

Mathematics. — *On the dissection of rectangles into squares.* (Second communication.) By C. J. BOUWKAMP. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of December 21, 1946.)

5. *Summary of the possible planar networks without wires in parallel or series connection ($T \leq 14$). Complexity of a network.*

Four different types of networks can be distinguished. In the first place there is a number of networks like that of fig. 3a; they all show one or more zero-currents after a source is placed in any of their wires (the remaining wires have equal resistances). The second type, for which fig. 3b may serve as an example, shows equal currents in a trivial manner after a source is placed in some of its wires. There are networks belonging to either of these two types. These networks, as far as they are not "compound", are tabulated in fig. 5; double arrows indicate pairs of dual networks. Networks without arrows are self-dual.

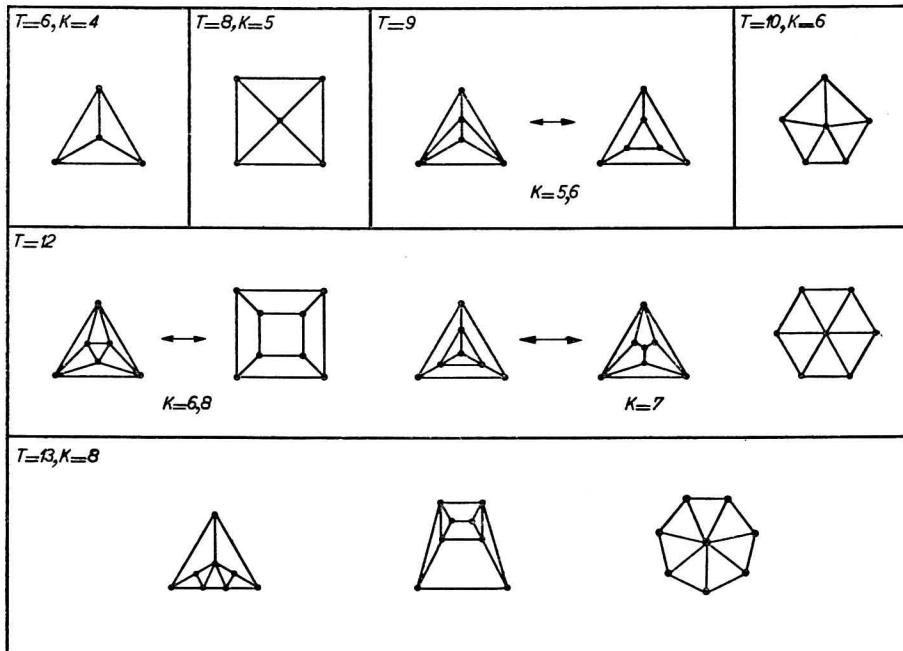


Fig. 5. Networks (not compound) giving either zero elements or trivially imperfect squarings.

The third type of networks is compound, having two or more smaller networks in series or parallel connection. They are gathered in fig. 6. Of equivalent networks only one is given.

None of the networks of figs 5, 6 gives rise to perfect or non-trivially imperfect squarings, except those in fig. 6 marked with α , β , γ . Each of the marked networks generates one of the non-trivially compound squarings mentioned in section 1; one of them is perfect (γ), two are non-trivially imperfect (α , β). The three exceptional squarings consist of 4 squares and 1 rectangle, the latter being dissected into nine squares according to one of the three possibilities of fig. 4. Their codes are

$$\begin{aligned}\alpha: & (14, 13) (1, 12) (6, 4, 5) (3, 1) (6) (5, 1) (4) \\ \beta: & (67, 65) (2, 63) (36, 33) (5, 28) (25, 9, 2) (7) (16) \\ \gamma: & (131, 130) (1, 129) (72, 60) (28, 32) (56, 16) (40, 4) (36),\end{aligned}$$

of which only the last corresponds to a perfect squaring.

Networks of the fourth type, which give rise to at least one squaring of the required character, are drawn in figs 7, 8. Their total number is 87, of which 17 are self-dual. It will be obvious that the sieving process must be carried out rather carefully in order to get them all¹⁾.

The networks in figs 7, 8 are arranged according to increasing "complexity". The *complexity C* of a network is the total number of "complete trees" (german: "volständiger Baum") that can be drawn in the network. A *complete tree* is a sub-network that (i) is connected, (ii) contains all the vertices and some (or all) of the wires of the original network, and (iii) does not contain any closed circuit.

The numerical value of the complexity of a given network can be found in the following way. Enumerate the K vertices arbitrarily: $P_1, P_2 \dots P_K$. Let further a square matrix of K rows be formed by elements a_{ik} , such that for diagonal elements $a_{ii} =$ total number of wires at the vertex P_i , and for non-diagonal elements ($i \neq k$) $a_{ik} = -1$ or 0, according as to whether or not P_i and P_k are connected by a wire. The determinant of this symmetrical matrix obviously vanishes. Furthermore its first sub-determinants have a common absolute value. This common value is the complexity of the network under consideration²⁾.

Dual networks have the same complexity; the inverse is not true, as can be seen from the networks $C = 1015, 1088$ in fig. 8, or from the networks α, β, γ , in fig. 6, which all have $C = 1040$ (there is in fig. 8 also a network of $C = 1040$).

This concept of complexity is very important with regard to the problem of squared rectangles. It was shown in paper A that *all squared rectangles derived from a network of complexity C can be drawn in such*

¹⁾ We are indebted to Prof. DE BRUIJN for constructing independently all the networks containing 14 wires. This diminishes to a large extent the probability of having overlooked one of the required graphs. With regard to the graphs involving 13 wires, it may be noticed here that our number of 67 perfect squared rectangles of order 12 is in full agreement with the result of paper A. Furthermore, in TELLEGREN's papers the possible networks are drawn for $T \leq 12$, none of which is missing here.

²⁾ For a proof of this theorem, see paper A. As was already remarked there, the theorem is quite trivial from known results in early Kirchhoff network theory.

a manner that their semi-perimeters are equal to C , whilst at the same time all the elements of all those squarings have integer sides.

