THE DEGREE-DIAMETER PROBLEM FOR SEVERAL
VARIETIES OF CAYLEY GRAPHS, I: THE ABELIAN CASE

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ABSTRACT. We address the degree-diameter problem for Cayley graphs of Abelian groups (Abelian graphs), both directed and undirected. The problem turns out to be closely related to the problem of finding efficient lattice coverings of Euclidean space by shapes such as octahedra and tetrahedra; we exploit this relationship in both directions. For 2 generators (dimensions) these methods yield optimal Abelian graphs with a given diameter k. (The results in two dimensions are not new; they are given in the literature of distributed loop networks.) We find an undirected Abelian graph with 3 generators and a given diameter k which we conjecture to be as large as possible; for the directed case, we obtain partial results. These results are connected to efficient lattice coverings of $\mathbf{R}^3$ by octahedra or by tetrahedra; computations on Cayley graphs lead us to such lattice coverings which we conjecture to be optimal. (The problem of finding such optimal coverings can be reduced to a finite number of nonlinear optimization problems.) We discuss the asymptotic behavior of the Abelian degree-diameter problem for large numbers of generators. The graphs obtained here are substantially better than traditional toroidal meshes, but, in the simpler undirected cases, retain certain desirable features such as good routing algorithms, easy constructibility, and the ability to host mesh-connected numerical algorithms without any increase in communication times.

Introduction. The degree-diameter problem for graphs is the following question: What is the largest number of vertices a graph (undirected or directed) can have if one is given upper bounds on the degree of each vertex and on the diameter of the graph (the maximum path-distance from any vertex to any other)? One application for such graphs is in the design of interconnection networks for parallel processors, where one wants to have a large number of processors without requiring a large number of wires at a single processor or incurring long delays in messages from one processor to another. For more information on

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the (undirected) degree-diameter problem, see Dinneen and Hafner [10]; up-to-date results can be found at the following Web site:

$$\texttt{http://www-mat.upc.es/grup\_de\_grafs/grafs/taula\_delta\_d.html}$$

A desirable extra property of such networks is that they look identical from any processor; this means that the graphs one uses should be vertex-transitive, i.e., for any two vertices x and y there is an automorphism of the graph which maps x to y. Here we will restrict ourselves to a special class of vertex-transitive graphs, called Cayley graphs. A Cayley graph is specified by a group and a set of generators for this group; the vertices of the graph are the elements of the group, and there is an edge from x to y if and only if there is a generator g such that $y = xg$. (It can be shown that every vertex-transitive graph is isomorphic to a generalized form of Cayley graph called a Cayley coset graph [19]. In this paper, though, we will look only at Cayley graphs. For Abelian groups, this is no loss of generality since every Cayley coset graph of an Abelian group is isomorphic to a Cayley graph of an Abelian group.) In a directed Cayley graph from a group on d generators, every vertex has in-degree and out-degree d; if d generators are used to form an undirected Cayley graph, then the degree of each vertex is the number of generators of order 2 plus twice the number of generators of order greater than 2 (unless there are redundant generators). So we will usually talk about Cayley graphs on a given number of generators rather than of a given degree; the cases where some generators have order 2 and hence only contribute 1 rather than 2 to the degree of an undirected Cayley graph will be handled separately.

The most straightforward approach for trying to find large Cayley graphs of small diameter on a given number d of generators is to examine various groups, look at some or all possible sets of d generators for such a group, and check whether each such set in fact generates the group and, if so, what the diameter of the graph is. But this can be a very large task even for relatively small groups and generating sets. In this paper, we will use a different approach which facilitates studying many groups and generating sets at once; it yields provably optimal results for some families of groups, and good lower and upper bounds for others.

In its most general form, the idea is as follows. Let F be the free (universal) group on d generators. Then, for any group G and any set of d generators for G, there is a homomorphism $\pi\colon F \to G$ which maps the canonical generators for F to the given generators for G; clearly π is surjective. Let N be the kernel of π. Then N is a normal subgroup of F, and $|F : N| = |G|$; in fact, G is isomorphic to F/N, and the Cayley graph of G with the given generators is isomorphic to the Cayley graph of F/N with the canonical generators for F. Let S be the set of elements of F which can be expressed as a word of length at most k in the generators. (If one is interested in undirected Cayley graphs, then such words may use inverse generators g^{-1} as well as generators; for the directed case, only words using generators and not inverse generators are allowed.)

Proposition 1. *The Cayley graph for G on the given generators has diameter at most k if and only if $SN = F$.*

Proof. First, suppose $SN = F$. Let a be an arbitrary element of G; then $a = \pi(x)$ for some x, and x can be written in the form wy with $w \in S$ and $y \in N$. Hence, $a = \pi(wy) = \pi(w)\pi(y) = \pi(w)$. Now w can be written as a word of length at most k in the generators of F, so $a = \pi(w)$ can be written as the same word in the corresponding generators of G. Since a was arbitrary, the Cayley graph has diameter at most k.

Now suppose the Cayley graph has diameter at most k. Let x be any element of F; then $\pi(x)$ can be written as a word w' of length at most k in the generators of G. Let w be the corresponding word in the generators of F; then $\pi(w) = w' = \pi(x)$, so $\pi(w^{-1}x)$ is the identity of G, so $w^{-1}x \in N$. Hence, $x = w(w^{-1}x) \in SN$, as desired. ∎

So finding a Cayley graph on d generators with diameter k whose size is as large as possible is equivalent to finding a normal subgroup N of F such that $SN = F$ and $|F : N|$ is as large as possible. Of course, we immediately get the upper bound $|F : N| \leq |S|$, but this is probably not attainable.

Unfortunately, the collection of normal subgroups of F is so large and varied as to be unmanageable. So what we will do instead is restrict ourselves to certain families (usually varieties) of groups; this allows us to replace F with a free group for the family in question, which may be much easier to work with. For instance, if we only consider the Cayley graphs of Abelian groups, then we can replace F with the free *Abelian* group on d generators, which is simply $\mathbf{Z}^d$; the normal subgroups of $\mathbf{Z}^d$ are well understood and relatively easy to work with. It turns out that this reduces the degree-diameter problem for Abelian Cayley graphs to interesting problems about lattice coverings of Euclidean space by various shapes. Some of these problems can be solved completely, giving optimal Abelian Cayley graphs; others are still open.

A simple path-counting argument gives upper bounds for the size n of a Cayley graph with d generators and diameter limit k: in the directed case,

$$n \leq 1 + d + d^2 + \cdots + d^k = \frac{d^{k+1} - 1}{d - 1},$$

and in the undirected case,

$$n \leq 1 + 2d + 2d(2d - 1) + \cdots + 2d(2d - 1)^{k-1} = \frac{d(2d - 1)^k - 1}{d - 1}.$$

(The formulas for $d = 1$ are $k + 1$ and $2k + 1$.) These limits are well-known and actually apply to the degree-diameter problem for arbitrary graphs; they are known as the Moore bounds. For $d = 1$, these limits are attained by simple cycle graphs, which are Cayley graphs of cyclic groups.

In most cases, we will find that the attainable values for n using Cayley graphs in the families we consider do not approach these upper bounds; the equations defining the families force many paths to be redundant. But the extra structure provided by the groups may provide compensating advantages in parallel computers, such as good routing algorithms, easy constructibility, and the ability to map common problems onto the architecture. In

particular, many of the Cayley graphs of Abelian groups that we discuss in this paper are multi-dimensional rectangular meshes with additional connections at the boundary. Thus, mesh calculations with natural boundary conditions are trivially mapped into these graphs, while the extra connections are utilized only when global communications are being carried out. In addition, the mesh nature of these graphs allows the physical construction of the network to be carried out with relatively short wires. This will be discussed further below.

In a separate paper, we will examine some other varieties of groups for which similar analyses of Cayley graphs are feasible.

We would like to (and hereby do) thank Michael Dinneen, whose computer work determining the diameters of many Cayley graphs yielded some of the specific groups listed here and suggested promising families of groups to examine. Thanks also to Francesc Comellas, for pointing us to the existing literature on directed loop networks (in particular, the very useful survey paper of Bermond, Comellas, and Hsu [3]), and to Alexander Hulpke for discussions on automorphisms of Abelian groups.

Abelian groups. In the rest of this paper, we will examine the Cayley graphs arising from Abelian groups. Toroidal meshes and hypercubes are examples of such graphs.

The degree-diameter problem for Abelian Cayley graphs has been considered by others. In particular, Annexstein and Baumslag [2] show that the number of generators d, diameter k, and size n of a directed Abelian Cayley graph satisfy

$$k \geq \Omega(n^{1/d});$$

in fact, if $d \leq n^{1/d}$, then

$$k \geq \Omega(dn^{1/d}).$$

They also discuss similar results for Cayley graphs of nilpotent groups.

In addition, Chung [6] has constructed directed Abelian Cayley graphs G with $n = p^t - 1$, $d = p$, and

$$k \leq \left\lceil 2t + \frac{4t \log t}{\log p - 2 \log(t-1)} \right\rceil$$

for any positive integer $t < \sqrt{p} + 1$, where p is a prime. (Note that Chung's examples have diameters which are small compared to the number of generators; in contrast, we will concentrate here on graphs with a small fixed number of generators and relatively large diameters.) Her methods involve estimates of the second eigenvalue of the Laplacian of the adjacency matrix of G. For more about estimating the diameter of general graphs from knowledge of this eigenvalue, see Chung, Faber, and Manteuffel [7]. This eigenvalue is also connected to the sphere *packing* problem for real lattices; see Urakawa [21].

The more specific case of Cayley graphs of *cyclic* groups has been studied more extensively; such graphs are usually referred to by some variant of the phrase 'loop networks.' The survey paper of Bernard, Comellas, and Hsu [3] is an excellent guide to the literature in this area.

We start by taking care of some generalities and notational matters. In this paper we will use the symbol $+$ for the group operation(s). Let $\mathbf{Z}_m$ be the cyclic group of order m

(for definiteness, the set $\{0, 1, \ldots, m-1\}$ with the operation of addition modulo m). The groups $\mathbf{Z}_m$ and the infinite cyclic group $\mathbf{Z}$ each have a canonical generator, 1; but they also have other sets of generators, and some of these will be important later.

