An efficient method to solve ODE with delta function

Denis Pleshkov, std.approach@gmail.com, 2022

This article and supporting scripts could be downloaded from:
https://github.com/stdapproach/sciArticle/tree/master/ODE_Delta

Abstract

This article is devoted to Linear Time-Invariant (LTI) Ordinal Differential Equations (ODE) with terms consisting of Dirac delta functions. The algorithm to exchange the original non-homogenous ODE with non-null initial condition (IC) to a homogenous one with different IC is provided. The resulting ODE could be solved analytical or numerical method with. Provided some examples with known analitycal solution to check the algorythm.

Keywords:

Impulse response function, time domain, linear ODE, Delta function

Introduction

The dynamics of evolving processes is often subjected to abrupt changes such as:
• impact by hammer to a beam,
• a bat striking a ball or a bolt of lightning striking a tower.

Often these short-term perturbations are treated as having acted instantaneously or in the form of “impulses”. Acording to Rao (p.381) impulsive force - a forse that has a large magnitude and acts for a very short time.
In this case, the output corresponding to this sudden force is referred to as the impulse response function (IFR).
Mathematically, an impulse can be modeled by an initial value problem (IVP) with a special type of function known as the Dirac delta function as the external force, i.e., the non-homogeneous term. The impulse response of a system is its response to the input $\delta(t)$ when the system is initially at rest.
According to Cohen (p.13) “The impulse function is useful when we are trying to model physical situations, such as the case of two billiard balls impinging, where we have a large force acting for a short time which produces a flnite change of momentum.”

We’ve tried to find a method for solve such kind systems. We only founded solutions for particular First and Second order’s ODE. So we decided to (re-)invent it.

To understand this paper there is only need to have a basic knowledge of ODE, Laplace Transform and Linear Algebra.

1 Definition & Terminology

Function:

$$y(t) \text{ - function takes an argument} \in \mathbb{R} \text{ and returns a result} \in \mathbb{R}$$

Derivative:

$$y' = \frac{dy}{dt} \ , y'' = \frac{d^2y}{dt^2} \ , y^{(n)} = \frac{d^ny}{dt^n} \ , y^{(0)} = y$$

Heaviside step function

$$H(t)$$

Dirac delta function

$$\delta(t) = \frac{dH(t)}{dt}$$

Delta function is a well known mathematical object. The property of it could be found at: Balakumar (p.287), Bottega (p.233), Chasnov (p.62), Finan (p.53), Nagy (p.185), Rao (p.381), Weber (p.86), Zill (p.292), Appendix A this article.

Initial Value Problem (IVP), Caushy problem

LTI ODE

$$L_n(\{a\}, y) = \sum_{i=0}^{n} a_i y^{(n-i)}(t) = f(t) \ , a_i = const \in \mathbb{R} \ , i \in 0 \dots n \tag{1.1}$$

and initial conditions (IC)

$$\{y\}|_{t_0} = IC|_{t_0} = \left\{ \begin{array}{c} y(t_0) \\ y'(t_0) \\ y''(t_0) \\ \vdots \\ y^{(n-1)}(t_0) \end{array} \right\} \tag{1.2}$$

named IVP which has an unique solution y(t) satisfied (1.1) and (1.2)
We use the following 3 equivalence short forms for IVP:

$$\begin{cases} L_n(y) = f(t) \\ \{y\}|_{t_0} \end{cases} \equiv \begin{cases} L_n(\{a\}, y) = f(t) \\ IC|_{t_0} = IC_0 \end{cases} \equiv IVP(\{a\}, f(t), t_0, IC_0)$$

Impulse Response (IR)

The impulse-response function g(t) is thus the response of a linear time-invariant system to a unit-impulse input when the initial conditions are zero. The Laplace transform of this function gives the transfer function (Ogata p.17).

2 First glimpse

Let's take a look at first order system
(order of a differential equation is the largest derivative present in the differential equation):

$$\begin{cases} x' + Ax = Bu(t) \\ x(0) = x_0 \end{cases}$$

We can rewrite it using our notation:

$$\begin{cases} L_n(\{1, A\}, x) = Bu(t) \\ IC|_{t_0=0} = x_0 \end{cases} \tag{2.1}$$

Solution of (2.1) is a function

$$x(t) = e^{-At}x_0 + \int_0^t e^{-A(t-\tau)}Bu(\tau)d\tau \tag{2.2}$$

The expression (2.2) delivers solution for the equation (2.1), which could be rewritten in a short form

$$x(t) = \begin{cases} L_n(\{1, A\}, x) = Bu(t) \\ IC|_{t_0=0} = x_0 \end{cases}$$

The solution of homogenous system (free response) is:

$$x_{free}(t) = \begin{cases} L_n(\{1, A\}, x) = 0 \\ IC|_{t_0=0} = x_0 \end{cases} = e^{-At}x_0$$

And substitute the Dirac delta function as load, so the system (2.1) becomes as

$$\begin{cases} x' + Ax = B\delta(t), \\ x(0) = x_0 \end{cases} \tag{2.3}$$

The solution is:

$$\begin{aligned} x_\delta(t) = x_0 e^{-At} + \int_0^t e^{-A(t-\tau)}B\delta(\tau)d\tau = \\ = x_0 e^{-At} + Be^{-At} = e^{-At}(x_0 + B) \end{aligned} \tag{2.4}$$

Obviosely the solution of the system (2.4) is the same as solution of next one:

$$\begin{cases} x' + Ax = 0, \\ x(0) = x_0 + B \end{cases} \tag{2.5}$$

So we can write a next statement:

$$\begin{cases} L_n(\{1,A\},y) = B\delta(t) \\ IC|_{t_0=0} = x_0 \end{cases} \equiv \begin{cases} L_n(\{1,A\},y) = \mathbf{0} \\ IC|_{t_0=0} = x_0 + \mathbf{B} \end{cases} \tag{2.6}$$

For the system (2.3) the solution is the same as free response of the same system but changed IC.

About changing IC

Some books provided an analitical solution for LTI ODE with delta function as load. For example: Finan (p.57), Nagy (pp.189-190), Ogata (p.190), Oliveira and Cortes (p.3), Rao (p.381), Zill (p.293).
Some other books noticed that the solution of IVP with delta function as load is the same as solution the similar homogenous ODE with defferernt IC.
Genta (p.180) provided the formulae for changing zero IC for second-order ODE in case of delta function loaded.
Rao (p.407) noticed that following systems are equal.

$$\begin{cases} y' + ay = F\delta(t), \\ y(0) = 0 \end{cases} \equiv \begin{cases} y' + ay = 0, \\ y(0) = F \end{cases}$$

Weber (p.733) noticed that following systems are equal.

$$\begin{cases} mx'' = P\delta(t), \\ y(0) = 0 \\ y'(0) = 0 \end{cases} \equiv \begin{cases} mx'' = 0, \\ y(0) = 0 \\ y'(0) = P/m \end{cases}$$

Kelly (p.315) noticed that following systems are equal.

$$\begin{cases} mx'' + cx' + kx = \delta(t), \\ y(0) = 0 \\ y'(0) = 0 \end{cases} \equiv \begin{cases} mx'' + cx' + kx = 0, \\ y(0) = 0 \\ y'(0) = 1/m \end{cases}$$

And Balachandran (pp.287-288), Beards (p.66), Bottega (pp.235-236), Genta (p.179), Meirovich (pp.160-161), Schiff (p.83) and Schmitz (p.118) noticed that the solution of following systems are equal.

$$\begin{cases} mx'' + cx' + kx = f_0\delta(t), \\ y(0) = 0 \\ y'(0) = 0 \end{cases} \equiv \begin{cases} mx'' + cx' + kx = 0, \\ y(0) = 0 \\ y'(0) = f_0/m \end{cases}$$

Chasnov (p.63) provided formula to change IC for second-order LTI ODE and Oliveira and Cortes (p.2) provided similar solution for particular systems with zero initial condition.
It looks like (but not proven yet) these two following systems are equal:

$$\begin{cases} L_n(\{a\},y) = b\delta(t) \\ IC|_{t_0} = IC_0 \end{cases} \equiv \begin{cases} L_n(\{a\},y) = 0 \\ IC|_{t_0} = IC_0 + \{0,0,\ldots,b/a_0\}^\mathsf{T} \end{cases} \tag{2.7}$$

3. A problem Type 0

This part regarding the following problems.

Type 0a

$$\begin{cases} L_n(y) = b\delta(t), \\ IC_0, n \geq 1 \end{cases}$$

Type 0b

$$\begin{cases} L_n(y) = b\delta(t-c), \\ IC_0, n \geq 1 \end{cases}$$

Type 0c

$$\begin{cases} L_n(y) = \sum_{i=0}^{k} b_i \delta(t-c_i), \\ IC_0, n \geq 1 \end{cases}$$

We suppose that $t_0 = 0$, otherwise for such time-invariant systems we could just change time variable as $t^* = t - t_0$.

3.1 Problem Type 0a

Find an IC for homogenous system which delivers the equivalence for non-homogenous one.
At this case 'equivalence' means:

$$z(t) = y(t), \forall t \geq t_0$$

for the next systems: a given non homogenous system

$$\begin{cases} L_n(\{a\}, y) = b\delta(t) \\ IC|_{t_0=0} = IC_y \end{cases}$$

the homogenous system which IC supposed to be found

$$\begin{cases} L_n(\{a\}, z) = 0 \\ IC_z \end{cases}$$

3.1.1 Laplace Transform

In mathematics, the Laplace transform (LT) is an integral transform. It takes a function of a real variable t (often time) to a function of a complex variable s (complex frequency).
LT are usually restricted to functions of t with t ≥ 0.
LT is invertible on a large class of functions. The inverse Laplace transform takes a function of a complex variable s (often frequency) and yields a function of a real variable t (time).

LT could be usefull solving differenfial equation, integral equation and so on.
So, for example, Laplace transformation from the time domain to the frequency domain transforms differential equations into algebraic equations and convolution into multiplication.

