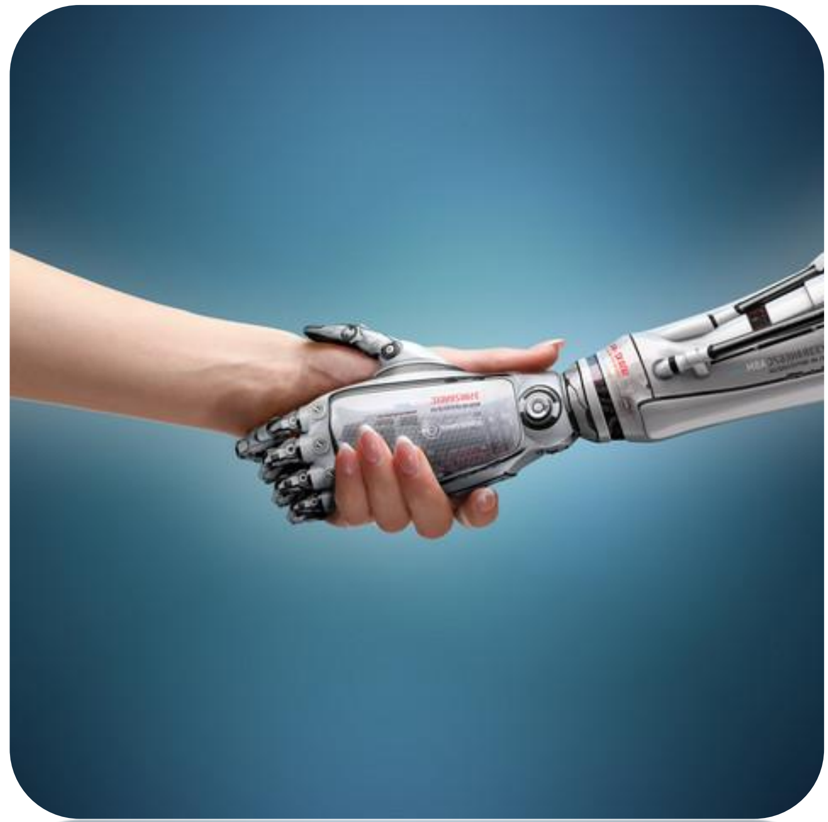
Zurich University
of Applied Sciences

Artificial Intelligence
V05: Constraint satisfaction problems

Introduction to CSPs
CSP solving
Solving CSPs in practice



Based on material by Stuart Russell, UC Berkeley

Zurich University of Applied Sciences and Arts
InIT Institute of Applied Information Technology (stdm)

Educational objectives

- **Remember** **what makes CSP** solving **more powerful** than pure search techniques

- **Explain** how CSPs are solved **on** the **algorithmic level** by **backtracking** using the **MRV** / **degree-** / **least constraining value** heuristics and **forward checking** / **constrained propagation**

- **Formulate** a suitable problem as a **CSP**

"In which we see how treating states as more than just little black boxes leads to the invention of a range of powerful new search methods and a deeper understanding of problem structure and complexity."

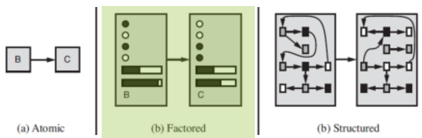
➔ Reading: AIMA, ch. 6

1. INTRODUCTION TO CSPS

Constraint satisfaction problems (CSPs)

Standard search problem

- State is a “**black box**“ – any data structure that supports Goal Test, Eval, Successor



A state thus has a **factored representation**, consisting of many variables and their **partial assignment** of values

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

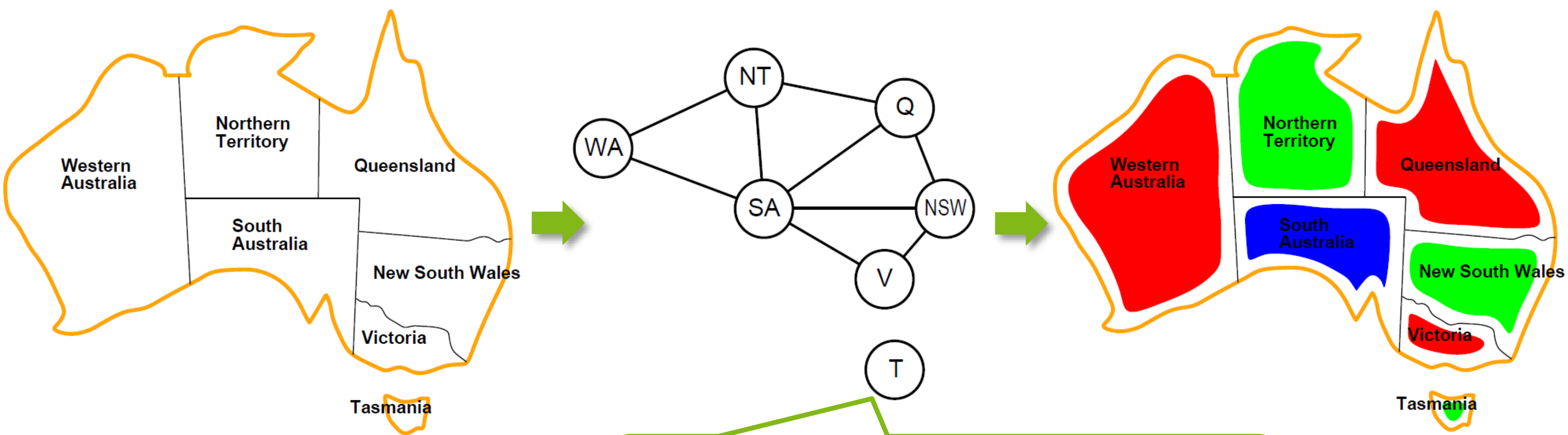
CSP

- **State** is defined by **variables** X_i with **values** from **domain** D_i
- **Goal Test** is a set of **constraints**: **allowable combinations of values** for **subsets of variables**

➔ Simple example of a **formal** representation **language**

➔ Allows useful **general-purpose algorithms** with **more power** than standard search

Example: Map-coloring



Binary CSPs (each constraint relates at most two variables) have a constraint graph. General-purpose CSP algorithms use the graph structure to **speed up search**: E.g., T is an independent subproblem!

