

ROTA-BAXTER ALGEBRAS AND NEW COMBINATORIAL IDENTITIES

KURUSCH EBRAHIMI-FARD, JOSÉ M. GRACIA-BONDÍA, AND FRÉDÉRIC PATRAS

ABSTRACT. The word problem for an arbitrary associative Rota–Baxter algebra is solved. This leads to a non-commutative generalization of the classical Spitzer identities. Links to other combinatorial aspects, particularly of interest in physics, are indicated.

Keywords: Rota–Baxter relation; free algebras; word problem; quasi-symmetric functions; noncommutative symmetric functions; Hopf algebra; pre-Lie relation; Dynkin idempotent; Spitzer’s identity; Bohnenblust–Spitzer identities; Magnus’ expansion

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1. INTRODUCTION AND DEFINITIONS

Nearly forty years ago, a class of combinatorial formulas for random variables were recast by Rota as identities in the theory of Baxter maps [3]. The key result was the solution of the word problem, for associative, commutative algebras endowed with such maps. This showed the equivalence of the combinatorics of fluctuations with that of classical symmetric functions. Since then, operators of the Baxter type kept showing up in all sorts of applications, and lately in the Hopf algebraic approach to renormalization [10]. In many instances, the algebra in question is not commutative. The time has come to revisit the word problem, and the corresponding identities, in the noncommutative case. Roughly speaking, we are led to replace symmetric functions of commuting variables by quasi-symmetric functions of non-commuting ones. Sequences of ‘noncommutative Spitzer identities’ ensue. In an applied vein, we explore the connection of our word problem with Lam’s approach to the Magnus expansion for ordinary differential equations.

Definition 1.1. Let \mathbb{K} be a field of characteristic zero. Let A be a \mathbb{K} -algebra, not necessarily associative nor commutative nor unital. An operator $R \in \text{End}(A)$ satisfying the relation

$$RaRb = R(Rab + aRb) + \theta R(ab), \quad \text{for all } a, b \in A, \tag{1}$$

is said Rota–Baxter of weight $\theta \in \mathbb{K}$. The pair (A, R) is a weight θ Rota–Baxter algebra (RBA).

The Rota–Baxter identity (1) prompts the definition of a new product $a *_R b := Rab + aRb + \theta ab$, $a, b \in A$.

Proposition 1.1. The linear space underlying A equipped with the product $*_R$ is again a RBA of the same weight with the same Rota–Baxter map. We denote it by (A_R, R) . If A is associative, so is A_R .

We call $*_R$ the Rota–Baxter double product. Clearly R becomes an algebra map from A_R to A . Note that $\tilde{R} := -\theta \text{id}_A - R$ is Rota–Baxter as well, and $*_{\tilde{R}} = -*_R$. One may think of Rota–Baxter operators as generalized integrals. Indeed, relation (1) for the weight $\theta = 0$ corresponds to the integration-by-parts identity for the Riemann integral; the reader will have no difficulty in checking duality of (1) with the ‘skewderivation’ rule

$$\delta(ab) = \delta a b + a \delta b + \theta \delta a \delta b.$$

For instance, the finite difference operator of step $-\theta$, given by $\delta f(x) := \theta^{-1}(f(x - \theta) - f(x))$, is a skewderivation. The summation operator $Zf(x) := \sum_{n \geq 1} \theta f(x + \theta n)$ is Rota–Baxter of weight θ , and we find $\delta Z = \text{id} = Z\delta$ on suitable classes of functions. Scaling $R \rightarrow \theta^{-1}R$ reduces the study of RBAs of nonvanishing weight to the case $\theta = 1$. For notational simplicity we proceed considering this one, returning to general weight when convenient. Also, henceforth we assume we are dealing with associative RBAs; non-associative RBAs will arise later in an ancillary role.

2. MAIN RESULT

We now extend to our noncommutative setting Rota’s notion of *standard* RBA, see [6, 25, 26]. Let $X = (x_1, \dots, x_n, \dots)$ be a countably infinite, ordered set of variables and $T(X)$ the tensor algebra over X . The elements of X are called noncommutative polynomials (over X). Consider the pair (\mathcal{A}, ρ) , where \mathcal{A} is the algebra

of countable sequences $\Upsilon \equiv (y_1, \dots, y_n, \dots)$ of elements $y_i \in T(X)$ with pointwise addition and product, and ρ given by

$$\rho\Upsilon = (0, y_1, y_1 + y_2, y_1 + y_2 + y_3, \dots).$$

By abuse of notation we regard X itself as an element of \mathcal{A} . The component y_p of Υ is denoted Υ_p .

Lemma 2.1. *The algebra \mathcal{A} together with $\rho \in \text{End}(\mathcal{A})$ defines a weight $\theta = 1$ Rota–Baxter algebra structure.*

This is a straightforward verification. We remark that ρ has a left inverse.

Theorem 2.1. *The Rota–Baxter subalgebra (\mathcal{R}, ρ) of \mathcal{A} generated by X is free on one generator in the category of \mathbb{K} -RBAs.*

In detail, our assertions are the following.

- $X \in \mathcal{R}$.
- The product in \mathcal{R} is associative.
- ρ is a Rota–Baxter operator.
- Let (A, R) be any associative RBA and $a \in A$. There is a unique algebra map $h : \mathcal{R} \rightarrow A$ with $h(X) = a$ and such that $R \circ h = h \circ \rho$.

