

On Potentially $(K_5 - C_4)$ -graphic Sequences *

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Abstract

In this paper, we characterize the potentially $(K_5 - C_4)$ -graphic sequences where $K_5 - C_4$ is the graph obtained from K_5 by removing four edges of a 4 cycle C_4 . This characterization implies a theorem due to Lai [6].

Key words: graph; degree sequence; potentially $(K_5 - C_4)$ -graphic sequences

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1 Introduction

An n -term nonincreasing nonnegative integer sequence $\pi = (d_1, d_2, \dots, d_n)$ is said to be graphic if it is the degree sequence of a simple graph G of order n ; such a graph G is referred to as a realization of π . We denote by $\sigma(\pi)$ the sum of all the terms of π . K_n is the complete graph on n vertices. C_n is the cycle of length n . $K_n - C_4$ is the graph obtained from K_n by removing 4 edges of a 4 cycle C_4 . Let H be a simple graph. A graphic sequence π is said to be potentially H -graphic if it has a realization G containing H as a subgraph.

Given a graph H , what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted

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$ex(n, H)$, and is known as the Turán number. This problem was proposed for $H = C_4$ by Erdős [1] in 1938 and generalized by Turán [16]. In terms of graphic sequences, the number $2ex(n, H) + 2$ is the minimum even integer l such that every n -term graphical sequence π with $\sigma(\pi) \geq l$ is forcibly H -graphical. In [3], Gould, Jacobson and Lehel considered the following variation of the classical Turán-type extremal problems: determine the smallest even integer $\sigma(H, n)$ such that every n -term positive graphic sequence $\pi = (d_1, d_2, \dots, d_n)$ with $\sigma(\pi) \geq \sigma(H, n)$ has a realization G containing H as a subgraph. They proved that $\sigma(pK_2, n) = (p-1)(2n-p)+2$ for $p \geq 2$; $\sigma(C_4, n) = 2[\frac{3n-1}{2}]$ for $n \geq 4$. In [5,6], Lai determined the values $\sigma(K_4 - e, n)$ for $n \geq 4$ and $\sigma(K_5 - C_4, n)$ for $n \geq 5$. Yin, Li, and Mao [14] determined the values $\sigma(K_{r+1} - e, n)$ for $r \geq 3$ and $r+1 \leq n \leq 2r$ and $\sigma(K_5 - e, n)$ for $n \geq 5$. Recently, Yin and Li [15] determined $\sigma(K_{r+1} - e, n)$. Erdős, Jacobson and Lehel [2] showed that $\sigma(K_k, n) \geq (k-2)(2n-k+1)+2$ and conjectured that the equality holds. They proved the conjecture is true for $k = 3$ and $n \geq 6$, i.e., $\sigma(K_3, n) = 2n$ for $n \geq 6$. The conjecture was confirmed in [3], [7], [8], [9] and [10].

Motivated by the above problems, we consider the following problem: given a graph H , characterize the potentially H -graphic sequences without zero terms. In [11], Luo characterized the potentially C_k -graphic sequences for each $k = 3, 4, 5$. Recently, Luo and Warner [12] characterized the potentially K_4 -graphic sequences. In [13], Eschen and Niu characterized the potentially $(K_4 - e)$ -graphic sequences.

In this paper, we characterize the potentially $(K_5 - C_4)$ -graphic sequences without zero terms. This characterization implies a theorem due to Lai [6].

2 Preparations

Let $\pi = (d_1, d_2, \dots, d_n)$ be a nonincreasing positive integer sequence. Then $\pi' = (d_1 - 1, d_2 - 1, \dots, d_{d_n} - 1, d_{d_n+1}, \dots, d_{n-1})$ is the residual sequence obtained by laying off d_n from π . We denote the nonincreasing sequence π' by $(d'_1, d'_2, \dots, d'_{n-1})$. From here on, denote π' the residual sequence obtained by laying off d_n from π and all the graphic sequences have no zero terms. In order to prove our main result, we need the following results.

Theorem 2.1 [3] If $\pi = (d_1, d_2, \dots, d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π .

The following corollary is obvious.

Corollary 2.2 Let H be a simple graph. If π' is potentially H -graphic, then π is potentially H -graphic.

We will use Corollary 2.2 repeatedly in the proofs of our main results.

Lemma 2.3 (Kleitman and Wang [4]) π is graphic if and only if π' is graphic.

3 Potentially $(K_5 - C_4)$ -graphic sequences

Our main result is as follows:

Theorem 3.1 Let $\pi = (d_1, d_2, \dots, d_n)$ be a graphic sequence with $n \geq 5$. Then π is potentially $(K_5 - C_4)$ -graphic if and only if the following conditions hold:

- (1) $d_1 \geq 4$.
- (2) $d_5 \geq 2$.
- (3) $\pi \neq ((n-2)^2, 2^{n-2})$ for $n \geq 6$, where the symbol x^y stands for y consecutive terms x .
- (4) $\pi \neq (n-k, k+i, 2^i, 1^{n-i-2})$ where $i = 3, 4, \dots, n-2k$ and $k = 1, 2, \dots, [\frac{n-1}{2}] - 1$.
- (5) If $n = 6$, then $\pi \neq (4, 2^5)$.
- (6) If $n = 7$, then $\pi \neq (4, 2^6)$.

Proof: First we assume that π is potentially $(K_5 - C_4)$ -graphic. In this case the necessary conditions (1) and (2) are obvious. we are going to prove the conditions (3) – (6) by way of contradiction.