Variables: WA, NT, Q, NSW, V, SA, T
Domains: $D_i = \{red, green, blue\}$
Constraints: adjacent regions must have different colors

- e.g. $WA \neq NT$ (if language allows this; otherwise $(WA, NT) \in \{(red, green), (red, blue), (green, red), (green, blue), \dots\}$)

Solutions: assignments satisfying all constraints

- e.g. $\{WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = geen\}$

Varieties of CSPs

Discrete variables

- Finite domains of **size $\boldsymbol{d}$** ➔ $O(d^n)$ complete assignments (n is number of variables)
- Other finite domains (integers, strings, etc.)
  - e.g., job scheduling: variables are days (or integer-minutes) for each job
  - need a **constraint language**, e.g., $StartJob_1 + 5 \leq StartJob_3$
  - linear constraints solvable, nonlinear undecidable

Continuous variables

- e.g., precise start/end times for Hubble Telescope observations
- linear constraints solvable in polynomial time by linear programming methods

Varieties of constraints

- **Unary** constraints: involve a single variable, e.g. $SA \neq green$
- **Binary** constraints involve variable pairs, e.g., $SA \neq WA$ (**all** constraints **can be made binary**)
- **Higher-order** constraints involve 3 or more variables, e.g. column constraints in Sudoku
- **Preferences** (soft) constraints, e.g. $red\ IS\_BETTER\_THAN\ green$

  ➔ often representable by a cost for each assignment: **constrained optimization problems (COP)**

Examples

Car assembly

(job scheduling, simplified)

- Variables: $Axle_F$, $Axle_B$, $Wheel_{RF}$, $Wheel_{LF}$, $Wheel_{RB}$, $Wheel_{LB}$, $Nuts_{RF}$, $Nuts_{LF}$, $Nuts_{RB}$, $Nuts_{LB}$, Cap_{RF}, Cap_{LF}, Cap_{RB}, Cap_{LB}, $Inspect$
- Domains: $D_i = \{1,2,3, \ldots, 27\}$
  (start time of tasks as integer, due to an overall runtime of 30 minutes)
- Constraints:
  (precedence constraints among tasks)
  - $Axle_F + 10 \leq Wheel_{RF}$; $Axle_F + 10 \leq Wheel_{LF}$
  - $Axle_B + 10 \leq Wheel_{RB}$; $Axle_B + 10 \leq Wheel_{LB}$
  - $Axle_F + 10 \leq Axle_B$ **or** $Axle_B + 10 \leq Axle_F$
  - …

Installing an axle takes 10 minutes and must be prior to wheel assembly

Only one shared tool for axle installing, so can´t be simultaneous

Cryptarithmetic

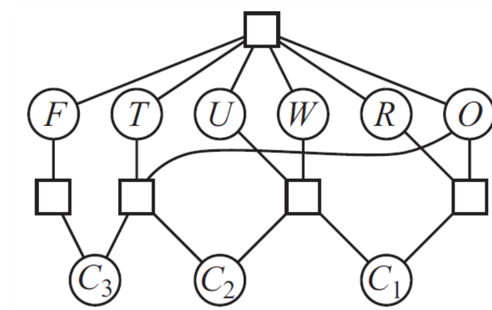
(which letter represents which digit?)

$$\begin{array}{rcccc} & & T & W & O \\ + & & T & W & O \\ \hline & F & O & U & R \end{array}$$

- Variables: $F, T, U, W, R, O, C_1, C_2, C_3$
- Domains: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- Constraints:
  - $alldiff(F, T, U, W, R, O)$
  - $O + O = R + 10C_1$
  - $C_1 + W + W = U + 10C_2$
  - $C_2 + T + T = O + 10C_3$
  - $C_3 = F$

C_1, C_2, C_3: auxiliary variables for carryover

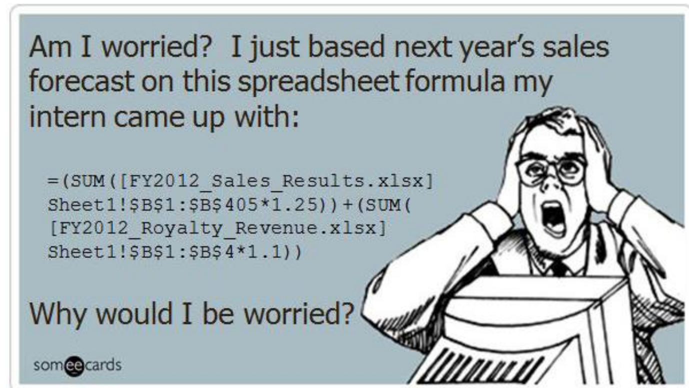
A so-called global constraint involves an **arbitrary number** of variables



Constraint hypergraphs have square (hyper-)nodes for n-ary constraints

Real-world CSPs

- **Assignment** problems
  e.g., who teaches what class
- **Timetabling** problems
  e.g., which class is offered when and where?
- **Optimization** with spreadsheets
  e.g., debugging (Abreu, Riboira & Wotawa, 2012)
- Other **scheduling** tasks
  e.g., in transportation or factory workflow
- Other **layout** tasks
  e.g., floor planning or hardware configuration



➔ Notice that many real-world problems involve real-valued variables

Exercise: Formulating Sudoku as a CSP
➔ see also P03

Sudoku puzzles are played on a 9x9 board and enjoyed by millions of people daily. The goal is to fill in each cell with a single digit, subject to several constraints:

- Each digit must be present in each row exactly once
- Each digit must be present in each column exactly once
- Each digit must be present in each box exactly once
  (the 9x9 board consists of 9 non-overlapping 3x3 boxes
  → see thicker lines below)
- Each digit must be consistent with any digit already placed on the original board by the riddle issuer

➔ Formulate the Sudoku riddle below as a CSP **using pen & paper** (i.e., decide on variables, domains and constraints)