The pair (\mathcal{R}, ρ) is what we call the standard RBA. The point of course is that the theorem allows us to prove the validity for any RBA A of an identity involving one element of A and R , by proving it for X in \mathcal{R} .

Only the last assertion in the list above asks for proof. We shall follow Rota and Smith [26] as far as possible. The adaptation to the noncommutative setting requires a bit of care. The lexicographical ordering $<_L$ for noncommutative monomials over X is useful; for any noncommutative polynomial P we write $\text{Sup } P$ for the highest monomial in P for $<_L$ and extend the lexicographical ordering of noncommutative monomials to a partial ordering on $T(X)$. Namely, we write $P <_L P'$ whenever $\text{Sup } P <_L \text{Sup } P'$. Note that, for P, P' homogeneous noncommutative polynomials and z, t in $T(X)$, we have

$$P <_L P' \Rightarrow Pz <_L P'z \quad \text{and} \quad z <_L t \Rightarrow Pz <_L Pt.$$

Henceforth we just employ the generic R for the Rota–Baxter map on the standard RBA; this should not lead to any confusion.

Proof. (Main steps.) Let us call End-algebra any associative algebra W provided with a distinguished endomorphism T_W , so that an End-algebra morphism f from W to W' satisfies $f \circ T_W = T_{W'} \circ f$. Write \mathcal{L} for the free End-algebra on one generator Z . The elements of \mathcal{L} are linear combinations of all symbols obtained from Z by iterative applications of the endomorphism T and of the associative product; they look like $ZT^2(TZT^3Z)$, and so on. We call these symbols \mathcal{L} -monomials. A RBA A is an End-algebra together with the relation (1) on $T_A \equiv R$. Denote by \mathcal{F} the free RBA on one generator Y . Between the three algebras $\mathcal{L}, \mathcal{F}, \mathcal{R}$ there are the following maps: unique End-algebra maps F, U from \mathcal{L} to \mathcal{F} , respectively \mathcal{R} , sending Z to Y respectively X ; and a unique onto Rota–Baxter map h' sending Y to X . Moreover $U = h' \circ F$.

We have to show the existence of an inverse for h' in the RBA category. Clearly $\ker F \subseteq \ker U$. We need only prove that $\ker U \subseteq \ker F$.

Any $l \in \mathcal{L}$ can be written uniquely as a linear combination of \mathcal{L} -monomials. We write $\text{Max } l$ for the maximal number of T 's occurring in the monomials, so that, say, $\text{Max}(ZT^2(ZTZ) + Z^3T^2ZZ) = 3$. We call α , a \mathcal{L} -monomial, *elementary* iff it can be written as either $Z^i, i \geq 0$ or as a product $Z^{i_1}Tb_1Z^{i_2}\cdots Tb_kZ^{i_{k+1}}$, where the b_i s are elementary, and i_2, \dots, i_k are strictly positive integers, while i_1 and i_{k+1} may be equal to zero; this definition makes sense by induction on $\text{Max } \alpha$. It turns out that every element l of \mathcal{L} can be written as the sum of a linear combination of elementary monomials with an element r_l such that $F(r_l) = 0$. This is due to the fact that, up to the addition of suitable elements in $\ker F$, products like $TcTd$ can be iteratively cancelled from the expression of l using relation (1).

We claim that for p large enough and $l \neq l'$, with l, l' elementary monomials, we have $\text{Sup } U(l)_p \neq \text{Sup } U(l')_p$, from which the required $\ker U \subseteq \ker F$ follows. Our assertion can be verified by induction on $\text{Max } l$, using that U is an End-algebra map. \square

Corollary 2.1. *The images of the elementary monomials of \mathcal{L} in \mathcal{R} form a linear basis of the free RBA on one generator.*

3. TWO INTERESTING HOPF ALGEBRAS

Inductively define in a general RBA (A, R) ,

$$(Ra)^{[n+1]} = R((Ra)^{[n]}a) \quad \text{and} \quad (Ra)^{\{n+1\}} = R(a(Ra)^{\{n\}}).$$

with the convention that $(Ra)^{[1]} = Ra = (Ra)^{\{1\}}$ and $(Ra)^{[0]} = 1 = (Ra)^{\{0\}}$, with the unit adjoined if need be. These iterated compositions with R appear in the context of Spitzer formulas. Of course there is no difference between $(Ra)^{[n]}$ and $(Ra)^{\{n\}}$ in the commutative context.

Coming back to the standard RBA (\mathcal{R}, R) , notice that:

$$(R(y_1, y_2, y_3, \dots))^{[2]} = R(R(y_1, y_2, y_3, \dots)(y_1, y_2, y_3, \dots)) = (0, 0, y_1y_2, y_1y_2 + y_1y_3 + y_2y_3, \dots)$$

This begins to give the game away. In general, the $(n+1)$ -th entry of $(R(y_1, y_2, y_3, \dots))^{[k]}$ is the elementary ‘symmetric’ function of degree k , restricted to the first n variables, the $(n+2)$ -th entry is given by the same, restricted to $n+1$ variables, and so on. The quotes on ‘symmetric’ remind us here that the y_i do not commute. The pertinent notion here is Hivert’s quasi-symmetric functions over a set of noncommuting variables [4, 19]. Denote as usual by $[n]$ the set of integers between 1 and n . Let f be a surjective map from $[n]$ to $[k]$. Then the quasi-symmetric function M_f over X associated to f is by definition

$$M_f X = \sum_{\phi} x_{\phi^{-1} \circ f(1)} \cdots x_{\phi^{-1} \circ f(n)},$$

where ϕ runs over the set of increasing bijections between subsets of \mathbb{N} of cardinality k and $[k]$. Let us represent f as the sequence of its values, $f = f(1), \dots, f(n)$, in the notation M_f . We also denote by M_f^l the image of M_f under the map sending x_i to 0 for $i > l$ and to itself otherwise. For example,

$$M_{1,3,3,2} X = x_1x_3x_3x_2 + x_1x_4x_4x_2 + x_1x_4x_4x_3 + x_2x_4x_4x_3 + \dots \quad \text{and} \quad M_{1,3,3,2}^3 X = x_1x_3x_3x_2.$$