If $\pi = ((n-2)^2, 2^{n-2})$ where $n \geq 6$ is potentially $(K_5 - C_4)$ -graphic, then according to theorem 2.1, there exists a realization G of π containing $K_5 - C_4$ as a subgraph so that the vertices of $K_5 - C_4$ have the largest degrees of π . Then the sequence $\pi^* = (n-4, n-6, 2^{n-5})$ obtained from $G - (K_5 - C_4)$ must be graphic and there must be no edge between two vertices with degree $n-4$ and $n-6$ for the realization of π^* , which is impossible. Thus, $\pi = ((n-2)^2, 2^{n-2})$ where $n \geq 6$ is not potentially $(K_5 - C_4)$ -graphic. Hence, (3) holds.

If $\pi = (n-k, k+i, 2^i, 1^{n-i-2})$ where $i = 3, 4, \dots, n-2k$ and $k = 1, 2, \dots, [\frac{n-1}{2}] - 1$ is potentially $(K_5 - C_4)$ -graphic, then according to theorem 2.1, there exists a realization G of π containing $K_5 - C_4$ as a subgraph so that the vertices of $K_5 - C_4$ have the largest degrees of π . Then the sequence $\pi^* = (n-k-4, k+i-2, 2^{i-3}, 1^{n-i-2})$ obtained from $G - (K_5 - C_4)$ must be graphic and there must be no edge between two vertices with degree $n-k-4$ and $k+i-2$ for the realization of π^* . Thus, π^* satisfies: $(n-k-4) + (k+i-2) \leq 2(i-3) + (n-i-2)$, that is, $0 \leq (-2)$, which is a contradiction. Hence, (4) holds.

If $\pi = (4, 2^5)$ is potentially $(K_5 - C_4)$ -graphic, then according to theorem 2.1, there exists a realization G of π containing $K_5 - C_4$ as a subgraph so that the vertices of $K_5 - C_4$ have the largest degrees of π . Then the sequence $\pi^* = (2)$ obtained from $G - (K_5 - C_4)$ must be the degree sequence of a

simple graph, which is a contradiction. Thus, $\pi = (4, 2^5)$ is not potentially $(K_5 - C_4)$ -graphic. Hence, (5) holds.

If $\pi = (4, 2^6)$ is potentially $(K_5 - C_4)$ -graphic, then according to theorem 2.1, there exists a realization G of π containing $K_5 - C_4$ as a subgraph so that the vertices of $K_5 - C_4$ have the largest degrees of π . Then the sequence $\pi^* = (2^2)$ obtained from $G - (K_5 - C_4)$ must be the degree sequence of a simple graph, which is a contradiction. Thus, $\pi = (4, 2^6)$ is not potentially $(K_5 - C_4)$ -graphic. Hence, (6) holds.

Now we prove the sufficient condition. Suppose the graphic sequence π satisfies the conditions (1) – (6). Our proof is by induction on n .

First we prove the sufficient condition for $n = 5$. Since $\pi \neq (4^2, 2^3)$, then π is one of the following sequences:

(4^5) , $(4^3, 3^2)$, $(4^2, 3^2, 2)$, $(4, 3^4)$, $(4, 3^2, 2^2)$, $(4, 2^4)$. It is easy to see that they are all potentially $(K_5 - C_4)$ -graphic. Therefore, π is potentially $(K_5 - C_4)$ -graphic for $n = 5$.

We now suppose that the sufficient condition holds for $(n - 1) \geq 5$. We will prove that it holds for n . Let $\pi = (d_1, d_2, \dots, d_n)$ be a graphic sequence with n terms that satisfies the conditions (1) – (6). We only need to show that π is potentially $(K_5 - C_4)$ -graphic. If π' satisfies the assumption, then π' is potentially $(K_5 - C_4)$ -graphic by the induction hypothesis. Therefore, π is potentially $(K_5 - C_4)$ -graphic by Corollary 2.2. Thus, we consider the following cases:

Case 1: If $\pi' = (4, 2^5)$, then $\pi = (5, 3, 2^5)$ or $\pi = (5, 2^5, 1)$. It is easy to see that both of them are potentially $(K_5 - C_4)$ -graphic.

Case 2: If $\pi' = (4, 2^6)$, then $\pi = (5, 3, 2^6)$ or $\pi = (5, 2^6, 1)$. It is easy to see that both of them are potentially $(K_5 - C_4)$ -graphic.

Case 3: $\pi' = ((n - 3)^2, 2^{n-3})$ where $n - 1 \geq 6$.

If $d_n = 2$, then $\pi = ((n - 2)^2, 2^{n-2})$, which is contradict to condition(3).

If $d_n = 1$, then $\pi = (n - 2, n - 3, 2^{n-3}, 1)$. We are going to prove that π is potentially $(K_5 - C_4)$ -graphic. First we show it is true for $n = 6$. In this case, $\pi = (4, 3, 2^3, 1)$. It is easy to see that π is potentially $(K_5 - C_4)$ -graphic. Now we prove that π is potentially $(K_5 - C_4)$ -graphic for $n \geq 7$. It is enough to show $\pi_1 = (n - 5, n - 6, 2^{n-6}, 1)$ is graphic and there exist no edge between two vertices with degree $n - 5$ and $n - 6$ for the realization of π_1 . Hence it is enough to show $\pi_2 = (n - 6, 1^{n-6})$ is graphic. Clearly, π_2 has a realization consisting of $n - 6$ edges and these edges have only one vertex in common.