You could found a lot of information regarding LT over here:
○ https://en.wikipedia.org/wiki/Laplace_transform
○Cohen (p.12)
○Nagy (p.196)
○Weber (p.693)
○table of LT for some function [Schiff p.210]

3.1.2 LT for ODE

The Laplace transform method for solving ODE could be summarized by the following steps:
○Take the Laplace transform of both sides of the equation.
○Obtain an equation $L(y) = F(s)$, where $F(s)$ is an algebraic expression in the variable s.
○Apply the inverse transform to yield the solution $y(t) = L^{-1}(F(s))$.
More information about solving ODE by LT you could found over here:
○ Kohen (p.7)
○ Schiff Schiff (p.59)
○ Xue p.380 provided a table for inverse LT for some functions

3.1.3 Find a solution for Type 0b

To solve the problem Type 0a let's perform Laplace Transform (LT) for $y(t)$ with respect to non-null initial condition.

$$LT\{y(t)\} = Y(s)$$

$$Y(s) = LT\left\{\sum_{i=0}^{n}\left(a_i y^{(n-i)}(t) - b\delta(t)\right)\right\} = \sum_{i=0}^{n}\left(a_i L\{y^{(n-i)}(t)\}\right) - bL\{\delta(t)\}$$

-

$$LT\{a_0 y^{(n-0)}\} = a_0 \Big[s^n Y - s^{n-1}y(0) - s^{n-2}y'(0) - s^{n-3}y''(0) - \ldots - s^1 y^{(n-2)}(0) - s^0 y^{(n-1)}(0)\Big]$$

$$LT\{a_1 y^{(n-1)}\} = a_1 \Big[s^{n-1} Y - s^{n-2}y(0) - s^{n-3}y'(0) - s^{n-2}y''(0) - \ldots - s^1 y^{(n-3)} - s^0 y^{(n-2)}(0)\Big]$$

$$LT\{a_2 y^{(n-2)}\} = a_2 \Big[s^{n-2} Y - s^{n-3}y(0) - s^{n-4}y'(0) - s^{n-5}y''(0) - \ldots - s^1 y^{(n-4)}(0) - s^0 y^{(n-3)}(0)\Big]$$

$$\ldots$$

$$LT\{a_{n-1} y'\} = a_{n-1}\Big[s' Y - s^0 y(0)\Big]$$

$$LT\{a_n y\} = a_n Y$$

$$-bLT\{\delta(t)\} = -be^{(-s)0} = -b$$

Let’s using (1.2) and rewrite at this way:

$$\begin{aligned} LT\{y(t)\} = & s^n (a_0 Y) + \\ & s^{n-1}\left(a_1 Y - a_0 y_0\right) + \\ & s^{n-2}\left(a_2 Y - a_1 y_0 - a_0 y_1\right) + \\ & s^{n-3}\left(a_3 Y - a_2 y_0 - a_1 y_1 - a_0 y_2\right) + \\ & \ldots \\ & s^1\left(a_{n-1} Y - \sum_{i=0}^{n-2} a_{n-2-i} y_i\right) + \\ & s^0\left(a_n Y - \sum_{i=0}^{n-1} a_{n-1-i} y_i + b\right) = \\ & = \sum_{i=0}^{n} s^{n-i} a_i Y - \sum_{i=1}^{n}\left(s^{n-i}\sum_{j=0}^{i-1} a_{(i-1)-j} y_j\right) - b \end{aligned}$$

Let’s perform LT for $z(t)$ with respect to non-null initial condition.

$$LT\{z(t)\} = Z(s)$$

$$LT\{z(t)\} = \sum_{i=0}^{n} s^{n-i} a_i Z - \sum_{i=1}^{n}\left(s^{n-i}\sum_{j=0}^{i-1} a_{(i-1)-j} z_j\right)$$

$$y(t)=z(t) \implies Y(s)=Z(s) \implies$$
$$\sum_{i=0}^{n} s^{n-i} a_i Y - \sum_{i=1}^{n}\left(s^{n-i}\sum_{j=0}^{i-1} a_{(i-1)-j} y_j\right) - b = \sum_{i=0}^{n} s^{n-i} a_i Z -$$
$$\sum_{i=1}^{n}\left(s^{n-i}\sum_{j=0}^{i-1} a_{(i-1)-j} z_j\right)$$

Due to

$$s^{n-i} a_i Y = s^{n-i} a_i Z, i=0...n \implies$$

$$\sum_{i=1}^{n}\left(s^{n-i}\sum_{j=0}^{i-1} a_{(i-1)-j} y_j\right) - b = \sum_{i=1}^{n}\left(s^{n-i}\sum_{j=0}^{i-1} a_{(i-1)-j} z_j\right) \implies$$

following equations:

$$\begin{aligned}
&i=1 \rightarrow (s^{n-1}) : a_0 y_0 = a_0 z_0 \\
&i=2 \rightarrow (s^{n-2}) : \sum_{j=0}^{1} a_{1-j} y_j = \sum_{j=0}^{1} a_{1-j} z_j \\
&\dots \\
&i=n \rightarrow (s^{n-n}) : \sum_{j=0}^{n-1} a_{n-1-j} y_j + b = \sum_{j=0}^{n-1} a_{n-1-j} z_j
\end{aligned}$$

Rewrite it at this way:

$$A = \begin{bmatrix} a_0 & 0 & 0 & \cdots & 0 \\ a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{bmatrix}, \{d\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b \end{Bmatrix} \tag{3.1}$$

$$[A]\begin{Bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{Bmatrix} + \{d\} = [A]\begin{Bmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{n-1} \end{Bmatrix}$$

$$[A]\{y\} + \{d\} = [A]\{z\}, det([A]) \neq 0 \Rightarrow$$

$$\{z\} = \{y\} + [A]^{-1}\{d\} \tag{3.2}$$

Due to $[A]$ is lower-triangle matrix and $\{d\} = \{0, 0, \ldots, b\}$ the main result is following:

$$\boxed{\begin{cases} L_n(\{a\}, y) = b\delta(t) \\ IC_0 \end{cases} \equiv \begin{cases} L_n(\{a\}, y) = \mathbf{0} \\ IC_0 + [\mathbf{A}]^{-1}\{\mathbf{d}\} \end{cases} \equiv \begin{cases} L_n(\{a\}, y) = 0 \\ IC_0 + \{0, 0, \ldots, b/a_0\}^{\mathsf{T}} \end{cases}} \tag{3.3}$$

The formula (3.3) is the mathematical notation of algorythm how to solve LTI ODE with Dirac function load. There is only need to change IC and solve the similar homogenous system using any method you like (analytical or numerical). A numerical one called Runge-Kutta fourth order method could be found at Butcher (p.98).
There has been proven (2.7). Indeed, (3.3) and (2.7) are the same.
The system Type 0a could be rewritten as simple impulse differential equation (look Benchohra, Henderson and Ntouyas “Impulsive Differential Equations and Inclusions”)

3.2 Problem Type 0b

The idea how to solve the Type 0b’s problem is very simple and well explained at the part 4.4-4.6.

3.3 Problem Type 0c

The idea how to solve the Type 0c’s problem is very simple and well explained at the part 4.7.

4. Verification by examples for Type 0

Let’s check the main result from previous chapter on examples Appendix B from (system Type0). To prove the method we’ve created number Python scripts performing the calculation and generating the charts with results. All scripts could be found at https://github.com/stdapproach/sciArticle/tree/develop/ODE_Delta/raw .

4.1 Example1 [Oliveira and Cortes, p.3], [Schiff, p.82]

Consider the following first order ODE (Type 0a)

$$\begin{cases} y'' + ay' = \delta(t), \\ y(0) = y'(0) = 0 \end{cases} \Rightarrow IVP(\{1\ a\ 0\}, \delta(t), t_0 = 0, y_0 = \{0, 0\}) \Rightarrow$$

according to (3.1)

$$A = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}, \{d\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \Rightarrow$$
$$A^{-1} = \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix}, A^{-1}\{d\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

According (3.2) and (3.3) these two following system are equal by solution

$$\begin{cases} y'' + ay' = \delta(t), \\ y(0) = y'(0) = 0 \end{cases} \equiv \begin{cases} y'' + ay' = 0, \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

In short form:

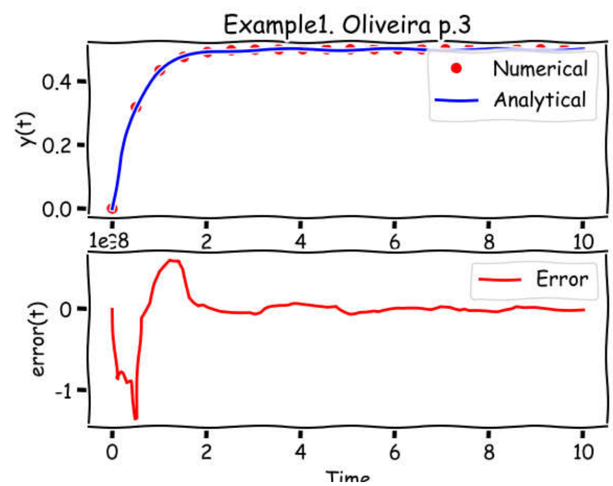
$$IVP(\{1\ a\ 0\}, \delta(t), t_0 = 0, y_0 = \{0, 0\})$$
$$\equiv IVP(\{1\ a\ 0\}, 0, t_0 = 0, y_0 = \{0, 1\})$$

