COMBINATORICS OF GEOMETRICALLY DISTRIBUTED RANDOM VARIABLES:
VALUE AND POSITION OF THE rTH LEFT–TO–RIGHT MAXIMUM

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ABSTRACT. For words of length n, generated by independent geometric random variables, we consider the average value and the average position of the rth left–to–right maximum, for fixed r and $n \to \infty$.

1. INTRODUCTION

For a permutation $\sigma_1 \sigma_1 \ldots \sigma_n$, a *left–to–right maximum* (*outstanding element, record,* ...) is an element σ_j with $\sigma_j > \sigma_i$ for all $i = 1, \ldots, j-1$. The number of left–to–right maxima was first studied by Rényi [9], compare also [7]. A survey of results on this topic can be found in [4].

Recently Wilf in [10] proved the formula $(1 - 2^{-r})n$ for the average *value* of the rth left–to–right maximum, for fixed r and $n \to \infty$; for the average *position* he obtained the asymptotic formula $(\log n)^{r-1}/(r-1)!$.

In [8] the number of left–to–right maxima was investigated in the model of *words* (strings) $a_1 \ldots a_n$, where the letters $a_i \in \mathbb{N}$ are independently generated according to the geometric distribution with $\mathbb{P}\{X = k\} = pq^{k-1}$, with $p + q = 1$. (We find it useful also to use the abbreviation $Q = q^{-1}$.) The motivation for this work came from Computer Science. Also, since equal letters are now allowed, there are two versions that should be considered in parallel, the standard version, and the weak version, where '$<$' is replaced by '$\leq$,' which means that a new maximum only has to be larger or equal to the previous ones. The paper [8] contains asymptotic results about the average and the variance of the number of left–to–right maxima in the context of geometric random variables. (H.–K. Hwang and his collaborators obtained further results about the limiting behaviour in [1].)

Motivated by Wilf's study we consider here the two parameters 'value' and 'position' of the rth left–to–right maximum for geometric random variables. Summarizing our results, we obtain the asymptotic formulæ $\frac{r}{p}$ and $\frac{1}{(r-1)!}\left(\frac{p}{q}\log_Q n\right)^{r-1}$ resp. $\frac{rq}{p}$ and $\frac{1}{(r-1)!}\left(p\log_Q n\right)^{r-1}$ in the weak case.

A certain knowledge of [8] might be beneficial to understanding the present derivations.

It should be noted that not all random strings of length n *have* r left–to–right maxima.

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Let us start with the *value.* The generating function of interest is

$$\frac{1}{1-z}\prod_{i=1}^{h-1}\left\{1+\frac{pq^{i-1}zu}{1-(1-q^i)z}\right\}zpq^{h-1},$$

which originates from the (unique) decomposition of a string as $a_1w_1\ldots a_{r-1}w_{r-1}a_rw$ where $a_1,\ldots,a_r$ are the left–to–right maxima, the w_i are the strings between them, and w can be anything. Note that if $a_k = l$, then this corresponds to a term $pq^{l-1}zu$, and thus w_k corresponds to $1/\big(1-(1-q^k)z\big)$. A value i must not necessarily occur as a left–to–right maximum; that is reflected by the $1+\ldots$ in the product. However, when we look for the coefficient of u^{r-1}, we have seen $r-1$ left–to–right maxima, and the rth has value h. What comes after that is irrelevant and covered by the factor $1/(1-z)$. (Compare [8] for similar generating functions.)

In the sequel we find it useful to use the abbreviation $[\![i]\!] := 1-(1-q^i)z$.

The coefficients of z^nu^{r-1}, call them $\pi_{n,h}^{(r)}$, are not probabilities, but $\pi_{n,h}^{(r)}/\pi_n^{(r)}$ are, where $\pi_n^{(r)}$ is the probability that a string of length n *has* r left–to–right maxima; we find it as

$$\pi_n^{(r)} := [z^nu^{r-1}]\frac{1}{1-z}\sum_{h\geq 1}\prod_{i=1}^{h-1}\left\{1+\frac{pq^{i-1}zu}{[\![i]\!]}\right\}zpq^{h-1} = \sum_{h\geq 1}\pi_{n,h}^{(r)}.$$