The linear span $\text{NCQSym}(X)$ of the M_f —a subalgebra of the completion of the algebra of noncommutative polynomials over X — is related to the Coxeter complex of type A_n and the corresponding Solomon–Tits and twisted descent algebras [22].

Finally, write $[n]$ for the identity map on $[n]$ and ω_n for the endofunction of $[n]$ reversing the ordering, so that $M_{\omega_n} = M_{n,n-1,\dots,1}$. We can regard X itself as an element of the standard RBA and then we have

$$(RX)^{[n]} = (0, M_{[n]}^1 X, M_{[n]}^2 X, \dots, M_{[n]}^l X, \dots), \quad n \geq 1,$$

where $M_{[n]}^l$ is at the $(l+1)$ -th position in the sequence. Similarly

$$(RX)^{\{n\}} = (0, M_{\omega_n}^1 X, M_{\omega_n}^2 X, \dots, M_{\omega_n}^l X, \dots), \quad n \geq 1.$$

Proposition 3.1. *The elements $(RX)^{[n]}$ generate freely a subalgebra of \mathcal{A} (respectively generate freely a subalgebra of \mathcal{A}_R).*

The proofs are omitted for the sake of brevity; the first uses the observation that, for l big enough, we find $\text{Sup}(M_{[n_1]}^l \cdots M_{[n_k]}^l) > \text{Sup}(M_{[m_1]}^l \cdots M_{[m_j]}^l)$ with $n_1 + \cdots + n_k = m_1 + \cdots + m_j$ iff the sequence (n_1, \dots, n_k) is smaller than the sequence (m_1, \dots, m_j) in the lexicographical ordering. The second is a bit more involved.

The algebra NCQSym of quasi-symmetric functions in noncommuting variables is naturally provided with a Hopf algebra structure [4]. On the elementary quasi-symmetric functions $M_{[n]}$, the coproduct Δ acts as on a sequence of divided powers: $\Delta(M_{[n]}) = \sum_{i=0}^n M_{[i]} \otimes M_{[n-i]}$. Thus the $M_{[n]}$ generate a free subalgebra of NCQSym naturally isomorphic as a Hopf algebra to the classical descent algebra, which is a convolution subalgebra of the endomorphism algebra of $T(X)$ [24] —or equivalently, to the algebra of noncommutative symmetric functions (NCSF) [14]. The same construction goes over to the free algebras over the $(RX)^{[n]}$ for the pointwise product and the Rota–Baxter double product $*_R$. The first one is naturally provided with a cocommutative Hopf algebra structure for which the $(RX)^{[n]}$ s form a sequence of divided powers, that is:

$$\Delta((RX)^{[n]}) = \sum_{0 \leq m \leq n} (RX)^{[m]} \otimes (RX)^{[n-m]},$$

this is just the structure inherited from the Hopf algebra structure on NCQSym . We call this algebra the free noncommutative Spitzer (Hopf) algebra on one generator, or the *Spitzer algebra* for short, and write \mathcal{S} for it. When dealing with the $*_R$ product, the right subalgebra to consider, as it will emerge soon, is the free algebra freely generated by the $(RX)^{[n]}X$. We also make it a Hopf algebra by requiring the free generators to form a sequence of divided powers, that is

$$\Delta_*((RX)^{[n]}X) = 1 \otimes (RX)^{[n]}X + \sum_{0 \leq m \leq n-1} (RX)^{[n-m-1]}X \otimes (RX)^{[m]}X + (RX)^{[n]}X \otimes 1.$$

Thus it is convenient to set $(RX)^{[-1]}X = 1$. We call this Hopf algebra the *double Spitzer algebra*, and write \mathcal{C} for it. We shall need the antipode for both Hopf algebras. For this, recourse to Atkinson’s theorem [2] seems the simplest method. Recall that we assume $\theta = 1$.

Theorem 3.1. (Atkinson [2]) Let (A, R) be a unital Rota–Baxter algebra. Fix $a \in A$ and let x and y be defined by $x = \sum_{n \in \mathbb{N}} t^n (Ra)^{[n]}$ and $y = \sum_{n \in \mathbb{N}} t^n (\tilde{R}a)^{\{n\}}$, that is, as the solutions of the equations

$$x = 1 + tR(xa) \quad \text{and} \quad y = 1 + t\tilde{R}(ay),$$

in $A[[t]]$. We have the following factorization

$$x(1+at)y = 1, \quad \text{so that} \quad 1+at = x^{-1}y^{-1}.$$

Corollary 3.1. Let (A, R) be an associative unital Rota–Baxter algebra. Fix $a \in A$ and assume x and y to solve the equations in the foregoing theorem. The inverses x^{-1} and y^{-1} solve the equations

$$x^{-1} = 1 - tR(ay) \quad \text{and} \quad y^{-1} = 1 - t\tilde{R}(xa),$$

in $A[[t]]$.

