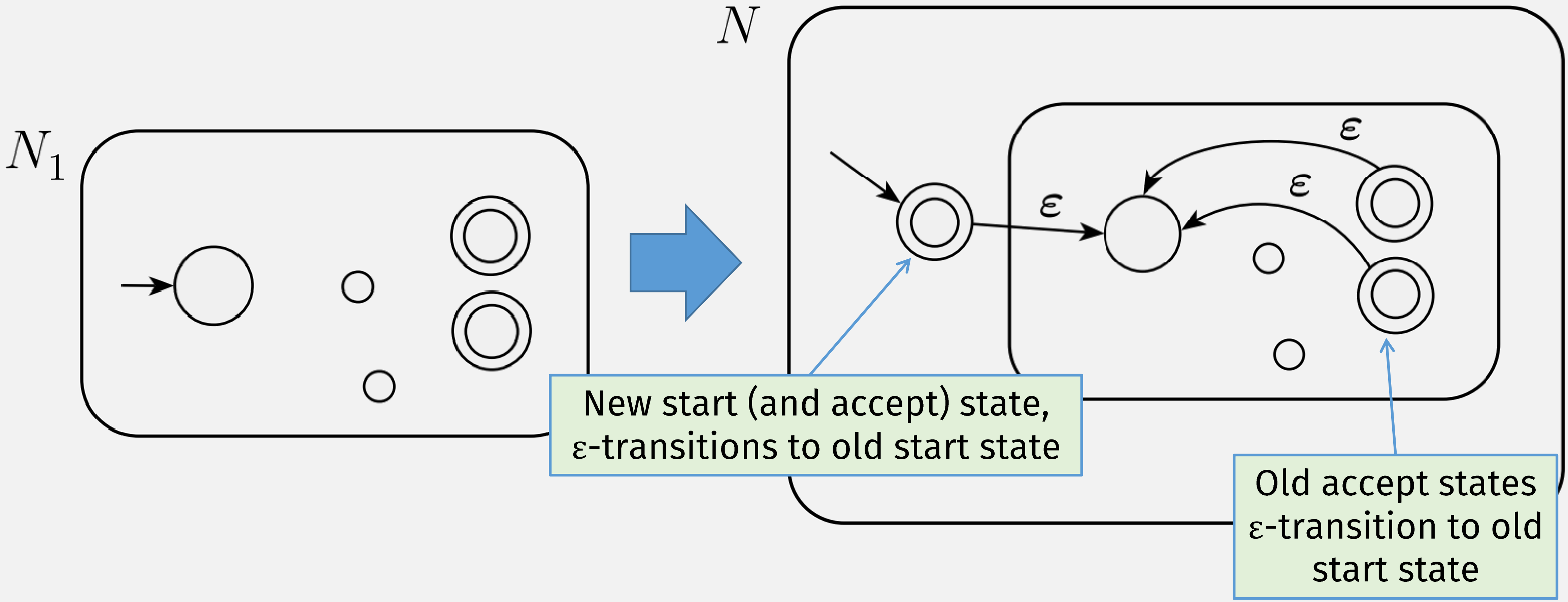
If $A = \{\mathtt{good}, \mathtt{bad}\}$ and $B = \{\mathtt{boy}, \mathtt{girl}\}$, then

$$A^* = \begin{array}{l}\{\varepsilon, \text{good, bad, goodgood, goodbad, badgood, badbad,} \\ \text{goodgoodgood, goodgoodbad, goodbadgood, goodbadbad, } \ldots\}\end{array}$$

Note: **repeat <u>zero or more times</u>**

(this is an infinite language!)

Kleene Star



N_1
N
ε
ε
ε
New start (and accept) state,
ε-transitions to old start state
Old accept states
ε-transition to old
start state

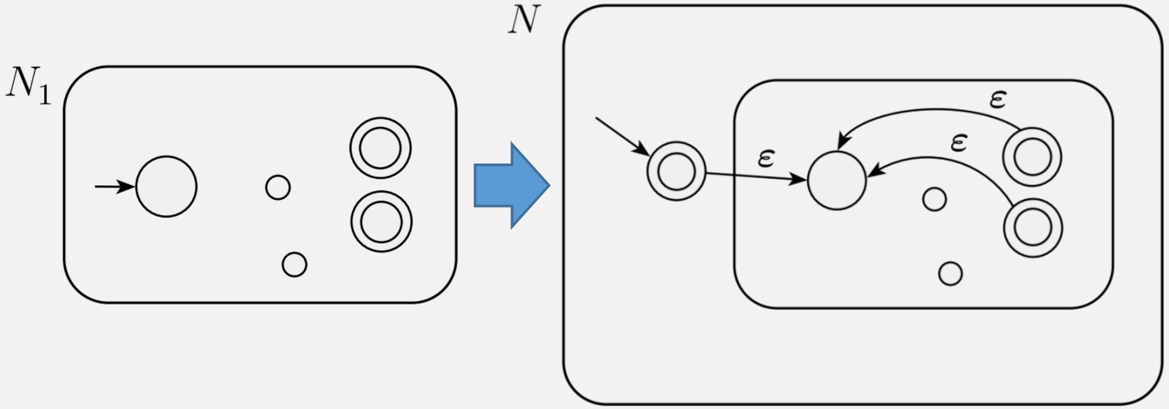
Kleene Star is Closed for Regular Langs

THEOREM

The class of regular languages is closed under the star operation.

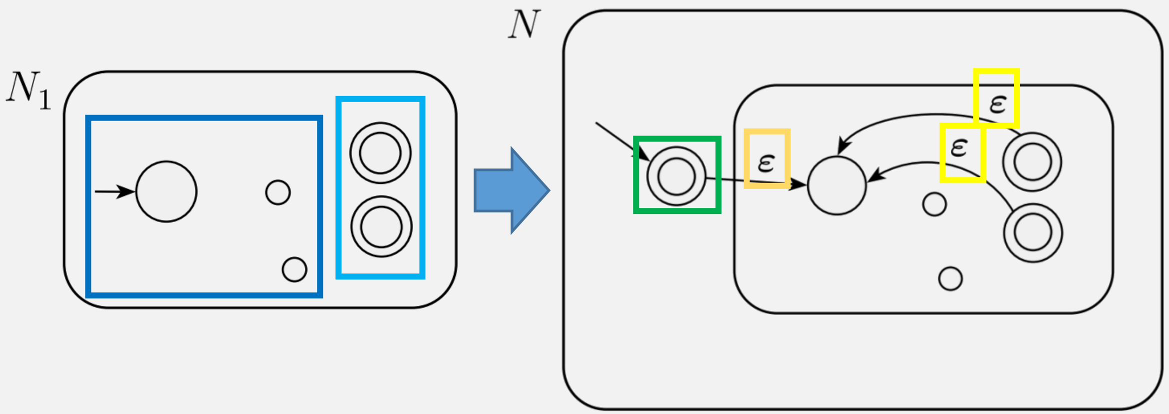
Kleene Star is Closed for Regular Langs

PROOF Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1. Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize A_1^*.



1. $Q = \{q_0\} \cup Q_1$
2. The state q_0 is the new start state.
3. $F = \{q_0\} \cup F_1$

Kleene Star is Closed for Regular Langs



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Kleene star of a language must accept the empty string!

Many More Closed Operations on Regular Languages!

- Complement
- Intersection
- Difference
- Reversal
- Homomorphism
- (See HW2)

Check-in Quiz 2/7

On gradescope