Now we turn to the *position.* Set

$$\sigma_{n,j}^{(r)} := [z^nu^{r-1}v^j]\frac{1}{1-z}\sum_{h\geq 1}\prod_{k=1}^{h-1}\left\{1+\frac{pq^{k-1}zvu}{1-(1-q^k)zv}\right\}zvpq^{h-1},$$

then $\sigma_{n,j}^{(r)}/\pi_n^{(r)}$ is the probability that a random string of length n has the rth maximum in position j. It is the same decomposition as before, however, we are not interested in the value h, so we sum over it. On the other hand, we label the position with the variable v, so we must make sure that every z that does not appear in the factor $1/(1-z)$ must be multiplied by a v. Computationally, we find it easier to work with the parameter "position $-r$," for which we have to consider

$$\frac{1}{1-z}\sum_{h\geq 1}\prod_{k=1}^{h-1}\left\{1+\frac{pq^{k-1}zu}{1-(1-q^k)zv}\right\}zpq^{h-1},$$

since the variable v appears in fewer places, as we don't have to multiply all those z's by v which count for the r left–to–right maxima.

2. Some technical lemmas

In order to read off coefficients, we state the obvious but nevertheless very useful formula

$$[w^n]\sum_i a_if(b_iw) = \sum_i a_ib_i^n\cdot[w^n]f(w).$$

In all our applications, $\sum_i a_ib_i^n$ can be summed in closed form.

Lemma 1. *Assume that we have power series*

$$A^{(j)}(w) = \sum_{n \geq 1} a_n^{(j)} w^n, \qquad j = 1, \ldots, s.$$

Then

$$[w^n] \sum_{1 \leq i_1 < i_2 < \cdots < i_s} A^{(1)}(wq^{i_1}) \ldots A^{(s)}(wq^{i_s}) = \sum_{0 = l_0 < l_1 < \cdots < l_{s-1} < l_s = n} \frac{a_{l_1 - l_0}^{(s)} \ldots a_{l_s - l_{s-1}}^{(1)}}{(Q^{l_1} - 1) \ldots (Q^{l_s} - 1)}.$$

Proof. For the sake of clarity, we treat the case $s = 3$ and leave it to the imagination of the reader to figure out the general case;

$$[w^n] \sum_{1 \leq i < j < h} A(wq^i) B(wq^j) C(wq^h) = \sum_l \sum_{1 \leq i < j < h} [w^{n-l}] A(wq^i) B(wq^j) \cdot [w^l] C(wq^h)$$

$$= \sum_l \frac{1}{Q^l - 1} \sum_{1 \leq i < j} [w^{n-l}] A(wq^i) B(wq^j) \cdot [w^l] C(w) q^{jl}$$

$$= \sum_l \frac{c_l}{Q^l - 1} \sum_{1 \leq i < j} [w^n] A(wq^i) B(wq^j) (wq^j)^l$$

$$= \sum_{l,m} \frac{c_l}{Q^l - 1} \sum_{1 \leq i < j} [w^{n-m}] A(wq^i) \cdot [w^m] B(wq^j) (wq^j)^l$$

$$= \sum_{l,m} \frac{c_l}{(Q^l - 1)(Q^m - 1)} \sum_{1 \leq i} [w^{n-m}] A(wq^i) q^{im} \cdot [w^m] B(w) w^l$$

$$= \sum_{l,m} \frac{c_l b_{m-l}}{(Q^l - 1)(Q^m - 1)} \sum_{1 \leq i} [w^n] A(wq^i) (wq^i)^m$$

$$= \sum_{l,m} \frac{c_l \, b_{m-l} \, a_{n-m}}{(Q^l - 1)(Q^m - 1)(Q^n - 1)}.$$

$\square$

Lemma 2.

$$[w^n] \sum_{1 \leq i_1 < i_2 < \cdots < i_s} A^{(1)}(wq^{i_1}) \ldots A^{(s)}(wq^{i_s}) \, i_s$$

$$= [t] \sum_{0 = l_0 < l_1 < \cdots < l_{s-1} < l_s = n} a_{l_1 - l_0}^{(s)} \ldots a_{l_s - l_{s-1}}^{(1)} \prod_{i=1}^{s} \left(\frac{1}{Q^{l_i} - 1} + t \frac{Q^{l_i}}{(Q^{l_i} - 1)^2} \right).$$

Proof. The proof is essentially the same as before, if we note that

$$\sum_{h > j} h \, q^{hl} = q^{jl} \left(\frac{Q^l}{(Q^l - 1)^2} + \frac{j}{Q^l - 1} \right).$$

$\square$

Our quantities will eventually come out as alternating sums, and the appropriate treatment of them is Rice's method which is surveyed in [3]; the key point is the following Lemma.