When one has groups $G_1, G_2, \ldots, G_l$ and a set of generators for each, then one can get a set of generators for the product $G_1 \times G_2 \times \cdots \times G_l$ by putting together the given generator sets. More precisely, for each $i \leq l$ and each generator g of G_i, let $\mathbf{e}_i(g)$ be the element of the product group which has the identity element of G_j as its j'th coordinate for all $j \leq l$ except i; the i'th coordinate is g. Then the set of all elements $\mathbf{e}_i(g)$ is a natural generating set for the product group. (The resulting diameter for the product group is the sum of the diameters of the groups G_i.) In the case where the groups G_i are cyclic groups with the canonical single generators, we write simply $\mathbf{e}_i$ for the l-tuple with 1 at the i'th coordinate and 0 elsewhere.

A two-dimensional toroidal mesh is simply the Cayley graph of the group $\mathbf{Z}_m \times \mathbf{Z}_n$ with the canonical generators $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$; higher-dimensional meshes are obtained from longer products. For the two-dimensional case, the number of vertices is mn, the degree is 2 in the directed case and 4 in the undirected case (assuming $m, n \geq 3$), and the diameter is $m + n - 2$ in the directed case, $\lfloor m/2 \rfloor + \lfloor n/2 \rfloor$ in the undirected case. The calculations in three or more dimensions are analogous.

The d-dimensional hypercube is the Cayley graph of the group $\mathbf{Z}_2^d$ with the canonical generators; since one gets bidirectional links in any case, we may as well just talk about the undirected version. In this case the size of the graph is exponential in the diameter, but only because the degree also grows with d: the size is 2^d, the degree is d (not $2d$, because the generators have order 2), and the diameter is also d.

We will see that with a fixed small number d of generators, one can obtain nearly optimal results for undirected Abelian Cayley graphs by using a twisted toroidal mesh; the twist allows one to multiply the number of nodes in the ordinary toroidal mesh by 2^{d-1} without increasing the diameter. For two dimensions we can get exactly optimal results by slightly adjusting this graph; for higher dimensions, the optimal size is still open. For the directed case, we can again get optimal results in two dimensions, but the higher-dimensional case is again unsolved (and rather strange even in three dimensions).

To get these results, we argue as in Proposition 1, but specialize to the case of Abelian groups. Hence, instead of the free group on d generators, we can use the free Abelian group on d generators, which is simply $\mathbf{Z}^d$ with the canonical generators $\mathbf{e}_i$, $1 \leq i \leq d$. For any Abelian group G generated by $g_1, \ldots, g_d$, there is a unique homomorphism from $\mathbf{Z}^d$ onto G which sends $\mathbf{e}_i$ to g_i for all i. Let N be the kernel of this homomorphism; then G is isomorphic to $\mathbf{Z}^d/N$, and the Cayley graph of G with the given generators is isomorphic to the Cayley graph of $\mathbf{Z}^d/N$ with the canonical generators for $\mathbf{Z}^d$.

Given a diameter limit k, let S_k be the set of elements of $\mathbf{Z}^d$ which can be expressed as a word of length at most k in the generators $\mathbf{e}_i$, which are allowed to occur positively or negatively. (The dimension d will be clear from context.) Then S_k can also be described as the set of points in $\mathbf{Z}^d$ at distance at most k from the origin under the l^1 (Manhattan) metric:

$$S_k = \{(x_1, \ldots, x_d) \in \mathbf{Z}^d : |x_1| + \cdots + |x_d| \leq k\}.$$

Let S_k' be the subset of S_k consisting of those elements whose coordinates are all nonnegative; these are the elements which can be expressed as words of length at most k in the generators $\mathbf{e}_i$ where only positive occurrences of the generators are allowed. Then S_k looks like a regular dual d-cube (a square for $d = 2$, an octahedron for $d = 3$), while S_k' looks like a right d-simplex (a triangle for $d = 2$, a tetrahedron for $d = 3$).

Now, by the same proof as for Proposition 1, we get:

Proposition 2. *Let G, N, and $g_1, \ldots, g_d$ be as above. Then the undirected Cayley graph for G and $g_1, \ldots, g_d$ has diameter at most k if and only if $S_k + N = \mathbf{Z}^d$, and the directed Cayley graph for G and $g_1, \ldots, g_d$ has diameter at most k if and only if $S_k' + N = \mathbf{Z}^d$.* ∎

So $|S_k|$ and $|S_k'|$ give upper bounds for the undirected and directed versions of this case of the degree-diameter problem. It is not hard to show that

$$|S_k'| = \binom{k + d}{d},$$

so $|S_k'| = k^d/d! + O(k^{d-1})$ for fixed d. For $|S_k|$, we easily get the asymptotic form $|S_k| = k^d 2^d/d! + O(k^{d-1})$ for fixed d, but exact formulas are harder; Stanton and Cowan [20] give several, such as

$$|S_k| = \sum_{i=0}^{d} 2^i \binom{d}{i}\binom{k}{i}.$$

In particular, when d is 1, 2, or 3, the formula for $|S_k|$ is $2k + 1$, $2k^2 + 2k + 1$, or $(4k^3 + 6k^2 + 8k + 3)/3$, respectively.

Lattice coverings and tilings. Proposition 2 tells us that, to find an optimal undirected (directed) Cayley graph of diameter k on d generators, we should look for a subgroup N of $\mathbf{Z}^d$ such that $S_k + N$ ($S_k' + N$) is all of $\mathbf{Z}^d$ and the index $|\mathbf{Z}^d : N|$ is as large as possible; the largest index we can hope for is $|S_k|$ ($|S_k'|$). But the structure of subgroups of $\mathbf{Z}^d$ of finite index (which are all normal, of course) is well known; they are precisely the d-dimensional lattices in $\mathbf{Z}^d$. Because of this, we will use the letter L instead of N for such subgroups of $\mathbf{Z}^d$ for the rest of this paper.

A d-dimensional lattice L in $\mathbf{Z}^d$ is specified by d linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_d$ in $\mathbf{Z}^d$; L is the set of all integral linear combinations of these vectors. We have $|\mathbf{Z}^d : L| = |\det M|$, where M is the $d \times d$ matrix whose i'th row is $\mathbf{v}_i$, for $i = 1, \ldots, d$.

Note that any bounded set contains only finitely many members of L. It follows that, if S is a bounded subset of $\mathbf{Z}^d$ and $\mathbf{x}$ is a point in $\mathbf{Z}^d$, then there are only finitely many $\mathbf{v} \in L$ such that $\mathbf{x} \in S + \mathbf{v}$.

Also note that L, or indeed the entire group $\mathbf{Z}^d$, has a linear ordering $\prec$ which is compatible with addition: $\mathbf{x} \prec \mathbf{y}$ implies $\mathbf{x} + \mathbf{z} \prec \mathbf{y} + \mathbf{z}$. To define $\prec$, first choose a direction (a nonzero vector $\mathbf{v}$ in $\mathbf{R}^d$), and put $\mathbf{x} \prec \mathbf{y}$ if $\mathbf{y}$ is farther in this direction than $\mathbf{x}$ is ($\mathbf{x} \cdot \mathbf{v} < \mathbf{y} \cdot \mathbf{v}$). If two vectors are at the same distance in this direction, then compare them in a second direction; repeat until all ties are broken. One example of this is lexicographic order: compare according to the first coordinate, then according to the

second coordinate if the first coordinates are equal, and so on. Or, in this discrete case, one can choose the initial direction so that it distinguishes all points and no tie-breaking is necessary; for instance, if $\mathbf{v} = (1, \pi, \pi^2, \ldots, \pi^{d-1})$, then we never have $\mathbf{x} \cdot \mathbf{v} = \mathbf{y} \cdot \mathbf{v}$ for distinct $\mathbf{x}, \mathbf{y}$ in $\mathbf{Z}^d$, so we can just define $\mathbf{x} \prec \mathbf{y}$ to mean $\mathbf{x} \cdot \mathbf{v} < \mathbf{y} \cdot \mathbf{v}$.

A *lattice covering* of $\mathbf{Z}^d$ by a set $S \subseteq \mathbf{Z}^d$ is a collection of translates of S by members of a lattice L (i.e., $\{S + \mathbf{v} : \mathbf{v} \in L\}$) which covers $\mathbf{Z}^d$. If the translates are disjoint, so that each point of $\mathbf{Z}^d$ is covered exactly once, then we have a *lattice tiling* of $\mathbf{Z}^d$ by S.

If we have a lattice covering as above, then $|S| \geq |\mathbf{Z}^d : L|$; if it is a tiling, then $|S| = |\mathbf{Z}^d : L|$. So we can measure the extent to which a covering is 'almost' a tiling by one of two numbers: the *density* of the covering, which is $|\mathbf{Z}^d : L| / |S| \geq 1$ (this is the average number of sets in the covering to which a random point of $\mathbf{Z}^d$ belongs), or the *efficiency* of the covering, which is $|S| / |\mathbf{Z}^d : L| \leq 1$. So Proposition 2 tells us that, in order to get the best possible Abelian Cayley graph on d generators with diameter k, we must find a lattice covering of $\mathbf{Z}^d$ by S_k or S_k' whose density (efficiency) is as small (large) as possible. We now give one more reformulation of the question.

Lemma 3. *Suppose we have a lattice covering of $\mathbf{Z}^d$ using a bounded set S and the lattice L. Then there is a set $T \subseteq S$ such that the translates of T by L form a lattice tiling of $\mathbf{Z}^d$.*

Proof. Let $\prec$ be a linear order of L compatible with addition. Now let T be the set of all points in S which are not in any of the sets $S + \mathbf{v}$ for $\mathbf{v} \in L$, $\mathbf{v} \succ \mathbf{0}$. We will see that every point $\mathbf{x}$ is in exactly one of the sets $T + \mathbf{v}$ for $\mathbf{v} \in L$.