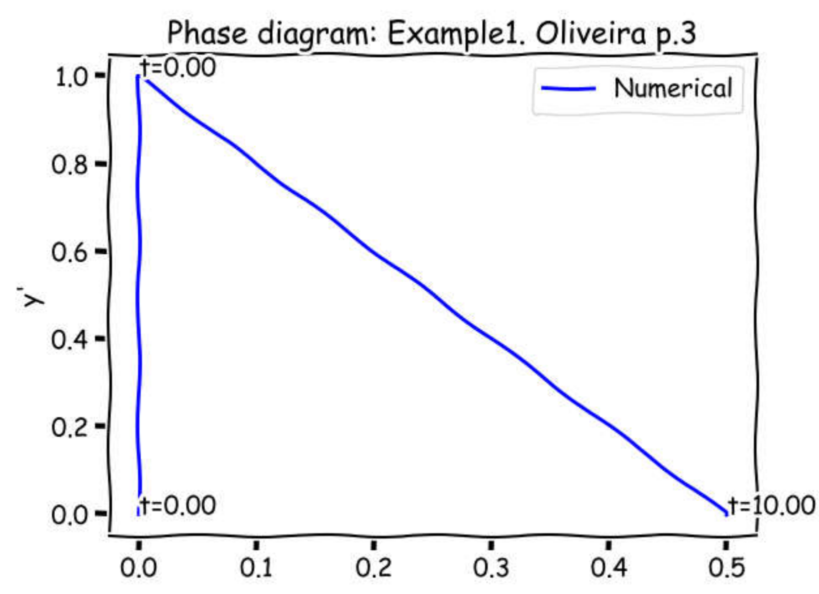
Let's check how to correspond the numerical solution for a=2 for homogenous sytem with non-null IC with analitical solution for the system

Analitycal solution taken from Appendix B:

$$y(t) = \frac{1}{a}\left(1 - e^{-at}\right)$$

Numerical solution, analytical solutions and error provided by Python's script (example1.py):





4.2 Example2 [Finan, pp.57-58]

Considering the following second order ODE (Type 0a)

$$\begin{cases} 2y'' + 4y' + 10y = \delta(t) \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \end{cases} \Rightarrow IVP(\{2, 4, 10\}, \delta(t), t_0 = 0, y_0 = \{0, 0\}) \Rightarrow$$

$$A = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix}, \{d\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \Rightarrow$$

$$A^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1 & 1/2 \end{bmatrix}, A^{-1}\{d\} = \begin{Bmatrix} 0 \\ 1/2 \end{Bmatrix} \Rightarrow$$

$$\{z\}_0 = \{y\}_0 + [A]^{-1}\{d\} = \begin{Bmatrix} 0 \\ 1/2 \end{Bmatrix}$$

These two following system are equal by solution

$$\begin{cases} 2y'' + 4y' + 10y = \delta(t) \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \end{cases} \equiv \begin{cases} 2z'' + 4z' + 10z = 0 \\ z_0 = z(0) = 0 \\ z_1 = z'(0) = 1/2 \end{cases}$$

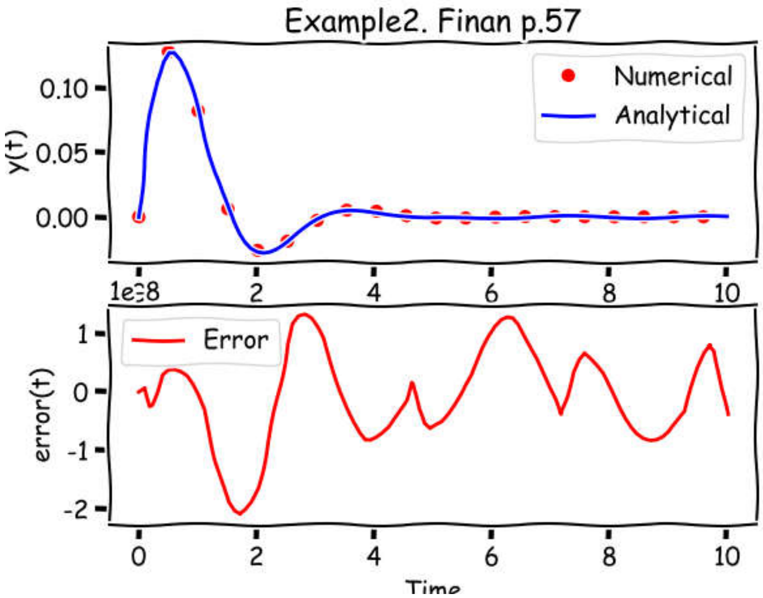
In short form:

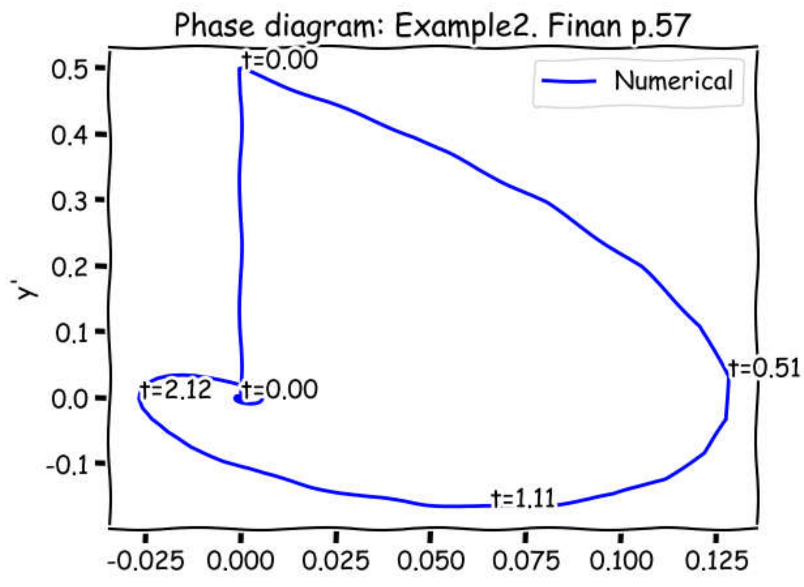
$$IVP(\{2, 4, 10\}, \delta(t), t_0 = 0, y_0 = \{0, 0\}) \equiv IVP(\{2, 4, 10\}, 0, t_0 = 0, y_0 = \{0, 1/2\})$$

Analitycal solution:

$$y(t) = \frac{1}{4} e^{-t} sin(2t)$$

Numerical solution, analytical solutions and error provided by Python’s script (example2.py):





4.3 Example3 [Nagy, p.189]

Considering the following second order ODE (Type 0a)

$$\begin{cases} y'' + 2y' + 2y = \delta(t) \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \end{cases} \Rightarrow IVP(\{1, 2, 2\}, \delta(t), t_0 = 0, y_0 = \{0, 0\}) \Rightarrow$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \{d\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \Rightarrow$$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, A^{-1}\{d\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \Rightarrow$$

$$\{z\}_0 = \{y\}_0 + [A]^{-1}\{d\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

These 2 folloing system are equal by solution:

$$\begin{cases} y'' + 2y' + 2y = \delta(t) \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \end{cases} \Rightarrow \begin{cases} z'' + 2z' + 2z = \delta(t) \\ z_0 = z(0) = 0 \\ z_1 = z'(0) = 1 \end{cases}$$

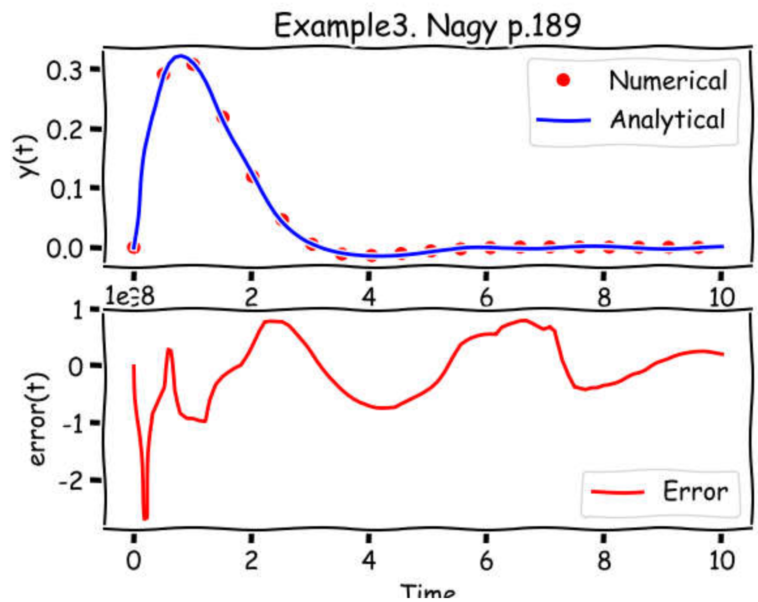
In short form:

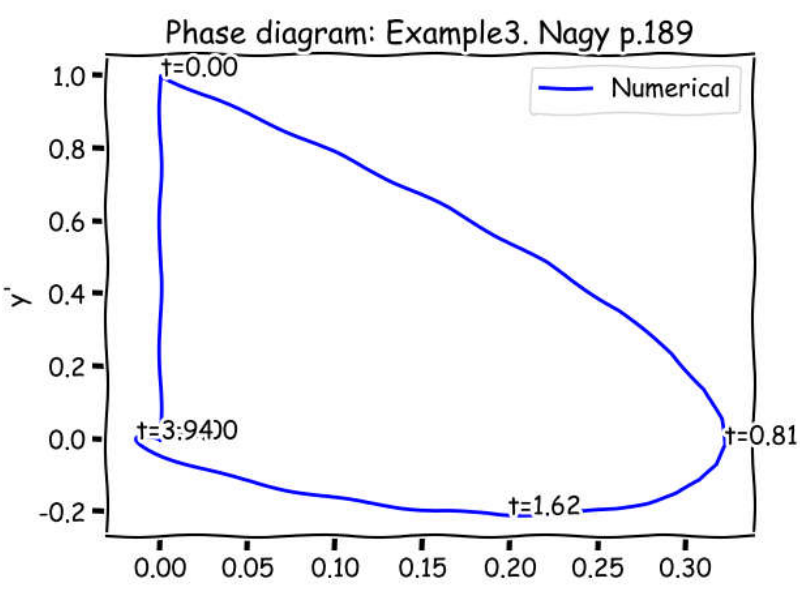
$$IVP(\{1, 2, 2\}, \delta(t), t_0 = 0, y_0 = \{0, 0\})$$
$$\equiv IVP(\{1, 2, 2\}, 0, t_0 = 0, y_0 = \{0, 1\})$$