Lemma 3. *Let $\mathcal{C}$ be a curve surrounding the points $1, 2, \ldots, n$ in the complex plane and let $f(z)$ be analytic inside $\mathcal{C}$. Then*

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k f(k) = -\frac{1}{2\pi i} \int_{\mathcal{C}} [n; z] f(z) dz,$$

where

$$[n; z] = \frac{(-1)^{n-1} n!}{z(z-1)\ldots(z-n)} = \frac{\Gamma(n+1)\Gamma(-z)}{\Gamma(n+1-z)}.$$

Extending the contour of integration it turns out that under suitable growth conditions on $f(z)$ (compare [3]) the asymptotic expansion of the alternating sum is given by

$$\sum \mathrm{Res}\big([n; z] f(z)\big) + \text{smaller order terms}$$

where the sum is taken over all poles z_0 different from $1, \ldots, n$. Poles that lie more to the left lead to smaller terms in the asymptotic expansion.

The range $1, \ldots, n$ for the summation is not sacred; if we sum, for example, over $k = 2, \ldots, n$, the contour must encircle $2, \ldots, n$, etc.

3. THE PROBABILITY THAT THERE ARE r MAXIMA

Now we want to read off the nth coefficients of the power series of interest. For this, it is beneficial to use the following formula:

$$[z^n] f(z) = (-1)^n [w^n] (1-w)^{n-1} f\Big(\frac{w}{w-1}\Big).$$

This form can be found in [6] and is based on ideas concerning the Euler transform in [2]. Then the quantities come out automatically as alternating sums, and Rice's method can be applied.

$$
\begin{aligned}
\pi_n^{(r)} &= [z^n] \frac{1}{1-z} \sum_{1 \le i_1 < \cdots < i_{r-1} < h} \frac{pq^{i_1-1}z}{[\![i_1]\!]} \cdots \frac{pq^{i_{r-1}-1}z}{[\![i_{r-1}]\!]} zpq^{h-1} \\
&= \Big(\frac{p}{q}\Big)^r [z^n] \frac{1}{1-z} \sum_{1 \le i_1 < \cdots < i_{r-1} < h} \frac{q^{i_1}z}{[\![i_1]\!]} \cdots \frac{q^{i_{r-1}}z}{[\![i_{r-1}]\!]} zq^h \\
&= (-1)^r \Big(\frac{p}{q}\Big)^r (-1)^n [w^n] (1-w)^{n-1} \sum_{1 \le i_1 < \cdots < i_{r-1} < h} \frac{q^{i_1}w \ldots q^{i_{r-1}}w}{(1-q^{i_1}w)\ldots(1-q^{i_{r-1}}w)} wq^h \\
&= (-1)^r \Big(\frac{p}{q}\Big)^r \sum_{k=r}^{n} \binom{n-1}{k-1} (-1)^k [w^k] \sum_{1 \le i_1 < \cdots < i_{r-1} < h} \frac{q^{i_1}w \ldots q^{i_{r-1}}w}{(1-q^{i_1}w)\ldots(1-q^{i_{r-1}}w)} wq^h.
\end{aligned}
$$

Now the evaluation of the inner sum can be done by our Lemma; $k = n$, $r = s$, $A^{(1)}(w) = \cdots = A^{(r-1)}(w) = \frac{w}{1-w}$, $A^{(r)}(w) = w$. Therefore

$$[w^k] \sum_{1 \leq i_1 < \cdots < i_{r-1} < h} \frac{q^{i_1} w \ldots q^{i_{r-1}} w}{(1 - q^{i_1} w) \ldots (1 - q^{i_{r-1}} w)} w q^h$$

$$= \sum_{0 = l_0 < 1 = l_1 < l_2 < \cdots < l_r = k} \frac{1}{(Q - 1)(Q^{l_2} - 1) \ldots (Q^{l_r} - 1)}$$

$$= \frac{q}{p} \sum_{2 \leq l_2 < \cdots < l_r = k} \frac{1}{(Q^{l_2} - 1) \ldots (Q^{l_r} - 1)}.$$