Fix $\mathbf{x}$. As noted before, $\mathbf{x}$ is in only finitely many of the translates $S + \mathbf{v}$, so let $\mathbf{w}$ be the $\prec$-greatest member of L such that $\mathbf{x} \in S + \mathbf{w}$. Let $\mathbf{y} = \mathbf{x} - \mathbf{w} \in S$. Then $\mathbf{y}$ cannot be in $S + \mathbf{v}$ for $\mathbf{v} \succ \mathbf{0}$ in L, because, if it were, we would have $\mathbf{x} = \mathbf{y} + \mathbf{w} \in S + \mathbf{v} + \mathbf{w}$ and $\mathbf{v} + \mathbf{w} \succ \mathbf{w}$, contradicting the maximality of $\mathbf{w}$. So $\mathbf{y} \in T$ and $\mathbf{x} \in T + \mathbf{w}$.

Now suppose $\mathbf{x} = \mathbf{y} + \mathbf{w} = \mathbf{y}' + \mathbf{w}'$ where $\mathbf{w}$ and $\mathbf{w}'$ are distinct members of L and $\mathbf{y}$ and $\mathbf{y}'$ are in T (and hence in S). Then $\mathbf{v} = \mathbf{y} - \mathbf{y}' = \mathbf{w}' - \mathbf{w}$ is a nonzero member of L, so either $\mathbf{v} \succ \mathbf{0}$ or $\mathbf{v} \prec \mathbf{0}$. In the former case, $\mathbf{y} = \mathbf{y}' + \mathbf{v} \in S + \mathbf{v}$ contradicts $\mathbf{y} \in T$; in the latter case, $\mathbf{y}' = \mathbf{y} - \mathbf{v} \in S + (-\mathbf{v})$ contradicts $\mathbf{y}' \in T$. ∎

Note that such a set T must be of cardinality $|\mathbf{Z}^d : L|$, which is the size of the Cayley graph of $\mathbf{Z}^d / L$. So the size of the largest undirected (directed) Abelian Cayley graph on d generators with diameter k is equal to the size of the largest subset T of S_k (S_k') such that there is a lattice tiling of $\mathbf{Z}^d$ using T.

Approximation by lattice coverings of real space. The study of lattice coverings and lattice tilings is more familiar for $\mathbf{R}^d$ than for $\mathbf{Z}^d$. We will show that real coverings can be approximated to some extent by integer coverings, and vice versa, so that known results from the real context can be transferred to the integer lattices we are interested in.

The definitions of lattice, lattice covering, and lattice tiling are the same in $\mathbf{R}^d$ as in $\mathbf{Z}^d$, except that we allow boundaries to be shared in the definition of a tiling. This lets us work throughout with closed sets (usually polyhedra with their interiors) instead of having to keep some of the boundary points and discard others. Most of the results above for integer

lattices go through verbatim for real lattices, including Lemma 3 and its proof (although we will probably use the closure of T rather than T itself in practice). The main difference is that the absolute determinant $|\det M|$ of the matrix formed from the generators of a lattice L is not $|\mathbf{R}^d : L|$ (which is infinite). Instead, this determinant is the measure of the parallelepiped determined by the generating vectors; this parallelepiped gives a lattice tiling of $\mathbf{R}^d$ using the lattice L. It follows easily that any other set S which gives a lattice tiling using the lattice L must have measure $|\det M|$ (barring pathological cases of non-measurable sets or positive-measure boundaries). Such a set S is called a *fundamental region* for the lattice L. One can now define the density or efficiency of a covering by dividing the measure of the covering set by this determinant or vice versa.

One can transform a lattice covering of $\mathbf{Z}^d$ using S into a lattice covering of $\mathbf{R}^d$ by replacing each point of S with a unit cube (i.e., replace S with $S + U$ where U is a fixed unit d-cube with edges parallel to the coordinate axes); the two coverings will have the same density. However, transforming results in the other direction is harder, because the real results usually involve actual triangles, octahedra, etc. rather than polycube approximations, and the lattices used will often not be integer lattices. We will now present results that allow us to get around these difficulties.

In $\mathbf{R}^d$, let $\bar{S}_k$ be the closed l^1-ball of radius k at the origin:

$$\bar{S}_k = \{(x_1, \ldots, x_d) : |x_1| + \cdots + |x_d| \le k\}.$$

Let $\bar{S}'_k$ be the set of nonnegative points in $\bar{S}_k$:

$$\bar{S}'_k = \{(x_1, \ldots, x_d) : x_1, \ldots, x_d \ge 0, x_1 + \cdots + x_d \le k\}.$$

Let L be any lattice in $\mathbf{R}^d$.

Proposition 4.
    (a) *If $S_k + L$ covers $\mathbf{Z}^d$, then $\bar{S}_{k+d/2} + L$ covers $\mathbf{R}^d$.*
    (b) *If $S'_k + L$ covers $\mathbf{Z}^d$, then $\bar{S}'_{k+d} + L$ covers $\mathbf{R}^d$.*

Proof. (a) By the triangle inequality for l^1 distance, we have $\bar{S}_k + \bar{S}_{d/2} \subseteq \bar{S}_{k+d/2}$, so $S_k + L + \bar{S}_{d/2} \subseteq \bar{S}_{k+d/2} + L$; therefore, it suffices to show that $\mathbf{Z}^d + \bar{S}_{d/2} = \mathbf{R}^d$. For any $\mathbf{x} \in \mathbf{R}^d$, let $\mathbf{y}$ be the element of $\mathbf{Z}^d$ nearest to $\mathbf{x}$ (i.e., round each coordinate of $\mathbf{x}$ to the nearest integer); then $\mathbf{x} - \mathbf{y}$ is in $\bar{S}_{d/2}$, so $\mathbf{x}$ is in $\mathbf{Z}^d + \bar{S}_{d/2}$.

(b) Similarly, this follows from the fact that $\mathbf{Z}^d + \bar{S}'_d = \mathbf{R}^d$, which is proved in the same way as above (round each coordinate of $\mathbf{x}$ downward instead of to the nearest integer). ∎

One can argue in the same way within $\mathbf{Z}^d$ to get:

Proposition 5. *If L is a lattice in $\mathbf{Z}^d$ and m is a positive integer, then:*
    (a) *If $S_k + L$ covers $\mathbf{Z}^d$, then $S_{mk+\lfloor m/2 \rfloor d} + mL$ covers $\mathbf{Z}^d$.*
    (b) *If $S'_k + L$ covers $\mathbf{Z}^d$, then $S'_{mk+(m-1)d} + mL$ covers $\mathbf{Z}^d$.*

Proof. For (a), clearly $S_{mk} + mL$ covers $m\mathbf{Z}^d$, so, as in the preceding proposition, it suffices to note that $m\mathbf{Z}^d + S_{\lfloor m/2 \rfloor d}$ covers $\mathbf{Z}^d$, because we can just round any member of $\mathbf{Z}^d$ to the nearest member of $m\mathbf{Z}^d$. Similarly, (b) holds because $m\mathbf{Z}^d + S'_{(m-1)d}$ covers $\mathbf{Z}^d$. ∎

Of course, one can get similar results for sets other than S_k and S'_k.

Using Proposition 4, it is easy to get from a covering of $\mathbf{Z}^d$ using an integer lattice to a covering of $\mathbf{R}^d$ using a real lattice (and using the real shape $\bar{S}_k$ or $\bar{S}'_k$); if k is large relative to d, then the two coverings have about the same efficiency. We will now show that one can move in the other direction as well.

First, we give a useful criterion for deciding whether one has a lattice covering of $\mathbf{R}^d$.

Proposition 6. *Suppose S is a nonempty subset of $\mathbf{R}^d$ and L is a lattice in $\mathbf{R}^d$. If there is an $\varepsilon > 0$ such that $S + L$ covers all points within distance ε of S, then $S + L$ covers $\mathbf{R}^d$.*

Proof. Clearly there is some point $\mathbf{x}_0$ in $S + L$. We will show that, if $\mathbf{x} \in S + L$ and the distance $\delta(\mathbf{x}, \mathbf{y})$ is less than ε, then $\mathbf{y} \in S + L$. Applying this once shows that all points within distance ε of $\mathbf{x}_0$ are in $S + L$; applying it again shows that all points within distance 2ε of $\mathbf{x}_0$ are in $S + L$; since this can be repeated forever, we find that all points of $\mathbf{R}^d$ are in $S + L$.

Let $\mathbf{x}$ and $\mathbf{y}$ be as above. Find $\mathbf{v} \in L$ such that $\mathbf{x} \in S + \mathbf{v}$. Then $\mathbf{y} - \mathbf{v}$ is within distance ε of $\mathbf{x} - \mathbf{v} \in S$, so there is $\mathbf{v}' \in L$ such that $\mathbf{y} - \mathbf{v} \in S + \mathbf{v}'$. Hence, $\mathbf{y} \in S + \mathbf{v}' + \mathbf{v}$, so $\mathbf{y} \in S + L$, as desired. ∎

Using this, we can now show that, if one has a lattice covering using a bounded subset of $\mathbf{R}^d$, then one can perturb the lattice slightly and still get a lattice covering using a slightly larger subset of $\mathbf{R}^d$.

Proposition 7. *Let S be a bounded subset of $\mathbf{R}^d$, and let L be a lattice in $\mathbf{R}^d$ such that $S + L = \mathbf{R}^d$; let $\mathbf{v}_1, \ldots, \mathbf{v}_d$ be a list of generators for L. Then there are positive numbers η and ρ such that, for all $r \in (0, 1)$, if the distance $\delta(\mathbf{v}_i, \mathbf{v}'_i)$ is less than $r\eta$ for all $i \leq d$, then $S^+ + L' = \mathbf{R}^d$, where L' is the lattice generated by $\mathbf{v}'_1, \ldots, \mathbf{v}'_d$ and S^+ is the set of points within distance $r\rho$ of S.*

Proof. The number of members of L within any bounded part of $\mathbf{R}^d$ is finite, so it only takes finitely many translates of S by members of L to cover any bounded part of $\mathbf{R}^d$. In particular, there is a number $M > 0$ such that the sets $S + a_1\mathbf{v}_1 + \cdots + a_d\mathbf{v}_d$ for $(a_1, \ldots, a_d) \in \mathbf{Z}^d$ with $|a_1| + \cdots + |a_d| \leq M$ cover all points within distance ρ of S, for some $\rho > 0$. Let $\eta = \rho/M$.