5	3			7				
6			1	9	5			
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2. CSP SOLVING

Standard search formulation
Seriously flawed, thus incremental

Let's start with the straightforward, dumb approach, then fix it

- **States** are defined by the **values assigned so far**
  - Initial state: the empty assignment {}
  - Successor function: assign a value to an unassigned variable without conflict with current assignment
    ➔ fail if no legal assignment (not fixable!)
  - Goal test: the current assignment is complete
- CSPs all have a common structure
  ➔ This is the same for all CSPs, **no domain-specific adaptations** (transition models etc.) needed! ☺

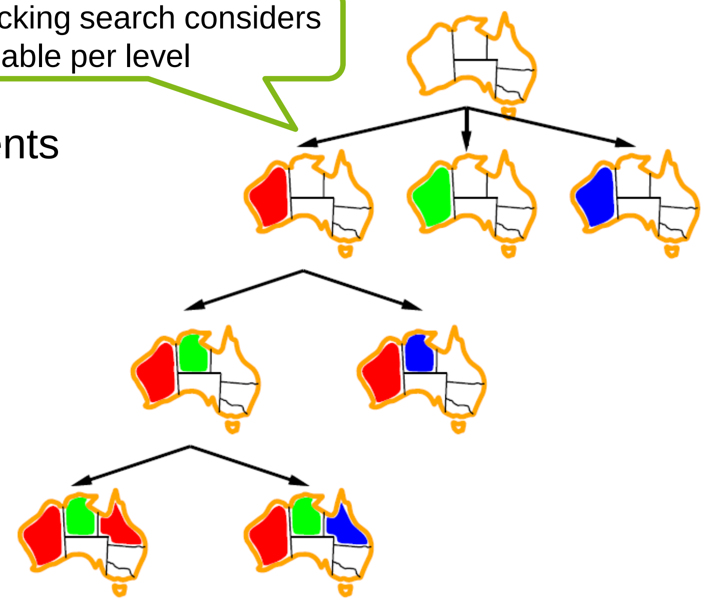
- Every solution appears at depth n (for n variables)
  ➔ use **depth-first search**
- Path is irrelevant, so can also use complete-state formulation (as with local search)
  ➔ i.e., **evolve one state** instead of creating new ones
- Branching factor $b = (n - l)d$ at depth l
  ➔ hence $\boldsymbol{n!\,d^n}$ **leaves**! ☹ ☹ ☹

Backtracking search

First improvement

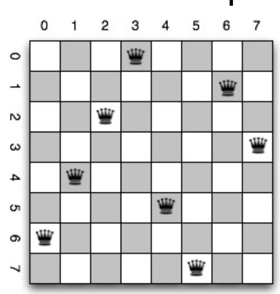
- Variable assignments are **commutative**
  e.g. $[WA = red, then\ NT\ = green]$ same as $[NT = green, then\ WA = red]$
  ➔ Only need to consider assignments to a single variable at each node
  ➔ $b = d$, thus there are d^n leaves

Backtracking search considers one variable per level



Backtracking search

- Using depth-first search with single-variable assignments for CSPs is called backtracking search
- It is the basic uninformed algorithm for CSPs
  ➔ Can solve n-queens for $n = 25$



Remember V04: simple heuristic solves 1’000’000-queens…

Backtracking search
Algorithm & suggested improvements

```
function Backtracking-Search(csp) returns solution/failure
    return Backtrack({}, csp)

function Backtrack(assignment, csp) returns solution/failure
    if assignment is complete then return assignment
    var ← Select-Unassigned-Variable(csp)
    for each value in Order-Domain-Values(var, assignment, csp) do
        if value is consistent with assignment then
            add {var = value} to assignment
            inferences ← Inference(csp, var, value)          #optional
            if inferences ≠ failure then                     #optional
                add inferences to assignment                 #optional
                result ← Backtrack(assignment, csp)
                if result ≠ failure then return result
        else remove {var = value} from assignment
    return failure
```

General-purpose methods can give huge gains in speed:

- **Which variable** should be assigned next?
- In what **order** should its **values** be tried?
- Can we **detect** inevitable **failure early**?
- Can we **take advantage** of **problem structure**?

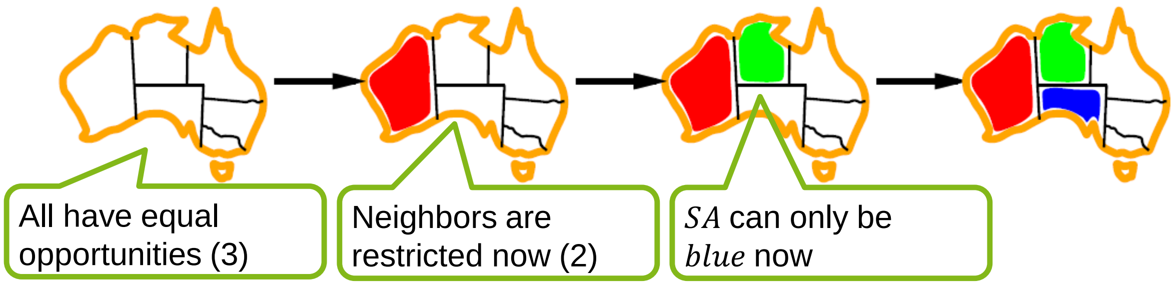
➔ can be achieved by implementing the ***bold/italic*** functions above

Which variable should be assigned next?