Thus, $\pi = (n - 2, n - 3, 2^{n-3}, 1)$ is potentially $(K_5 - C_4)$ -graphic for $n \geq 6$.

Case 4: $\pi' = (n - 1 - k, k + i, 2^i, 1^{n-i-3})$ where $i = 3, 4, \dots, n - 1 - 2k$ and $k = 1, 2, \dots, [\frac{n-2}{2}] - 1$.

If $d_n = 2$, then $n - i - 3 = 0$ and $\pi = (n - k, k + i + 1, 2^{i+1})$, which is contradict to condition(4).

If $d_n = 1$, then $\pi = (n - k', k' + i, 2^i, 1^{n-i-2})$, which is contradict to condition(4) .

Case 5: $d_n \geq 4$. In this case, π' satisfies the conditions (1) – (6). Thus, π' is potentially $(K_5 - C_4)$ -graphic. Therefore, π is potentially $(K_5 - C_4)$ -graphic by Corollary 2.2.

Case 6: $d_n = 3$.

If $d_1 \geq 5$, then π' satisfies the conditions (1) – (6). Thus, π' is potentially $(K_5 - C_4)$ -graphic. Therefore, π is potentially $(K_5 - C_4)$ -graphic by Corollary 2.2.

If $d_1 = 4$, there are two subcases: $d_4 = 4$ and $d_4 = 3$.

Subcase 1: $d_4 = 4$. In this case, $d_1 = d_2 = d_3 = d_4 = 4$. Obviously, π' satisfies the conditions (1) – (6). Thus, π' is potentially $(K_5 - C_4)$ -graphic. Therefore, π is potentially $(K_5 - C_4)$ -graphic by Corollary 2.2.

Subcase 2: $d_4 = 3$.

Subcase 2.1: $d_3 = 4$. Then $\pi = (4^3, 3^{n-3})$. Since $\sigma(\pi)$ is even, n must be odd. We are going to prove that π is potentially $(K_5 - C_4)$ -graphic. It is easy to see that $\pi = (4^3, 3^4)$ is potentially $(K_5 - C_4)$ -graphic. If $n \geq 9$, then $(4^3, 3^{n-3})$ has a realization containing a $K_5 - C_4$ (see Figure 1).

Thus, $\pi = (4^3, 3^{n-3})$ where n is odd is potentially $(K_5 - C_4)$ -graphic.

Subcase 2.2: $d_3 = 3$.

If $d_2 = 4$, then $\pi = (4^2, 3^{n-2})$. Since $\sigma(\pi)$ is even, n must be even. We are going to prove that π is potentially $(K_5 - C_4)$ -graphic. It is easy to see that $\pi = (4^2, 3^4)$ and $\pi = (4^2, 3^6)$ are potentially $(K_5 - C_4)$ -graphic. If $n \geq 10$, then $(4^2, 3^{n-2})$ has a realization containing a $K_5 - C_4$ (see Figure 2).

Thus, $\pi = (4^2, 3^{n-2})$ where n is even is potentially $(K_5 - C_4)$ -graphic.

If $d_2 = 3$, then $\pi = (4, 3^{n-1})$. Since $\sigma(\pi)$ is even, n must be odd . We are going to prove that π is potentially $(K_5 - C_4)$ -graphic. It is easy to see that $\pi = (4, 3^6)$ is potentially $(K_5 - C_4)$ -graphic. If $n \geq 9$, then $(4, 3^{n-1})$ has a realization containing a $K_5 - C_4$ (see Figure 3).

Thus, $\pi = (4, 3^{n-1})$ where n is odd is potentially $(K_5 - C_4)$ -graphic.

Case 7: $d_n = 2$ and $\pi' \neq ((n-3)^2, 2^{n-3})$ where $n-1 \geq 6$, $\pi' \neq (n-1-k, k+i, 2^i, 1^{n-i-3})$ where $i = 3, 4, \dots, n-1-2k$ and $k = 1, 2, \dots, [\frac{n-2}{2}] - 1$. $\pi' \neq (4, 2^5)$, $\pi' \neq (4, 2^6)$.

If $d_1 \geq 5$, then π' satisfies the conditions (1) – (6). Thus, π' is potentially $(K_5 - C_4)$ -graphic. Therefore, π is potentially $(K_5 - C_4)$ -graphic by Corollary 2.2.

If $d_1 = 4$, there are three subcases: $d_2 = 4$, $d_2 = 3$ and $d_2 = 2$.

Subcase 1: $d_2 = 4$.

If $d_3 = 4$, then π' satisfies the conditions (1) – (6). Thus, π' is potentially $(K_5 - C_4)$ -graphic. Therefore, π is potentially $(K_5 - C_4)$ -graphic by Corollary 2.2.

If $d_3 = 3$, then $\pi = (4^2, 3^a, 2^{n-2-a})$ where $a \geq 1$ and $n-2-a \geq 1$. Since $\sigma(\pi)$ is even, a must be even. We are going to prove that π is potentially $(K_5 - C_4)$ -graphic.

First, we consider $\pi = (4^2, 3^2, 2^{n-4})$. It is easy to see that $\pi = (4^2, 3^2, 2^2)$ and $\pi = (4^2, 3^2, 2^3)$ are potentially $(K_5 - C_4)$ -graphic. If $n \geq 8$, then $(4^2, 3^2, 2^{n-4})$ has a realization containing a $K_5 - C_4$ (see Figure 4). Thus, we are done.