Let r be any positive number less than 1; we must see that, if S^+ and L' are defined as above, then $S^+ + L' = \mathbf{R}^d$. By the preceding proposition, it will suffice to show that $S^+ + L'$ covers all points within distance $(1 - r)\rho$ of S^+. Suppose $\mathbf{y}$ is within distance $(1 - r)\rho$ of S^+; then $\mathbf{y}$ is within distance $(1 - r)\rho + r\rho = \rho$ of S, so there exist integers $a_1, \ldots, a_d$ with $|a_1| + \cdots + |a_d| \leq M$ and a point $\mathbf{x} \in S$ such that $\mathbf{y} = \mathbf{x} + a_1\mathbf{v}_1 + \cdots + a_d\mathbf{v}_d$. Let $\mathbf{x}' = \mathbf{x} + \sum_{i=1}^{d} a_i(\mathbf{v}_i - \mathbf{v}'_i)$; then

$$\delta(\mathbf{x}, \mathbf{x}') \leq \sum_{i=1}^{d} |a_i| \delta(\mathbf{v}_i, \mathbf{v}'_i) < \sum_{i=1}^{d} |a_i| r\eta \leq Mr\eta = r\rho,$$

so $\mathbf{x}' \in S^+$. Since $\mathbf{y} = \mathbf{x}' + a_1\mathbf{v}'_1 + \cdots + a_d\mathbf{v}'_d$, we have $\mathbf{y} \in S^+ + L'$, as desired. ■

Note that, in the above propositions, 'distance' need not be Euclidean distance; it can be any metric arising from a norm on $\mathbf{R}^d$. For our present purposes, it will be most convenient (but not essential) to use l^∞-distance: $\delta(\mathbf{x}, \mathbf{y}) = \max_i |x_i - y_i|$.

Theorem 8. *Suppose one has a lattice L in $\mathbf{R}^d$ such that $\bar{S}_k + L$ covers $\mathbf{R}^d$; let $\mathbf{v}_1, \ldots, \mathbf{v}_d$ be generators for L. Then there is a constant c such that, for all sufficiently large real numbers t, if $\mathbf{w}_i$ is obtained from $t\mathbf{v}_i$ by rounding all coordinates to the nearest integer, and $\bar{L}$ is the lattice generated by $\mathbf{w}_1, \ldots, \mathbf{w}_d$, then $\bar{S}_{tk+c} + \bar{L}$ covers $\mathbf{R}^d$. The same statement holds for $\bar{S}'_k$ and $\bar{S}'_{tk+c}$ instead of $\bar{S}_k$ and $\bar{S}_{tk+c}$.*

Proof. For the $\bar{S}_k$ case, let $S = \bar{S}_k$ and find η and ρ as in the preceding proposition, letting the distance δ be the l^∞ metric. Let c be any fixed number greater than $d\rho/2\eta$. Then, for any $t > c/d\rho$, if we let $r = c/d\rho t$, then $r < 1$ and $1/2t < r\eta$. If we define $\mathbf{w}_i$ and $\bar{L}$ as above, and let $\mathbf{v}'_i = \mathbf{w}_i/t$ and $L' = \bar{L}/t$, then $\delta(\mathbf{v}_i, \mathbf{v}'_i) \leq 1/2t$ for each i, so we can conclude that $S^+ + L'$ covers $\mathbf{R}^d$, where S^+ is the set of points within distance $r\rho$ of $\bar{S}_k$. It is easy to see that $S^+ \subseteq \bar{S}_{k+dr\rho}$, so $\bar{S}_{k+dr\rho} + L'$ covers $\mathbf{R}^d$. Hence, $\bar{S}_{tk+tdr\rho} + tL'$ covers $\mathbf{R}^d$; but $tdr\rho = c$ and $tL' = \bar{L}$, so we are done.

The proof for $\bar{S}'_k$ is almost the same. Let $S = \bar{S}'_k$ and apply the preceding proposition (using the l^∞ metric again) to get η and ρ. Fix $c > d\rho/\eta$. For any $t > c/2d\rho$, if we let $r = c/2d\rho t$, then $r < 1$ and $1/2t < r\eta$. Now define $\mathbf{w}_i$, $\bar{L}$, $\mathbf{v}'_i$, L', and S^+ as above, and conclude again that $S^+ + L'$ covers $\mathbf{R}^d$. One can easily check that $S^+ \subseteq \bar{S}'_{k+2dr\rho} - r\rho\mathbf{u}$, where $\mathbf{u} = (1, \ldots, 1)$. Hence, $\bar{S}'_{k+2dr\rho} - r\rho\mathbf{u} + L'$ covers $\mathbf{R}^d$, so $\bar{S}'_{k+2dr\rho} + L'$ covers $\mathbf{R}^d + r\rho\mathbf{u} = \mathbf{R}^d$. Now multiply by t to see that $\bar{S}'_{tk+c} + \bar{L}$ covers $\mathbf{R}^d$. ■

Again, it is easy to modify this proof to work for other sets in place of $\bar{S}_k$ or $\bar{S}'_k$. Also, the proof is quite effective, allowing one to compute specific values of c and t which work for a given lattice L (assuming it is feasible to compute M and ρ).

The covering $\bar{S}_{tk} + tL$ has the same efficiency as the covering $\bar{S}_k + L$; since $\bar{S}_{tk+c} + \bar{L}$ is a relatively slight perturbation of $\bar{S}_{tk} + tL$ when t is large, it has almost the same efficiency. Since $\bar{L}$ is an integer lattice, the fact that $\bar{S}_{tk+c} + \bar{L}$ covers $\mathbf{R}^d$ implies that $S_{\lfloor tk+c\rfloor} + \bar{L}$ covers $\mathbf{Z}^d$; again the efficiency is almost the same if t is large. Therefore, we can construct integer lattice coverings as nearly efficient as desired to a given real lattice covering, thus giving asymptotic results for the present case of the degree-diameter problem. The precise result is as follows.

Theorem 9. *Let $\varepsilon_{\mathbf{R}}$ be the best possible efficiency for a lattice covering of $\mathbf{R}^d$ by $\bar{S}_1$, and let $\varepsilon_{\mathbf{Z}}(k)$ be the best possible efficiency for a lattice covering of $\mathbf{Z}^d$ by S_k. Then $\varepsilon_{\mathbf{Z}}(k) = \varepsilon_{\mathbf{R}} + O(k^{-1})$. The same applies to $\bar{S}'_1$ and S'_k.*

Proof. Let L be a lattice giving a lattice covering of $\mathbf{R}^d$ by $\bar{S}_1$ with efficiency $\varepsilon_{\mathbf{R}}$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_d$ and c be as in Theorem 8. Given a large integer k, let $t = k - c$, and let $\bar{L}$ be the integer lattice approximating tL as in Theorem 8, generated by $\mathbf{w}_1, \ldots, \mathbf{w}_d$. Since $\bar{L}$ is an integer lattice and $\bar{S}_{t+c} + \bar{L} = \mathbf{R}^d$, we have $S_k + \bar{L} = \mathbf{Z}^d$. Let M and $\bar{M}$ be the $d \times d$ matrices whose rows are $\mathbf{v}_i$ and $\mathbf{w}_i$, respectively; then $\det M = (2^d/d!)\varepsilon_{\mathbf{R}}$ and $\det \bar{M} \leq |S_k|\varepsilon_{\mathbf{Z}}(k)$,

which implies $\det(k^{-1}\bar{M}) \le (2^d/d! + O(k^{-1}))\varepsilon_{\mathbf{Z}}(k)$. But $\bar{M} = tM + O(1) = kM + O(1)$, so $k^{-1}\bar{M} = M + O(k^{-1})$, so $\det(k^{-1}\bar{M}) = \det M + O(k^{-1})$; this implies $\varepsilon_{\mathbf{Z}}(k) \ge \varepsilon_{\mathbf{R}} + O(k^{-1})$.

On the other hand, Proposition 4(a) states that any lattice that gives a covering of $\mathbf{Z}^d$ by S_k also gives a covering of $\mathbf{R}^d$ by $\bar{S}_{k+d/2}$. If the efficiency of the former is $\varepsilon_{\mathbf{Z}}(k)$, then the efficiency of the latter is $(|S_k|/\operatorname{vol}(\bar{S}_{k+d/2}))\varepsilon_{\mathbf{Z}}(k) = (1 + O(k^{-1}))\varepsilon_{\mathbf{Z}}(k)$; hence, $(1 + O(k^{-1}))\varepsilon_{\mathbf{Z}}(k) \le \varepsilon_{\mathbf{R}}$, so $\varepsilon_{\mathbf{Z}}(k) \le \varepsilon_{\mathbf{R}} + O(k^{-1})$.