Thus

$$\pi_n^{(r)} = (-1)^{r-1} \left(\frac{p}{q}\right)^{r-1} \sum_{k=r-1}^{n-1} \binom{n-1}{k} (-1)^k f(k)$$

with

$$f(k) = \sum_{2 \leq l_2 < \cdots < l_r = k+1} \frac{1}{(Q^{l_2} - 1) \ldots (Q^{l_r} - 1)}.$$

In order to apply Rice's method one needs the continuation of $f(k)$ to the complex plane. Using *symmetric functions,* one can always represent such iterated summations by powersums [7]

$$\vartheta(k) := \sum_{l=2}^{k} \frac{1}{(Q^l - 1)^d},$$

and the task is reduced to continue this quantity $\vartheta(k)$ to the complex plane. For this, the standard way of doing it is via

$$\vartheta(z) := \sum_{l \geq 2} \frac{1}{(Q^l - 1)^d} - \sum_{l \geq 1} \frac{1}{(Q^{l+z} - 1)^d}.$$

However, we only need the values $f(0), \ldots, f(r-2)$.

In [5] we learn how such a sum has to be interpreted; we thus find $f(1) = \cdots = f(r-2) = 0$ and

$$f(0) = (-1)^r \left(\frac{1}{Q-1}\right)^{r-1}.$$

Hence Rice's method and the pole at $z = 0$ give us

$$\pi_n^{(r)} = 1 + O\left(\frac{1}{n}\right),$$

which is intuitively clear.

Note that there are poles at $z = -1 + 2\pi i k / \log Q$, $k \in \mathbb{Z}$, and they lead to a periodic fluctuation of order $\frac{1}{n}$; this phenomenon is well–known and appears in many places (compare [3] and some other references).

4. The average value of the rth maximum

Now we can safely deal with the quantities $\pi_{n,h}^{(r)}$ alone, and the so computated average value $E_n^{(r)}$ will be correct within an error term of the form $1 + O(\frac{1}{n})$. We compute

$$
\begin{aligned}
E_n^{(r)} &= [z^n]\frac{1}{1-z} \sum_{1 \le i_1 < \cdots < i_{r-1} < h} \frac{pq^{i_1-1}z}{[\![i_1]\!]} \cdots \frac{pq^{i_{r-1}-1}z}{[\![i_{r-1}]\!]} zpq^{h-1}h \\
&= \left(\frac{p}{q}\right)^r [z^n]\frac{1}{1-z} \sum_{1 \le i_1 < \cdots < i_{r-1} < h} \frac{q^{i_1}z}{[\![i_1]\!]} \cdots \frac{q^{i_{r-1}}z}{[\![i_{r-1}]\!]} zq^h h \\
&= (-1)^r \left(\frac{p}{q}\right)^r (-1)^n [w^n](1-w)^{n-1} \sum_{1 \le i_1 < \cdots < i_{r-1} < h} \frac{q^{i_1}w \ldots q^{i_{r-1}}w}{(1-q^{i_1}w)\ldots(1-q^{i_{r-1}}w)} wq^h h \\
&= (-1)^r \left(\frac{p}{q}\right)^r \sum_{k=r}^{n} \binom{n-1}{k-1}(-1)^k [w^k] \sum_{1 \le i_1 < \cdots < i_{r-1} < h} \frac{q^{i_1}w \ldots q^{i_{r-1}}w}{(1-q^{i_1}w)\ldots(1-q^{i_{r-1}}w)} wq^h h.
\end{aligned}
$$