One checks $xx^{-1} = x^{-1}x = 1$ by using the definitions and the Rota–Baxter property. Similarly for y^{-1} .

Corollary 3.2. The action of the antipode S on the Spitzer algebra \mathcal{S} , is given by

$$S((RX)^{[n]}) = -R(X(\tilde{R}X)^{\{n-1\}}).$$

Indeed, the Spitzer bialgebra is naturally graded. The series $\sum_{n \in \mathbb{N}} (RX)^{[n]}$ is a group-like element in \mathcal{S} . The inverse series computes the action of the antipode on the terms of the series. The corollary follows, since

$$\left(\sum_{n \in \mathbb{N}} (RX)^{[n]} \right)^{-1} = 1 - R\left(X\left(\sum_{n \in \mathbb{N}} (\tilde{R}X)^{\{n\}} \right) \right).$$

Corollary 3.3. The action of the antipode S on the double Spitzer algebra \mathcal{C} is given by

$$S((RX)^{[n]}X) = -(X(\tilde{R}X)^{\{n\}}). \tag{2}$$

For the proof, one can observe that the operator R induces an isomorphism of free graded algebras between \mathcal{C} and \mathcal{S} (which is the identity on scalars). That is, for any sequence of integers i_1, \dots, i_k , we have:

$$R((RX)^{[i_1]}X *_R \cdots *_R (RX)^{[i_k]}X) = (RX)^{[i_1+1]} \cdots (RX)^{[i_k+1]}.$$

Hence, this implies (2).

Corollary 3.4. The free $*_R$ subalgebras of A generated by the $(RX)^{[n]}X$ and the $X(\tilde{R}X)^{\{n\}}$ are canonically isomorphic. The antipode exchanges the two families of generators. In particular, the $X(\tilde{R}X)^{\{n\}}$ form also a sequence of divided powers in the double Spitzer algebra.

4. ENTER THE DYNKIN MAP

The Dynkin operator is usually defined as the multilinear map from an associative algebra B into itself given by the left-to-right iteration of the associated Lie bracket,

$$D(x_1, \dots, x_n) = [\cdots [[x_1, x_2], x_3] \cdots, x_n],$$

where $[x, y] := xy - yx$. Specializing to $B = T(X)$, the Dynkin operator can be shown to become a quasi-idempotent—that is, its action on an homogeneous element of degree n satisfies $D^2 = nD$. The associated projector D/n sends $T_n(X)$ to the component of degree n of the free Lie algebra over X , see the monograph [24]. Now, D can be rewritten in purely Hopf algebraic terms as $S \star N$, where N is the grading operator and \star the convolution product in $\text{End}(T(X))$. This definition generalizes to any graded connected cocommutative or commutative Hopf algebra [23]. One actually deals there with a more general phenomenon, namely the possibility to define an action of the classical descent algebra on any graded connected commutative or cocommutative Hopf algebra [21].

Theorem 4.1. Let H be an arbitrary graded connected cocommutative Hopf algebra over a field of characteristic zero. The Dynkin operator $D \equiv S \star N$ induces a bijection between the group $G(H)$ of group-like elements of H and the Lie algebra $\text{Prim}(H)$ of primitive elements in H . The inverse morphism from $\text{Prim}(H)$ to $G(H)$ is given by

$$h = \sum_{n \in \mathbb{N}} h_n \longmapsto \Gamma(h) := \sum_{n \in \mathbb{N}} \sum_{\substack{i_1 + \cdots + i_k = n, \\ i_1, \dots, i_k > 0}} \frac{h_1 \cdots h_k}{i_1(i_1 + i_2) \cdots (i_1 + \cdots + i_k)}. \tag{3}$$

This corresponds to Theorem 4.1 in our [12], establishing the same formula for characters and infinitesimal characters of graded connected commutative Hopf algebras. The proof follows from the one in that reference by dualizing the notions and identities, and can be omitted. In the particular case where H is a free associative algebra over a set of graded generators y_1, \dots, y_n, \dots and H is provided with the structure of a cocommutative Hopf algebra by requiring the y_i to be a sequence of divided powers, the images of the generators y_i under the action of D forms a sequence of primitive elements of H that generate freely H as an associative algebra. This result is a direct consequence of our theorem. Two particular examples of such a situation are well known. If H is the NCSF Hopf algebra, then H is generated as a free associative algebra by the complete homogeneous NCSF, which form a sequence of divided powers, and the corresponding primitive elements under the action of the Dynkin operator are known as the power sums NCSF of the first kind [14]. Second, in the classical descent algebra the abstract Dynkin operator sends the identity of $T(X)$ to the classical Dynkin operator. This was put to use in [24] to rederive classical identities of the Lie type.