The same argument works for $\bar{S}'_1$ and S'_k. ∎

Again, the same applies to other shapes as well. Combining this with the known sizes of the sets S_k and S'_k gives:

Corollary 10. *Let d be a fixed positive integer.*
(a) If $\varepsilon_{\mathbf{R}}$ is the best possible efficiency for a lattice covering of $\mathbf{R}^d$ by $\bar{S}_1$, then the size of the largest possible undirected Cayley graph of an Abelian group on d generators with diameter at most k is $(2^d\varepsilon_{\mathbf{R}}/d!)k^d + O(k^{d-1})$.
(b) If $\varepsilon'_{\mathbf{R}}$ is the best possible efficiency for a lattice covering of $\mathbf{R}^d$ by $\bar{S}'_1$, then the size of the largest possible directed Cayley graph of an Abelian group on d generators with diameter at most k is $(\varepsilon'_{\mathbf{R}}/d!)k^d + O(k^{d-1})$. ∎

The above assumes that the upper limit $\varepsilon_{\mathbf{R}}$ on the efficiency of a lattice covering of $\mathbf{R}^d$ by $\bar{S}_1$ is actually attained. To see that this is the case, first note that there is certainly a lattice covering with positive efficiency, say ε_0. Now consider the possible sets of generating vectors $\mathbf{v}_1, \ldots, \mathbf{v}_d$ for a lattice L giving a covering of efficiency at least ε_0. We may assume that, if $\mathbf{v}_i$ is one of the generators and $\mathbf{w}$ is an integral linear combination of the other generators, then $|\mathbf{v}_i| \le |\mathbf{v}_i + \mathbf{w}|$; otherwise, just replace $\mathbf{v}_i$ with $\mathbf{v}_i + \mathbf{w}$ to get a smaller set of generating vectors for L, and iterate until no more such reductions are possible. It follows that, if, say, $\mathbf{v}_1$ is the longest of the vectors $\mathbf{v}_i$, then the angle between $\mathbf{v}_1$ and the hyperplane P spanned by $\mathbf{v}_2, \ldots, \mathbf{v}_d$ is bounded below by a positive number ($\pi/3$ for $d = 2$, somewhat less for higher d). But the distance from $\mathbf{v}_1$ to P is at most the diameter of $\bar{S}_1$ (i.e., 2), since otherwise $\bar{S}_1 + L$ would consist of 'hyperplanes' of copies of $\bar{S}_1$ with gaps in between. Putting these together, we get a fixed upper bound B on the length $|\mathbf{v}_1|$, and hence on all of the lengths $\mathbf{v}_i$. But we also have a positive lower bound b on the determinant $\det(\mathbf{v}_1, \ldots, \mathbf{v}_d)$, namely $\varepsilon_0 \operatorname{vol}(\bar{S}_1)$. One can now show that there is a fixed number M such that, if $m_1, \ldots, m_d \in \mathbf{Z}$ and $|m_1| + \cdots + |m_d| > M$, then $|m_1\mathbf{v}_1 + \cdots + m_d\mathbf{v}_d| > 3$. Hence, the finitely many translates $\bar{S}_1 + m_1\mathbf{v}_1 + \cdots + m_d\mathbf{v}_d$ with $|m_1| + \cdots + |m_d| \le M$ will have to cover all points at distance ≤ 2 from the origin. Now, the set of sequences $\mathbf{v}_1, \ldots, \mathbf{v}_d$ with all $|\mathbf{v}_i| \le B$ such that the translates $\bar{S}_1 + m_1\mathbf{v}_1 + \cdots + m_d\mathbf{v}_d$ for $|m_1| + \cdots + |m_d| \le M$ cover all points at distance ≤ 2 from $\mathbf{0}$ is a compact set, so there is a sequence $\mathbf{v}_1, \ldots, \mathbf{v}_d$ in this set for which $\det(\mathbf{v}_1, \ldots, \mathbf{v}_d)$ is maximal; this sequence of vectors generates a lattice covering of $\mathbf{R}^d$ by $\bar{S}_1$ (by Proposition 6) with maximal possible efficiency. The same argument works for any compact shape of positive volume, such as $\bar{S}'_1$. (Presumably this argument is well known, but the authors were not able to find a reference for it.)

As we will see in a later section (for the case $d = 3$, but the argument is general), the determination of actual values for $\varepsilon_{\mathbf{R}}$ and $\varepsilon'_{\mathbf{R}}$ for a particular dimension d can in principle

be reduced to the solution of a finite number of nonlinear optimization problems (each of which requires maximizing a degree-d polynomial function over a region which is a convex polytope in $\mathbf{R}^{d^2}$). Unfortunately, this finite number is extremely large, so the actual values are not known for $d > 2$. (For $d = 1$ we trivially have $\varepsilon_{\mathbf{R}} = \varepsilon'_{\mathbf{R}} = 1$. For $d = 2$ we will see below that $\varepsilon_{\mathbf{R}} = 1$ and $\varepsilon'_{\mathbf{R}} = 2/3$.) The computations described in later sections lead us to conjecture that, for $d = 3$, $\varepsilon_{\mathbf{R}} = 8/9$ and $\varepsilon'_{\mathbf{R}} = 63/125$.

Two generators, undirected. We now begin to consider the results that can be obtained for specific values of d. As noted previously, the case $d = 1$ is trivial, so we will start with the case of undirected Abelian Cayley graphs on two generators and a given bound k on the diameter. The results in this subsection are not new.

As noted before, the diameter of the ordinary toroidal mesh $\mathbf{Z}_m \times \mathbf{Z}_n$ is $\lfloor m/2 \rfloor + \lfloor n/2 \rfloor$. Hence, the largest such mesh with diameter $\leq k$ is the one where m is k rounded up to the nearest odd integer and n is $k+1$ rounded up to the nearest odd integer. This corresponds to the lattice covering of $\mathbf{Z}^2$ by S_k using the lattice $L = m\mathbf{Z} \times n\mathbf{Z}$; the efficiency of this covering is $mn/(2k^2 + 2k + 1)$, which tends to $1/2$ as k grows large. So one can hope for better results.

To get these, consider the real rotated square $\bar{S}_k$. There is obviously a lattice tiling using this square; it is just a rotated orthogonal grid with spacing $\sqrt{2}k$ between lines. The lattice L_1 for this tiling is generated by the vectors (k, k) and $(-k, k)$; as expected, the corresponding determinant is $2k^2$, which is equal to the area of $\bar{S}_k$.

It now follows from the approximation results that we can get lattice coverings of $\mathbf{Z}^2$ using S_k with efficiencies that approach 1 for large k. However, we do not need the general results here; since L_1 is already an integer lattice, we can simply note that $\bar{S}_k + L_1 = \mathbf{R}^2$ implies $S_k + L_1 = \mathbf{Z}^d$. So this gives a lattice covering using S_k whose index is $|\mathbf{Z}^2 : L_1| = 2k^2$, which is better than that from the best toroidal mesh for all $k \geq 3$; for large k, the efficiency approaches 1.

The corresponding Cayley graph $\mathbf{Z}^2/L_1$ turns out to be quite simple to describe. The $2k \times k$ rectangle $\{1, \ldots, 2k\} \times \{1, \ldots, k\}$ contains exactly one point from each coset of L_1, so it can serve as a set of vertices for the graph. Adjacent points in the rectangle (horizontally and vertically) are connected as in the usual mesh. Horizontally, one has the usual toroidal connections at the ends: $(1, j)$ is connected to $(2k, j)$. But vertically, there is an offset of k: $(i, 1)$ is connected to $(i + k, k)$ if $i \leq k$, or to $(i - k, k)$ if $i > k$. This is just like a $2k \times k$ toroidal mesh, except that the torus is twisted halfway around before the long edges are glued together. This twist allows one to double the number of vertices in a $k \times k$ toroidal mesh while increasing the diameter by at most 1 (there is no increase if k is even).

A number of the useful properties of ordinary toroidal meshes apply with very little change to twisted toroidal meshes. For instance, since the new mesh is still just a rectangular mesh with extra connections at the boundary, it is easy to map a simple rectangular grid into the mesh, by simply ignoring the boundary connections.

Another nice property of toroidal meshes is that it is easy to find a shortest route from one node to another: just check for each coordinate separately which of the two possible directions gives a shorter path, and put the results together. Finding optimal routes is

only slightly more complicated for the twisted toroidal mesh. To see this, consider the given $2k \times k$ rectangle as half of a $2k \times 2k$ rectangle; each node (i, j) in the first half has a copy $(i \pm k, j \pm k)$ in the other half. This larger rectangle is then copied periodically without further twists to cover $\mathbf{Z}^2$; in other words, the $2k \times k$ twisted toroidal mesh is just a $2k \times 2k$ toroidal mesh where (i, j) and $(i \pm k, j \pm k)$ are identified as a single node. Therefore, to find an optimal route from (i, j) to (i', j') in the twisted mesh, apply the ordinary $2k \times 2k$ toroidal mesh routing algorithm to find optimal routes from (i, j) to (i', j') and to $(i' \pm k, j' \pm k)$, and choose the shorter of the two.

In the real case the lattice L_1 gave a perfect tiling of $\mathbf{R}^2$ using $\bar{S}_k$, since boundary overlap did not count; but in the integer case the boundary overlap reduces the efficiency slightly, from 1 to $2k^2/(2k^2 + 2k + 1)$. It turns out that if one uses a slightly modified lattice, namely the lattice L_2 with generating vectors $(k, k + 1)$ and $(-k - 1, k)$, then one gets a covering of $\mathbf{Z}^2$ by copies of S_k with efficiency 1 (i.e., a tiling). See Figure 1. We therefore get:

Theorem 11 (multiple authors). *The largest possible size for the undirected Cayley graph of an Abelian group on two generators having diameter k is $2k^2 + 2k + 1$.*   ■

This result has appeared in various forms in a number of places (usually stated so as to apply only to cyclic Cayley graphs, but since the optimal Abelian Cayley graphs turn out to be cyclic, the results are basically equivalent). See, for instance, Boesch and Wang [4] or Yebra el al. [23]; the Bermond-Comellas-Hsu survey [3] has many additional references.

The tiling in Figure 1 is in Yebra el al. [23], among other places; it even appears in native artwork of the southwestern United States, and may date back to the ancient Aztecs, who did use the stepped diamond shape in temple ornamentation. (This shape is now commonly known as the Aztec diamond, a term coined by J. Propp.) However, it is unlikely that the Aztecs were motivated by the desire to construct efficient parallel computation networks.

It is easy to see that the lattice tiling of $\mathbf{Z}^2$ by S_k, or of $\mathbf{R}^2$ by the Aztec diamond, is unique except for a possible reflection about the line $x = y$; this just corresponds to interchanging the two generators for the Cayley graph. Therefore, the Cayley graph attaining the bound in Theorem 11 is unique up to isomorphism.

Since the point $(2k + 1, 1)$ is in L_2, we have $\mathbf{e}_2 + L_2 = (-2k - 1)(\mathbf{e}_1 + L_2)$ in $\mathbf{Z}^2/L_2$. Hence, $\mathbf{Z}^2/L_2$ is a cyclic group, generated by $\mathbf{e}_1 + L_2$ alone. It is isomorphic (not only as a group but as a Cayley graph) to $\mathbf{Z}_{2k^2+2k+1}$ with the generating set $\{1, 2k^2\}$. One may choose to replace the second generator by its inverse, making the generating set $\{1, 2k+1\}$; other generating sets can be used as well.