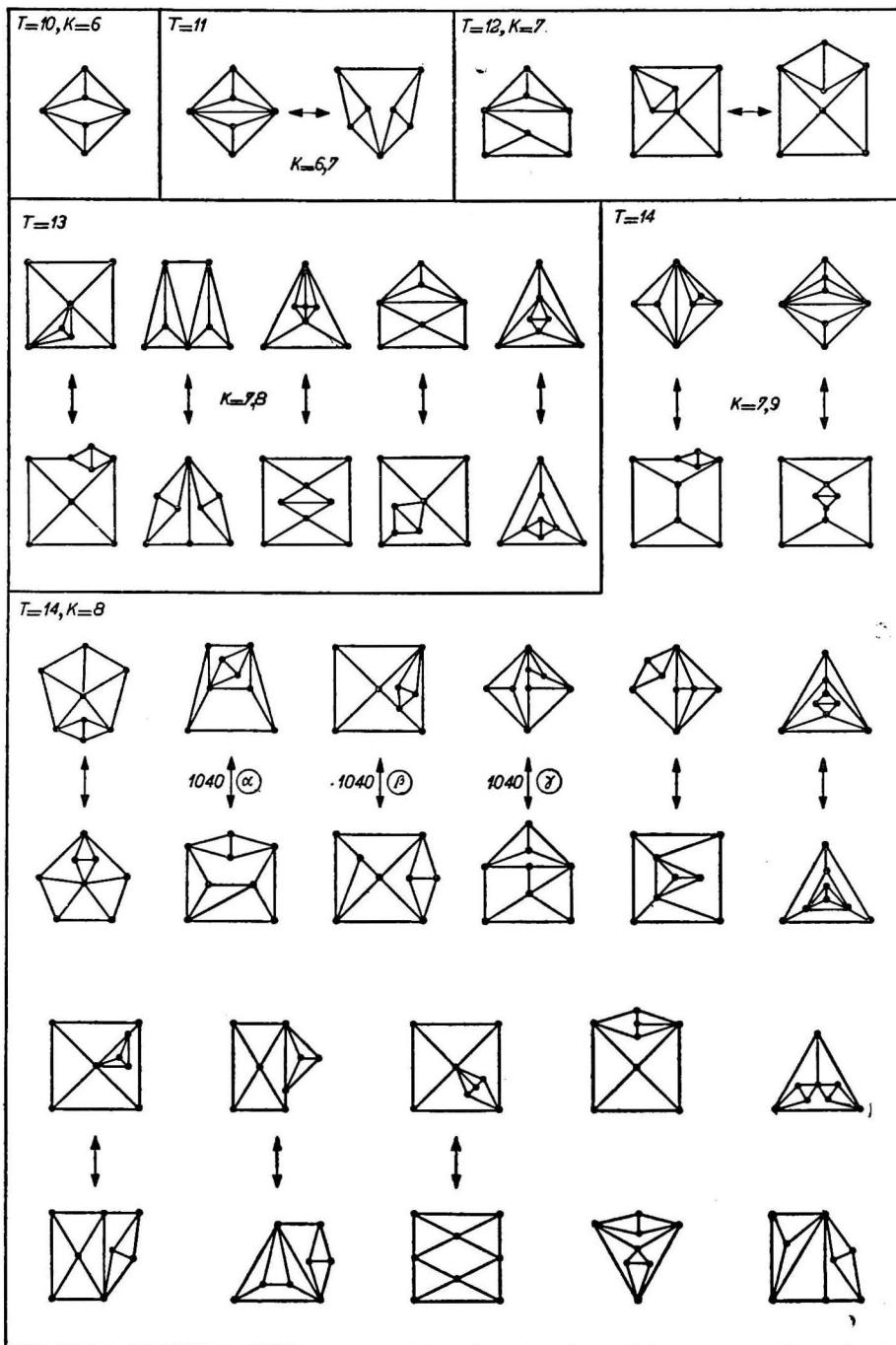


Fig. 6. Compound networks. The only networks that lead to non-trivially compound squarings are marked with α , β or γ .

The last part of this theorem shows that any squared rectangle has commensurable sides (and elements). The converse is also true. In paper A

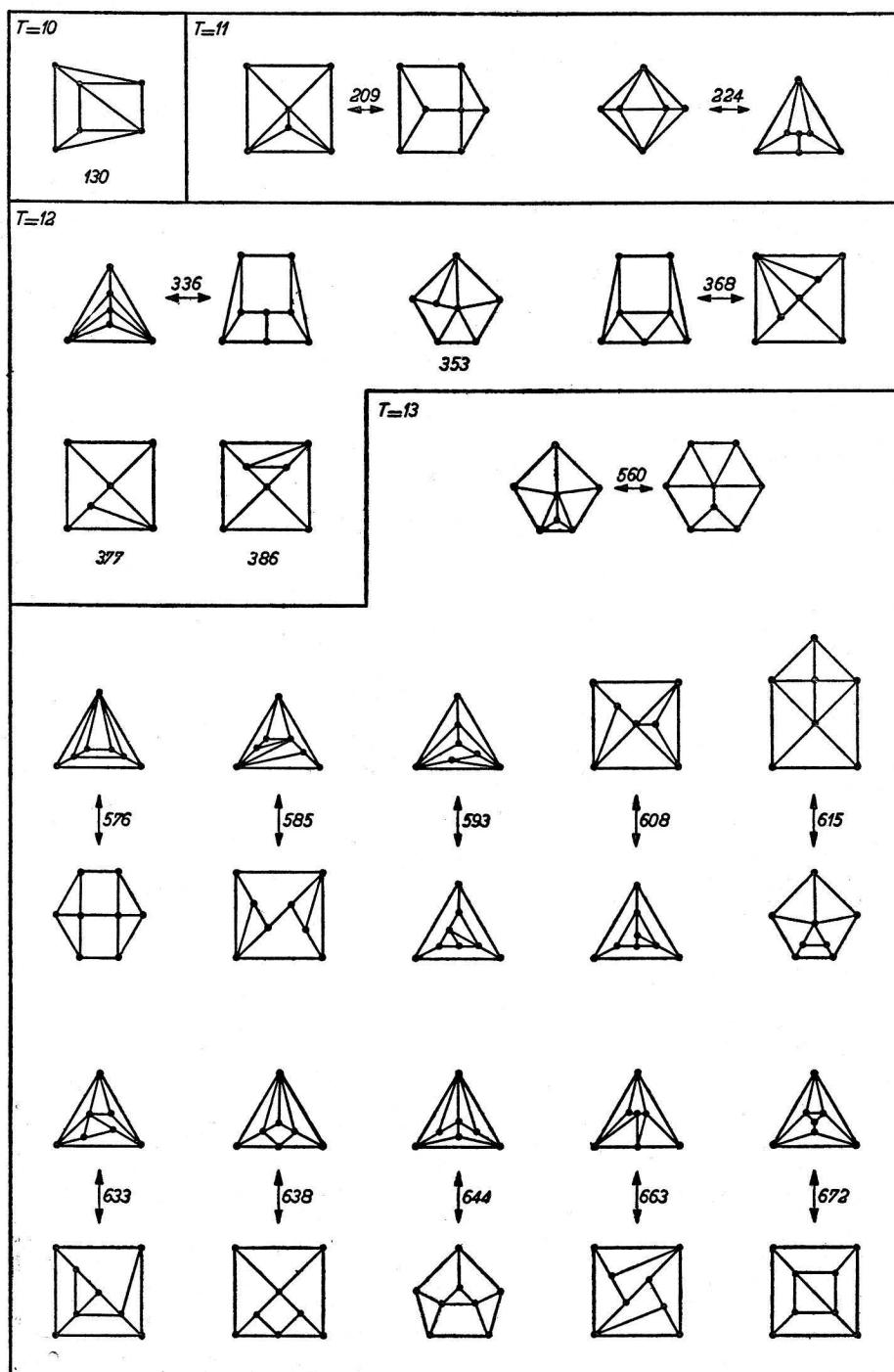


Fig. 7. Networks giving at least one simple squaring of order up to 12. The numbers denote corresponding values of complexity.

it is shown that any rectangle of commensurable sides can be dissected into unequal squares, even in an infinite number of essentially different ways. It is not known, however, whether every such rectangle is perfectible and *simple*.

The integer sides referred to above are called the *full* elements of the squaring. It often happens that the full elements of a particular squaring have a common factor. After division by the highest common factor one