The evaluation of the inner sum is now done by the (second) Lemma:

$$
\begin{aligned}
[w^k] \sum_{1 \le i_1 < \cdots < i_{r-1} < h} & \frac{q^{i_1}w \ldots q^{i_{r-1}}w}{(1-q^{i_1}w)\ldots(1-q^{i_{r-1}}w)} wq^h h \\
&= [t] \sum_{1 = l_1 < l_2 < \cdots < l_r = k} \prod_{i=1}^{r} \left(\frac{1}{Q^{l_i}-1} + t\frac{Q^{l_i}}{(Q^{l_i}-1)^2}\right).
\end{aligned}
$$

Or,

$$
E_n^{(r)} = (-1)^{r-1}\left(\frac{p}{q}\right)^r \sum_{k=r-1}^{n-1} \binom{n-1}{k}(-1)^k f(k)
$$

with

$$
f(k) = [t] \sum_{1 = l_1 < l_2 < \cdots < l_r = k+1} \prod_{i=1}^{r} \left(\frac{1}{Q^{l_i}-1} + t\frac{Q^{l_i}}{(Q^{l_i}-1)^2}\right).
$$

Again, $f(1) = \cdots = f(r-2) = 0$ and

$$
\begin{aligned}
f(0) &= [t] \sum_{1 = l_1 < l_2 < \cdots < l_r = 1} \prod_{i=1}^{r} \left(\frac{1}{Q^{l_i}-1} + t\frac{Q^{l_i}}{(Q^{l_i}-1)^2}\right) \\
&= (-1)^r [t]\left(\frac{1}{Q-1} + t\frac{Q}{(Q-1)^2}\right)^r = (-1)^r r\frac{Q}{(Q-1)^{r+1}}.
\end{aligned}
$$

Thus we have proved the following theorem

Theorem 1. *The average value $E_n^{(r)}$ of the rth left–to–right maximum in a random sequence of n elements, generated by geometric random variables is given by*

$$E_n^{(r)} = \frac{r}{p} + O\left(\frac{1}{n}\right) \qquad \text{for fixed } r \text{ and } n \to \infty.$$

A full asymptotic expansion would be available, at least in principle, with more involved computations, as well as the variance.

Again, as in all the examples that will follow, the lower order terms contain periodic fluctuations of the form $\delta(\log_Q n)$.

5. The average position of the rth maximum

In order to compute this parameter (or rather the modified version), we have to differentiate the generating function from the Introduction and plug in $v = 1$. The desired quantity is then obtained via

$$[z^n]\left(\frac{p}{q}\right)^r \frac{1}{1-z}[t] \sum_{1 \le i_1 < \cdots < i_{r-1} < h} \prod_{j=1}^{r-1} \left(\frac{q^{i_j} z}{[\![i_j]\!]} + t\frac{q^{i_j} z(1-q^{i_j})z}{[\![i_j]\!]^2}\right) zq^h$$

$$= (-1)^n [w^n](1-w)^{n-1}\left(\frac{p}{q}\right)^r (-1)^r \times$$

$$\times [t] \sum_{1 \le i_1 < \cdots < i_{r-1} < h} \prod_{j=1}^{r-1} \left(\frac{q^{i_j} w}{1-q^{i_j} w} - t\frac{q^{i_j} w(1-q^{i_j})w}{(1-q^{i_j} w)^2}\right) wq^h$$

$$= \left(\frac{p}{q}\right)^r (-1)^{r-1} \sum_{k=r-1}^{n-1} \binom{n-1}{k}(-1)^k f(k)$$

where

$$f(k) = [w^{k+1}][t] \sum_{1 \le i_1 < \cdots < i_{r-1} < h} \prod_{j=1}^{r-1} \left(\frac{q^{i_j} w}{1-q^{i_j} w} - t\frac{q^{i_j} w(1-q^{i_j})w}{(1-q^{i_j} w)^2}\right) wq^h = f_1(k) + f_2(k)$$

with

$$f_1(k) = [t][w^k] \sum_{1 \le i_1 < \cdots < i_{r-1} < h} \prod_{j=1}^{r-1} \left(\frac{q^{i_j} w}{1-q^{i_j} w} - t\frac{q^{i_j} w}{(1-q^{i_j} w)^2}\right) wq^h$$

and

$$f_2(k) = [t][w^{k+1}] \sum_{1 \le i_1 < \cdots < i_{r-1} < h} \prod_{j=1}^{r-1} \left(\frac{q^{i_j} w}{1-q^{i_j} w} + t\frac{(q^{i_j} w)^2}{(1-q^{i_j} w)^2}\right) wq^h.$$

Now the two sums are in a form where our technical lemma applies!