We contend that the same machinery can be used to rederive the already known formulas for commutative RBAs, and moreover prove new formulas in the noncommutative framework. We compute inductively the action of D on the generators of \mathcal{C} ; that will give the action on the generators of \mathcal{S} , too. Let us denote for the purpose by π_* the product on \mathcal{C} . Using $N(1) = 0$ and $N(X) = 1$, there follows $D((RX)^{[0]}X) = (S \star N)(X) = \pi_* \circ (S \otimes N)\Delta_*(X) = \pi_* \circ (S \otimes N)(X \otimes 1 + 1 \otimes X) = X$. We then find:

$$\begin{aligned} D((RX)^{[n-1]}X) &= (S \star N)((RX)^{[n-1]}X) = \pi_* \circ (S \otimes N)\left(\sum_{0 \leq p \leq n} (RX)^{[p-1]}X \otimes (RX)^{[n-p-1]}X\right) \\ &= \sum_{0 \leq p \leq n} S((RX)^{[p-1]}X) *_R N((RX)^{[n-p-1]}X) \\ &= \sum_{0 \leq p \leq n-1} S((RX)^{[p-1]}X) *_R N((RX)^{[n-p-1]})X + S((RX)^{[p-1]}X) *_R (RX)^{[n-p-1]}X \\ &= \sum_{0 \leq p \leq n-1} S((RX)^{[p-1]}X) *_R N((RX)^{[n-p-1]})X - S((RX)^{[n-1]}X) \\ &= \sum_{0 \leq p \leq n-1} R\left(S((RX)^{[p-1]}X) *_R N((RX)^{[n-p-2]}X)\right)X \\ &\quad - \sum_{1 \leq p \leq n-1} S((RX)^{[p-1]}X)\tilde{R}\left(R(N(R^{[n-p-2]}X))X\right) - S((RX)^{[n-1]}X). \end{aligned}$$

In the fourth line we used vanishing of $(S \star \text{id})((RX)^{[n-1]}X)$, then $a *_R (Rb c) = R(a *_R b)c - a\tilde{R}(Rb c)$; the rest should be clear. After further simple manipulations, using (2) it comes

$$D((RX)^{[n-1]}X) = R(D((RX)^{[n-2]}X))X + X\tilde{R}(D((RX)^{[n-2]}X)).$$

The calculation suggests we introduce a new product.

Definition 4.1. Let (A, R) be an associative Rota–Baxter algebra. Introduce the binary operation

$$a \bullet_R b := Ra b - bRa - ba = [Ra, b] - ba = Ra b + b\tilde{R}a, \tag{4}$$

and the elements $c^{(n)}(a_1, \dots, a_n) := (\dots ((a_1 \bullet_R a_2) \bullet_R a_3) \dots \bullet_R a_{n-1}) \bullet_R a_n$, for $n > 1$, and $c^{(1)}(a_1) := a_1$.

We define $c^{(n)}(a)$ as the n -times iterated product $c^{(n)}(a, \dots, a) = (\dots ((a \bullet_R a) \bullet a) \dots \bullet_R a) \bullet_R a$. All these parenthesis are unavoidable, as the composition \bullet_R is not associative, see next section. As well we define $C^{(n)}(a) := R(c^{(n)}(a))$. In conclusion, we have proved

Theorem 4.2. The action of the Dynkin operator, D , on the generators $(RX)^{[n]}$ of the Spitzer algebra (respectively on the generators $(RX)^{[n]}X$ of the double Spitzer algebra) is given by

$$D((RX)^{[n]}) = C^{(n)}(X), \quad \text{respectively by} \quad D((RX)^{[n]}X) = c^{(n)}(X).$$

This immediately implies

Corollary 4.1. We have the following identity in the Spitzer algebra \mathcal{S}

$$(RX)^{[n]} = \sum_{\substack{i_1 + \dots + i_k = n, \\ i_1, \dots, i_k > 0}} \frac{C^{(i_1)}(X) \cdots C^{(i_k)}(X)}{i_1(i_1 + i_2) \cdots (i_1 + \dots + i_k)}. \tag{5}$$

Corollary 4.2. *We have the following identity in the double Spitzer algebra \mathcal{C}*

$$(RX)^{[n-1]}X = \sum_{\substack{i_1+\dots+i_k=n, \\ i_1,\dots,i_k>0}} \frac{c^{(i_1)}(X) *_R \dots *_R c^{(i_k)}(X)}{i_1(i_1+i_2)\cdots(i_1+\dots+i_k)}.$$

The corollaries follow readily from our Theorem 4.1 by applying the inverse Dynkin map (3).

5. THE GENERALIZED BOHNENBLUST–SPITZER IDENTITIES

If (A, R) is a *commutative* Rota–Baxter algebra of weight one with Rota–Baxter operator R , then on $A[[t]]$ the following identity by Spitzer holds [3, 28]:

$$\sum_{m \in \mathbb{N}} t^m (Ra)^{[m]} = \exp(R \log(1 + at)). \quad (6)$$

In the framework of the commutative standard RBA this becomes Waring’s formula relating elementary and power symmetric functions [27, Chapter 4]. From (6) follows

$$n! (Ra)^{[n]} = \sum_{\sigma} (-1)^{n-k(\sigma)} Ra^{|\tau_1|} Ra^{|\tau_2|} \dots Ra^{|\tau_{k(\sigma)}|}.$$

Here the sum is over all permutations σ of $[n]$ and $\sigma = \tau_1 \tau_2 \dots \tau_{k(\sigma)}$ is the decomposition of σ into disjoint cycles [26]. We denote by $|\tau_i|$ the number of elements in τ_i . By polarization one obtains

$$\sum_{\sigma} R\left(R(\dots(Ra_{\sigma(1)})a_{\sigma(2)}\dots)a_{\sigma(n)}\right) = \sum_{\sigma} (-1)^{n-k(\sigma)} R\left(\prod_{j_1 \in \tau_1} a_{j_1}\right) \dots R\left(\prod_{j_{k(\sigma)} \in \tau_{k(\sigma)}} a_{j_{k(\sigma)}}\right).$$