Then we consider $\pi = (4^2, 3^a, 2^{n-2-a})$ where $a \geq 4$ and $n-2-a \geq 1$. It is easy to see that $\pi = (4^2, 3^4, 2)$ and $\pi = (4^2, 3^4, 2^2)$ are potentially $(K_5 - C_4)$ -graphic. If $a = 4$ and $n \geq 9$, then $(4^2, 3^4, 2^{n-6})$ has a realization containing a $K_5 - C_4$ (see Figure 5). If $a \geq 6$, then $(4^2, 3^a, 2^{n-2-a})$ has a realization containing a $K_5 - C_4$ (see Figure 6).

If $d_3 = 2$, then $\pi = (4^2, 2^{n-2})$. Since $\pi \neq (4^2, 2^4)$, we must have $n \geq 7$. We are going to prove that π is potentially $(K_5 - C_4)$ -graphic. It is enough to show $\pi_1 = (2^{n-4})$ is graphic. Clearly, C_{n-4} is a realization of π_1 . Thus, we are done.

Subcase 2: $d_2 = 3$. Then $\pi = (4, 3^a, 2^{n-1-a})$ where $a \geq 1$ and $n-1-a \geq 1$. Since $\sigma(\pi)$ is even, a must be even. We are going to prove that π is potentially $(K_5 - C_4)$ -graphic.

First, we consider $\pi = (4, 3^2, 2^{n-3})$. It is enough to show $\pi_1 = (2^{n-5}, 1^2)$ is graphic. Clearly, π_1 is graphic. Thus, $\pi = (4, 3^2, 2^{n-3})$ is potentially $(K_5 - C_4)$ -graphic.

Second, we consider $\pi = (4, 3^4, 2^{n-5})$. It is easy to see that $\pi = (4, 3^4, 2)$ and $\pi = (4, 3^4, 2^2)$ are potentially $(K_5 - C_4)$ -graphic. If $n \geq 8$, then $(4, 3^4, 2^{n-5})$ has a realization containing a $K_5 - C_4$ (see Figure 7). Thus, we are done.

Then we consider $\pi = (4, 3^a, 2^{n-1-a})$ where $a \geq 6$ and $n-1-a \geq 1$. It is easy to see that $\pi = (4, 3^6, 2)$ is potentially $(K_5 - C_4)$ -graphic. If $a = 6$ and $n \geq 9$, then $(4, 3^6, 2^{n-7})$ has a realization containing a $K_5 - C_4$ (see Figure 8). If $a \geq 8$ and $n-1-a = 1$, then $(4, 3^a, 2)$ has a realization containing a $K_5 - C_4$ (see Figure 9). If $a \geq 8$ and $n-1-a \geq 2$, then $(4, 3^a, 2^{n-1-a})$ has a realization containing a $K_5 - C_4$ (see Figure 10). Thus, we are done.

Subcase 3: $d_2 = 2$. Then $\pi = (4, 2^{n-1})$. Since $\pi \neq (4, 2^5)$ and $\pi \neq (4, 2^6)$, we must have $n \geq 8$. We are going to prove that π is potentially $(K_5 - C_4)$ -graphic. It is enough to show $\pi_1 = (2^{n-5})$ where $n \geq 8$ is graphic. Obviously, C_{n-5} is a realization of π_1 . Thus, $\pi = (4, 2^{n-1})$ is potentially $(K_5 - C_4)$ -graphic.

Case 8: $d_n = 1$ and $\pi' \neq ((n-3)^2, 2^{n-3})$, $\pi' \neq (n-1-k, k+i, 2^i, 1^{n-i-3})$ where $i = 3, 4, \dots, n-1-2k$ and $k = 1, 2, \dots, [\frac{n-2}{2}] - 1$. $\pi' \neq (4, 2^5)$, $\pi' \neq (4, 2^6)$.

If $d_1 \geq 5$, then π' satisfies the conditions (1) – (6). Thus, π' is potentially $(K_5 - C_4)$ -graphic. Therefore, π is potentially $(K_5 - C_4)$ -graphic by Corollary 2.2.

If $d_1 = 4$, there are three subcases: $d_2 = 4$, $d_2 = 3$ and $d_2 = 2$.

Subcase 1: $d_2 = 4$. In this case, π' satisfies the conditions (1) – (6). Thus, π' is potentially $(K_5 - C_4)$ -graphic. Therefore, π is potentially $(K_5 - C_4)$ -graphic by Corollary 2.2.

Subcase 2: $d_2 = 3$. Then $\pi = (4, 3^a, 2^b, 1^{n-1-a-b})$ where $a \geq 1$, $a+b \geq 4$ and $n-1-a-b \geq 1$. Since $\sigma(\pi)$ is even, $n-1-b$ must be even. We are going to prove that π is potentially $(K_5 - C_4)$ -graphic.

Subcase 2.1: $a = 1$. Then $\pi = (4, 3, 2^b, 1^{n-2-b})$. It is enough to show $\pi_1 = (2^{b-3}, 1^{n-1-b})$ is graphic. Clearly, π_1 is graphic. Thus, we are done.

Subcase 2.2: $a = 2$. Then $\pi = (4, 3^2, 2^b, 1^{n-3-b})$. It is enough to show $\pi_1 = (2^{b-2}, 1^{n-1-b})$ is graphic. Clearly, π_1 is graphic. Thus, we are done.

