

# Sensitivity to initial conditions and nonextensivity in biological evolution

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## Abstract

We consider biological evolution as described within the Bak and Sneppen 1993 model. We exhibit, at the self-organized critical state, a power-law sensitivity to the initial conditions, calculate the associated exponent, and relate it to the recently introduced nonextensive thermostatistics. The scenario which here emerges *without tuning* strongly reminds that of the *tuned* onset of chaos in say logistic-like onedimensional maps. We also calculate the dynamical exponent  $z$ .

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There is nowadays a massive evidence of fractals and scale invariant phenomena in nature. They appear in an impressive variety of inanimate systems such as the geological (e.g., earthquakes) or climatic (e.g., atmospheric turbulence) ones, as well as of biological or living systems (e.g., biological evolution, cell growth, economic phenomena, among others). In most of the naturally occurring cases, no particular tuning is perceived. Per Bak and collaborators have advanced [1,2] the hypothesis that, for many if not all the cases, this is so because the microscopic dynamics of the system makes it to spontaneously evolve towards a critical, scale-invariant, state. This is largely known today as *self-organized criticality* (SOC). Models have been formulated and experiments have been performed which profusely exhibit this interesting type of behaviour in sandpiles, ricepiles, earthquakes and others (see for instance [3–5] and references therein). One such model is that introduced in 1993 by Bak and Sneppen [6] to paradigmatically describe biological species evolution. This is the model that we focus herein. A variety of its properties are already known. Nevertheless, there is a crucial one that has never been addressed, namely the *sensitivity to the initial conditions*, which is known to be most relevant in nonlinear dynamical systems (quantities intensively studied such as Liapunov exponents, spread of damage, are in fact nothing but specific expressions of this concept). The study of this important property is the basic aim of the present work.

Before describing our particular approach of this evolution model, let us introduce some preliminary notions by using, as a simple illustration, the following one-dimensional logistic-like map (see [7] and references therein)

$$x_{t+1} = 1 - a |x_t|^\zeta, \quad (\zeta > 1; 0 < a \leq 2; t = 0, 1, 2, \dots) \quad (1)$$

This map recovers, for  $\zeta = 2$ , the usual logistic map (in its centered representation). For fixed  $\zeta$  there is a critical value  $a_c(\zeta)$  such that, for  $a < a_c(\zeta)$ , we observe a regular evolution (finite-cycle attractors), whereas, for  $a > a_c(\zeta)$ , chaos becomes possible. Approaching  $a_c(\zeta)$  from below we can see the celebrated doubling-period road to chaos, with its successive bifurcations. Topological properties of the evolution (such as the nature of the successive

attractors while varying  $a$ ) do not depend on  $\zeta$ , but metrical properties (such as the Feigenbaum's constants, characterizing geometrical rates of approach of the bifurcations leading to the chaotic critical point  $a_c(\zeta)$ ) do depend on  $\zeta$ . (Let us anticipate that the quantity we shall focus herein, the entropic index  $q$ , belongs to this second class of properties). If we consider, at  $t = 0$ , two values of  $x_0$  which slightly differ by  $\Delta x(0)$  and follow their time evolution, we typically observe the following exponential behaviour for  $\Delta x(t)$

$$\lim_{\Delta x(0) \rightarrow 0} \frac{\Delta x(t)}{\Delta x(0)} = \exp[\lambda_1 t] \quad (2)$$

If  $\lambda_1 < 0$  (which is in fact the case for most values of  $a$  *below*  $a_c(\zeta)$ ) we shall say that the system is *strongly insensitive* to the initial conditions. If  $\lambda_1 > 0$  (which is in fact the case for most values of  $a$  *above*  $a_c(\zeta)$ ) we shall say that the system is *strongly sensitive* to the initial conditions. Finally, if  $\lambda_1$  vanishes we shall speak of a *marginal* case. This is what happens, in particular, for the chaotic critical point. For this value of  $a$ , the sensitivity is not characterized by an exponential-law, but rather by a power-law. Eq. (2) can be generalized as follows [8,9]

$$\lim_{\Delta x(0) \rightarrow 0} \frac{\Delta x(t)}{\Delta x(0)} = [1 + (1 - q) \lambda_q t]^{\frac{1}{1-q}} \quad (q \in \mathcal{R}) \quad (3)$$