Analitycal solution:

$$y(t) = e^{-t} sin(t)$$

Numerical solution, analytical solutions and error provided by Python's script (example3.py):





4.4 Example4 [Nagy, p.189]

Considering the following second order ODE (Type 0b)

$$\begin{cases} y'' + 2y' + 2y = \delta(t-c) \\ y(0) = y'(0) = 0 \\ c = 2 \end{cases}$$

This system could be separated on two systems and the IC for a second system based on results first one:

$$\begin{cases} y_1'' + 2y_1' + 2y_1 = 0 \\ y_1(0) = 0 \\ y_1'(0) = 0 \\ 0 \leq t \leq c \end{cases}$$

$$\begin{cases} y_2'' + 2y_2' + 2y_2 = \delta(t-c) \\ y_2(c) = y_1(c) \\ y_2'(c) = y_1'(c) \\ t \geq c \end{cases}$$

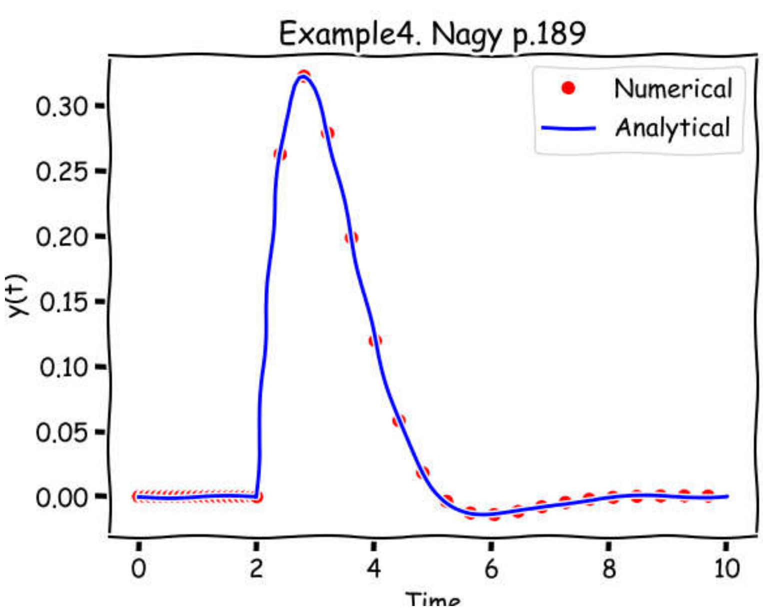
So, the solution of the original system is:

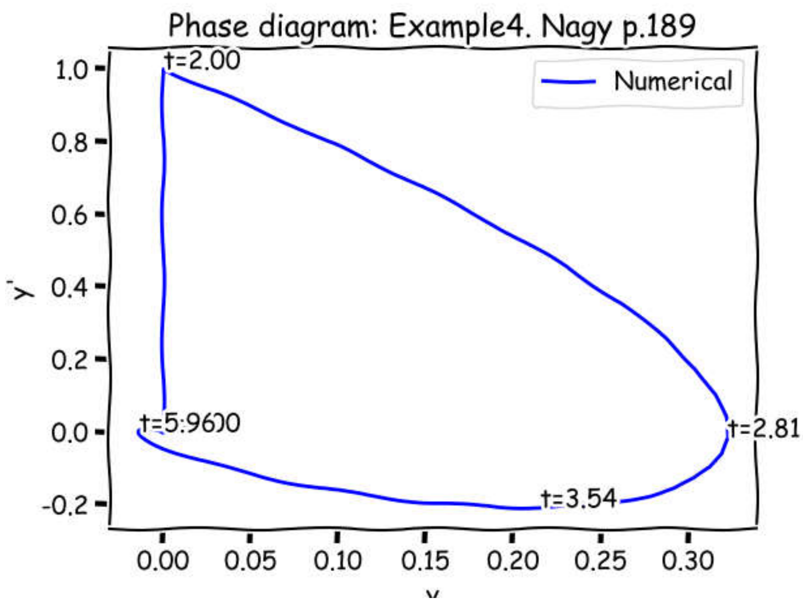
$$y(t) = \begin{cases} y_1(t), & \text{if } 0 \leq t \leq c \\ y_2(t), & \text{if } t \geq c \end{cases}$$

Analitycal solution:

$$y(t) = H(t-c)e^{-(t-c)}sin(t-c)$$

Numerical solution and analytical solutions provided by Python's script (example4.py):





4.5 Example5 [Chasnov, p.65]

Considering the following second order ODE (Type 0b)

$$\begin{cases} 2y'' + y' + 2y = \delta(t-c) \\ y(0) = y'(0) = 0 \\ c = 2 \end{cases}$$

This system could be separated on two systems and the IC for a second system based on results first one:

$$\begin{cases} 2y_1'' + y_1' + 2y_1 = 0 \\ y_1(0) = 0 \\ y_1'(0) = 0 \\ 0 \le t \le c \end{cases}$$

$$\begin{cases} 2y_2'' + y_2' + 2y_2 = \delta(t-c) \\ y_2(c) = y_1(c) \\ y_2'(c) = y_1'(c) \\ t \ge c \end{cases}$$

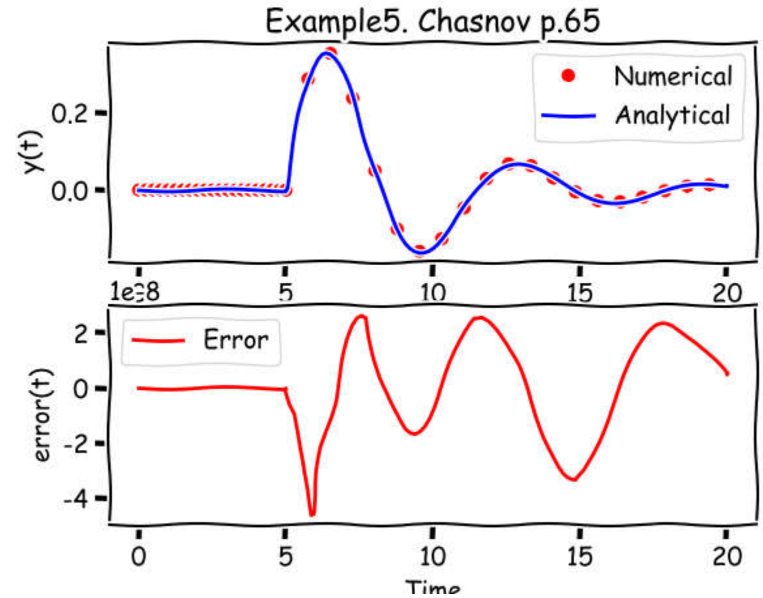
So, the solution of the original system is:

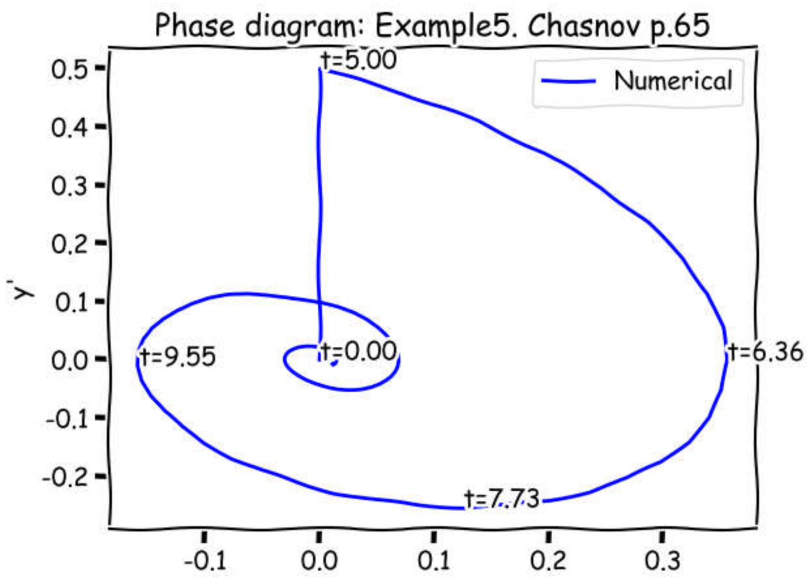
$$y(t) = \begin{cases} y_1(t), & \text{if } 0 \le t \le c \\ y_2(t), & \text{if } t \ge c \end{cases}$$

Analitycal solution:

$$y(t) = \frac{2}{\sqrt{15}} H(t-5) e^{-(t-5)/4} sin(\sqrt{15}(t-5)/4)$$

Numerical solution and analytical solutions provided by Python’s script (example5.py):





4.6 Example6 [Zill, p.293]

Considering the following second order ODE (Type 0b)

$$\begin{cases} y'' + y = 4\delta(t - 2\pi), \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \end{cases}$$

This system could be separated on two systems, and the IC for a second system based on results first one:

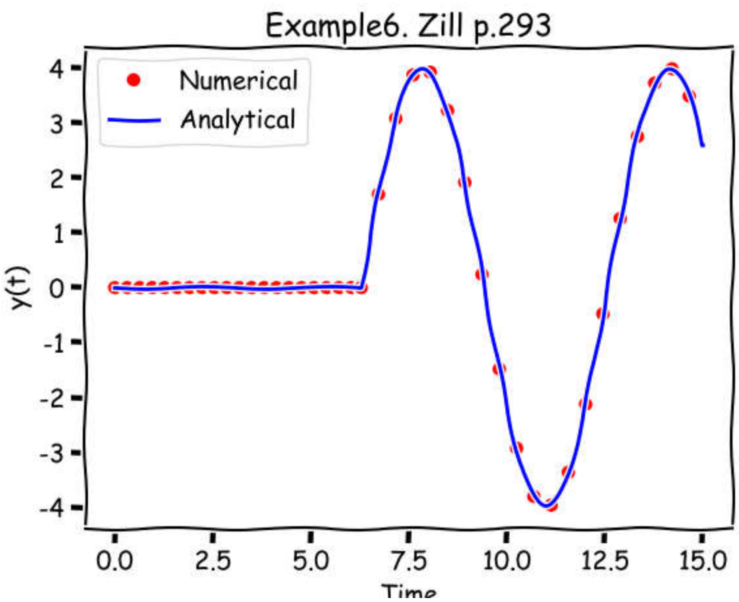
$$\begin{cases} y_1'' + y_1 = 0, \\ y_1(0) = 0 \\ y_1'(0) = 0 \\ 0 \le t \le 2\pi \end{cases}$$