Subcase 2.3: $a = 3$. Then $\pi = (4, 3^3, 2^b, 1^{n-4-b})$. First, we consider $\pi = (4, 3^3, 2, 1^{n-5})$ where n is even. It is easy to see that $\pi = (4, 3^3, 2, 1)$ is potentially $(K_5 - C_4)$ -graphic. If $n \geq 8$, then $(4, 3^3, 2, 1^{n-5})$ has a realization containing a $K_5 - C_4$ (see Figure 11). Second, we consider $\pi = (4, 3^3, 2^2, 1^{n-6})$ where n is odd. It is easy to see that $\pi = (4, 3^3, 2^2, 1)$ is potentially $(K_5 - C_4)$ -graphic. If $n \geq 9$, then $(4, 3^3, 2^2, 1^{n-6})$ has a realization containing a $K_5 - C_4$ (see Figure 12). Third, we consider $\pi = (4, 3^3, 2^3, 1^{n-7})$ where n is even. It is easy to see that $\pi = (4, 3^3, 2^3, 1)$ is potentially $(K_5 - C_4)$ -graphic. If $n \geq 10$, then $(4, 3^3, 2^3, 1^{n-7})$ has a realization containing a $K_5 - C_4$ (see Figure 13). Then, we consider $\pi = (4, 3^3, 2^b, 1^{n-4-b})$ where $b \geq 4$. In this case, $(4, 3^3, 2^b, 1^{n-4-b})$ has a realization containing a $K_5 - C_4$ (see Figure 14). Thus, we are done.

Subcase 2.4: $a = 4$. Then $\pi = (4, 3^4, 2^b, 1^{n-5-b})$. There are two subcases: $b \geq 1$ and $b = 0$.

Suppose $b \geq 1$. It is easy to see that $\pi = (4, 3^4, 2, 1^{n-6})$ and $\pi = (4, 3^4, 2^2, 1^{n-7})$ are potentially $(K_5 - C_4)$ -graphic (see Figure 15 and Figure 16, respectively). If $b \geq 3$, then $(4, 3^4, 2^b, 1^{n-5-b})$ has a realization containing a $K_5 - C_4$ (see Figure 17). Thus, we are done.

Suppose $b = 0$. Then $\pi = (4, 3^4, 1^{n-5})$. Since $\sigma(\pi)$ is even, $n-5$ must be even. Clearly, $(4, 3^4, 1^{n-5})$ has a realization containing a $K_5 - C_4$ (see Figure 18). Thus, we are done.

Subcase 2.5: $a \geq 5$. Then $\pi = (4, 3^a, 2^b, 1^{n-1-a-b})$ where $a \geq 5$ and $n-1-a-b \geq 1$. There are two subcases: $b \geq 1$ and $b = 0$.

Suppose $b \geq 1$.

If a is even, it is easy to see that $\pi = (4, 3^6, 2, 1^{n-8})$ has a realization containing a $K_5 - C_4$ (see Figure 19). If $a = 6$ and $b \geq 2$, then $(4, 3^6, 2^b, 1^{n-7-b})$ has a realization containing a $K_5 - C_4$ (see Figure 20). If $a \geq 8$ and $b = 1$, then $(4, 3^a, 2, 1^{n-2-a})$ has a realization containing a $K_5 - C_4$ (see Figure 21). If $a \geq 8$ and $b \geq 2$, then $(4, 3^a, 2^b, 1^{n-1-a-b})$ has a realization containing a $K_5 - C_4$ (see Figure 22).

If a is odd, it is easy to see that $\pi = (4, 3^5, 2, 1^{n-7})$ has a realization containing a $K_5 - C_4$ (see Figure 23). If $a = 5$ and $b \geq 2$, then $(4, 3^5, 2^b, 1^{n-6-b})$

has a realization containing a $K_5 - C_4$ (see Figure 24). If $a \geq 7$ and $b = 1$, then $(4, 3^a, 2, 1^{n-2-a})$ has a realization containing a $K_5 - C_4$ (see Figure 25). If $a \geq 7$ and $b \geq 2$, then $(4, 3^a, 2^b, 1^{n-1-a-b})$ has a realization containing a $K_5 - C_4$ (see Figure 26). Thus, we are done.

Suppose $b = 0$. Then $\pi = (4, 3^a, 1^{n-1-a})$. Since $\sigma(\pi)$ is even, $n - 1$ must be even.

If a is even, it is easy to see that $\pi = (4, 3^6, 1^2)$ is potentially $(K_5 - C_4)$ -graphic. If $a = 6$ and $n \geq 11$, then $(4, 3^6, 1^{n-7})$ has a realization containing a $K_5 - C_4$ (see Figure 27). If $a \geq 8$, then $(4, 3^a, 1^{n-1-a})$ has a realization containing a $K_5 - C_4$ (see Figure 28).

If a is odd, it is easy to see that $\pi = (4, 3^5, 1)$ and $\pi = (4, 3^7, 1)$ are potentially $(K_5 - C_4)$ -graphic. If $a = 5$ and $n \geq 9$, then $(4, 3^5, 1^{n-6})$ has a realization containing a $K_5 - C_4$ (see Figure 29). If $a = 7$ and $n \geq 11$, then $(4, 3^7, 1^{n-8})$ has a realization containing a $K_5 - C_4$ (see Figure 30). If $a \geq 9$, then $(4, 3^a, 1^{n-1-a})$ has a realization containing a $K_5 - C_4$ (see Figure 31). Thus, we are done.