Ideas for `Select-Unassigned-Variable(csp)`

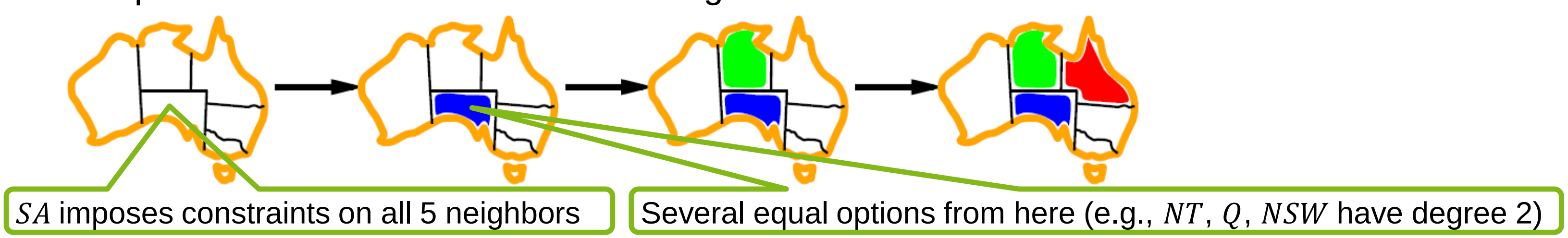
Minimum remaining values (MRV):

- **Choose** the variable with the **fewest legal values**
  → **failing fast** prunes large portions of the tree
- Can work up to $1'000$ times better than picking just the next (or a random) unassigned variable (very problem dependent)



Degree heuristic

- **Choose** the variable that adds **most constraints on remaining** variables
  → In practice: Used as **tie-breaker** among MRV variables

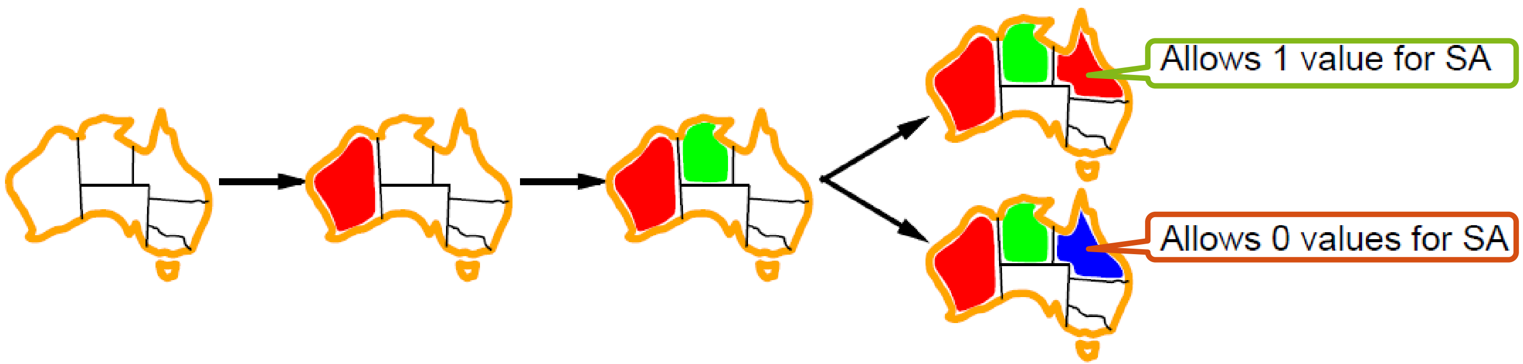


In what order should its values be tried?

Ideas for `Order-Domain-Values(var, assignment, csp)`

Least constraining value

- Given var, **choose** the value that **rules out the fewest values** in the remaining var
  → Combining this with the previous 2 heuristics makes $1'000$-queens feasible (instead 25)

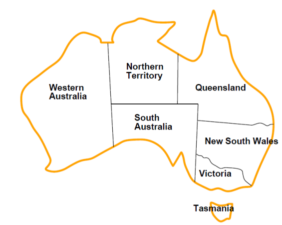


Can we detect inevitable failure early?

Ideas for *`Inference(csp, var, value)`*

Forward checking

- Idea: Keep **track** of **remaining legal values** for unassigned variables
  ➔ Terminate search when any variable has no legal values



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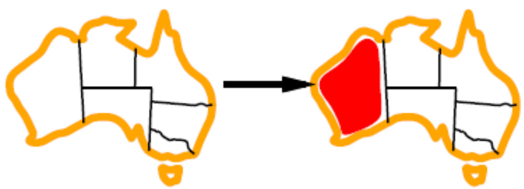

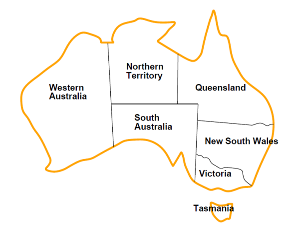

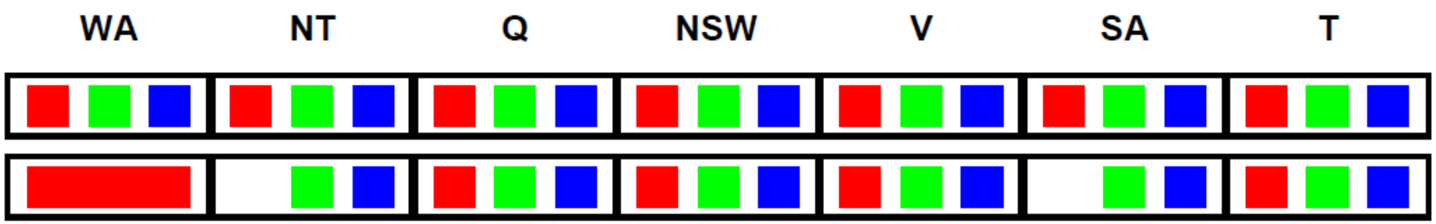

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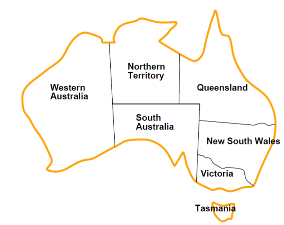