This leads to the classical formula [26]

$$\sum_{\sigma} R\left(R(\dots(Ra_{\sigma(1)})a_{\sigma(2)}\dots)a_{\sigma(n)}\right) = \sum_{\pi \in \mathcal{P}_n} (-1)^{n-|\pi|} \prod_{\pi_i \in \pi} (m_i - 1)! R\left(\prod_{j \in \pi_i} a_j\right). \quad (7)$$

Here π now runs through all unordered set partitions \mathcal{P}_n of $[n]$; by $|\pi|$ we denote the number of blocks in π ; and $m_i := |\pi_i|$ is the size of the particular block π_i . Those are often called Bohnenblust–Spitzer formulas. The generalization to *noncommutative* Bohnenblust–Spitzer formulas springs from Corollaries 4.1, respectively 4.2. Moreover, we arrive at the following theorem.

Theorem 5.1. *Let (A, R) be an associative Rota–Baxter algebra. For $a_i \in A$, $i = 1, \dots, n$, we have*

$$\begin{aligned} \sum_{\sigma} R\left(R(\dots(Ra_{\sigma(1)})a_{\sigma(2)}\dots)a_{\sigma(n)}\right) &= \sum_{\sigma} R\left(a_{\sigma(1)} \diamond_1 a_{\sigma(2)} \diamond_2 \dots \diamond_n a_{\sigma(n)}\right), \quad \text{where} \\ a_{\sigma(i)} \diamond_i a_{\sigma(i+1)} &= \begin{cases} a_{\sigma(i)} *_R a_{\sigma(i+1)}, & \max(\sigma(j) | j \leq i) < \sigma(i+1) \\ a_{\sigma(i)} \bullet_R a_{\sigma(i+1)}, & \text{otherwise;} \end{cases} \end{aligned} \quad (8)$$

furthermore consecutive \bullet_R products should be performed from left to right, and always before the $*_R$ product.

The reader might wish to perform a few checks here. One readily finds

$$R(Ra_1 a_2) + R(Ra_2 a_1) = Ra_1 Ra_2 + R(a_2 \bullet_R a_1) = R(a_1 *_R a_2 + a_2 \bullet_R a_1) = R(a_2 *_R a_1 + a_1 \bullet_R a_2).$$

This is a fancy way to write the Bohnenblust–Spitzer identity in terms of the non-associative Rota–Baxter product \bullet_R and the associative Rota–Baxter double product $*_R$. To check by direct calculation that

$$\begin{aligned} \sum_{\sigma \in S_3} R\left(R(Ra_{\sigma(1)} a_{\sigma(2)})a_{\sigma(3)}\right) &= R(a_1 *_R a_2 *_R a_3) + R(a_1 *_R (a_3 \bullet_R a_2)) + R(a_2 *_R (a_3 \bullet_R a_1)) \\ &\quad + R((a_2 \bullet_R a_1) *_R a_3) + R((a_3 \bullet_R a_2) \bullet_R a_1) + R((a_3 \bullet_R a_1) \bullet_R a_2) \\ &= Ra_1 Ra_2 Ra_3 + Ra_1 R(a_3 \bullet_R a_2) + Ra_2 R(a_3 \bullet_R a_1) \\ &\quad + R(a_2 \bullet_R a_1) Ra_3 + R((a_3 \bullet_R a_2) \bullet_R a_1) + R((a_3 \bullet_R a_1) \bullet_R a_2) \end{aligned}$$

is already somewhat tedious. We give a practical rule for the decomposition in Theorem 5.1. Given any permutation σ of $[n]$, we place a vertical bar to the left of σ_{i+1} iff it is bigger than all numbers to its left. For instance, for $n = 3$ we obtain in the one-line notation the ‘cut permutations’ $(1|2|3)$, $(21|3)$, (312) , $(1|32)$, (321) , $(2|31)$. The cuts indicate where the $*_R$ products, if any, should be placed. Of course, as the left hand side of (8) is symmetrical in its arguments, alternative rules could be devised. For the decomposition of $\sum_{\sigma} R(a_{\sigma(1)} R(a_{\sigma(2)} \dots R(a_{\sigma(n)} \dots))$ our rule is: place a vertical bar to the right of σ_i iff it is smaller than all numbers to its right. For $n = 3$ we then obtain the ‘cut permutations’ $(1|2|3)$, $(21|3)$, $(31|2)$, $(1|32)$, (321) , $(23|1)$; note the differences. Moreover, in this case the \bullet_R product is defined by $aRb - Rba - ba$ and consecutive \bullet_R products are performed from right to left.

As advertised, in the commutative case, when $a \bullet_R b$ reduces to $-ab$, we recover the classical Bohnenblust–Spitzer identities from any of the two previous forms.

6. REMARKS AND APPLICATIONS

1. Although the composition \bullet_R in (4) is not associative, it is Vinberg or (left) pre-Lie. Recall that a left pre-Lie algebra V is a vector space, together with a bilinear product $\bullet : V \otimes V \rightarrow V$, satisfying the left pre-Lie relation

$$(a \bullet b) \bullet c - a \bullet (b \bullet c) = (b \bullet a) \bullet c - b \bullet (a \bullet c), \quad a, b, c \in V.$$

This is enough for the commutator $[a, b] := a \bullet b - b \bullet a$ to satisfy the Jacobi identity. Hence the algebra of commutators L_V is a Lie algebra, justifying the nomenclature. Of course, every associative algebra is pre-Lie. See [7] for more details on pre-Lie structures.