Subcase 3: $d_2 = 2$. Then $\pi = (4, 2^a, 1^{n-1-a})$ where $a \geq 4$ and $n - 1 - a \geq 1$. Since $\sigma(\pi)$ is even, $n - 1 - a$ must be even. We are going to prove that π is potentially $(K_5 - C_4)$ -graphic. If $a = 4$, then $\pi = (4, 2^4, 1^{n-5})$ where $n - 5$ is even. It is enough to show $\pi_1 = (1^{n-5})$ is graphic. Clearly, π_1 has a realization consisting of $\frac{n-5}{2}$ disjoint edges. Thus, $\pi = (4, 2^4, 1^{n-5})$ is potentially $(K_5 - C_4)$ -graphic. If $a \geq 5$, it is enough to show $\pi_1 = (2^{a-4}, 1^{n-1-a})$ is graphic. Clearly, π_1 is graphic. Thus, we are done.

4 Application

Using Theorem 3.1, we give a simple proof of the following theorem due to Lai [6]:

Theorem 4.1 (Lai [6]) For $n \geq 5$, $\sigma(K_5 - C_4, n) = 4n - 4$.

Proof: First we claim that for $n \geq 5$, $\sigma(K_5 - C_4, n) \geq 4n - 4$. It is enough to show that there exist π_1 with $\sigma(\pi_1) = 4n - 6$, such that π_1 is not potentially $(K_5 - C_4)$ -graphic. Take $\pi_1 = ((n-1)^2, 2^{n-2})$, then $\sigma(\pi_1) = 4n - 6$, and it is easy to see that π_1 is not potentially $(K_5 - C_4)$ -graphic by Theorem 3.1.

Now we show that if π is an n -term ($n \geq 5$) graphical sequence with $\sigma(\pi) \geq 4n - 4$, then there exist a realization of π containing a $K_5 - C_4$. Hence, it suffices to show that π is potentially $(K_5 - C_4)$ -graphic.

If $d_5 = 1$, then $\sigma(\pi) = d_1 + d_2 + d_3 + d_4 + (n-4)$ and $d_1 + d_2 + d_3 + d_4 \leq 12 + (n-4) = n+8$. Therefore, $\sigma(\pi) \leq 2n+4 < 4n-4$, which is a contradiction. Thus, $d_5 \geq 2$.

If $d_1 \leq 3$, then $\sigma(\pi) \leq 3n < 4n-4$, which is a contradiction. Thus, $d_1 \geq 4$.

Since $\sigma(\pi) \geq 4n - 4$, then π is not one of the following:
 $((n-2)^2, 2^{n-2})$ for $n \geq 6$, $(n-k, k+i, 2^i, 1^{n-i-2})$ where $i = 3, 4, \dots, n-2k$ and $k = 1, 2, \dots, [\frac{n-1}{2}]-1$, $(4, 2^5), (4, 2^6)$. Thus, π satisfies the conditions (1) – (6) in Theorem 3.1. Therefore, π is potentially $(K_5 - C_4)$ -graphic.

References

- [1] P. Erdős, On sequences of integers no one of which divides the product of two others and some related problems, Izv. Naustno-Issl. Mat. i Meh. Tomsk 2(1938), 74-82.
- [2] P. Erdős, M.S. Jacobson and J. Lehel, Graphs realizing the same degree sequences and their respective clique numbers, in Graph Theory, Combinatorics and Application, Vol. 1(Y. Alavi et al., eds.), John Wiley and Sons, Inc., New York, 1991, 439-449.
- [3] R.J. Gould, M.S. Jacobson and J. Lehel, Potentially G -graphic degree sequences,in Combinatorics, Graph Theory and Algorithms,Vol. 2 (Y. Alavi et al.,eds.), New Issues Press, Kalamazoo, MI, 1999, 451-460.
- [4] D.J. Kleitman and D.L. Wang , Algorithm for constructing graphs and digraphs with given valences and factors,Discrete Math., 6(1973),79-88.
- [5] Chunhui Lai, A note on potentially $K_4 - e$ graphical sequences, Australasian Journal of Combinatorics 24(2001), 123-127. math.CO/0308105
- [6] Chunhui Lai, An extremal problem on potentially $K_m - C_4$ -graphic sequences, Journal of Combinatorial Mathematics and Combinatorial Computing, to appear in 2007. math.CO/0409041
- [7] Jiong-Sheng Li and Zi-Xia Song, An extremal problem on the potentially P_k -graphic sequences, Discrete Math., (212)2000, 223-231.
- [8] Jiong-Sheng Li and Zi-Xia Song, The smallest degree sum that yields potentially P_k -graphical sequences, J. Graph Theory, 29(1998), 63-72.
- [9] Jiong-sheng Li, Zi-Xia Song and Rong Luo, The Erdős-Jacobson-Lehel conjecture on potentially P_k -graphic sequence is true, Science in China(Series A), 41(5)(1998), 510-520.
- [10] Jiong-Sheng Li and Zi-Xia Song, On the potentially P_k -graphic sequences, Discrete Math., (195)1999, 255-262.
- [11] Rong Luo, On potentially C_k -graphic sequences, Ars Combinatoria 64(2002), 301-318.

- [12] Rong Luo, Warner Morgan. On potentially K_k -graphic sequences, *Ars Combinatoria* 75(2005), 233-239.
- [13] Elaine M. Eschen and Jianbing Niu, On potentially $K_4 - e$ -graphic sequences, *Australasian Journal of Combinatorics*, 29(2004), 59-65.
- [14] Jianhua Yin, Jiongsheng Li and Rui Mao, An extremal problem on the potentially $K_{r+1} - e$ -graphic sequences, *Ars Combinatoria*, 74(2005), 151-159.
- [15] Jianhua Yin and Jiongsheng Li, Two sufficient conditions for a graphic sequence to have a realization with prescribed clique size, *Discrete Math.*, 301(2005) 218-227.
- [16] P. Turán, On an extremal problem in graph theory, *Mat. Fiz. Lapok* 48(1941), 436-452.