For $f_1(k)$ note that $A_1(w) = \cdots = A_{r-1}(w) = \frac{w}{1-w} - \frac{tw}{(1-w)^2}$ and $A_r(w) = w$. Thus

$$f_1(k) = [t] \sum_{1=l_1<l_2<\cdots<l_r=k} \frac{\big(1 - t(l_2 - l_1)\big) \ldots \big(1 - t(l_r - l_{r-1})\big)}{(Q^{l_1} - 1)\ldots(Q^{l_r} - 1)}$$

$$= -(k-1) \sum_{1=l_1<l_2<\cdots<l_r=k} \frac{1}{(Q^{l_1} - 1)\ldots(Q^{l_r} - 1)}.$$

For $f_2(k)$ note that $A_1(w) = \cdots = A_{r-1}(w) = \frac{w}{1-w} + \frac{tw^2}{(1-w)^2}$ and $A_r(w) = w$. Thus

$$f_2(k) = [t] \sum_{1=l_1<l_2<\cdots<l_r=k+1} \frac{\big(1 + t(l_2 - l_1 - 1)\big) \ldots \big(1 + t(l_r - l_{r-1} - 1)\big)}{(Q^{l_1} - 1)\ldots(Q^{l_r} - 1)}$$

$$= (k - r) \sum_{1=l_1<l_2<\cdots<l_r=k+1} \frac{1}{(Q^{l_1} - 1)\ldots(Q^{l_r} - 1)}.$$

As we know from before, it is the "value" (the behaviour) of $f(0)$ that is required. It is $f_1(0)$ that is dominant here: Since in general

$$\sum_{2\leq l_2<\cdots<l_{r-1}<0} a_{l_2} \ldots a_{l_{r-1}} = (-1)^r \frac{a_0^{r-1} - a_1^{r-1}}{a_0 - a_1},$$

we find that as $z \to 0$

$$f(z) \sim \frac{(-1)^r}{(Q - 1)(Q^z - 1)^{r-1}}.$$

Thus, according to the theory in [3], where it is explained in detail what kind of contribution an rth order pole at $z = 0$ gives, we have proved that

Theorem 2. *The average position of the rth left–to–right maximum in a random sequence of n elements, generated by geometric random variables is given by*

$$\frac{1}{(r-1)!} \left(\frac{p}{q} \log_Q n\right)^{r-1} + O\big(\log^{r-2} n\big) \qquad \text{for fixed } r \text{ and } n \to \infty.$$

6. WEAK LEFT–TO–RIGHT MAXIMA; THE VALUE

We mention here briefly the analogous developments for the instance of weak left–to–right maxima.

The generating function of interest is

$$\frac{1}{1 - z} \prod_{i=1}^{h-1} \left\{1 - \frac{pq^{i-1}zu}{[\![i-1]\!]}\right\}^{-1} pq^{h-1}z,$$

and the coefficient of u^{r-1} therein is

$$\frac{1}{1 - z} \sum_{1\leq i_1\leq\cdots\leq i_{r-1}\leq h} \frac{pq^{i_1-1}z}{[\![i_1-1]\!]} \cdots \frac{pq^{i_{r-1}-1}z}{[\![i_{r-1}]\!]} pq^{h-1}z$$

$$= p^r \frac{1}{1 - z} \sum_{0\leq i_1\leq\cdots\leq i_{r-1}\leq h} \frac{q^{i_1}z}{[\![i_1]\!]} \cdots \frac{q^{i_{r-1}}z}{[\![i_{r-1}]\!]} q^h z.$$

The technical lemmas that we need now are

Lemma 4.