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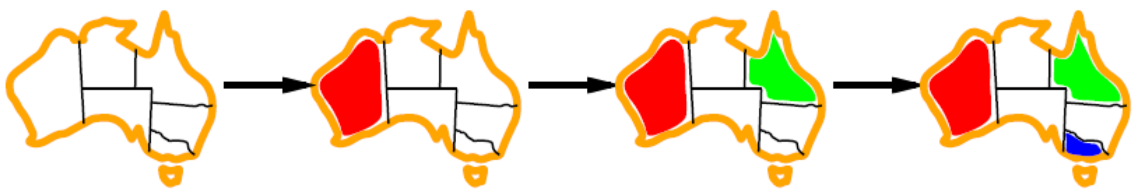
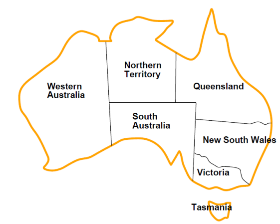
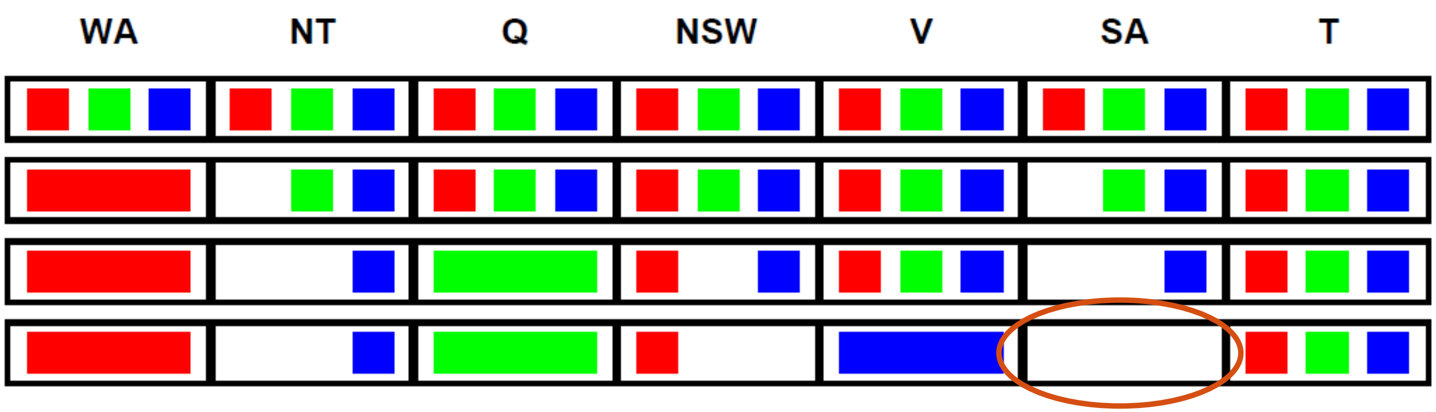


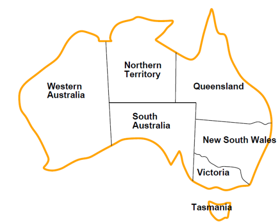


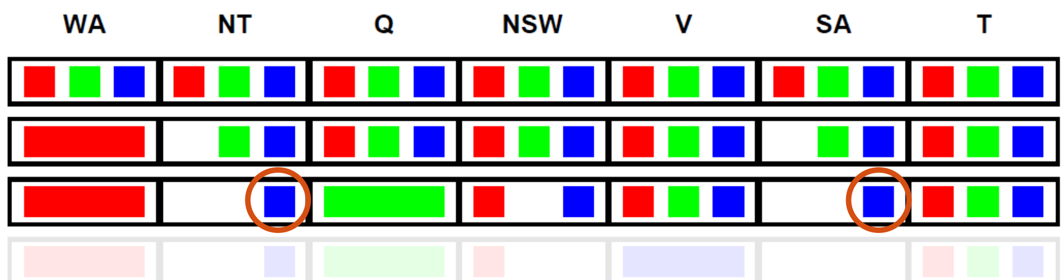


Can we detect inevitable failure early? (contd.)
Ideas for `Inference(csp, var, value)`

Constraint propagation

- **Forward checking** propagates information from assigned variables **only to immediate neighbours** (i.e., fails to do so recursively after a change in some domain)
  → e.g., NT and SA cannot both be $blue$!



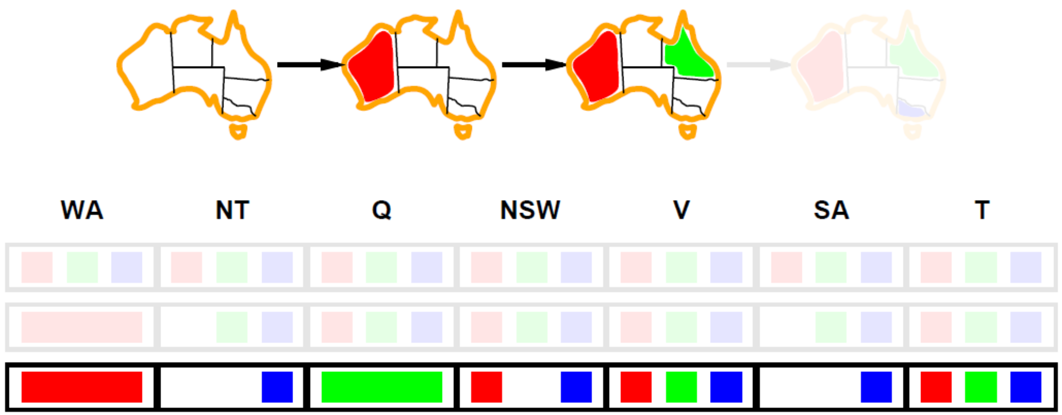


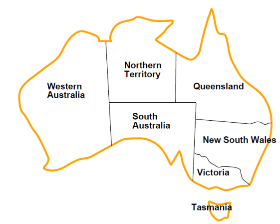
➔ Constraint propagation would repeatedly **enforce constraints locally**

Can we detect inevitable failure early? (contd.)