Appendix

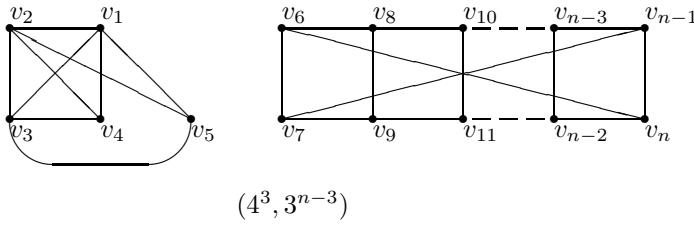
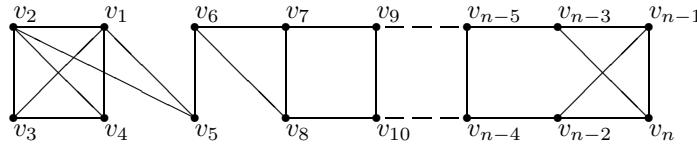
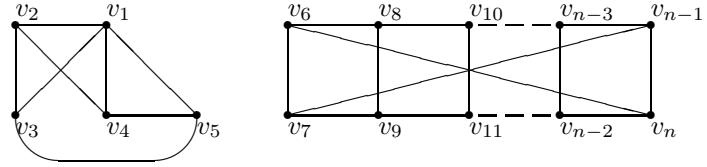