$$[w^n] \sum_{0 \le i_1 \le i_2 \le \cdots \le i_s} A^{(1)}(wq^{i_1}) \ldots A^{(s)}(wq^{i_s}) = \sum_{0=l_0<l_1<\cdots<l_{s-1}<l_s=n} \frac{a^{(s)}_{l_1-l_0} \ldots a^{(1)}_{l_s-l_{s-1}}}{(1-q^{l_1}) \ldots (1-q^{l_s})}$$

and

$$[w^n] \sum_{0 \le i_1 \le i_2 \le \cdots \le i_s} A^{(1)}(wq^{i_1}) \ldots A^{(s)}(wq^{i_s})\,(i_s+1)$$

$$= [t] \sum_{0=l_0<l_1<\cdots<l_{s-1}<l_s=n} a^{(s)}_{l_1-l_0} \ldots a^{(1)}_{l_s-l_{s-1}} \prod_{i=1}^{s} \left(\frac{1}{1-q^{l_i}} + t\frac{q^{l_i}}{(1-q^{l_i})^2} \right).$$

We find

$$E_n^{(r)} = (-1)^{r-1}p^r \sum_{k=r-1}^{n-1} \binom{n-1}{k}(-1)^k f(k)$$

with

$$f(k) = [t] \sum_{1=l_1<l_2<\cdots<l_r=k+1} \prod_{i=1}^{r} \left(\frac{1}{1-q^{l_i}} + t\frac{q^{l_i}}{(1-q^{l_i})^2} \right).$$

Also, $f(0) = (-1)^r r \frac{q}{(1-q)^{r+1}}$ and thus

Theorem 3. *The average value $E_n^{(r)}$ of the rth left–to–right maximum (in the weak sense) in a random sequence of n elements, generated by geometric random variables is given by*

$$E_n^{(r)} = \frac{r\,q}{p} + O\left(\frac{1}{n}\right) \qquad \text{for fixed } r \text{ and } n \to \infty.$$

7. Weak left–to–right maxima; the position

The probability generating funtion of interest (up to normalization by a factor that is basically 1, as before) is given by

$$\frac{1}{1-z} \sum_{h \ge 1} \prod_{i=1}^{h-1} \left\{ 1 - \frac{pq^{i-1}zvu}{1-(1-q^i)zv} \right\}^{-1} zvpq^{h-1},$$

and the desired expectated value is

$$[z^n]p^r\frac{1}{1-z}\sum_{0\leq i_1\leq\cdots\leq i_{r-1}\leq h}\prod_{j=1}^{r-1}\left(\frac{q^{i_j}z}{[\![i_j]\!]}+t\frac{q^{i_j}z(1-q^{i_j})z}{[\![i_j]\!]^2}\right)zq^h$$

$$=p^r(-1)^{r-1}\sum_{k=r-1}^{n-1}\binom{n-1}{k}(-1)^k f(k)$$

with $f(k)=f_1(k)+f_2(k)$ and

$$f_1(k)=[t][w^k]\sum_{0\leq i_1\leq\cdots\leq i_{r-1}\leq h}\prod_{j=1}^{r-1}\left(\frac{q^{i_j}w}{1-q^{i_j}w}-t\frac{q^{i_j}w}{(1-q^{i_j}w)^2}\right)wq^h$$

and

$$f_2(k)=[t][w^{k+1}]\sum_{0\leq i_1\leq\cdots\leq i_{r-1}\leq h}\prod_{j=1}^{r-1}\left(\frac{q^{i_j}w}{1-q^{i_j}w}+t\frac{(q^{i_j}w)^2}{(1-q^{i_j}w)^2}\right)wq^h.$$

We find

$$f_1(k)=-(k-1)\sum_{1=l_1<l_2<\cdots<l_r=k}\frac{1}{(1-q^{l_1})\dots(1-q^{l_r})}$$

and

$$f_2(k)=(k-r)\sum_{1=l_1<l_2<\cdots<l_r=k+1}\frac{1}{(1-q^{l_1})\dots(1-q^{l_r})}.$$

As $z\to 0$

$$f(z)\sim\frac{(-1)^r}{(1-q)(1-q^z)^{r-1}},$$

and thus

Theorem 4. *The average position of the rth left–to–right maximum (in the weak sense) in a random sequence of n elements, generated by geometric random variables is given by*

$$\frac{1}{(r-1)!}\big(p\log_Q n\big)^{r-1}+O\big(\log^{r-2}n\big)\qquad\text{for fixed }r\text{ and }n\to\infty.$$

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