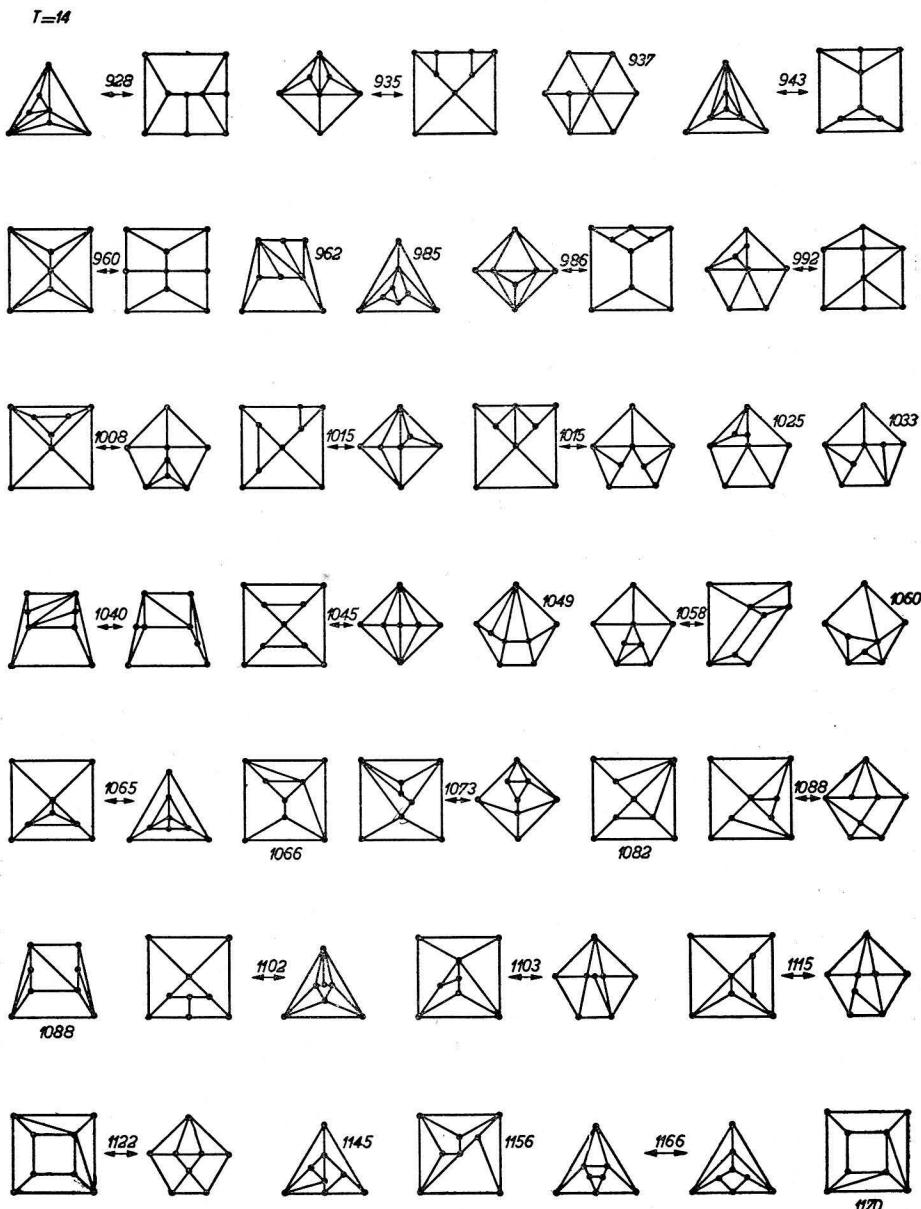


Fig. 8. Networks giving at least one simple squaring of order 13. The numbers denote corresponding values of complexity.

gets the *reduced elements*; this factor is called *reduction factor*. In paper A many theorems on reduction were given; for example, all non-trivially imperfect squarings are reducible. The inverse of this is not true. Further, if C is prime, all the squarings are irreducible. For details and proofs we may refer the reader to the fundamental paper A.

6. Summary and classification of all simple squared rectangles of order less than 14.

Even if the required networks have been constructed, it is still a matter of patience to derive from them the possible squared rectangles, though this is by no means difficult. It is not necessary to apply the solution of the Kirchhoff equations in the well-known form of a quotient of two determinants. It is much more simple to assume a certain number of unknown currents to be equal to x, y, z, \dots , and then calculate — by use of the current- and voltage-equations — the remaining currents, which are integral linear combinations of the x, y, z, \dots . Usually, not more than three or four unknown currents are already sufficient. Finally, a system of independent equations is obtained, homogeneous in x, y, z, \dots . The number of equations is one less than the total number of assumed quantities x, y, z, \dots . This diophantine system can be easily solved in smallest numbers. By repeating the process with these numerical values of x, y, z, \dots , the reduced elements of the squaring are easily found.

The squarings will now be *classified* as follows. Undoubtedly, the most important feature is the order of the squaring. The second important property is the least³⁾ possible value of the complexity of all networks that will generate it. Therefore the squarings are conveniently arranged in increasing order, and further in classes according to increasing complexity. The order will be denoted by *roman*, the complexity by *arabic*, figures. Squarings in the same class are arranged in such a way that the difference of full horizontal and vertical sides decreases; so the rectangles tend more and more to a square. The elements in a class are distinguished by letters a, b, c, \dots , a prime denoting (non-trivial) imperfection. It may be noticed that in our coding system only the reduced elements occur.

With this in mind, we have got the following *classification of all simple squared rectangles of order less than 14*:

Order IX.

$C = 130$	a'	(6, 4, 5) (3, 1) (6) (5, 1) (4)
	b	(36, 33) (5, 28) (25, 9, 2) (7) (16)
	c	(18, 15) (7, 8) (14, 4) (10, 1) (9)

a' has reduction factor 5; that of c is 2.

³⁾ As follows from the reduction of the order in the case of zero-currents (cf. the end of section 2), there may exist "higher" networks that generate the squared rectangle in question. Cf. the concept of "normal" polar networks in paper A.

Order X.

$C = 209$	a	(45, 44, 41) (3, 38) (12, 35) (34, 11) (23)
	b	(60, 55) (16, 39) (34, 15, 11) (4, 23) (19)
	c	(57, 54) (3, 7, 44) (41, 15, 4) (11) (26)
	d	(60, 45) (19, 26) (44, 16) (12, 7) (33) (28)
$C = 224$	a	(25, 17, 23) (11, 6) (5, 24) (22, 3) (19)
	b	(30, 27) (3, 11, 13) (25, 8) (17, 2) (15)

Both squarings have reduction factor 2.

Order XI.

$C = 336$	a	(72, 71, 66) (5, 61) (1, 19, 56) (55, 18) (37)
	b	(95, 90) (5, 24, 61) (56, 25, 19) (6, 37) (31)
$C = 353$	a	(85, 57, 67) (47, 10) (77) (59, 26) (7, 40) (33)
	b	(100, 94) (29, 65) (59, 25, 16) (9, 7) (36) (34)
	c	(97, 94) (26, 28) (65, 32) (9, 17) (33, 8) (25)
	d	(99, 88) (10, 78) (1, 9) (67, 25, 8) (17) (42)
	e	(100, 85) (43, 42) (68, 32) (1, 41) (4, 40) (36)
	f	(99, 78) (21, 57) (77, 43) (16, 41) (34, 9) (25)
$C = 368$	a	(89, 49, 71) (27, 22) (5, 88) (32) (70, 19) (51)
	b	(105, 94) (19, 75) (64, 33, 8) (27) (31, 2) (29)
	c	(51, 47) (8, 39) (35, 11, 5) (1, 7) (6) (24)
	d	(102, 89) (40, 49) (75, 27) (48, 19) (10, 39) (29)
	e	(105, 80) (33, 47) (78, 27) (19, 14) (5, 56) (51)

c is of reduction 2.

$C = 377$	a	(92, 64, 53) (11, 42) (44, 31) (76, 16) (73) (60)
	b	(102, 97) (16, 81) (76, 15, 11) (4, 23) (19) (42)
	c	(102, 92) (31, 23, 38) (81, 21) (8, 15) (60) (53)
$C = 386$	a	(43, 29, 40) (19, 10) (9, 1) (41) (38, 5) (33)
	b	(96, 56, 57) (55, 1) (58) (81, 15) (66, 4) (62)
	c	(105, 100) (6, 13, 81) (76, 28, 1) (7) (20) (48)
	d	(50, 48) (7, 19, 22) (45, 5) (12) (28, 3) (25)
	e	(105, 90) (15, 31, 44) (86, 34) (18, 13) (57) (52)
	f	(56, 41) (17, 24) (40, 14, 2) (12, 7) (31) (26)

a, d and f are of reduction 2.