Ideas for *Inference(csp, var, value)*

Arc





Can we detect inevitable failure early? (contd.)
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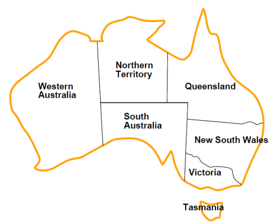
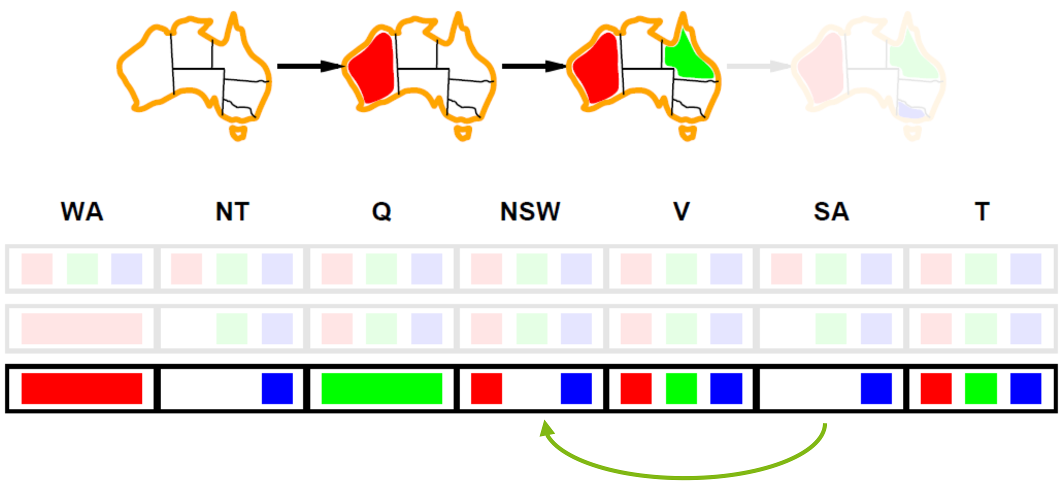


Arc



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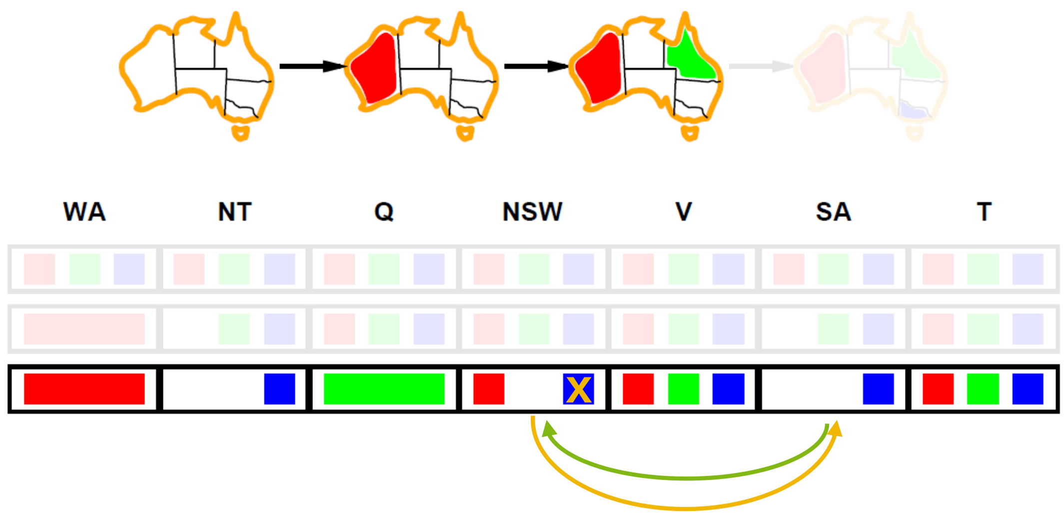

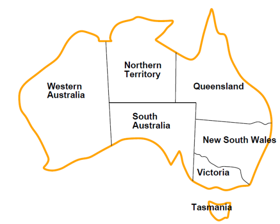

- If X loses a value, neighbors of X need to be rechecked (→ see AC-3 algorithm in appendix)

Can we detect inevitable failure early? (contd.)

Ideas for `Inference(csp, var, value)`

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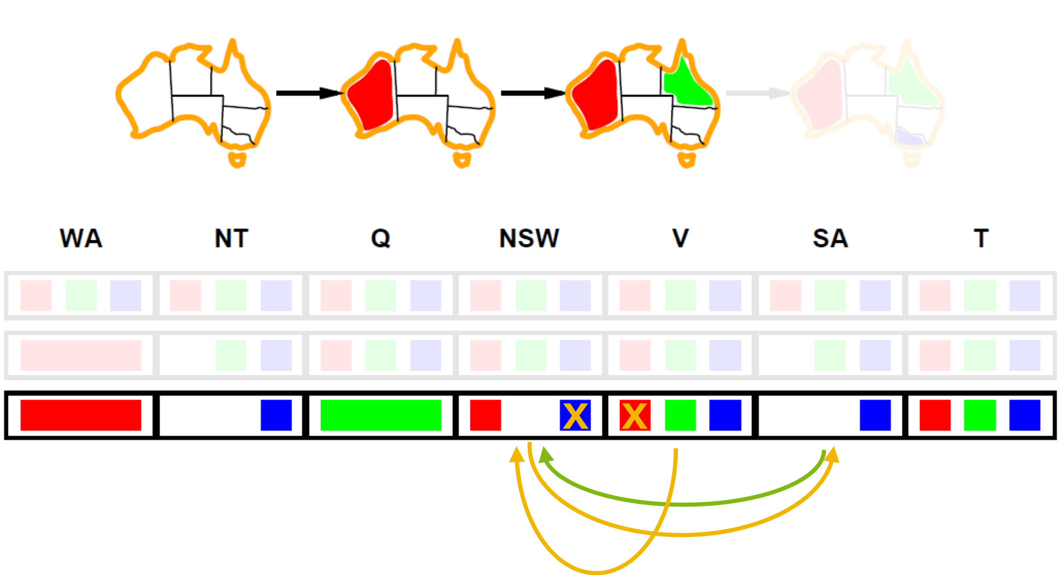