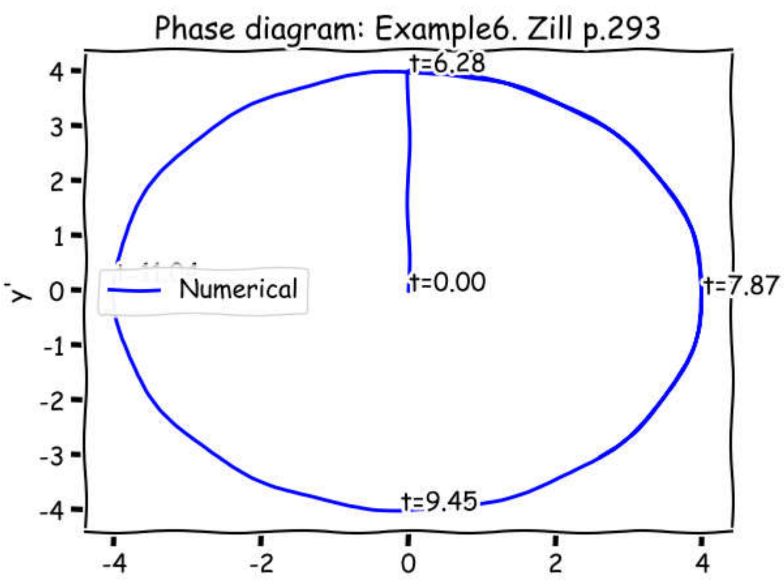
$$
\begin{cases}
y_2'' + y2 = 4\delta(t - 2\pi), \\
y_2(2\pi) = y_1(2\pi) \\
y_2'(2\pi) = y_1'(2\pi) \\
t \geq 2\pi
\end{cases}
$$

Analitycal solution:

$$y(t) = H(t - 2\pi)4sin(t)$$

Numerical solution and analytical solutions provided by Python's script (example6.py):





4.7 Example7 [Zill, p.293]

Considering the following second order ODE (Type 0b)

$$\begin{cases} y'' + y = 4\delta(t - 2\pi), \\ y_0 = y(0) = 1 \\ y_1 = y'(0) = 0 \end{cases}$$

This system could be separated on two systems, moreover the IC for a second system based on results first one:

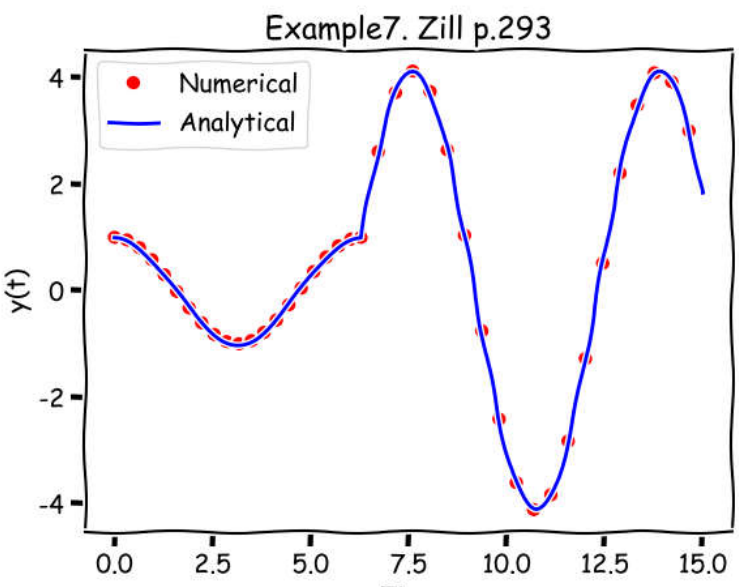
$$\begin{cases} y_1'' + y_1 = 0, \\ y_1(0) = 1 \\ y_1'(0) = 0 \\ 0 \le t \le 2\pi \end{cases}$$

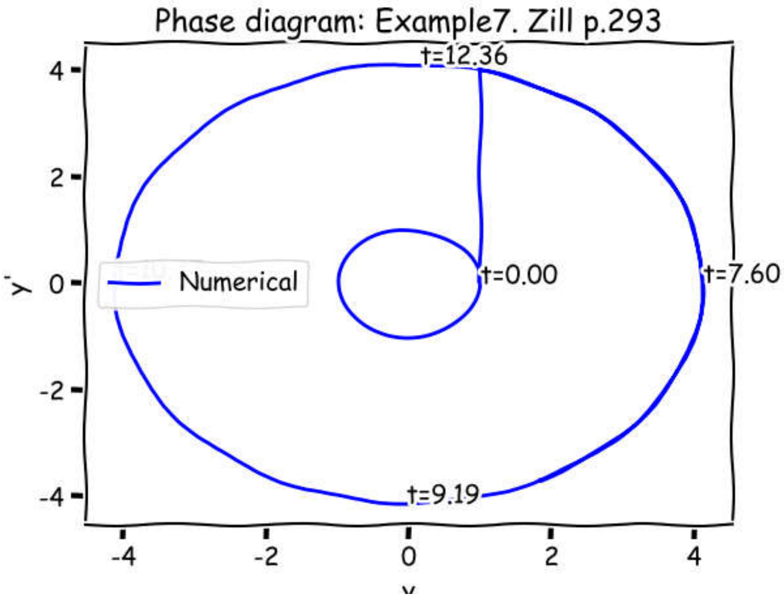
$$\begin{cases} y_2'' + y_2 = 4\delta(t - 2\pi), \\ y_2(2\pi) = y_1(2\pi) \\ y_2'(2\pi) = y_1'(2\pi) \\ t \ge 2\pi \end{cases}$$

Analitycal solution:

$$y(t) = cos(t) + 4H(t, 2\pi)sin(t)$$

Numerical solution and analytical solutions provided by Python’s script (example7.py):





4.8 Example8 [Nagy, p.190]

Considering the following second order ODE (Type 0c)

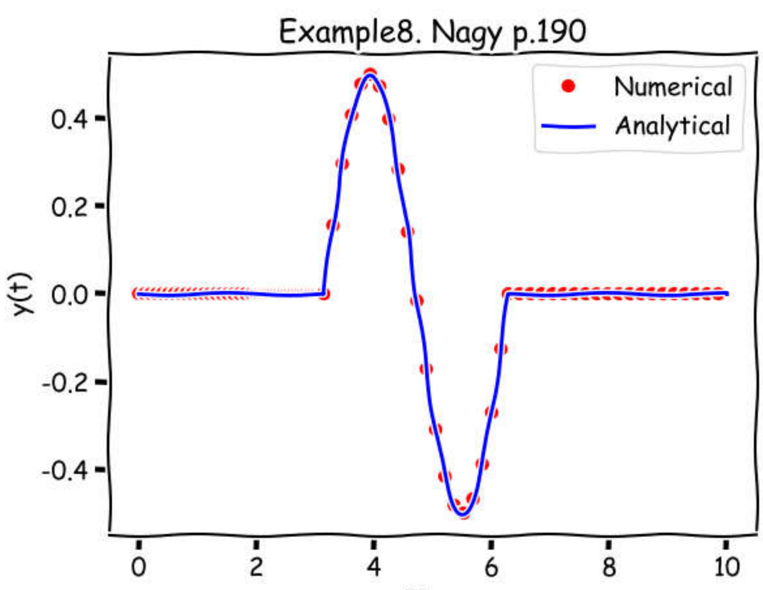
$$\begin{cases} y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \\ y(0) = y'(0) = 0 \end{cases}$$

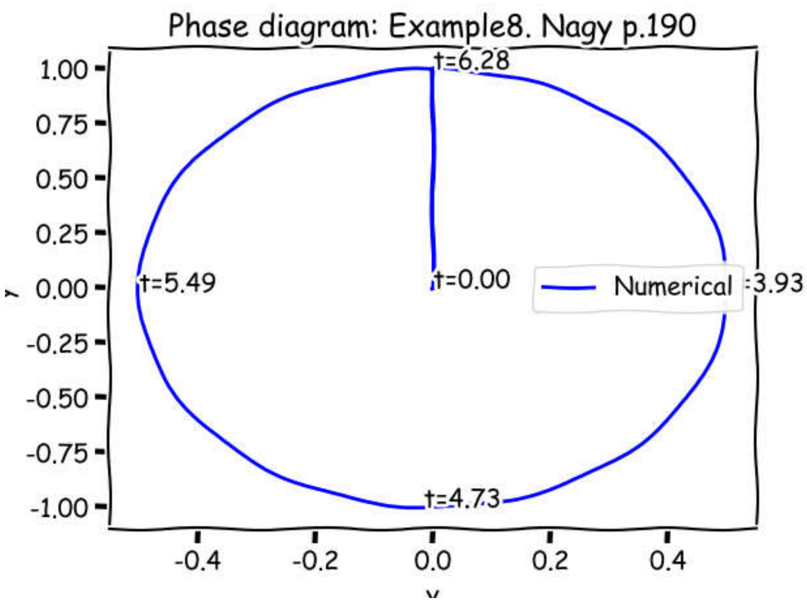
To solve this type of system we recommend to use the same approach as used for Type 0b (see 4.4). I.e. separate the orginal system on time line to some number similar system. And the IC for the next system should be taken from previous one.