Order XII.

$C = 560$	a	(124, 120, 109) (11, 98) (44, 87) (83, 41) (1, 43) (42)
	b	(164, 149) (51, 98) (83, 41, 40) (4, 47) (1, 43) (42)
	c	(154, 149) (41, 108) (103, 51) (14, 27) (1, 13) (52, 40)
	d	(154, 143) (11, 12, 120) (109, 41, 14, 1) (13) (27) (68)
	e'	(41, 31) (14, 17) (27, 14) (11, 3) (20) (13, 1) (12)

e' has reduction 4.

$C = 576$	a'	(15, 16, 15) (11, 4) (4, 11) (3, 10, 3) (7) (7)
	b'	(21, 19) (6, 13) (11, 6, 4) (3, 7) (5, 1) (4)

Both of reduction 8.

$C = 585$	a'	(9, 8, 5, 7) (3, 2) (9) (3, 8) (7, 2) (5)
	b	(132, 128, 93) (35, 58) (36, 104, 23) (100, 32) (81) (68)
	c	(171, 152) (42, 110) (91, 37, 20, 23) (17, 3) (68) (54)
	d'	(33, 31) (9, 22) (20, 6, 7) (5, 1) (4, 13) (9)
	e	(155, 152) (26, 126) (123, 32) (9, 17) (33, 8) (25) (58)
	f	(55, 44) (12, 11, 21) (41, 14) (1, 10) (13) (31) (27)

Reductions of a', d' and f are 13, 5 and 3, respectively.

$C = 593$	<i>a</i>	(113, 76, 96, 101) (56, 20) (111, 5) (106) (94, 19) (75)
	<i>b</i>	(121, 129, 118) (11, 107) (104, 17) (9, 35, 96) (26) (61)
	<i>c</i>	(123, 129, 101) (28, 73) (117, 6) (111, 152) (7, 66) (59)
	<i>d</i>	(143, 73, 120) (43, 30) (13, 17) (56) (137) (114, 29) (85)
	<i>e</i>	(166, 160) (6, 47, 107) (101, 43, 28) (15, 13) (60) (58)
	<i>f</i>	(175, 149) (29, 120) (94, 43, 35, 3) (32) (8, 59) (51)
	<i>g</i>	(166, 155) (11, 27, 117) (106, 55, 16) (43) (51, 4) (47)
	<i>h</i>	(160, 155) (5, 8, 19, 123) (118, 44, 3) (11) (30) (74)
	<i>i</i>	(165, 149) (19, 130) (114, 44, 7) (4, 15) (11) (26) (70)
	<i>j</i>	(176, 137) (67, 70) (104, 44, 28) (16, 76, 3) (73) (60)
	<i>k</i>	(165, 143) (67, 76) (120, 45) (75, 28, 9) (19, 66) (47)
	<i>l</i>	(175, 130) (45, 85) (113, 55, 52) (12, 73) (3, 61) (58)
	<i>m</i>	(176, 121) (47, 74) (20, 27) (120, 44, 12) (32) (101) (76)
$C = 608$	<i>a</i>	(136, 123, 118) (5, 113) (20, 108) (95, 34, 7) (27) (61)
	<i>b'</i>	(31, 33, 28) (13, 15) (29, 2) (27, 8) (19, 2) (17)
	<i>c</i>	(129, 132, 92) (33, 59) (7, 26) (126, 3) (123, 19) (104)
	<i>d'</i>	(37, 24, 23) (6, 17) (19, 5) (11) (31, 6) (25) (28)
	<i>e</i>	(166, 163) (3, 44, 116) (113, 56) (15, 29) (57, 14) (43)
	<i>f</i>	(83, 79) (4, 12, 61) (59, 19, 9) (1, 11) (10) (40)
	<i>g</i>	(85, 77) (12, 65) (57, 17, 7, 4) (3, 13) (10) (40)
	<i>h</i>	(163, 158) (5, 24, 129) (124, 25, 19) (6, 37) (31) (68)
	<i>i'</i>	(19, 10, 11) (9, 1) (12) (17, 7, 4) (3, 13) (10)
	<i>j</i>	(165, 154) (19, 135) (124, 33, 8) (27) (31, 2) (29) (60)
	<i>k</i>	(165, 148) (52, 37, 59) (130, 35) (15, 22) (95, 7) (88)
	<i>l</i>	(170, 135) (35, 100) (133, 72) (7, 25, 68) (61, 18) (43)
	<i>f, g</i>	have reduction 2; <i>b', d'</i> have 4 and <i>i'</i> has 8.
$C = 615$	<i>a</i>	(144, 97, 127) (67, 30) (37, 120) (103, 41) (21, 83) (62)
	<i>b</i>	(142, 84, 127) (41, 43) (39, 2) (37, 135) (120, 22) (98)
	<i>c</i>	(174, 164) (51, 113) (103, 43, 28) (15, 13) (2, 62) (60)
	<i>d</i>	(58, 48) (20, 28) (41, 17) (7, 13) (24) (5, 23) (18)
	<i>e</i>	(165, 143) (22, 67, 54) (142, 45) (13, 41) (97, 28) (69)
	<i>d</i>	has reduction 3.
$C = 633$	<i>a</i>	(138, 116, 132) (51, 49, 16) (33, 115) (109, 29) (82) (80)
	<i>b</i>	(141, 113, 123) (28, 75, 10) (133) (115, 54) (7, 68) (61)
	<i>c</i>	(139, 93, 136) (59, 34) (25, 9) (16, 129) (126, 13) (113)
	<i>d</i>	(48, 28, 44) (12, 16) (8, 4) (17, 47) (43, 13) (30)
	<i>e</i>	(144, 116, 93) (23, 70) (28, 64, 47) (136, 36) (117) (100)
	<i>f</i>	(180, 164) (39, 125) (109, 48, 23) (25, 37) (61, 12) (49)
	<i>g</i>	(57, 56) (2, 12, 42) (41, 11, 4, 1) (3) (7) (30)
	<i>h</i>	(170, 168) (47, 59, 62) (125, 45) (80, 12) (68, 3) (65)
	<i>i</i>	(56, 27, 29) (25, 2) (31) (17, 8) (43, 13) (39) (30)
	<i>j</i>	(170, 164) (29, 135) (129, 25, 16) (9, 7) (36) (34) (70)
	<i>k</i>	(171, 158) (13, 64, 81) (133, 51) (82, 33) (16, 65) (49)
	<i>l</i>	(168, 158) (10, 49, 37, 62) (139, 39) (12, 25) (100) (87)
	<i>m</i>	(60, 46) (18, 28) (45, 15) (11, 7) (4, 3) (31) (30)
	<i>d, g, i, m</i>	of reduction 3.
$C = 638$	<i>a</i>	(71, 60, 62) (27, 31, 2) (64) (55, 16) (39, 4) (35)
	<i>b</i>	(150, 92, 111) (73, 19) (54, 76) (135, 15) (120, 22) (98)
	<i>c</i>	(175, 164) (10, 19, 135) (1, 9) (124, 44, 8) (36) (80)
	<i>d</i>	(175, 160) (15, 73, 72) (128, 62) (1, 71) (4, 70) (66)
	<i>e</i>	(82, 80) (2, 11, 31, 36) (75, 9) (20) (46, 5) (41)
	<i>f</i>	(88, 72) (16, 18, 38) (71, 31, 2) (20) (9, 49) (40)

a, e, f of reduction 2.

$C = 644$	<i>a</i>	(72, 52, 69) (35, 17) (26, 60) (57, 15) (42, 8) (34)
	<i>b</i>	(89, 88) (8, 23, 57) (56, 26, 7) (15) (4, 34) (30)
	<i>c</i>	(88, 77) (13, 28, 36) (69, 17, 2) (15) (52, 8) (44)
	<i>d</i>	(79, 74) (15, 23, 36) (70, 26, 8) (18, 13) (49) (44)
	<i>e</i>	(44, 37) (7, 9, 21) (36, 13, 2) (11) (23, 1) (22)

c of reduction 4; the rest of reduction 2.