We immediately see that the  $q = 1$  case reproduces Eq. (2), whereas for  $q \neq 0$  we have

$$\lim_{\Delta x(0) \rightarrow 0} \frac{\Delta x(t)}{\Delta x(0)} \sim [(1 - q) \lambda_q]^{\frac{1}{1-q}} t^{\frac{1}{1-q}} \quad (t \rightarrow \infty) \quad (4)$$

The  $q < 1$  and the  $q > 1$  cases respectively correspond to *weakly sensitive* and *weakly insensitive* to the initial conditions [8]. The coefficient  $\lambda_q$  is a generalized Liapunov exponent, and satisfies [8,9]  $K_q = \lambda_q$  if  $\lambda_q \geq 0$  and  $K_q = 0$  if  $\lambda_q < 0$  (generalization, for arbitrary  $q$ , of the well known Pesin equality) , where the generalized Kolmogorov-Sinai entropy  $K_q$  is defined [8,9] analogously to the usual Kolmogorov-Sinai entropy ( $K_1$  herein). More precisely, in the same way  $K_1$  essentially is the increase per unit time of the Boltzmann-Gibbs-Shannon entropy  $S_1 \equiv -\sum_i p_i \ln p_i$  ,  $K_q$  essentially is the increase per unit time of the generalized, nonextensive, entropic form [10]

$$S_q \equiv \frac{1 - \sum_i p_i^q}{q - 1} \quad (q \in \mathcal{R}) \quad (5)$$

The nonextensivity of this form can be seen from the fact that if  $A$  and  $B$  are two independent systems (in the sense that the probabilities associated with  $A + B$  factorize into those of  $A$  and  $B$ ), then

$$S_q(A + B) = S_q(A) + S_q(B) + (1 - q) S_q(A) S_q(B) \quad (6)$$

We immediately verify that, since  $S_q$  is nonnegative,  $q = 1$ ,  $q < 1$  and  $q > 1$  respectively correspond to the extensive, superextensive and subextensive cases. This generalized entropy (5) has generated a generalized thermostatistics [10,11] (which recovers the usual, extensive, Boltzmann-Gibbs statistics as the  $q = 1$  particular case), and has received applications in a variety of situations such as self-gravitating systems [12,13], two-dimensional-like turbulence in pure-electron plasma [13,14], Lévy-like [15] and correlated-like [16] anomalous diffusions, solar neutrino problem [17], peculiar velocity distribution of galaxy clusters [18], cosmology [19], linear response theory [20], long-range fluid and magnetic systems [21], optimization techniques [22], among others (including of course the nonlinear maps (1) [8,9]).

Now that we have introduced all the needed ingredients, we can close the illustration associated with Eq. (1) by mentionning that the  $\zeta$ -dependence of the entropic index  $q$  and of the fractal dimension  $d_f$  have been calculated [9] at the chaotic critical point  $a_c(\zeta)$ . It was exhibited that, while  $d_f$  varies from 0 to 1 (or close to it),  $q$  increases from  $-\infty$  to 1 (or close to it). Therefore, the Boltzmann-Gibbs limit  $q = 1$  is attained when the attractor has an euclidean (nonfractal) dimension  $d_f = 1$ . But, when the system lives on a fractal ( $d_f < 1$  in the present case), nonextensive behaviour is revealed.