Analitycal solution:

$$y(t) = \frac{1}{2}\ [H(t-\pi) - H(t-2\pi)]\, sin(2t)$$

Numerical solution and analytical solutions provided by Python's script (example8.py):





Note: this example shows that impulse load could be using to generate of vibration and to damper it.

4.9 Example9

Considering the following third order ODE (Type 0a)

$$\begin{cases} y''' + 2y'' + 2y' = \delta(t) \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \\ y_2 = y''(0) = 0 \end{cases} \Rightarrow IVP(\{1,2,2,0\}, \delta(t), t_0 = 0, y_0 = \{0,0,0\})$$

$$\Rightarrow$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}, \{d\} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \Rightarrow$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}, A^{-1}\{d\} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \Rightarrow$$

$$\{z\}_0 = \{y\}_0 + [A]^{-1}\{d\} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

These 2 folloing system are equal by solution

$$\begin{cases} y''' + 2y'' + 2y' = \delta(t) \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \\ y_2 = y''(0) = 0 \end{cases} \equiv \begin{cases} y''' + 2y'' + 2y' = 0 \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \\ y_2 = y''(0) = 1 \end{cases}$$

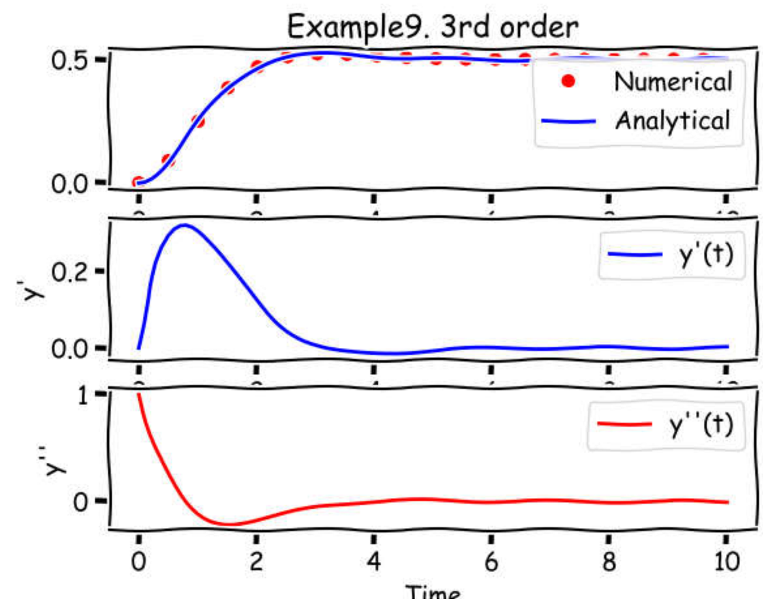
In short form:

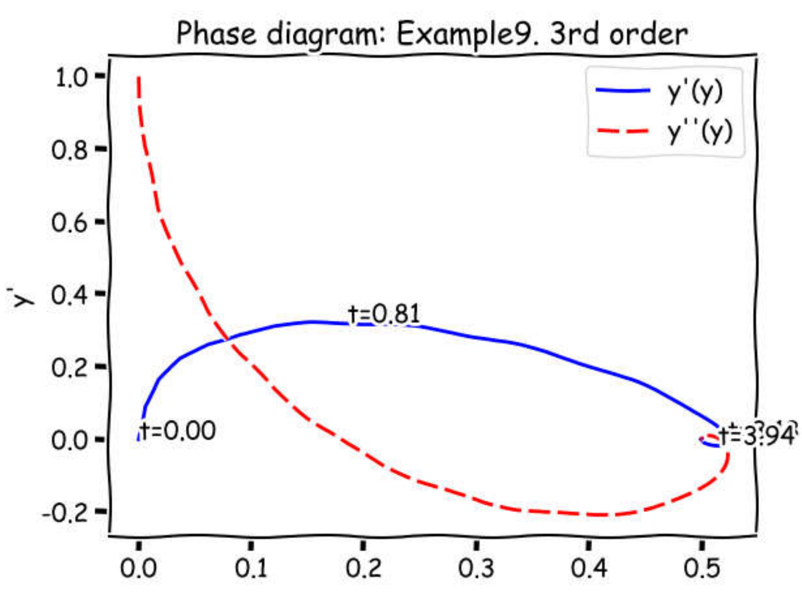
$$IVP(\{1, 2, 2, 0\}, \delta(t), t_0 = 0, y_0 = \{0, 0, 0\})$$
$$\equiv IVP(\{1, 2, 2, 0\}, 0, t_0 = 0, y_0 = \{0, 0, 1\})$$

Analitycal solution:

$$y(t) = \frac{1}{2} - \frac{1}{2}\, e^{-t}\, (sin(t) + cos(t))$$

Numerical solution and analytical solutions provided by Python’s script (example9.py):





5. A problem Type 1

This part ragarding the following problem, we called as Type 1:

$$\sum_{i=0}^{n} a_i y^{(n-i)}(t) = \sum_{j=0}^{m} b_j \delta^{(m-j)}(t), \tag{5.1}$$
$$m < n$$

$$\{y\}|_{t_0} = IC|_{t_0} = \begin{Bmatrix} y(t_0) \\ y'(t_0) \\ y''(t_0) \\ \vdots \\ y^{(n-1)}(t_0) \end{Bmatrix} \tag{5.2}$$

Again, (5.1) and (5.2) together delivered IVP and we suppose that $t_0 = 0$, otherwise ... you know how to handle with it. We explain later why m should be less than n.

5.1 Change IC for IVP

Again, we're trying to find a solution for change IC for IVP such as (5.1, 5.2) which delivered an eqivalence solution for honogenous system.

To solve this problemlet's perform LT for $y(t)$, and right side with respect to non-null initial condition.
Skipping the trivial part we've got the similar for (3.1) equation:

Rewrite it at this way:

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{bmatrix}, \{d\} = \begin{Bmatrix} 0 \\ \vdots \\ b_0 \\ \vdots \\ b_{m-1} \\ b_m \end{Bmatrix} \tag{5.3}$$

The main result for Type1 is following:

$$\boxed{\begin{cases} L_n(\{a\}, y) = L_m(\{b\}, \delta) \\ IC_0 \end{cases} \equiv \begin{cases} L_n(\{a\}, y) = \mathbf{0} \\ IC_0 + [A]^{-1} \{0 \quad \cdots \quad b_0 \quad b_1 \cdots b_m\ \}^{\mathsf{T}} \end{cases}} \tag{5.4}$$

Now it's obvious why m should be less than n. Otherwise the system (5.3) couldn't been even written. Phisically it means that if m great or equal n the solution would contain Delta function (and/or it's derivatives) which is phisically impossible.

5.2 Connection for Control Theory

In control theory the impulse response is the response of a system to a Dirac delta input. This proves useful in the analysis of dynamic systems; the Laplace transform of the delta function is 1, so the impulse response is equivalent to the inverse Laplace transform of the system's transfer function
(https://en.wikipedia.org/wiki/Impulse_response#Control_systems)
This kind of knowledge also could be found at some books:
○ The impulse response completely characterizes the system [Genta p.180];
○ the impulse response is equal to the Inverse Laplace Transform of the transfer function [Meirovich p.180];
○ the transfer function is also the Laplace transform of its impulsive response, which is the response due to a unit impulse [Kelly p.314];
○ the response of a system due to a unit impulse can be determined as the free response with zero initial displacement and an initial velocity equal to velocity imparted by the impulse [Kelly p.370];
○ the transfer function and impulse-response function of a linear, time-invariant system contain the same information about the system dynamics. It is hence possible to obtain complete information about the dynamic characteristics of the system by exciting it with an impulse input and measuring the response [Ogata p.17].

$$IR = g(t) = L^{-1}(W(s));$$

$$W(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

$$W(s) = \frac{Y(s)}{X(s)} \iff \begin{cases} a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = \\ = b_0 x^{(m)} + b_1 s^{(m-1)} + \cdots + b_{m-1} x' + b_m x; \\ IC = \{0 \cdots 0\}^{\mathsf{T}} \end{cases}$$

$\Rightarrow$

$$IR = \begin{cases} L_n(\{a\}, y) = L_m(\{b\}, \delta) \\ IC_0 \end{cases} \tag{5.5}$$

6. Verification by examples for Type 1

Let's check the main result from previous chapter (5.5) on examples Appendix B from. First example regarding first order system (Type0a), the rest are system of Type1). To prove the method we've created number Python scripts performing the calculation and generating the charts with results.