Figure 1



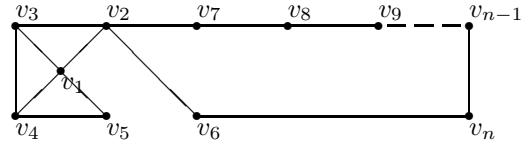
$(4^2, 3^{n-2})$

Figure 2



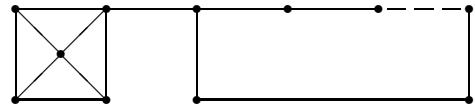
$$(4, 3^{n-1})$$

Figure 3



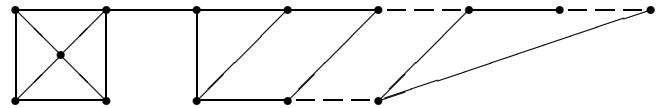
$$(4^2, 3^2, 2^{n-4})$$

Figure 4



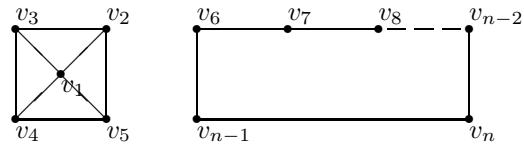
$$(4^2, 3^4, 2^{n-6})$$

Figure 5



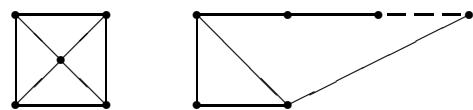
$$(4^2, 3^a, 2^{n-2-a})$$

Figure 6



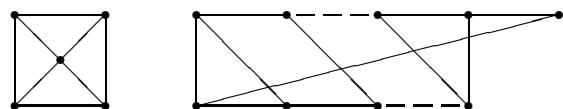
$$(4, 3^4, 2^{n-5})$$

Figure 7



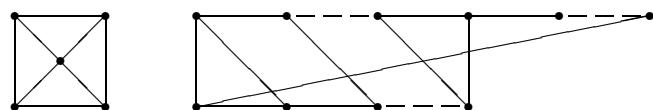
$$(4, 3^6, 2^{n-7})$$

Figure 8



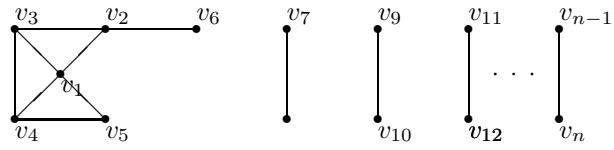
$$(4, 3^a, 2)$$

Figure 9



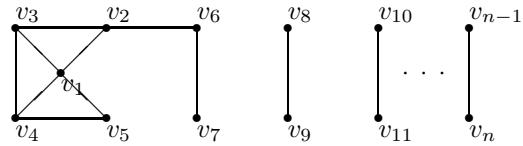
$$(4, 3^a, 2^{n-1-a})$$

Figure 10



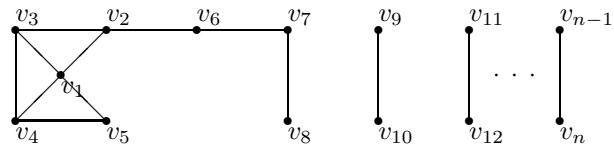
$$(4, 3^3, 2, 1^{n-5})$$

Figure 11



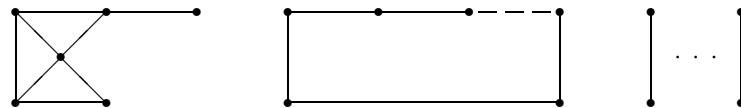
$$(4, 3^3, 2^2, 1^{n-6})$$

Figure 12



$$(4, 3^3, 2^3, 1^{n-7})$$

Figure 13



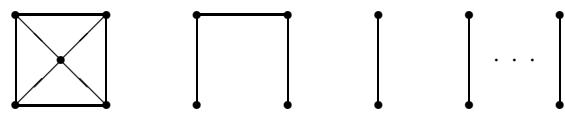
$$(4, 3^3, 2^b, 1^{n-4-b})$$

Figure 14



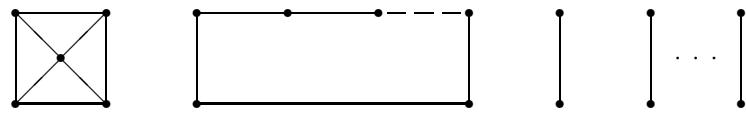
$$(4, 3^4, 2, 1^{n-6})$$

Figure 15



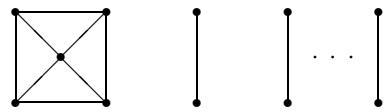
$$(4, 3^4, 2^2, 1^{n-7})$$

Figure 16



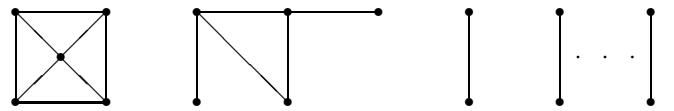
$$(4, 3^4, 2^b, 1^{n-5-b})$$

Figure 17



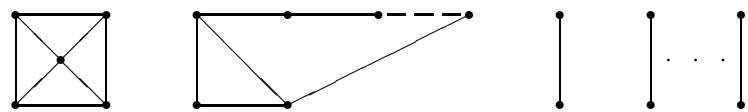
$$(4, 3^4, 1^{n-5})$$

Figure 18



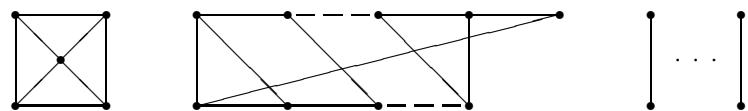
$$(4, 3^6, 2, 1^{n-8})$$

Figure 19



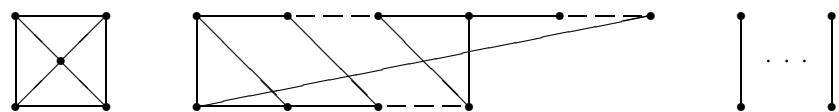
$$(4, 3^6, 2^b, 1^{n-7-b})$$

Figure 20



$$(4, 3^a, 2, 1^{n-2-a})$$

Figure 21



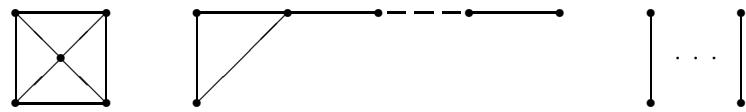
$$(4, 3^a, 2^b, 1^{n-1-a-b})$$

Figure 22



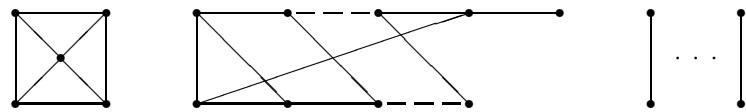
$$(4, 3^5, 2, 1^{n-7})$$

Figure 23



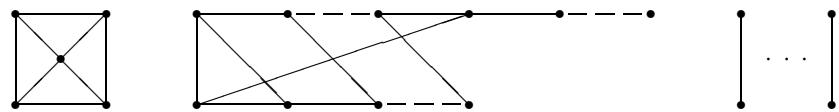
$$(4, 3^5, 2^b, 1^{n-6-b})$$

Figure 24



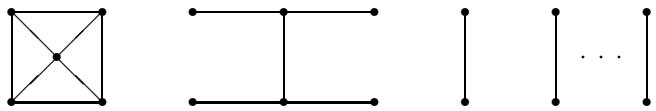
$$(4, 3^a, 2, 1^{n-2-a})$$

Figure 25



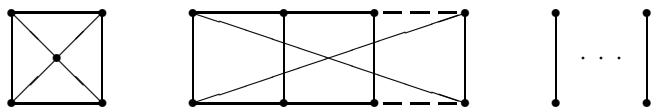
$$(4, 3^a, 2^b, 1^{n-1-a-b})$$

Figure 26



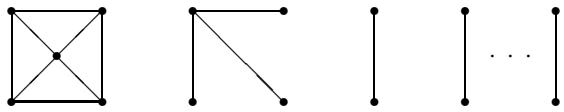
$$(4, 3^6, 1^{n-7})$$

Figure 27



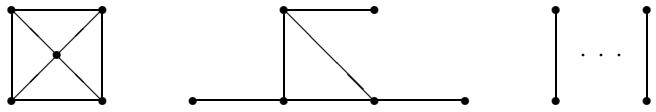
$$(4, 3^a, 1^{n-1-a})$$

Figure 28



$$(4, 3^5, 1^{n-6})$$

Figure 29



$$(4, 3^7, 1^{n-8})$$

Figure 30

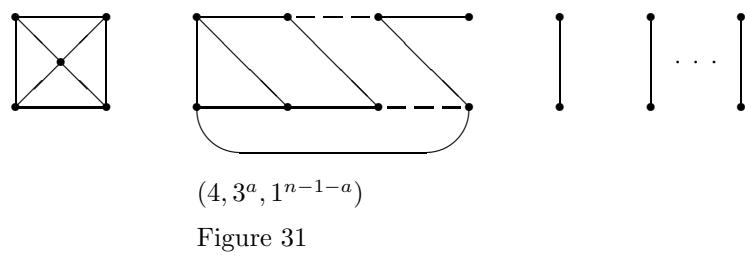


Figure 31