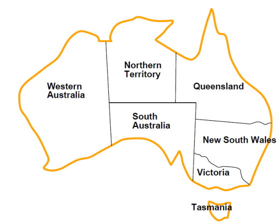

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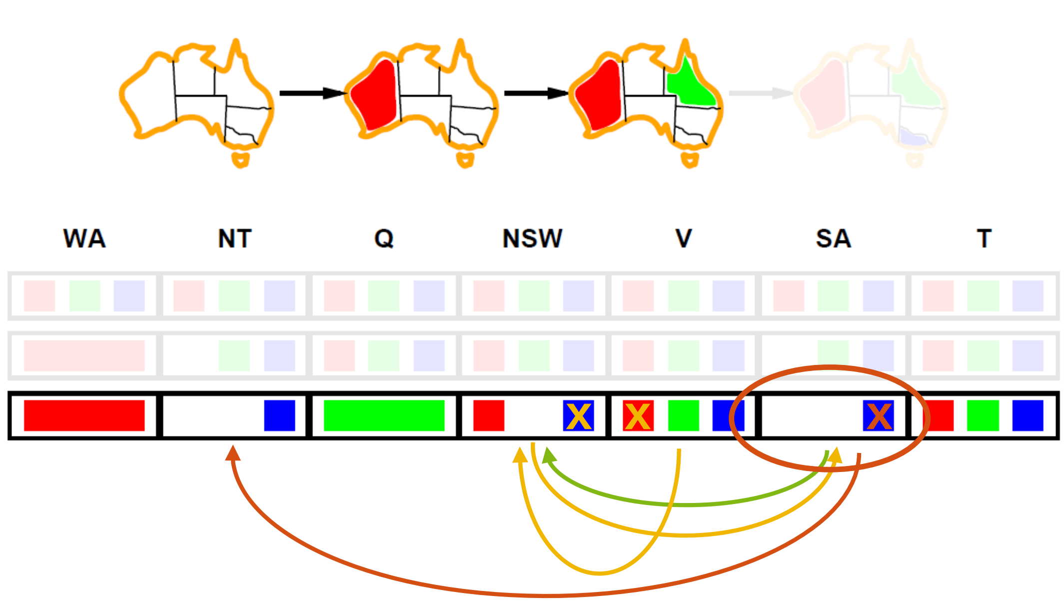

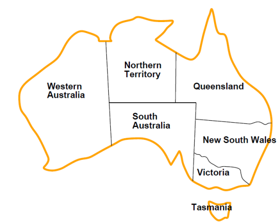

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Backtracking search
Revisiting suggested improvements