6.1 Example10 [Ogata p.163]

$$C(s) = \frac{1}{Ts+1}$$

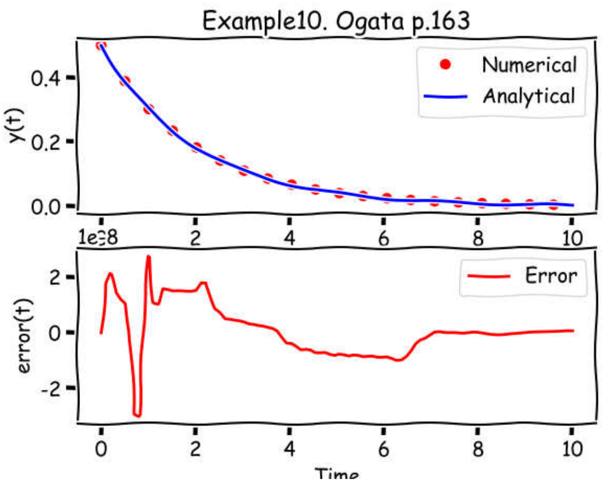
Impulse function for this transfer function corresponds to this system:

$$\begin{cases} Ty' + y = \delta(t), \\ y(0) = 0 \end{cases} \Rightarrow g(t) = IVP(\{T, 1\}, \delta(t), t_0 = 0, y_0 = \{0\})$$

Analitycal solution (impulse response) taken from Appendix B:

$$g(t) = \frac{1}{T} e^{-t/T}$$

Let’s check how to correspond the numerical solution for T=2 for homogenous sytem with zero IC with analitical solution for the system.
Numerical solution, analytical solutions and error provided by Python’s script (example10.py):



6.2 Example11 [Xue p.380]

$$TF = \frac{s}{(s+a)(s+b)} = \frac{s}{s^2 + (a+b)s + ab}$$

Impulse function for this transfer function corresponds to this system:

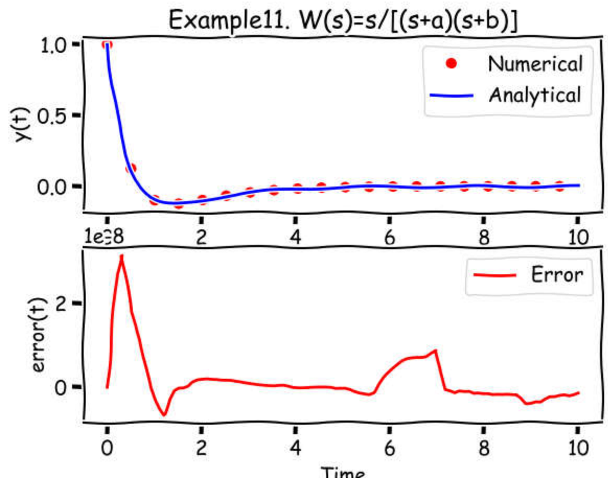
$$\begin{cases} y'' + (a+b)y' + ab \cdot y = \delta'(t) + 0 \cdot \delta(t), \\ y(0) = y'(0) = 0 \end{cases} \Rightarrow$$

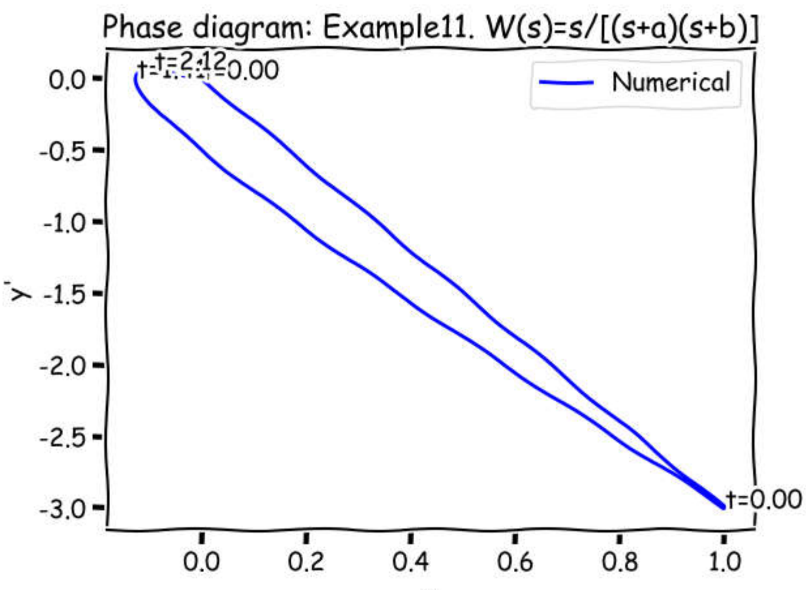
$$g(t) = IVP(\{1, (a+b), ab\}, \{1, 0\}, t_0 = 0, y_0 = \{0, 0\})$$

Analitycal solution (impulse response) taken from Appendix B:

$$g(t) = L^{-1}\left\{\frac{s}{(s+a)(s+b)}\right\} = \frac{1}{a-b}\left[ae^{-at} - be^{-bt}\right]$$

Let’s check how to correspond the numerical solution for a=1 and b=2 for homogenous sytem with zero IC with analitical solution for the system.
Numerical solution, analytical solutions and error provided by Python’s script (example11.py):





6.3 Example12 [Xue p.380]

$$TF = \frac{s+d}{(s+a)(s+b)} = \frac{s+d}{s^2+(a+b)s+ab}$$

Impulse function for this transfer function corresponds to this system:

$$\begin{cases} y'' + (a+b)y' + ab \cdot y = \delta'(t) + d \cdot \delta(t), \\ y(0) = y'(0) = 0 \end{cases} \Rightarrow$$

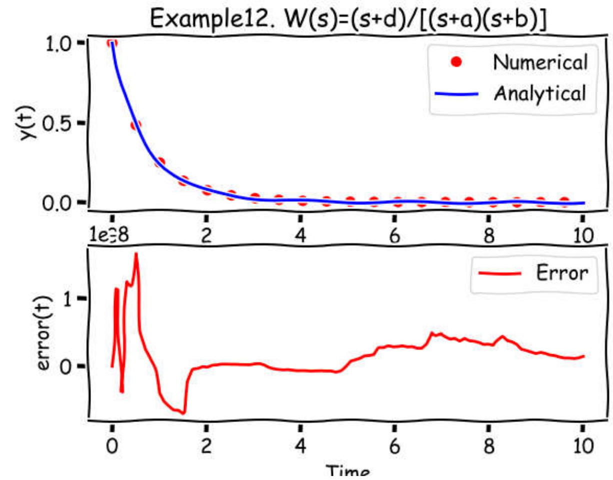
$$g(t) = IVP(\{1, (a+b), ab\}, \{1, d\}, t_0 = 0, y_0 = \{0, 0\})$$

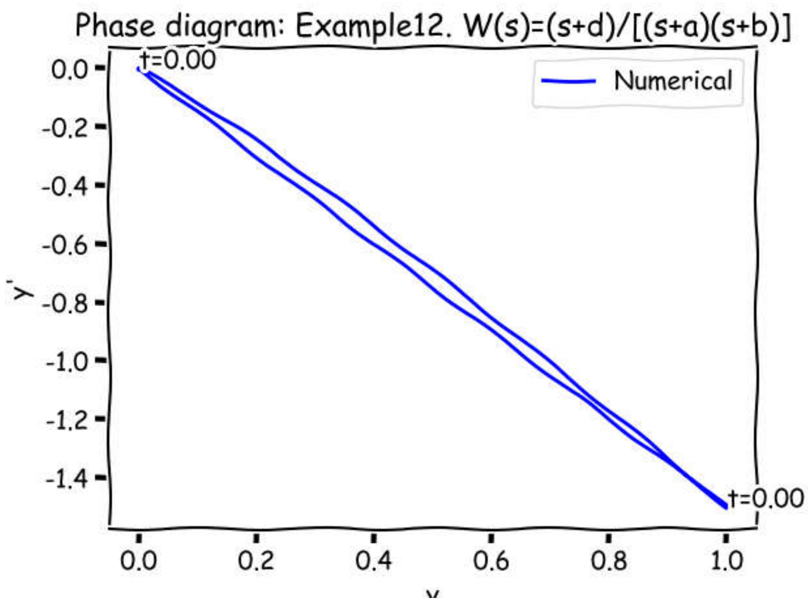
Analitycal solution (impulse response) taken from Appendix B:

$$g(t) = L^{-1}\left\{\frac{s+d}{(s+a)(s+b)}\right\} = \frac{1}{b-a}\left[(d-a)e^{-at} - (d-b)e^{-bt}\right]$$

Let’s check how to correspond the numerical solution for a=1 and b=2 and d=1.5 for homogenous sytem with zero IC with analitical solution for the system.

Numerical solution, analytical solutions and error provided by Python's script (example12.py):





6.4 Example13 [Xue p.380]

$$TF = \frac{s+d}{s(s+a)(s+b)} = \frac{s+d}{s^3+(a+b)s^2+ab\cdot s}$$

Impulse function for this transfer function corresponds to this system:

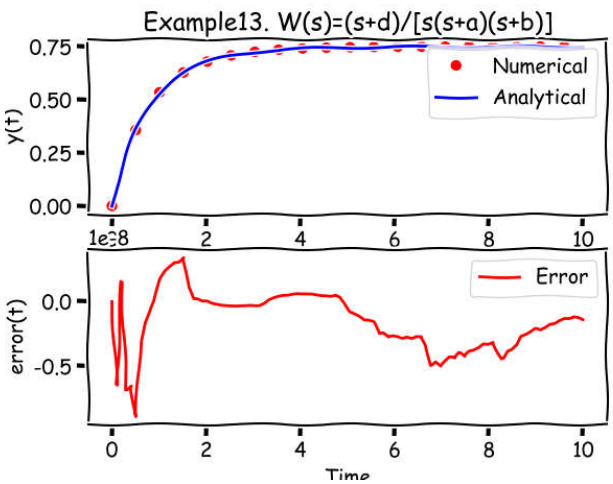
$$\begin{cases} y''' + (a+b)y'' + ab\cdot y' = \delta'(t) + d\cdot\delta(t), \\ y(0) = y'(0) = y''(0) = 0 \end{cases} \Rightarrow$$
$$g(t) = IVP(\{1,\,(a+b),\,ab,\,0\},\{1,\,d\},t_0 = 0, y_0 = \{0,0,0\})$$

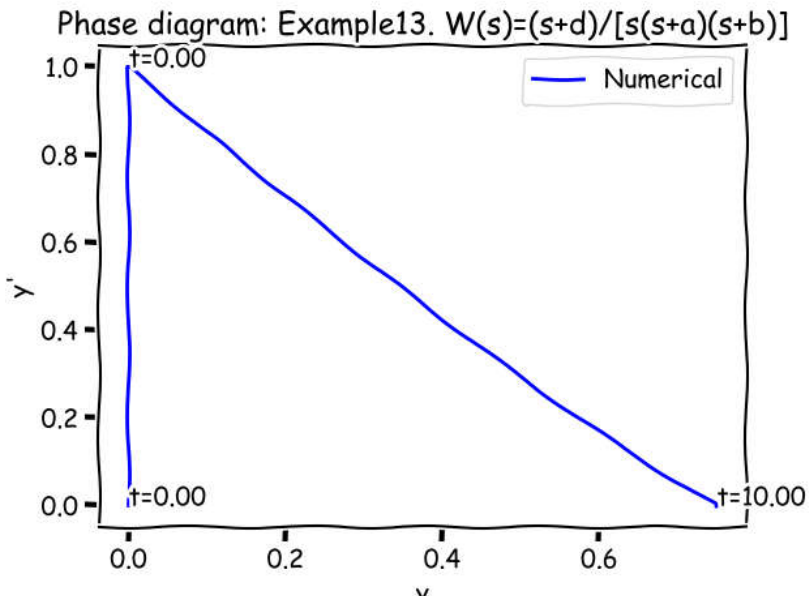
Analitycal solution (impulse response) taken from Appendix B:

$$g(t) = L^{-1}\left\{\frac{s+d}{s(s+a)(s+b)}\right\} = \frac{1}{ab}\left[d - \frac{b(d-a)}{b-a}\,e^{-at} + \frac{a(d-b)}{b-a}\,e^{-bt}\right]$$

Let's check how to correspond the numerical solution for a=1 and b=2 and d=1.5 for homogenous sytem with zero IC with analitical solution for the system.