$C = 663$	<i>a</i>	(148, 108, 130) (59, 27, 22) (5, 147) (32) (129, 19) (110)
	<i>b'</i>	(12, 7, 10) (4, 3) (1, 12) (1, 4) (10, 3) (7)
	<i>c</i>	(160, 113, 95) (18, 77) (72, 59) (135, 25) (13, 123) (110)
	<i>d</i>	(52, 36, 32) (4, 28) (16, 24) (49, 19) (11, 41) (30)
	<i>e</i>	(60, 58) (6, 7, 45) (43, 13, 4) (9, 1) (8) (30)
	<i>f</i>	(180, 155) (25, 53, 77) (148, 57) (29, 24) (5, 96) (91)
	<i>g</i>	(174, 160) (33, 53, 74) (155, 19) (52) (32, 21) (95) (84)

d, e of reduction 3; *b'* of 13.

$C = 672$	<i>a</i>	(76, 47, 70) (24, 23) (20, 73) (5, 19) (67, 14) (53)
	<i>b</i>	(97, 72) (29, 43) (70, 23, 2) (19, 14) (5, 52) (49)

Both of reduction 2.

Order XIII.

$C = 928$	<i>a</i>	(168, 164, 152, 149) (3, 146) (12, 143) (45, 131) (127, 41) (86)
	<i>b</i>	(196, 204, 193) (51, 142) (139, 49, 8) (41, 131, 40) (91) (90)
	<i>c</i>	(204, 200, 181) (19, 162) (4, 72, 143) (139, 69) (1, 71) (70)
	<i>d'</i>	(24, 25, 23) (2, 21) (20, 3, 1) (2, 7, 19) (5) (12)
	<i>e'</i>	(52, 51, 37) (14, 23) (1, 14, 41, 9) (40, 13) (32) (27)
	<i>f'</i>	(52, 31, 49) (13, 18) (8, 5) (3, 18, 51) (48, 15) (33)
	<i>g</i>	(280, 241) (75, 166) (127, 61, 56, 36) (20, 91) (5, 71) (66)
	<i>h</i>	(135, 123) (40, 83) (71, 36, 28) (8, 17, 43) (35, 9) (26)
	<i>i</i>	(133, 125) (8, 36, 81) (73, 40, 28) (19, 45) (33, 7) (26)
	<i>j</i>	(266, 247) (19, 60, 168) (149, 61, 34, 41) (27, 7) (108) (88)
	<i>k</i>	(250, 247) (3, 8, 52, 184) (181, 49, 18, 5) (13) (31) (132)
	<i>l</i>	(246, 241) (41, 200) (195, 51) (14, 27) (1, 13) (52) (40) (92)
	<i>m'</i>	(35, 25) (10, 15) (21, 10, 9, 5) (4, 16) (1, 12) (11)
	<i>n</i>	(270, 195) (75, 120) (193, 69, 38, 45) (31, 7) (24, 148) (124)

Reductions 2(*h, i*), 4(*e', f'*) and 8(*d', m'*).

$C = 935$	<i>a</i>	(199, 210, 184) (26, 158) (143, 56) (45, 59, 132) (87, 14) (73)
	<i>b</i>	(39, 42, 31) (11, 20) (36, 3) (33, 14, 9) (5, 24) (19)
	<i>c</i>	(269, 254) (15, 81, 158) (143, 71, 70) (4, 77) (1, 73) (72)
-	<i>d</i>	(269, 243) (26, 37, 180) (154, 71, 59, 11) (48) (12, 95) (83)
	<i>e</i>	(254, 243) (11, 12, 25, 195) (184, 66, 14, 1) (13) (52) (118)
	<i>f</i>	(264, 208) (56, 52, 100) (4, 48) (199, 81, 44) (37, 155) (118)

Reduction of *b* is 5.

$C = 937$	<i>a</i>	(229, 154, 177) (131, 23) (200) (148, 81) (39, 92) (67, 14) (53)
	<i>b</i>	(268, 253) (90, 163) (148, 67, 53) (14, 39) (81) (17, 73) (56)
	<i>c</i>	(259, 256) (90, 166) (163, 96) (14, 76) (67, 24, 5) (19) (43)
	<i>d</i>	(259, 253) (81, 172) (166, 93) (38, 43) (73, 20) (53, 5) (48)
	<i>e</i>	(268, 229) (119, 110) (172, 96) (9, 101) (36, 92) (76, 20) (56)
	<i>f</i>	(264, 232) (23, 209) (9, 14) (177, 67, 24, 5) (19) (43) (110)
	<i>g</i>	(264, 209) (55, 154) (200, 119) (53, 101) (81, 38) (5, 48) (43)

$C = 943$	<i>a</i>	(188, 164, 105, 151) (59, 46) (13, 184) (65, 171) (147, 41) (106)
	<i>b</i>	(207, 194, 184) (10, 174) (13, 27, 164) (151, 55, 14) (41) (96)

c (275, 246) (70, 176) (147, 55, 32, 41) (23, 9) (14, 106) (92)
d (263, 253) (10, 69, 174) (164, 50, 59) (41, 9) (32, 105) (73)
e (275, 205) (70, 135) (188, 92, 65) (27, 50, 123) (96, 23) (73)

$C = 960 \quad a'$ (28, 25, 23) (2, 21) (8, 19) (16, 7, 5) (2, 11) (9)

Reduction 8.

$C = 962 \quad a'$ (17, 9, 7, 12) (2, 5) (11) (17) (12, 5) (2, 9) (7)
b (138, 127) (38, 89) (78, 31, 15, 14) (1, 13) (16) (51) (47)
c (265, 254) (65, 189) (178, 87) (33, 32) (1, 31) (91, 30) (61)
d (276, 221) (127, 94) (189, 87) (33, 61) (15, 117, 28) (102) (89)

Reductions 2(b), 13(a').

$C = 985 \quad a$ (209, 205, 194) (11, 183) (44, 172) (168, 41) (1, 43) (42) (85)
b (204, 220, 169) (71, 98) (188, 16) (172, 64) (44, 27) (125) (108)
c (43, 44, 25) (11, 14) (8, 3) (17) (42, 1) (41, 12) (29)
d (269, 264) (5, 71, 188) (183, 91) (24, 47) (1, 23) (92) (70)
e (269, 253) (16, 27, 210) (194, 76, 15) (4, 23) (19) (42) (118)
f (264, 253) (11, 27, 215) (204, 55, 16) (43) (51, 4) (47) (98)
g (304, 209) (91, 118) (64, 27) (168, 76, 60) (145) (16, 108) (92)

Reduction of *c* is 5.

$C = 986 \quad a$ (91, 62, 84, 85) (40, 22) (18, 87, 1) (86) (80, 11) (69)
b (219, 158, 216) (61, 97) (39, 177) (174, 70, 36) (34, 138) (104)
c (281, 264) (26, 61, 177) (160, 68, 44, 9) (35) (24, 116) (92)
d (136, 135) (1, 12, 35, 87) (86, 40, 11) (23) (6, 52) (46)
e (281, 244) (39, 97, 106) (182, 80, 58) (22, 124, 9) (115) (102)
f (132, 121) (11, 17, 40, 53) (108, 29, 6) (23) (79, 13) (66)

Reduction 2 (*a*, *d*, *f*).