Let us now return to the Bak and Sneppen evolution model. We shall exhibit that, at its self-organized critical state, weak sensitivity to the initial conditions (i.e., a power-law of the type indicated in Eq. (4)) occurs very similarly to the one just described for the chaotic critical point of the map (1). The model consists in a  $N$ -site ring (linear chain with periodic boundary conditions); on each site ( $j = 1, 2, \dots, N$ ) we locate a real variable

$B_j$  ( $0 \leq B_j \leq 1, \forall j$ ) which corresponds to a "fitness barrier" separating two connected (first-neighboring) "species of living organisms". We start by randomly and independently attributing the set of values  $\{B_j\}$ . At each successive *elementary* time step we identify the *smallest*  $B_j$ , and randomly change ("mutate") it as well as its two nearest neighbors. After some transient, a peculiar self-organized state emerges [6], rich in avalanches of all sizes and other scale invariant properties which makes the system to exhibit a variety of power laws. In the present work we have focused, as follows, the sensitivity to the initial conditions of the Bak and Sneppen model. Once SOC has been achieved, we consider that system as replica 1 ( $\{B_j^{(1)}\}$ ) and create a replica 2 ( $\{B_j^{(2)}\}$ ) by randomly choosing one of its  $N$  sites, and exchanging the value associated with this site and that with the smallest barrier; we consider this moment as the *collective* time step  $t = 1$  (we define a collective time step as  $N$  times the elementary time step, i.e., each site is going to be updated only once in average during a unit collective time step). From now on, we apply for both replicas the model rules of identifying the smallest barrier and updating that particular one and its two first-neighbors. We use (as in usual damage spreading techniques) the *same random numbers for both replicas* (hence, three different and independent random numbers are involved in the operation, since we change a particular barrier and its first-neighbors).

We define now the *Hamming distance* between the two replicas as

$$D(t) \equiv \frac{1}{N} \sum_{j=1}^N |B_j^{(1)}(t) - B_j^{(2)}(t)| \quad (7)$$

One such realization is shown in Fig. 1 for  $N = 1000$ . We then average  $N_r$  realizations (we have used typically  $N_r N = 10^5$ ) and obtain  $\langle D \rangle(t)$ : see Fig. 2, where a power-law is evident. These results enable the determination of the slope  $1/(1 - q)$  (see Eq. (4)), hence of  $q$ ; we obtained  $q = -2.1$  (to be compared, for instance, with the  $\zeta = 2$  logistic map value 0.24... [8]). Finally, for fixed  $N$ , if we denote by  $\tau$  the value of  $t$  at which the increasing regime *crossovers* onto the saturation regime (intersection, in Fig. 2, of two straight lines, namely those defined by the linearly increasing branch of the curve and the horizontal branch. The proportionality  $\tau(N) \propto N^z$  defines the dynamical exponent  $z$  ([23] and references therein)).

We obtained (see Fig.3)  $z = 1.56$ , to be compared, for instance, with 2.16 obtained [23] for the square-lattice Ising ferromagnet.

Since  $(q - 1)$  measures the degree of entropy nonextensivity (see Eq. (6)), it is an important index to be analyzed whenever discussing universality classes. Consistently, the determination of  $q$  for other SOC models would be very welcome. Indeed, it will provide an insight on the fractal nature of the attractor towards which the system is spontaneously driven.

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## REFERENCES

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- [1] P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988).
- [2] P. Bak and K. Chen, *Self-organized criticality*, in Scientific American (January 1991) page 26.
- [3] Z. Olami, H.J.S. Feder and K. Christensen, Phys. Rev. Lett. **68**, 1244 (1992).
- [4] V. Frette, K. Christensen, A. Malthe-Sorensen, J. Feder, T. Jossang and P. Meakin, Nature **379**, 49 (1996).
- [5] K. Christensen, A. Corral, V. Frette, J. Feder and T. Jossang, Phys. Rev. Lett. **77**, 107 (1996).
- [6] P. Bak and K. Sneppen, Phys. Rev. Lett. **71**, 4083 (1993).
- [7] P.R. Hauser, C. Tsallis and E.M.F. Curado, Phys. Rev. A **30**, 2074 (1984); B.-L. Hao, *Elementary symbolic dynamics* (World Scientific, Singapore, 1989).
- [8] C. Tsallis, A.R. Plastino and W.-M. Zheng, Chaos, Solitons and Fractals (1996), in press.
- [9] U.M.S. Costa, M.L. Lyra, A.R. Plastino and C. Tsallis, preprint (1996).
- [10] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
- [11] E.M.F. Curado and C. Tsallis, J.Phys. A **24**, L69 (1991) [Corrigenda: **24**, 3187 (1991) and **25**, 1019 (1992)]; C. Tsallis, Phys. Lett. A **206**, 389 (1995).
- [12] A. R. Plastino and A. Plastino, Phys. Lett. A **174**, 384 (1993); J.J. Aly, in *N-body problems and gravitational dynamics*, Proceedings of the Meeting held at Aussois-France (21-