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    if assignment is complete then return assignment
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                result ← Backtrack(assignment, csp)
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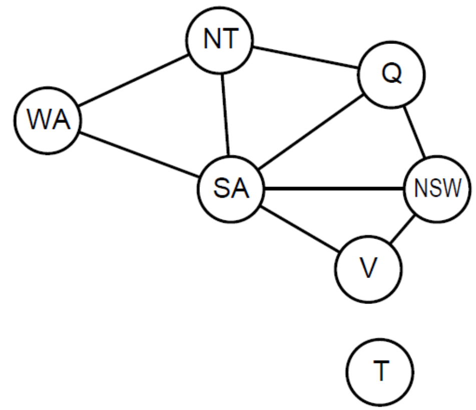
        else remove {var = value} from assignment
    return failure
```

General-purpose methods can give huge gains in speed:

- **Which variable** next? MRV (fewest legal values), degree heuristic (most constraints on rest) on tie
- What **value first**? Least constraining value
- How **detect failure early**? Constraint propagation via arc consistency

- Can we **take advantage** of **problem structure**? → next

Can we take advantage of problem structure?
Exploiting structure in the constraint graph



Example

- Tasmania and mainland are independent **subproblems,** identifiable as connected components of constraint graph
  ➔ can be solved individually, and solution combined

- Suppose each subproblem has c variables (out of n total)
  ➔ Worst-case solution cost is $n/c \cdot d^c$ (linear in n)

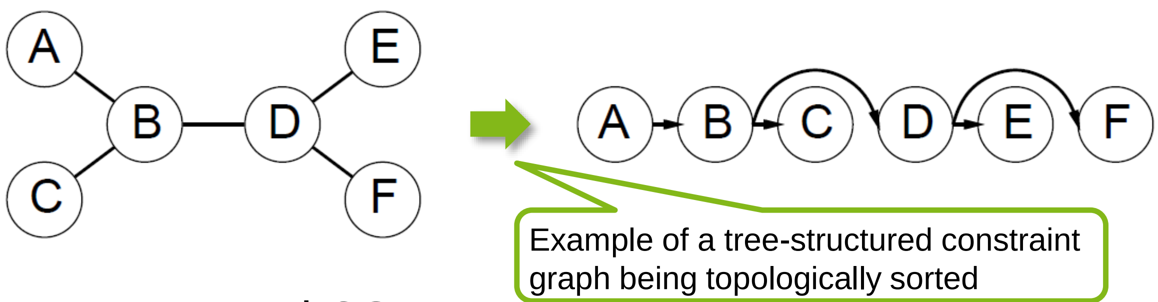
- This is a **dramatic improvement**!
  - E.g., $n = 80$, $d = 2$, $c = 20$:
    ➔ $2^{80} =$ 4 billion years (at 10 million nodes/second)
    ➔ $4 \cdot 2^{20} =$ 0.4 seconds (at 10 million nodes/second)

Can we take advantage of problem structure?

Exploiting structure in the constraint graph (contd.)

Tree-structured CSPs

- A (constraint) graph is a **tree if** any **2 variables** are **connected by only 1 path** (i.e., no loops)
- **Theorem**: If the constraint graph has **no loops**, the CSP can be solved in $\boldsymbol{O(nd^2)}$ time
  ➔ Compare to **general CSPs**, where **worst-case** time is $\boldsymbol{O(d^n)}$
  ➔ Also applies to logical and probabilistic reasoning
  ➔ Important example of the relation between **syntactic restrictions** and the **complexity of reasoning**



Algorithm for tree-structured CSPs

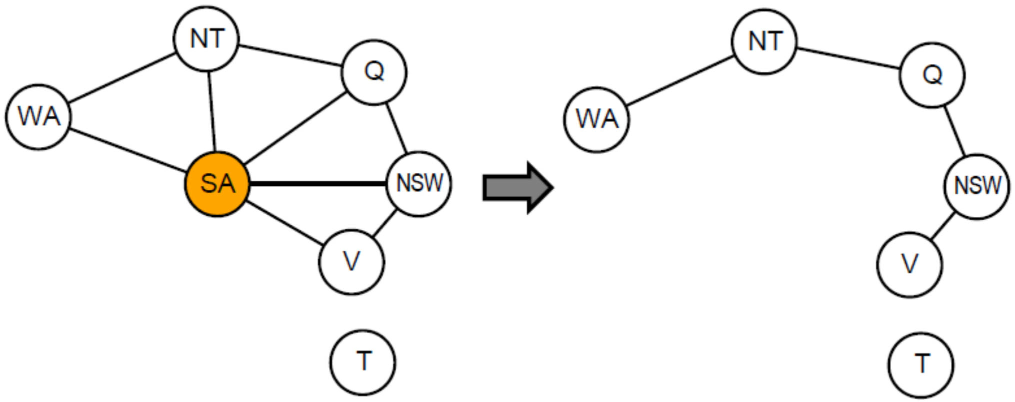
- Do a topological sort: **Choose** a variable as **root**, then **order variables** from root to leaves such that every node's parent precedes it in the ordering
- Create directed arc-consistency by: For j from n down to 2, make $\left(Parent(X_j), X_j\right)$ arc consistent
- For j from 1 to n, **assign** X_j consistently with $Parent(X_j)$

3. SOLVING CSPS IN PRACTICE

Exploiting non-optimal structure

Nearly tree-structured CSPs

- Many real-world CSPs can be converted to tree-structured problems
  ➔ then solved by divide & conquer
- …by choosing a cycle cutset: a set of variables that if removed make the graph a tree



- …and subsequent cutset conditioning: instantiate (in all ways) the variables in the cutset, then prune choices from remaining variables in the tree
  ➔ Very fast for small cutset size c: Runtime is $O\left(d^c \cdot (n-c)d^2\right)$ (linear in n)

Other advice

- **Exploiting structure in** the **values by breaking symmetry** reduces search space up to $d!$
  (e.g., we have to give WA, NT, SA 3 different colors, but have $3!$ options to do so
  → can be reduced by adding a symmetry-breaking constraint like $NT < SA < WA$)

- **Local search** (→ see V04) is **very effective** for CSPs

  Applicable because CSPs also work with complete-state formulations

  ➔ Min-conflicts heuristic very useful: start with random full assignment, subsequently change the variable that minimizes remaining conflicts

  ➔ E.g., hill climbing search with min-conflicts solves n-queens in constant time with high probability (even for $n = 10'000'000$)

- **Constraint learning** (→ see appendix) is one of the **most important technique**s in modern CSP solvers
  (together with backtracking search, the MRV / degree- / least constraining value heuristics, and forward checking / arc consistency)

- **Trade-off** between the **cost of enforcing consistency** and the **reduction in search time**
  (some researchers favor pure forward checking, some full arc consistency after each assignment
  ➔ full arc consistency pays off for harder CSPs)

- Comparing CSP algorithms is done empirically (**no algorithm dominates** on all CSPs)

Where's the intelligence?
Man vs. machine

- If classical **search is brute force**…

- …**CSP** solving **enhances** it using the following powerful ingredients:
  - **General-purpose** heuristics
    (MRV etc. ➔ not problem- or domain specific!)
  - **Inference** over constraints
    (constraint propagation ➔ allows e.g. for intelligent backjumping)
  - **Exploiting structure** in the problem definition to vastly prune the search space
    (e.g. symmetric values, tree-like constraint graph ➔ implements a **general divide & conquer** approach)

- CSP solving thus can reduce the **time complexity** of some problems **from exponential to linear**, by **act**ing **more "clever"**

- **Human** intelligence goes into **stating the task** as a CSP

Review

- CSPs are a special kind of problem:
  - **states** defined by values of a **fixed set of variables**
  - **goal test** defined by **constraints on** variable **values**

- **Backtracking** = **depth-first search** with one variable assigned per node
  - **Variable ordering** and **value selection heuristics** help significantly
  - **Forward checking prevents** assignments that guarantee **later failure**
  - **Constraint propagation** (e.g., arc consistency) does **additional** work to constrain values and detect **inconsistencies**

- The CSP representation allows **analysis of problem structure**
  - **Tree-structured** CSPs can be solved in **linear time**
  - **Iterative min-conflicts** is usually effective **in practice**

- **Methods** can **handle** problems with **up to 100’000 variables**, and up to **1’000’000 constraints** in practice

APPENDIX

Arc consistency
AC-3 Algorithm

```
function AC-3(csp) returns the CSP, possibly with reduced domains
    inputs: csp, a binary CSP with variables (X, D, C)
    local variables: queue, a queue of arcs, initially all the arcs in csp
    while queue is not empty do
        (X_i, X_j) ← Remove-First(queue)
        if Revise(csp, X_i, X_j) then
            if size of D_i = 0 then return false
            for each X_k in X_i.Neighbors - {X_j} do
                add(X_k, X_i) to queue
    return true

function Revise(csp, X_i, X_j) returns true iff we revise the domain of X_i
    revised ← false
    for each x in D_i do
        if no value y in D_j allows (x,y) to satisfy the constraint X_i and X_j then
            delete x from D_i
            revised ← true
    return revised
```

- After applying AC-3, either every arc is consistent or some variable has an empty domain
  ➔ CSP not solvable
- Time complexity: $O(n^2 d^3)$ (can be reduced to $O(n^2 d^2)$, but detecting all is NP-hard)
- Trivia: Name stems from this algorithm being the third one in the paper (Mackworth, 1977)

Can we detect inevitable failure early? (contd.)

Ideas for *Inference(csp, var, value)*

Constraint learning

- If `Backtrack()` fails on X_i, it **backs up to** the **last variable** and tries another value
  ➔ would be more intelligent to track back to one of the variables that caused $D_i = \{\}$
- Forward checking etc. already has this information
  ➔ can be stored in a conflict set
- Constraint learning **adds new constraints** on the fly for sets of assignments (so-called no-goods) that **repeatedly** caused `Backtrack()` to fail