$C = 992 \quad a$ (224, 180, 211) (104, 76) (45, 166) (153, 71) (121) (11, 93) (82)
b' (31, 17, 24) (10, 7) (3, 28) (13) (21, 10) (1, 12) (11)
c' (61, 31, 48) (14, 17) (11, 3) (8, 60) (19) (47, 14) (33)
d (288, 263) (85, 178) (153, 83, 52) (44, 8) (93) (70, 13) (57)
e (279, 266) (85, 181) (168, 68, 43) (25, 18) (7, 11) (100) (96)
f (137, 133) (40, 93) (89, 48) (12, 28) (41, 19) (3, 25) (22)
g (143, 127) (28, 99) (83, 48, 12) (40) (35, 13) (9, 31) (22)
h (274, 263) (71, 192) (181, 93) (26, 45) (7, 19) (88, 12) (76)
i (275, 254) (45, 209) (188, 56, 31) (13, 32) (25, 6) (19) (132)
j' (60, 33, 39) (27, 6) (45) (56, 31) (13, 32) (25, 6) (19)
k (275, 244) (108, 136) (198, 77) (121, 64) (43, 93) (57, 7) (50)
l (286, 211) (83, 128) (209, 77) (38, 45) (31, 7) (24, 156) (132)

Reductions 2 (*f*, *g*); 4 (*c'*, *j'*) and 8 (*b'*).

$C = 1008 \quad a'$ (57, 56, 39) (14, 25) (3, 11) (15, 44) (43, 14) (36) (29)
b (71, 72, 52) (19, 33) (70, 1) (5, 14) (69, 4) (9) (56)
c (285, 266) (75, 191) (172, 57, 56) (15, 116) (43, 14) (29) (72)
d (266, 263) (3, 44, 216) (213, 56) (15, 29) (57, 14) (43) (100)
e (95, 76) (24, 19, 33) (70, 25) (5, 14) (20, 9) (56) (45)

Reductions 3 (*b*, *e*); 4 (*a'*).

$C = 1015 \quad a$ (180, 143, 167, 173) (37, 82, 24) (185, 6) (179) (172, 45) (127)
b (219, 215, 199) (16, 183) (17, 47, 167) (163, 43, 13) (30) (120)
c' (41, 43, 39) (4, 16, 19) (39, 2) (37, 12) (25, 3) (22)
d (244, 157, 207) (107, 50) (57, 200) (163, 61, 20) (41, 143) (102)
e (235, 178, 195) (57, 104, 17) (212) (172, 73, 47) (26, 125) (99)

f' (47, 30, 43) (17, 13) (16, 40) (36, 16, 12) (4, 24) (20)
 g (222, 164, 207) (80, 41, 43) (39, 2) (37, 215) (200, 22) (178)
 h (247, 154, 192) (116, 38) (230) (175, 72) (49, 67) (103, 18) (85)
 i (284, 274) (10, 81, 183) (173, 73, 48) (25, 23) (2, 102) (100)
 j (244, 164, 144) (20, 124) (80, 104) (219, 81, 24) (57, 195) (138)
 k (284, 268) (16, 57, 195) (179, 80, 41) (39, 59) (99, 20) (79)
 l (294, 253) (38, 215) (3, 35) (174, 91, 32) (67) (83, 8) (75)
 m' (55, 54) (4, 10, 40) (39, 9, 4, 3) (1, 6) (5) (30)
 n (274, 268) (6, 26, 31, 205) (199, 61, 20) (41, 5) (36) (138)
 o (285, 253) (29, 224) (8, 21) (192, 70, 18, 5) (13) (52) (122)
 p (275, 253) (22, 107, 124) (212, 85) (127, 48, 17) (31, 110) (79)
 q (280, 247) (125, 122) (208, 72) (52, 20) (3, 119) (32, 116) (84)
 r (270, 253) (17, 82, 59, 95) (222, 65) (23, 36) (157, 13) (144)
 s' (57, 46) (25, 21) (43, 14) (4, 17) (29, 14) (1, 16) (15)

Reduction 5 (c' , f' , m' , s').

$C = 1025$

a'	(52, 36, 29) (7, 22) (28, 15) (37) (36, 16) (4, 24) (20)
b'	(57, 55) (36, 10, 11) (17, 38) (9, 1) (8, 4) (21) (17)
c	(51, 36, 25) (11, 14) (24, 20, 3) (17) (42, 9) (37) (33)
d	(56, 55) (16, 39) (38, 18) (3, 4, 9) (20, 1) (5) (14)
e	(57, 52) (20, 13, 19) (39, 18) (7, 6) (25) (3, 24) (21)
f'	(55, 53) (9, 44) (42, 6, 7) (5, 1) (4, 13) (9) (22)
g	(55, 51) (15, 8, 9, 19) (44, 11) (7, 1) (10) (33) (29)

All have reductions 5.

$C = 1033$

a	(241, 169, 198) (140, 29) (227) (184, 57) (46, 11) (35, 116) (81)
b	(236, 128, 116, 113) (3, 110) (12, 107) (140) (217) (204, 32) (172)
c	(288, 259) (32, 227) (198, 76, 11, 3) (8, 27) (19) (46) (122)
d	(304, 241) (119, 122) (184, 76, 44) (32, 12) (128, 3) (125) (108)
e	(285, 259) (29, 230) (204, 43, 35, 3) (32) (8, 59) (51) (110)
f	(285, 236) (107, 59, 70) (227, 58) (48, 11) (81) (169, 44) (125)
g	(288, 230) (58, 172) (227, 119) (16, 43, 113) (108, 11) (27) (70)

$C = 1040$

a'	(16, 12, 10, 13) (7, 3) (7, 5) (16) (13, 3) (12) (10)
b	(252, 144, 197) (91, 53) (250) (27, 64) (195, 47, 10) (37) (148)
c	(304, 263) (53, 210) (169, 73, 50, 12) (65) (23, 27) (96) (92)
d	(290, 263) (39, 224) (197, 65, 28) (16, 23) (37, 7) (30) (132)
e'	(29, 25) (12, 13) (21, 8) (13, 5, 2) (1, 12) (3) (8)
f'	(65, 37, 30) (7, 23) (28, 16) (39) (63, 30) (3, 36) (33)

Reductions 4 (f'); 10 (e') and 13 (a').

$C = 1045$

a	(225, 203, 205) (22, 82, 97, 2) (207) (187, 60) (127, 15) (112)
b	(280, 278) (2, 77, 97, 102) (207, 75) (132, 20) (112, 5) (107)

$C = 1049$

a	(236, 168, 229) (65, 103) (42, 187) (3, 62) (180, 59) (145) (121)
b	(261, 150, 197) (103, 47) (244) (8, 95) (180, 89) (2, 93) (91)
c	(298, 266) (47, 219) (187, 96, 15) (62) (19, 43) (91, 24) (67)
d	(293, 266) (42, 224) (197, 59, 24, 13) (11, 2) (44) (35) (138)
e	(261, 138, 153) (123, 15) (168) (236, 96, 52) (44, 8) (176) (140)
f	(293, 244) (123, 121) (219, 74) (54, 67) (145, 52) (93, 13) (80)
g	(298, 229) (89, 140) (224, 74) (54, 35) (19, 16) (3, 153) (150)

$C = 1058$

a	(248, 169, 216) (115, 54) (7, 209) (61) (177, 71) (35, 141) (106)
b	(233, 157, 225) (99, 58) (41, 17) (7, 218) (24) (210, 23) (187)
c	(273, 161, 159) (2, 157) (163) (192, 81) (8, 149) (30, 141) (111)
d	(253, 123, 83, 126) (40, 43) (163) (169) (220, 33) (187, 9) (178)
e	(300, 281) (85, 196) (177, 80, 43) (20, 23) (17, 3) (111) (97)

<i>f</i>	(145, 143) (45, 98) (96, 26, 23) (3, 20) (29) (12, 53) (41)
<i>g</i>	(286, 281) (71, 210) (205, 81) (15, 56) (55, 41) (14, 83) (69)
<i>h</i>	(290, 273) (67, 99, 107) (205, 85) (35, 32) (123, 8) (120) (115)
<i>i</i>	(143, 138) (4, 7, 17, 110) (1, 3) (105, 39) (10) (27) (66)
<i>j</i>	(150, 124) (56, 68) (105, 45) (15, 29, 12) (17, 63) (60) (46)
<i>k</i>	(286, 253) (33, 58, 55, 107) (233, 86) (3, 52) (61) (159) (147)
<i>l</i>	(312, 225) (78, 147) (9, 69) (209, 112) (67, 149) (97, 15) (82)
<i>m</i>	(156, 109) (43, 66) (4, 16, 23) (108, 40, 12) (28) (89) (68)
<i>n'</i>	(12, 11) (1, 3, 7) (11, 2) (5) (2, 5) (4, 1) (3)

Reductions 2 (*f*, *i*, *j*, *m*) and 23 (*n'*); *n'* is a square.