Lemma 6.1. *Let (A, R) be an associative Rota–Baxter algebra. The binary composition (4) defines a left pre-Lie structure on A , which we call left Rota–Baxter pre-Lie product.*

The lemma follows by direct inspection. It may also be related to more recondite properties of RBAs [8]. Let $(D, *)$ be an associative algebra and assume that it is represented on itself, from the left and from the right, with commuting actions. We write \prec and \succ for the left and right actions, respectively. Assume moreover that we have $a * b = a \prec b + a \succ b$; then D is by definition a dendriform dialgebra. In detail, the dendriform properties are

$$(x \prec y) \prec z = x \prec (y \prec z + y \succ z); \quad (x \succ y) \prec z = x \succ (y \prec z); \quad (x \prec y + x \succ y) \succ z = x \succ (y \succ z). \quad (9)$$

Conversely, the latter relations are enough to ensure associativity of $(D, *)$. We refer to [15] for information on the subject.

Now, D gives rise to a pre-Lie algebra and, in two different ways, to the same Lie algebra. The pre-Lie algebra structure is given by $x \bullet y := x \succ y - y \prec x$. As observed already in [8], generalizing an observation made by Aguiar for the weight-zero case [1], the notion applies in particular to weight $\theta \neq 0$ RBAs, since the associative and pre-Lie products $*_R$ and \bullet_R , respectively, are composed from sums and differences of the binary operations

$$a \prec_R b := -a\tilde{R}(b) \quad \text{and} \quad a \succ_R b := R(a)b,$$

that satisfy equations (9) and define therefore a dendriform dialgebra structure on any associative Rota–Baxter algebra. In the case of the Rota–Baxter pre-Lie composition, we find

$$[a, b]_{\bullet_R} = [R(a), b] + [a, R(b)] + \theta[a, b] = [a, b]_{*_R}. \quad (10)$$

Proposition 6.1. *Let (A, R) be an associative Rota–Baxter algebra. The left pre-Lie algebra (A, \bullet_R) with the left Rota–Baxter pre-Lie product is a Rota–Baxter pre-Lie algebra of the same weight, with Rota–Baxter map R .*

The proof of this is left as an exercise.

2. It should be obvious now that, in the language of NCSF [14], if $X_a(t) := \sum_{n=0}^{\infty} t^n (Ra)^{[n]}$ solves the initial value problem $d/dt X_a(t) = X_a(t) \psi_a(t)$, $X_a(0) = 1$, then $\psi_a(t) := \sum_{n>0}^{\infty} t^{n-1} C^{(n)}(a)$.

3. The formulae developed in this paper actually apply without restriction to any associative RBA, in particular to the solution of differential equations. We actually drew inspiration for this paper from that subject: mainly from the path-breaking papers by Lam [16, 17] and recent work by two of us [5]. To reestablish general weight in the pre-Lie product formulas amounts simply to replace in (4) the product ba by θba , and thus the case $\theta = 0$ is included in our considerations. In fact, Corollary 4.1 yields the most efficient way to organize the terms coming from the standard methods to solve differential equations, the Dyson–Chen expansion and the Magnus series. Lam did obtain our formulas for $(Ra)^{\{n\}}$ for the case $\theta = 0$; part of the magic of the subject is how little needs to be changed when $\theta \neq 0$. It is worth mentioning that this arose from the need to prove deep theorems with strong physical roots, on approximations to quantum chromodynamics. In respect to the previous remark, if we define the Magnus series coefficients K_n by $d/dt \log X_a(t) = \sum_{n>0}^{\infty} t^n K_n(a)$, then the relation between the $C^{(n)}$ and the K_n is precisely the relation between power sums NCSF of the first and of the second kind [14]. The advantage of writing the Magnus series in this way has been recently recognized by the practitioners [20]. Eventually, pointing to the following remark we should underline that the NCSF picture implies an exponential solution to Atkinson’s recursion in Theorem 3.1.

4. It would be nice to be able to derive the new Bohnenblust–Spitzer identities at one stroke from an equation like the commutative Spitzer formula (6). One of us participated in an attempt in this direction a few years ago by [9], with the net result that in the noncommutative case $\sum_m t^m (Ra)^{[m]}$ is still a functional of $\log(1 + at)$, through a non-linear recursion (for which existence and unicity were proven) called, for want of a better name, the Baker–Campbell–Hausdorff recursion, e.g. see [11]. In practice, work with this functional was painful. There is a direct link between that recursion and the Magnus expansion. Explicit expressions for all the terms in the

latter are known; and so we are now forced to conclude that the ‘solution’ to the Baker–Campbell–Hausdorff recursion has been staring at us for a while. However, these formulas are rather clumsy and will be presented elsewhere; the matter is under investigation.

5. As shown in [12], the Dynkin operator is a key ingredient for the mathematical understanding of the combinatorial processes underlying the Bogoliubov recursion for renormalization in perturbative quantum field theory. Use of general Spitzer-like identities for noncommutative Rota–Baxter algebras is bound to deepen this algebraic understanding of renormalization. From the foregoing remarks it is clear that one can solve completely the Bogoliubov recursion with this kind of Lie algebraic tools; this will appear in a forthcoming work [13].