For layout purposes, one may just arrange the nodes in the form of the diamond S_k and connect the boundary nodes as specified by L_2, but it is probably more convenient to use the almost-rectangular shape outlined in Figure 1 (a $(2k + 1) \times k$ rectangle with an extra partial row of length $k+1$). The boundary connections are similar to those for the twisted toroidal mesh given before, but now there is also a slight twist when connecting the short sides; there is a drop of one row when wrapping around from right to left. This layout shows that one can embed a rectangular grid into this graph so as to use almost all of the nodes.
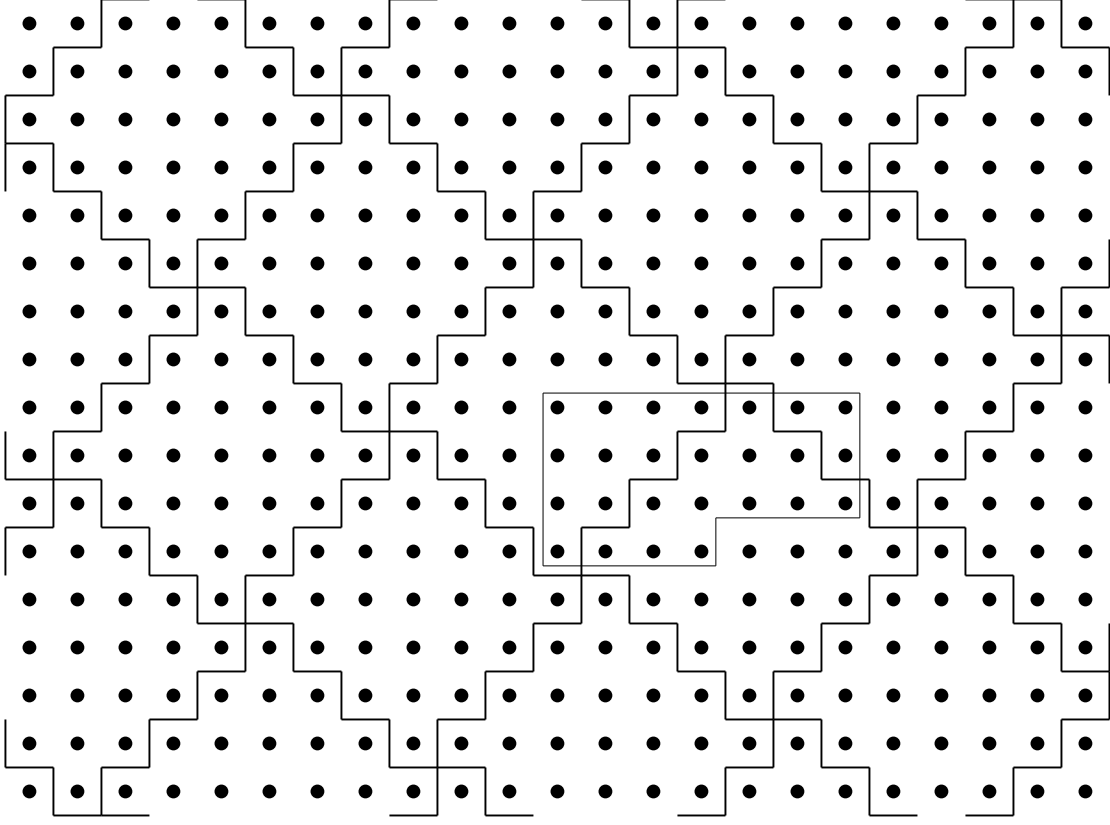
FIGURE 1. Lattice tiling of $\mathbf{Z}^2$ using S_k (shown for $k = 3$).

Two generators, directed. We now describe the largest possible directed Cayley graph of an Abelian group on two generators with diameter bounded by k. As in the preceding subsection, the two-generator results here are already known.

The best toroidal mesh in this case is $\mathbf{Z}_m \times \mathbf{Z}_{m'}$, where $m = \lfloor k/2 \rfloor + 1$ and $n = \lceil k/2 \rceil + 1$; this gives size $mm' = \lfloor (k+2)^2/4 \rfloor$, which is about $1/2$ of $|S'_k|$, so one can hope to do better.

However, one is not going to get perfect efficiency in this case. One can easily tile the plane with triangles such as $\bar{S}'_k$ if one is allowed to rotate them, but this is not possible using only a lattice of translated copies of a triangle. The exact minimum density for a lattice covering of the plane by triangles was computed by I. Fáry in 1950; we will give a different proof of his result here, and then give the analogue for $\mathbf{Z}^2$.

Theorem 12 (Fáry [13]). *The minimum density for a lattice covering of $\mathbf{R}^2$ by triangles is $3/2$. Equivalently, the maximum efficiency is $2/3$.*

Proof. Since the density and efficiency of a lattice covering are invariant under affine transformations, it does not matter which triangle we work with, so, for slight convenience, let us work with the isosceles right triangle $\bar{S}'_1$.

One can attain the efficiency $2/3$ by using the lattice with generating vectors $(1/3, 1/3)$ and $(2/3, -1/3)$. This corresponds to a tiling of the plane using an L-tromino that takes
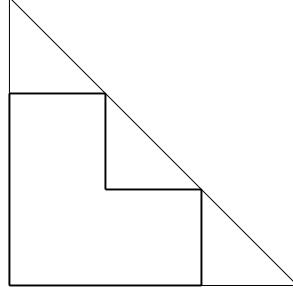
FIGURE 2. A subset of the triangle $\bar{S}_1'$ which tiles the plane.

up $2/3$ of $\bar{S}_1'$, as shown in Figure 2. Or one can cut off all three corners of the triangle to get a hexagon that tiles the plane.

Now suppose that we have a lattice covering of $\mathbf{R}^2$ using $\bar{S}_1'$ and the lattice L; we must show that the efficiency of the covering is at most $2/3$. Let $\prec$ be a linear ordering of L compatible with addition which is defined by primarily ordering points (x, y) according to the sum $x + y$ (so points that are farther out in the direction $(1, 1)$ come later in the ordering) and breaking ties (if any) by distance in some other direction.

Let $\overline{AB}$ be the hypotenuse of $\bar{S}_1'$. Only finitely many of the L-translates of $\bar{S}_1'$ lie near $\bar{S}_1'$; of these, the ones of the form $\bar{S}_1' + \mathbf{v}$ for $\mathbf{v} \succ \mathbf{0}$ must cover the points which are near $\overline{AB}$ on the side away from $\bar{S}_1'$. Since the union of finitely many translates of $\bar{S}_1'$ is closed, $\overline{AB}$ itself is covered by finitely many translates $\bar{S}_1' + \mathbf{v}$ with $\mathbf{v} \succ \mathbf{0}$. Find such a covering of $\overline{AB}$ with as few translates as possible, say $\bar{S}_1' + \mathbf{v}_1, \ldots, \bar{S}_1' + \mathbf{v}_m$, where $\mathbf{v}_i \succ \mathbf{0}$.

Note that, since $\bar{S}_1' + \mathbf{v}_i$ must intersect $\overline{AB}$, it contains one of the endpoints A and B if and only if the coordinates of $\mathbf{v}_i$ are not both positive. We may assume that at most one of the vectors $\mathbf{v}_i$ has both coordinates positive. For if there are two such, let them be $\bar{S}_1' + \mathbf{v}_i$ and $\bar{S}_1' + \mathbf{v}_j$ where $\mathbf{v}_i \prec \mathbf{v}_j$. Then $\mathbf{v}_j - \mathbf{v}_i \succ \mathbf{0}$; since $\mathbf{v}_i$ has both coordinates positive, we have

$$(\bar{S}_1' + \mathbf{v}_j) \cap \overline{AB} \subseteq (\bar{S}_1' + \mathbf{v}_j - \mathbf{v}_i) \cap \overline{AB}.$$

Furthermore, $\mathbf{v}_j - \mathbf{v}_i$ cannot have both coordinates positive, because, if it did, we would have

$$(\bar{S}_1' + \mathbf{v}_j) \cap \overline{AB} \subseteq (\bar{S}_1' + \mathbf{v}_i) \cap \overline{AB},$$

so $\bar{S}_1' + \mathbf{v}_j$ would not have been needed in the covering of $\overline{AB}$, contradicting the minimality of m. So we can replace $\bar{S}_1' + \mathbf{v}_j$ with $\bar{S}_1' + \mathbf{v}_j - \mathbf{v}_i$ to get another covering of $\overline{AB}$ using fewer vectors with both coordinates positive. Repeat this until only one such vector is left.

Since each translate $\bar{S}_1' + \mathbf{v}_i$ is convex, its intersection with $\overline{AB}$ is a segment or a point. Therefore, at most one of these translates can contain A, since otherwise one of the intersections $(\bar{S}_1' + \mathbf{v}_i) \cap \overline{AB}$ would include another such intersection, making the latter translate superfluous and contradicting the minimality of m. Similarly, at most one of the translates $\bar{S}_1' + \mathbf{v}_i$ contains B. Putting these facts together, we conclude that we needed at most three of the translates $\bar{S}_1' + \mathbf{v}$ with $\mathbf{v} \succ \mathbf{0}$ to cover the segment $\overline{AB}$.

In other words, there are points P and Q on $\overline{AB}$ such that each of the three segments $\overline{AP}$, $\overline{PQ}$, and $\overline{QB}$ is covered by one of the translates $\bar{S}'_1 + \mathbf{v}$ with $\mathbf{v} \succ \mathbf{0}$. Let l_1, l_2, l_3 be the lengths of these three segments; then $l_1 + l_2 + l_3 = \sqrt{2}$. Note that, if $\bar{S}'_1 + \mathbf{v}$ covers $\overline{AP}$, then $\bar{S}'_1 + \mathbf{v}$ covers the entire isosceles right triangle below $\overline{AP}$ whose hypotenuse is $\overline{AP}$; the area of this triangle is $l_1^2/4$. Similar statements hold for $\overline{PQ}$ and $\overline{QB}$. So we have three disjoint triangles included in $\bar{S}'_1$ which are covered by translates $\bar{S}'_1 + \mathbf{v}$ with $\mathbf{v} \succ \mathbf{0}$, and the total area of these triangles is $(l_1^2 + l_2^2 + l_3^2)/4$.