$C = 1060$	<i>a</i> (122, 95, 102) (37, 51, 7) (109) (89, 33) (23, 14) (65) (56)
	<i>b</i> (152, 137) (33, 104) (89, 38, 25) (7, 26) (13, 19) (51) (45)
	<i>c</i> (147, 134) (8, 11, 115) (5, 3) (14) (102, 38, 12) (26) (64)
	<i>d'</i> (27, 15, 14) (1, 13) (16) (23, 4) (4, 9) (19, 5) (14)
	<i>e</i> (76, 61) (29, 32) (52, 24) (10, 16, 3) (35) (28, 6) (22)
	<i>f</i> (147, 127) (20, 48, 59) (109, 58) (37, 11) (51, 7) (70) (44)
	<i>g</i> (135, 134) (7, 127) (59, 51, 19, 6) (13) (32) (8, 75) (67)

Reductions 2, except *d'* (10) and *e* (4).

$C = 1065$	<i>a</i> (73, 75, 63) (20, 43) (71, 2) (69, 8) (5, 38) (33)
	<i>b</i> (247, 129, 232) (71, 58) (13, 45) (84) (52, 225) (210, 37) (173)
	<i>c</i> (240, 188, 165) (47, 118) (52, 112, 24) (71) (232, 60) (189, 172)
	<i>d</i> (284, 278) (6, 47, 225) (219, 43, 28) (15, 13) (60) (58) (118)
	<i>e</i> (285, 248) (37, 112, 99) (247, 75) (28, 71) (172, 15) (43) (114)

Reduction of *a* is 3.

$C = 1066$	<i>a</i> (121, 86, 115) (38, 48) (19, 96) (90, 31) (28, 10) (77) (59)
	<i>b</i> (264, 162, 167) (157, 5) (172) (209, 55) (154, 58) (38, 134) (96)
	<i>c</i> (153, 137) (31, 106) (90, 44, 19) (4, 27) (23) (2, 48) (46)
	<i>d</i> (297, 280) (22, 49, 209) (192, 100, 5) (27) (76) (92, 8) (84)
	<i>e</i> (153, 121) (50, 71) (106, 47) (29, 21) (25, 67) (59, 17) (42)
	<i>f</i> (140, 132) (25, 48, 59) (121, 19) (2, 23) (21) (81, 11) (70)
	<i>g</i> (297, 242) (55, 69, 118) (230, 88, 34) (20, 49) (54) (167) (142)

Reductions 2 (*a*, *c*, *e*, *f*).

$C = 1073$	<i>a</i> (244, 192, 197) (100, 92) (87, 110) (196, 48) (156, 23) (148) (133)
	<i>b</i> (252, 156, 89, 111) (67, 22) (133) (135, 88) (221) (213, 39) (174)
	<i>c</i> (259, 133, 216) (79, 54) (25, 29) (100, 4) (249) (206, 53) (153)
	<i>d</i> (261, 168, 164) (4, 41, 119) (135, 37) (78) (219, 42) (197) (177)
	<i>e</i> (272, 168, 145) (23, 122) (92, 99) (12, 80) (221) (216, 68) (148)
	<i>f</i> (301, 275) (53, 222) (196, 78, 27) (80) (49, 29) (10, 89) (69)
	<i>g</i> (296, 275) (48, 227) (206, 68, 22) (17, 5) (12, 41) (29) (138)
	<i>h</i> (293, 274) (42, 232) (213, 37, 20, 23) (17, 3) (68) (54) (122)
	<i>i</i> (290, 274) (39, 235) (219, 48, 23) (25, 37) (61, 12) (49) (110)
	<i>j</i> (272, 116, 164) (68, 48) (20, 192) (88) (249, 111) (27, 165) (138)
	<i>k</i> (301, 244) (64, 61, 119) (227, 74) (3, 58) (67) (12, 165) (153)
	<i>l</i> (293, 252) (99, 79, 74) (235, 58) (5, 69) (20, 64) (177) (133)

$C = 1082$	<i>a</i> (126, 82, 114) (50, 32) (41, 105) (93, 33) (27, 23) (64) (60)
	<i>b</i> (239, 162, 232) (77, 85) (15, 217) (210, 106) (100) (104, 2) (102)
	<i>c</i> (241, 146, 228) (64, 82) (46, 18) (89, 239) (226, 15) (61) (150)
	<i>d</i> (154, 140) (33, 107) (93, 41, 20) (1, 32) (21) (52, 10) (42)
	<i>e</i> (154, 126) (53, 73) (107, 47) (22, 31) (11, 62) (60, 9) (51)
	<i>f</i> (312, 241) (77, 164) (217, 89, 6) (83) (128, 44) (40, 124) (84)
	<i>g</i> (156, 116) (41, 75) (113, 32, 11) (10, 31) (21) (81, 3) (78)

Reductions 2 (*a*, *d*, *e*, *g*).

$C = 1088$	a	(242, 172, 219) (99, 43, 30) (13, 17) (56) (236) (213, 29) (184)
	b'	(33, 24, 19) (6, 13) (15, 8, 1) (7) (28) (27, 6) (21)
	c	(280, 168, 145) (23, 122) (92, 99) (85, 7) (228) (215, 65) (150)
	d	(310, 281) (65, 216) (187, 87, 36) (38, 63) (13, 25) (100) (88)
	e	(296, 281) (51, 230) (215, 41, 40) (4, 47) (1, 43) (42) (132)
	f	(298, 279) (23, 24, 232) (213, 81, 4) (26, 1) (25) (51) (132)
	g'	(35, 21, 16) (5, 11) (20, 6) (17) (29, 6) (23, 3) (20)
	h'	(37, 35) (12, 10, 13) (27, 10) (7, 3) (17, 5) (16) (12)
	i'	(37, 35) (6, 29) (27, 6, 4) (3, 7) (5, 1) (4) (16)
	j'	(7, 6, 5) (1, 4) (4, 3) (4, 3) (7) (1, 6) (5)
	k'	(37, 33) (12, 8, 13) (29, 8) (3, 5) (1, 2) (21) (20)
	l'	(57, 41, 42) (40, 1) (43) (33, 24) (9, 51, 4) (47) (42)
	m'	(81, 59) (26, 33) (51, 26, 4) (23, 7) (40) (25, 1) (24)
	n	(280, 279) (29, 250) (104, 63, 85, 28) (57) (41, 22) (164) (145)
	o	(149, 125) (24, 40, 61) (121, 52) (19, 21) (17, 2) (84) (69)
	p	(155, 119) (52, 67) (115, 40) (24, 13, 15) (11, 2) (84) (75)
	q	(324, 221) (87, 134) (40, 47) (219, 81, 24) (57, 7) (188) (138)

Reductions 2 (o, p); 4 (l', m'); 8 (b', g', h', i', k') and 32 (j').