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REFERENCES

- [1] M. Aguiar, “Prepoisson algebras”, *Lett. Math. Phys.* **54** (2000) 263–277.
- [2] F. V. Atkinson, “Some aspects of Baxter’s functional equation”, *J. Math. Anal. Appl.* **7** (1963) 1–30.
- [3] G. Baxter, “An analytic problem whose solution follows from a simple algebraic identity”, *Pac. J. Math.* **10** (1960) 731–742.
- [4] N. Bergeron and M. Zabrocki, “The Hopf algebras of symmetric functions and quasisymmetric functions in non-commutative variables are free and cofree”, [ArXiv:math.CO/0509265](https://arxiv.org/abs/math/0509265).
- [5] J. Cariñena, K. Ebrahimi-Fard, H. Figueroa and J. M. Gracia-Bondía, “Hopf algebras in dynamical systems theory”, submitted, [ArXiv:math.CA/0701010](https://arxiv.org/abs/math/0701010).
- [6] P. Cartier, “On the structure of free Baxter algebras”, *Adv. Math.* **9** (1972) 253–265.
- [7] F. Chapoton and M. Livernet, “Pre-Lie algebras and the rooted trees operad”, *Int. Math. Res. Notices* **8** (2001) 395–408.
- [8] K. Ebrahimi-Fard, “Loday-type algebras and the Rota–Baxter relation”, *Lett. Math. Phys.* **61** (2002) 139–147.
- [9] K. Ebrahimi-Fard, L. Guo and D. Kreimer, “Integrable Renormalization II: the General case”, *Ann. Henri Poincaré* **6** (2005) 369–395.
- [10] K. Ebrahimi-Fard and L. Guo, “Rota–Baxter Algebras in Renormalization of Perturbative Quantum Field Theory”, to appear in *Fields Institute Communications*. [ArXiv:hep-th/0604116](https://arxiv.org/abs/hep-th/0604116).
- [11] K. Ebrahimi-Fard, L. Guo and D. Manchon, “Birkhoff type decompositions and the Baker–Campbell–Hausdorff recursion”, *Commun. Math. Phys.* **267** (2006) 821–845.
- [12] K. Ebrahimi-Fard, J. M. Gracia-Bondía and F. Patras, “A Lie theoretic approach to renormalization”, [ArXiv:hep-th/0609035](https://arxiv.org/abs/hep-th/0609035).
- [13] K. Ebrahimi-Fard, J. M. Gracia-Bondía and F. Patras, “The Bohnenblust–Spitzer identity for noncommutative Rota–Baxter algebras solves Bogoliubov’s counterterm recursion”, preprint.
- [14] I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh and J.-Y. Thibon, “Noncommutative symmetric functions”, *Adv. Math.* **112** (1995) 218–348.
- [15] J.-L. Loday, “Dialgebras”, Lecture Notes in Mathematics 1763, Springer, Berlin, 2001; pp. 7–66.
- [16] C. S. Lam and K. F. Liu, “Consistency of the baryon-multipleson amplitudes for large- N_c QCD Feynman diagrams”, *Phys. Rev. Lett.* **79** (1997) 597–600.
- [17] C. S. Lam, “Decomposition of time-ordered products and path-ordered exponentials”, *J. Math. Phys.* **39** (1998) 5543–5558.
- [18] W. Magnus, “On the exponential solution of differential equations for a linear operator”, *Commun. Pure Appl. Math.* **7** (1954) 649–673.
- [19] J.-C. Novelli and J.-Y. Thibon, “Polynomial realizations of some trialgebras” [ArXiv:math.CO/0605061](https://arxiv.org/abs/math/0605061).
- [20] J. A. Oteo and J. Ros, “From time-ordered products to Magnus expansion”, *J. Math. Phys.* **41** (2000) 3268–3277.
- [21] F. Patras, “L’algèbre des descentes d’une bigèbre graduée”, *J. Algebra* **170** (1994) 547–566.
- [22] F. Patras and M. Schocker, “Trees, set compositions and the twisted descent algebra”, *J. Alg. Comb.*, to appear. [ArXiv:math.CO/0512227](https://arxiv.org/abs/math/0512227).
- [23] F. Patras and C. Reutenauer, “On Dynkin and Klyachko idempotents in graded bialgebras”, *Adv. Appl. Math.* **28** (2002) 560–579.
- [24] C. Reutenauer, Free Lie algebras, Oxford University Press, Oxford, 1993.
- [25] G.-C. Rota, “Baxter algebras and combinatorial identities. I”, *Bull. Amer. Math. Soc.* **75** (1969) 325–329.
- [26] G.-C. Rota and D. A. Smith, “Fluctuation theory and Baxter algebras”, *Symposia Mathematica* **IX** (1972) 179–201.
- [27] B. S. Sagan, The symmetric group, Springer, New York, 2001.
- [28] F. Spitzer, “A combinatorial lemma and its application to probability theory”, *Trans. Amer. Math. Soc.* **82** (1956) 323–339.

MAX-PLANCK-INSTITUT FÜR MATHEMATIK, VIVATSGASSE 7, D-53111 BONN, GERMANY.

E-mail address: kurusch@mpim-bonn.mpg.de

URL: <http://www.th.physik.uni-bonn.de/th/People/fard/>

DEPARTAMENTO DE FÍSICA TEÓRICA I, UNIVERSIDAD COMPLUTENSE, MADRID 28040, SPAIN

LABORATOIRE J.-A. DIEUDONNÉ UMR 6621, CNRS, PARC VALROSE, 06108 NICE CEDEX 02, FRANCE

E-mail address: patras@math.unice.fr

URL: www.math.unice.fr/~patras