Numerical solution, analytical solutions and error provided by Python’s script (example13.py):





6.5 Example14 [Xue p.380]

$$TF = \frac{s+a}{s^2+\omega^2}$$

Impulse function for this transfer function corresponds to this system:

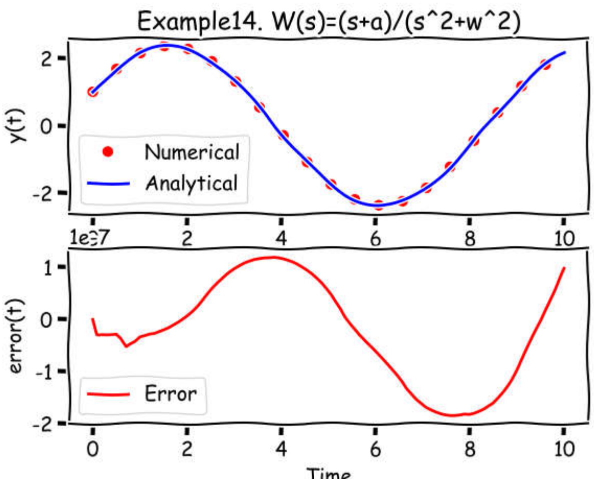
$$\begin{cases} y'' + 0 \cdot y' + \omega^2 \cdot y = \delta'(t) + a \cdot \delta(t), \\ y(0) = y'(0) = 0 \end{cases} \Rightarrow$$

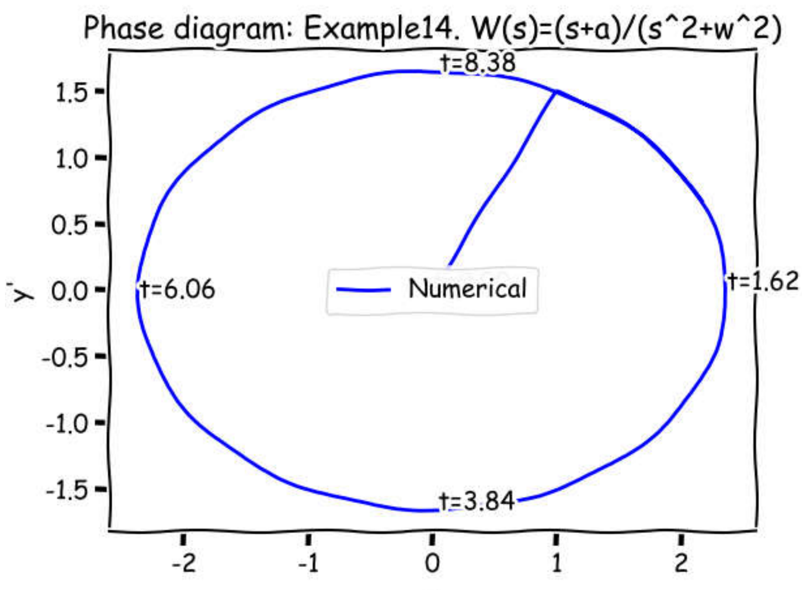
$$g(t) = IVP(\{1, 0, \omega^2\}, \{1, a\}, t_0 = 0, y_0 = \{0, 0\})$$

Analitycal solution (impulse response) taken from Appendix B:

$$g(t) = L^{-1}\left\{\frac{s+a}{s^2+\omega^2}\right\} = \frac{\sqrt{a^2+\omega^2}}{\omega}\, sin(\omega t + tan^{-1}(\frac{\omega}{a}))$$

Let's check how to correspond the numerical solution for a=1.5 and $\omega = 0.7$ for homogenous sytem with zero IC with analitical solution for the system.
Numerical solution, analytical solutions and error provided by Python's script (example14.py):





Apendix A

General formulas

Unit impulse function (Dirac's function)

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a)$$

$$\int_{-\infty}^{\infty} \delta(x-a)dx = \int_{-\infty}^{\infty} \delta(x)dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta^{(n)}(x-a)dx = (-1)^n f^{(n)}(a)$$

$$\int_{-\infty}^{\infty} f(x)\delta'(x-a)dx = -f'(a)$$

$$\delta(-x) = \delta(x)$$

$$\delta(ax) = \frac{1}{|a|}\delta(x), a \neq 0$$

$$x\delta(x) = 0$$

Laplace transform

$$LT\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t)dt = F(s)$$

$$LT\{\delta(t-t_0)\} = e^{-st_0}$$

$$LT\{\delta(t)\} = 1$$

$$LT\{f'\} = sL\{f\} - f(0)$$

$$LT\{f''\} = s^2 L\{f\} - sf(0) - f'(0)$$

$$\begin{aligned} LT\{f^{(n)}\} &= s^n LT\{f\} - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s^0 y^{(n-1)}(0) \\ &= s^n LT\{f\} - \sum_{i=0}^{n-1} s^i f^{(n-1-i)}(0) \end{aligned}$$

Apendix B

Examples

Example1 [Oliveira and Cortes, p.3]

$$\begin{cases} y'' + ay' = \delta(t), \\ y(0) = y'(0) = 0 \end{cases}$$

Solution:

$$y(t) = \frac{1}{a}\left(1 - e^{-at}\right)$$

Example2 [Finan, pp.57-58]

$$\begin{cases} 2y'' + 4y' + 10y = \delta(t), \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \end{cases}$$

Solution (! there was a typo at original book, here's the proper formula which we've checked by MathCad14 and WolframAlfa!):

$$y(t) = \frac{1}{4}\, e^{-t} sin(2t)$$

Example3 [Nagy, p.189]

$$\begin{cases} y'' + 2y' + 2y = \delta(t), \\ y(0) = y'(0) = 0 \end{cases}$$

Solution:

$$y(t) = e^{-t} sin(t)$$

Example4 [Nagy, p.189]

$$\begin{cases} y'' + 2y' + 2y = \delta(t - c), \\ y(0) = y'(0) = 0 \end{cases}$$

Solution:

$$y(t) = H(t - c)e^{-(t-c)} sin(t - c)$$

Example5 [Chasnov, p.65]

$$\begin{cases} 2y'' + y' + 2y = \delta(t - 5), \\ y(0) = y'(0) = 0 \end{cases}$$

Solution:

$$y(t) = \frac{2}{\sqrt{15}}\, H(t - 5)e^{-(t-5)/4} sin(\sqrt{15}\,(t - 5)/4)$$

Example6 [Zill, p.293]

$$\begin{cases} y'' + y = 4\delta(t - 2\pi), \\ y_0 = y(0) = 0 \\ y_1 = y'(0) = 0 \end{cases}$$

Solution:

$$y(t) = \begin{cases} 0, & 0 \le t < 2\pi \\ 4sin(t), & t \ge 2\pi \end{cases}$$

Example7 [Zill, p.293]

$$\begin{cases} y'' + y = 4\delta(t - 2\pi), \\ y_0 = y(0) = 1 \\ y_1 = y'(0) = 0 \end{cases}$$

Solution:

$$y(t) = cos(t) + 4H(t, 2\pi)sin(t)$$

Example8 [Nagy p.190]

$$\begin{cases} y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), \\ y(0) = y'(0) = 0 \end{cases}$$

Solution:

$$y(t) = \frac{1}{2}\ [H(t - \pi) - H(t - 2\pi)]\, sin(2t)$$

Example9

$$\begin{cases} y''' + 2y'' + 2y = \delta(t), \\ y(0) = y'(0) = y''(0) = 0 \end{cases}$$

Solution, getting from WolframAlfa

$$y(t) = \frac{1}{2} - \frac{1}{2}\, e^{-t}\, (sin(t) + cos(t))$$

Example10 [Ogata p.163]

$$C(s) = \frac{1}{Ts + 1}$$

Impulse Response:

$$g(t) = \frac{1}{T}\, e^{-t/T}$$

Example11 [Xue p.380]

$$L^{-1}\left\{\frac{s}{(s+a)(s+b)}\right\}=\frac{1}{a-b}\ \left[ae^{-at}-be^{-bt}\right]$$

Example12 [Xue p.380]

$$L^{-1}\left\{\frac{s+d}{(s+a)(s+b)}\right\}=\frac{1}{b-a}\ \left[(d-a)e^{-at}-(d-b)e^{-bt}\right]$$

Example13 [Xue p.380]

$$L^{-1}\left\{\frac{s+d}{s(s+a)(s+b)}\right\}=\frac{1}{ab}\left[d-\frac{b(d-a)}{b-a}\,e^{-at}+\frac{a(d-b)}{b-a}\,e^{-bt}\right]$$

Example14 [Xue p.380]

$$L^{-1}\left\{\frac{s+a}{s^2+\omega^2}\right\}=\frac{\sqrt{a^2+\omega^2}}{\omega}\ sin(\omega t+tan^{-1}(\frac{\omega}{a}))$$

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