By the proof of (the real version of) Lemma 3, if we let T be the part of $\bar{S}'_1$ which is not covered by any translate $\bar{S}'_1 + \mathbf{v}$ with $\mathbf{v} \succ \mathbf{0}$, then T gives a lattice tiling of $\mathbf{R}^2$ using L, so the efficiency of the covering using $\bar{S}'_1$ and L is $\text{Area}(T)/\text{Area}(\bar{S}'_1)$. We have $\text{Area}(\bar{S}'_1) = 1/2$ and $\text{Area}(T) \le 1/2 - (l_1^2 + l_2^2 + l_3^2)/4$. A standard minimization shows that, if $l_1 + l_2 + l_3 = \sqrt{2}$, then $l_1^2 + l_2^2 + l_3^2 \ge 2/3$ (with equality only when $l_1 = l_2 = l_3 = \sqrt{2}/3$). Therefore, $\text{Area}(T) \le 1/3$, so the efficiency of the covering by $\bar{S}'_1$ and L is at most $2/3$. ∎

It now follows from Theorem 9 that the largest possible index $|\mathbf{Z}^2 : L|$ for an integer lattice L giving a lattice covering of $\mathbf{Z}^2$ by S'_k is approximately $(2/3)|S'_k|$, or about $k^2/3$, for large k. However, we can actually get an exact answer rather than an approximation.

Theorem 13 (mainly Wong and Coppersmith [22]). *The largest possible index $|\mathbf{Z}^2 : L|$ for a lattice L giving a lattice covering of $\mathbf{Z}^2$ by S'_k is $\lfloor (k+2)^2/3 \rfloor$.*

Proof. We will give a discrete form of the proof of Theorem 12. Let a be $(k+2)/3$ rounded to the nearest integer, and let $b = k + 2 - 2a$ (so b is also about $(k+2)/3$). Let T_k be the set of (i,j) in $\mathbf{Z}^2$ such that $i, j \ge 0$, $\min(i,j) < a$, and $\max(i,j) < a + b$. Then $T_k \subseteq S'_k$, since any (i,j) in T_k satisfies $i + j \le a - 1 + a + b - 1 = k$. The set T_k looks like the L-tromino from Figure 2, and it tiles $\mathbf{Z}^2$ using the lattice with generating vectors (a, a) and $(a + b, -b)$. Therefore, this lattice gives a covering of $\mathbf{Z}^2$ using S'_k, and its index is $a(a + 2b)$, which works out to be $\lfloor (k+2)^2/3 \rfloor$.

Now, suppose we have a lattice covering of $\mathbf{Z}^2$ using S'_k and a lattice L; we must show that $|\mathbf{Z}^2 : L| \le \lfloor (k+2)^2/3 \rfloor$. Define the linear order $\prec$ of L as before. Let A and B be the points $(0, k+1)$ and $(k+1, 0)$; then the segment $\overline{AB}$ contains $k + 2$ integral points, which must be covered by translates $S'_k + \mathbf{v}$ where $\mathbf{v} \in L$ and $\mathbf{v} \succ \mathbf{0}$.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be a list of as few vectors as possible in L such that $\mathbf{v}_i \succ \mathbf{0}$ and the translates $S'_{k+1} + \mathbf{v}_i$ of S'_{k+1} cover all of the integral points on $\overline{AB}$. Then the same argument as for Theorem 12 shows that m is at most 3. Hence, $\overline{AB}$ can be broken up into three segments $\overline{AP}$, $\overline{P'Q}$, and $\overline{Q'B}$ (where P and P' are adjacent integral points on $\overline{AB}$, as are Q and Q'), each of whose integral points are covered by one of the translates $S'_{k+1} + \mathbf{v}$ with $\mathbf{v} \succ \mathbf{0}$. Let l_1, l_2, l_3 be the numbers of integral points on these segments; then $l_1 + l_2 + l_3 = k + 2$.

If $S'_{k+1} + \mathbf{v}$ covers the integral points on $\overline{AP}$, then it covers all of the integral points in the isosceles right triangle below $\overline{AP}$ and having $\overline{AP}$ as its hypotenuse. In fact, all of these points other than those on $\overline{AP}$ itself are covered by $S'_k + \mathbf{v}$; there are $(l_1^2 - l_1)/2$ such points, and they are all in S'_k. Similarly, the segments $\overline{P'Q}$ and $\overline{Q'B}$ give $(l_2^2 - l_2)/2 + (l_3^2 - l_3)/2$ more points of S'_k which are covered by translates $S'_k + \mathbf{v}$ with $\mathbf{v} \succ \mathbf{0}$.

As in the proof of Lemma 3, let T be the set of points in S'_k that are not in $S'_k + \mathbf{v}$ for any $\mathbf{v} \succ \mathbf{0}$; then $|T| = |Z^2 : L|$. The calculations above show that

$$|T| \leq |S'_k| + (l_1 + l_2 + l_3)/2 - (l_1^2 + l_2^2 + l_3^2)/2.$$

Here $|S'_k| = (k+1)(k+2)/2$, and $l_1 + l_2 + l_3$ is just $k+2$. Given $l_1 + l_2 + l_3$, we minimize $l_1^2 + l_2^2 + l_3^2$ by making the numbers l_1, l_2, l_3 as close to equal as possible; in this case, this means that the minimum occurs when two of them are a and the third is b. Therefore,

$$|T| \leq \frac{(k+1)(k+2)}{2} + \frac{k+2}{2} - \frac{2a^2 + b^2}{2},$$

which simplifies to $|T| \leq \lfloor (k+2)^2/3 \rfloor$, as desired. ∎

Corollary 14 (mainly Wong and Coppersmith [22]). *The largest possible size for the directed Cayley graph of an Abelian group on two generators having diameter k is $\lfloor (k+2)^2/3 \rfloor$.* ∎

Again it is hard to be historically accurate here, because different authors have presented results in quite different ways; see the Bermond-Comellas-Hsu survey [3] for more information and references.

Let a, b, and T_k be as in the proof of Theorem 13; then the set T_k gives a suitable layout for a network realizing this Cayley graph. In addition to the mesh connections from (i, j) to $(i+1, j)$ and $(i, j+1)$ within T_k, one will also need wraparound connections from $(a+b-1, j)$ to $(0, j+b)$ for $j < a$, from $(a-1, j+a)$ to $(0, j)$ for $j < b$, from $(i, a+b-1)$ to $(i+b, 0)$ for $i < a$, and from $(i+a, a-1)$ to $(i, 0)$ for $i < b$.

In the case $a = b$, one can use an alternate layout in the form of a $3a \times a$ rectangle, with wraparound connections from $(3a, j)$ to $(1, j)$ and from (i, a) to $((i+a) \bmod a, 1)$. This is just a variant of the twisted toroidal mesh where the long dimension is twisted by a factor of $1/3$ rather than $1/2$; it is convenient for construction and for embedding a rectangular grid without boundary connections into the network (although this is not particularly useful in the directed case). If $a \neq b$, then one gets a rectangle with some missing nodes or extra nodes along part of one edge, and the cross-connections are slightly messier.

If $k \equiv 1 \pmod 3$, so that $a = b = (k+2)/3$, then one can see from the proof of Theorem 13 that the lattice L with generating vectors (a, a) and $(a+b, -b)$ is the unique lattice attaining the bound in the theorem, and hence the Cayley graph attaining the bound in Corollary 14 is also unique. However, if $k \not\equiv 1 \pmod 3$, so that a and b differ by 1, then there are two more lattices attaining the bound: the lattice $\tilde{L}$ with generating vectors (a, b) and $(2a, -a)$, and the mirror image with generating vectors (b, a) and $(-a, 2a)$. The latter two give Cayley graphs that are isomorphic to each other, but not to the Cayley graph of $\mathbf{Z}^2/L$ (if $k > 1$), because the Cayley graph of $\mathbf{Z}^2/L$ has cycles of length $2a$ while that of $\mathbf{Z}^2/\tilde{L}$ does not. Therefore, if $k > 0$ and $k \not\equiv 1 \pmod 3$, then there are exactly two Cayley graphs meeting the bound of Corollary 14.

If $k > 1$ and $k \equiv 1 \pmod 3$, then the optimal group $\mathbf{Z}^2/L$ is not cyclic; it is isomorphic to $\mathbf{Z}_{3a} \times \mathbf{Z}_a$ by an isomorphism sending $\mathbf{e}_1$ and $\mathbf{e}_2$ to $(1,0)$ and $(3a-1,1)$. On the other hand, if $k \not\equiv 1 \pmod 3$, then $(2a+b, a-b)$ is in L and $a - b = \pm 1$, so $\mathbf{e}_2$ is a multiple of $\mathbf{e}_1$ in $\mathbf{Z}^2/L$ and hence $\mathbf{Z}^2/L$ is cyclic. Similarly, $\mathbf{Z}^2/\tilde{L}$ is cyclic, since $(3a, b - a) \in \tilde{L}$. One can get the corresponding Cayley graphs directly from the cyclic group $\mathbf{Z}_{\lfloor (k+2)^2/3 \rfloor}$ by using the generator pairs $\{1, (2a+b)/(b-a)\}$ and $\{1, 3a/(a-b)\}$, respectively.

Three generators, undirected. For $d = 3$, we must consider three-dimensional lattice tilings by the regular octahedron $\bar{S}_k$ and its discrete approximation S_k. These shapes do not tile space perfectly, and the best possible efficiency for a lattice covering of $\mathbf{R}^3$ by $\bar{S}_k$ appears to be still open (although there is a good guess, as we shall see). So we will apply our results in reverse, using computed results about the degree-diameter problem to obtain information about lattice tilings by octahedra.

The best three-dimensional toroidal mesh with diameter k is $\mathbf{Z}_{2b_0+1} \times \mathbf{Z}_{2b_1+1} \times \mathbf{Z}_{2b_2+1}$, where $b_i = \lfloor (k+i)/3 \rfloor$; this has about $(8/27)k^3$ vertices for large k. This corresponds to the covering of $\mathbf{R}^3$ by $\bar{S}_1$ using the cubic lattice $(2/3)\mathbf{Z}^3$; this covering has efficiency $2/9$.