$C = 1102$	a	(120, 101, 115) (44, 43, 14) (29, 100) (95, 25) (1, 71) (70)
	b	(239, 192, 232) (47, 105, 40) (65, 207) (200, 86) (28, 142) (114)
	c	(119, 90, 113) (24, 43, 23) (20, 116) (5, 19) (110, 14) (96)
	d	(249, 154, 230) (78, 76) (2, 65, 239) (17, 63) (220, 46) (174)
	e	(249, 202, 164) (38, 126) (47, 105, 88) (238, 58) (17, 197) (180)
	f	(152, 127) (25, 39, 63) (120, 43, 14) (29, 24) (5, 82) (77)

*Reductions 2 (a, c, f).

$C = 1103$	a	(258, 204, 176) (28, 148) (105, 127) (207, 51) (156) (7, 141) (134)
	b	(254, 148, 231) (73, 75) (71, 2) (69, 8) (239) (216, 38) (178)
	c	(248, 204, 163) (41, 122) (52, 112, 81) (240, 8) (60) (203) (172)
	d	(285, 137, 171) (103, 34) (69, 136) (105, 67) (225, 60) (203) (165)
	e	(305, 286) (60, 226) (207, 57, 41) (16, 85) (73) (4, 81) (77)
	f	(301, 285) (19, 34, 232) (216, 78, 7) (4, 15) (11) (60) (138)
	g	(286, 128, 171) (85, 43) (42, 172) (127) (232, 54) (178, 3) (175)
	h	(296, 286) (51, 235) (225, 43, 28) (15, 13) (2, 62) (60) (122)
	i	(296, 285) (81, 76, 128) (226, 70) (5, 71) (156) (19, 109) (90)
	j	(286, 285) (38, 247) (137, 112, 37) (75) (53, 134) (109, 28) (81)
	k	(305, 258) (64, 75, 119) (235, 70) (53, 11) (42, 44) (165) (163)
	l	(312, 248) (67, 62, 119) (5, 57) (231, 78, 3) (75) (176) (153)
	m	(301, 254) (101, 153) (247, 54) (103, 52) (64, 141) (90, 13) (77)
	n	(312, 240) (44, 60, 136) (28, 16) (76) (239, 101) (37, 175) (138)

$C = 1115$	a	(239, 205, 219) (60, 131, 14) (233) (213, 26) (86) (15, 116) (101)
	b	(262, 154, 222) (86, 68) (35, 255) (69, 17) (52) (215, 47) (168)
	c	(260, 145, 228) (88, 57) (31, 26) (254) (27, 92) (222, 65) (157)
	d'	(53, 34, 36) (25, 9) (7, 29) (16) (47, 6) (41, 6) (35)
	e'	(52, 51) (21, 30) (32, 20) (12, 20, 9) (39) (36, 8) (28)
	f'	(61, 58) (5, 7, 46) (43, 13, 5) (3, 2) (9) (8) (30)
	g	(292, 144, 157) (131, 13) (170) (79, 52) (230, 62) (27, 195) (168)
	h	(304, 288) (13, 40, 235) (3, 10) (219, 81, 7) (17) (57) (138)
	i	(292, 290) (47, 243) (122, 125, 45) (92) (119, 3) (116, 12) (104)
	j	(304, 274) (30, 100, 144) (233, 101) (31, 69) (132) (25, 119) (94)
	k	(324, 254) (100, 154) (213, 81, 30) (51, 79) (132) (25, 129) (104)
	l	(305, 262) (105, 157) (243, 62) (87, 80) (28, 129) (7, 101) (94)
	m	(324, 239) (101, 138) (228, 68, 28) (12, 52, 37) (40) (175) (160)
	n	(288, 274) (14, 51, 87, 122) (265, 37) (88) (52, 35) (157) (140)

Reductions 5 (d', e', f').

$C = 1122$	<i>a</i>	(42, 31, 39) (20, 11) (3, 36) (33, 9) (14) (24, 5) (19)
	<i>b</i>	(124, 80, 118) (42, 38) (35, 121) (11, 31) (115, 9) (20) (86)
	<i>c</i>	(287, 142, 164) (120, 22) (186) (25, 95) (242, 70) (7, 179) (172)
	<i>d</i>	(329, 248) (88, 160) (216, 106, 7) (95) (23, 137) (4, 114) (110)
	<i>e</i>	(329, 234) (88, 146) (30, 58) (230, 76, 23) (53) (25, 179) (154)
	<i>f</i>	(154, 127) (27, 29, 71) (126, 53, 2) (31) (20, 11) (82) (73)

Reductions 2 (*b*, *f*); 6 (*a*).

$C = 1145$	<i>a</i>	(257, 169, 237) (120, 49) (30, 19) (11, 245) (41) (225, 32) (193)
	<i>b</i>	(277, 180, 176) (45, 131) (139, 41) (86) (235, 42) (12, 205) (193)
	<i>c</i>	(62, 60) (4, 9, 47) (45, 15, 2) (6) (1, 8) (7) (30)
	<i>d</i>	(300, 149, 159) (139, 10) (169) (12, 87, 40) (237, 75) (209) (162)
	<i>e</i>	(60, 28, 32) (24, 4) (36) (8, 16) (49, 19) (11, 41) (30)
	<i>f</i>	(310, 268) (42, 86, 140) (257, 95) (51, 35) (16, 19) (162) (159)
	<i>g</i>	(300, 277) (55, 91, 131) (268, 32) (87) (51, 40) (11, 160) (149)

Reductions 5 (*c*, *e*).

$C = 1156$	<i>a</i>	(68, 46, 47) (30, 16) (15, 32) (14, 17) (60, 8) (52) (49)
	<i>b</i>	(138, 84, 97) (71, 13) (42, 68) (121, 17) (16, 26) (104) (94)
	<i>c</i>	(153, 152) (8, 23, 121) (120, 26, 7) (15) (4, 34) (30) (64)
	<i>d</i>	(141, 78, 85) (71, 7) (92) (133, 8) (51, 28) (23, 97) (74)
	<i>e</i>	(152, 141) (13, 60, 68) (133, 17, 2) (15) (32) (84, 8) (76)
	<i>f</i>	(76, 69) (15, 17, 37) (68, 8) (21, 2) (19) (39, 1) (38)

All have reductions 2, except *a*, *f* having 4.

$C = 1166$	<i>a</i>	(264, 152, 247) (66, 86) (46, 20) (97, 9) (256) (239, 71) (168)
	<i>b</i>	(140, 96, 86) (10, 76) (63, 43) (20, 23) (121, 19) (102) (99)
	<i>c</i>	(271, 192, 170) (22, 148) (88, 126) (262, 9) (97) (59, 215) (156)
	<i>d</i>	(324, 271) (59, 84, 128) (247, 71, 6) (65) (40, 44) (176) (172)
	<i>e</i>	(154, 140) (33, 43, 64) (135, 19) (42, 10) (32, 21) (85) (74)
	<i>f</i>	(162, 131) (21, 32, 78) (10, 11) (128, 44) (43) (84, 3) (81)

Reductions 2 (*b*, *e*, *f*).

$C = 1170$	<i>a</i>	(44, 27, 41) (14, 13) (12, 1) (3, 11) (42) (39, 8) (31)
	<i>b</i>	(91, 62, 58) (4, 54) (29, 37) (88, 32) (24, 13) (67) (56)
	<i>c</i>	(56, 42) (16, 26) (41, 13, 2) (11, 7) (4, 3) (29) (28)

Reductions 3 (*b*) and 6 (*a*, *c*).

$C = 1176$	<i>a</i>	(47, 38) (7, 12, 19) (2, 5) (36, 13) (10, 7) (26) (23)
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Reduction 7.