- 25 March 1993), eds. F. Combes and E. Athanassoula (Publications de l'Observatoire de Paris, Paris, 1993), p. 19; A.R. Plastino and A. Plastino, Phys. Lett. A **193**, 251 (1994).
- [13] B.M. Boghosian, Phys. Rev. E **53**, 4754 (1996).
- [14] C. Anteneodo and C. Tsallis, J. Mol. Liq. (1996), in press.
- [15] P.A. Alemany and D.H. Zanette, Phys. Rev. E **49**, R956 (1994); D.H. Zanette and P.A. Alemany, Phys. Rev. Lett. **75**, 366 (1995); C. Tsallis, S.V.F. Levy, A.M.C. de Souza and R. Maynard, Phys. Rev. Lett. **75**, 3589 (1995); M.O. Caceres and C.E. Budde, Phys. Rev. Lett. **77**, 2589 (1996); D.H. Zanette and P.A. Alemany, Phys. Rev. Lett. **77**, 2590 (1996).
- [16] A.R. Plastino and A. Plastino, Physica A **222**, 347 (1995); C. Tsallis and D.J. Bukman, Phys. Rev. E **54**, R2197 (1996); A. Compte and D. Jou, J. Phys. A **29**, 4321 (1996); D.A. Stariolo, Phys. Rev. E (1996), in press.
- [17] G. Kaniadakis, A. Lavagno and P. Quarati, Phys. Lett. B **369**, 308 (1996).
- [18] A. Lavagno, G. Kaniadakis, M. Rego-Monteiro, P. Quarati and C. Tsallis, preprint (1996) [astro-phy/9607147].
- [19] V.H. Hamity and D.E. Barraco, Phys. Rev. Lett. **76**, 4664 (1996).
- [20] A.K. Rajagopal, Phys. Rev. Lett. **76**, 3469 (1996).
- [21] P. Jund, S.G. Kim and C. Tsallis, Phys. Rev. B **52**, 50 (1995); J.R. Grigera, Phys. Lett. A **217**, 47 (1996); S.A. Cannas and F.A. Tamarit, Phys. Rev B **54**, R12661 (1996); L.C. Sampaio, M.P. de Albuquerque and F.S. de Menezes, Phys. Rev. E (1996), in press.
- [22] D.A. Stariolo and C. Tsallis, Ann. Rev. Comp. Phys., vol. II, ed. D. Stauffer (World Scientific, Singapore, 1995), p. 343; T.J.P. Penna, Phys. Rev. E **51**, R1 (1995) and Computers in Physics **9**, 341 (1995); K.C. Mundim and C. Tsallis, Int. J. Quantum

Chem. **58**, 373 (1996); J. Schulte, Phys. Rev. E **53**, 1348 (1996); I. Andricioaei and J.E. Straub, Phys. Rev. E **53**, R3055 (1996); P. Serra, A.F. Stanton and S. Kais, Phys. Rev. E (1996), in press.

[23] F. Wang, N. Hatano and M. Suzuki, J. Phys. A **28**, 4543 (1995).

## FIGURES

FIG. 1. Time evolution of the damage associated with one realization of a typical system with  $N = 1000$ .

FIG. 2. Average of  $N_r$  realizations such as those of figure 1 for three different sizes:  $N = 1000$  (top),  $N = 500$  (middle) and  $N = 250$  (bottom). The slope  $1/(1 - q)$  equals  $0.32 \pm 0.01$ .

FIG. 3. Log-log plot of  $\tau$  versus  $N$  for the three curves of figure 2.