It turns out that a good lattice to use for coverings with regular octahedra is the body-centered cubic lattice, defined most simply as the set L_{bcc} of points in $\mathbf{Z}^3$ whose coordinates are all odd or all even. If $\mathbf{x}$ is an arbitrary point of $\mathbf{R}^3$, then $\mathbf{x}$ lies in or on one of the unit cubes with vertices in $\mathbf{Z}^3$; two opposite corners of this cube will be in L_{bcc}, say $\mathbf{v}$ and $\mathbf{w}$. Then each coordinate of $\mathbf{x}$ lies between (inclusively) the corresponding coordinates of $\mathbf{v}$ and $\mathbf{w}$, so, letting δ be the l^1 metric on $\mathbf{R}^3$, we have $\delta(\mathbf{v}, \mathbf{x}) + \delta(\mathbf{x}, \mathbf{w}) = \delta(\mathbf{v}, \mathbf{w}) = 3$. Hence, either $\delta(\mathbf{v}, \mathbf{x}) \leq 3/2$ or $\delta(\mathbf{w}, \mathbf{x}) \leq 3/2$. This shows that $|\bar{S}_{3/2}| + L_{\text{bcc}} = \mathbf{R}^3$. Now, L_{bcc} has generators $(2,0,0)$, $(0,2,0)$, and $(1,1,1)$, giving a matrix with determinant 4, while the volume of $\bar{S}_{3/2}$ is $9/2$, so this lattice covering of $\mathbf{R}^3$ has efficiency $8/9$. A fundamental region for the lattice can be obtained by truncating each of the corners of the octahedron, giving an Archimedean solid whose faces are eight regular hexagons and six squares.

The same reasoning shows that, for $k \geq 1$, one can get a lattice covering of $\mathbf{Z}^3$ by S_k using the slightly distorted body-centered cubic lattice $L_{\text{bcc}}(a_1, a_2, a_3)$ with generating vectors $(2a_1, 0, 0)$, $(0, 2a_2, 0)$, and (a_1, a_2, a_3), where $a_i = \lfloor (2k+i)/3 \rfloor$. This gives a Cayley graph of size $4a_1 a_2 a_3$, or approximately $(32/27)k^3$ for large k. This is an improvement over the best toroidal mesh of diameter k; it is about 4 times as good for large k.

One can lay out the Cayley graph for $\mathbf{Z}^3/L_{\text{bcc}}(a_1, a_2, a_3)$ in the form of a $2a_1 \times 2a_2 \times a_3$ mesh. Opposite $2a_i \times a_3$ sides are connected to each other as in the usual toroidal mesh, but the toroidal connections between the top and bottom $2a_1 \times 2a_2$ sides are twisted in two directions: node (j_1, j_2, a_3) is connected to node $(j_1 \pm a_1, j_2 \pm a_2, 1)$, where the signs are chosen to give numbers between 1 and $2a_i$, inclusive. Routing algorithms and embeddings of rectangular grids work here just as they did in the two-dimensional version.

Two questions now arise. First, can one improve the efficiency by making small adjustments to the discrete lattice, as we did in the two-generator cases? Second, can one get better results by using a completely different lattice? The answers to these questions are not immediately clear, so we will approach the problem from another direction.

One can write computer programs to examine various groups, choose all (or at least

many) possible sets of a certain number of generators for the group, and compute the resulting diameters. M. Dinneen has performed many such computations, some using exhaustive search of generator sets and others using random sampling, on a number of different kinds of groups, resulting in new best-known graphs for the degree-diameter problem; see, for instance, Dinneen and Hafner [10]. Some of Dinneen's earlier unpublished computations were for Abelian (usually cyclic) groups of diameter up to 10 on various numbers of generators.

The authors have written a program to extend these calculations. The program does an exhaustive search of generating sets for each Abelian group, but avoids examining many generating sets which give Cayley graphs isomorphic to ones already examined; for instance, in the case of a cyclic group $\mathbf{Z}_n$, one may assume that the first generator is a divisor of n. Here 'exhaustive search' means that all Abelian groups of size up to $|S_k| = (4k^3+6k^2+8k+3)/3$ were examined, so the results definitely give the largest possible Cayley graph of an Abelian group with diameter k. The program uses bit manipulations adapted from (but simpler than) those of Dougherty and Janwa [11], which gave algorithms for diameter computations for Cayley graphs of Abelian groups of exponent 2.

It turns out that, for each k for which the calculation has been done so far (up to 14), the best Abelian Cayley graph has been obtained from a cyclic group. The results of the computation are shown in Table 1. This extends (and corrects an erroneous final entry in) a similar table given by Chen and Jia [5].

k	$|S_k|$	Toroidal	Twisted	n_c	Generators	$n_c/|S_k|$	$n_c/\operatorname{vol}(\bar{S}_{k+3/2})$
0	1	1		1		1	.222222
1	7	3	4	7	$1, 2, 3$	1	.336000
2	25	9	16	21	$1, 2, 8$	.840000	.367347
3	63	27	48	55	$1, 5, 21$	.873016	.452675
4	129	45	108	117	$1, 16, 22$	.906977	.527423
5	231	75	192	203	$1, 7, 57$	.878788	.554392
6	377	125	320	333	$1, 9, 73$	.883289	.592000
7	575	175	500	515	$1, 46, 56$	.895652	.628944
8	833	245	720	737	$1, 11, 133$	.884754	.644700
9	1159	343	1008	1027	$1, 13, 157$	.886109	.665371
10	1561	441	1372	1393	$1, 92, 106$	.892377	.686940
11	2047	567	1792	1815	$1, 15, 241$	.886663	.696960
12	2625	729	2304	2329	$1, 17, 273$	.887238	.709953
13	3303	891	2916	2943	$1, 154, 172$	.891008	.724015
14	4089	1089	3600	3629	$1, 19, 381$	.887503	.730892

TABLE 1. Best undirected Cayley graphs of cyclic groups, three generators.

The first column is the desired diameter k. The second column gives the largest size one could hope for of an undirected Cayley graph of an Abelian group on 3 generators. The next two columns give the sizes attained by the best possible ordinary toroidal mesh and the twisted toroidal mesh described above. Next comes n_c, the computed largest n such

that $\mathbf{Z}_n$ has three generators giving it an undirected diameter of k. Then comes a triple of generators of $\mathbf{Z}_{n_c}$ attaining this diameter (this is not always unique, but only one generator set is given here). The final two columns gives the efficiencies of the corresponding lattice coverings of $\mathbf{Z}^3$ by S_k and of $\mathbf{R}^3$ by $\bar{S}_{k+3/2}$ (see Proposition 4).

Some interesting observations can be made from Table 1. First, note that the twisted toroidal meshes do almost as well as the optimal cyclic groups. Also note that the numbers in the second-to-last column do seem to be getting close to $8/9$ for larger k; this provides evidence that the body-centered cubic lattice gives the best lattice covering of $\mathbf{R}^3$ by S_1.

One can confirm this more strongly by reconstructing the lattices L for which $\mathbf{Z}^3/L$ gives these optimal cyclic groups. For instance, look at $k = 10$, for which we have the cyclic group $\mathbf{Z}_{1393}$ with generating set $\{1, 92, 106\}$. There is a unique homomorphism from $\mathbf{Z}^3$ to $\mathbf{Z}_{1393}$ which sends $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $1, 92, 106$, and the desired lattice L is just the kernel of this homomorphism; this means that

$$L = \{(x_1, x_2, x_3) \in \mathbf{Z}^3: \quad x_1 + 92x_2 + 106x_3 \equiv 0 \pmod{1393}\}.$$

One can easily find three vectors in L, namely $(1393, 0, 0)$, $(92, -1, 0)$, and $(106, 0, -1)$; the matrix with these three vectors as rows has determinant $1393 = |Z^3 : L|$, so these vectors generate L. Now one can perform elementary operations to reduce these vectors to a smaller set of generators for L, such as $(7, 7, 7)$, $(8, -7, 6)$, and $(6, 8, -7)$. These vectors are quite close to the vectors $(7, 7, 7)$, $(7, -7, 7)$, and $(7, 7, -7)$, which generate a scaled-up body-centered cubic lattice (in fact, the latter lattice gives the twisted toroidal mesh of size 1372 mentioned in the table). Similarly, one finds that the other lattices corresponding to the generators in Table 1 are almost body-centered cubic.

There are definite patterns in Table 1; every third k gives groups and generators of the same form. These patterns can be generalized, giving the following result.

Theorem 15. *For all $k \geq 0$, there is an undirected Cayley graph on three generators of an Abelian (in fact, cyclic) group which has diameter k and size n, where*

$$n = \begin{cases} (32k^3 + 48k^2 + 54k + 27)/27 & \text{if } k \equiv 0 \pmod 3, \\ (32k^3 + 48k^2 + 78k + 31)/27 & \text{if } k \equiv 1 \pmod 3, \\ (32k^3 + 48k^2 + 54k + 11)/27 & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

Proof. We will show the existence of lattices $L_k \subseteq \mathbf{Z}^3$ such that $\mathbf{Z}^3/L_k$ is cyclic, $S_k + L_k = \mathbf{Z}^3$, and $|\mathbf{Z}^3 : L|$ is the n specified in the theorem.

Let $a = \lceil 2k/3 \rceil$. For each k, we define L_k by specifying three generating vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ for it, as follows:

$$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 = \begin{cases} (a{+}1, a, a), \ (a, -a, a{+}1), \ (a{+}1, a{-}1, -a{-}1) & \text{if } k \equiv 0 \pmod 3, \\ (a, a, a), \ (a{+}1, -a, a{-}1), \ (a{-}1, a{+}1, -a) & \text{if } k \equiv 1 \pmod 3, \\ (a, a, a{-}1), \ (a{-}1, -a, a), \ (a, a{-}1, -a) & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

A simple determinant computation shows that $|\mathbf{Z}^3 : L_k|$ is $(2a^2+a+1)(2a+1)$, $4a^3+3a$, or $(2a^2 - a + 1)(2a - 1)$ in the respective cases $k \equiv 0$, $k \equiv 1$, or $k \equiv 2 \pmod 3$. Since a is respectively $2k/3$, $(2k + 1)/3$, or $(2k + 2)/3$, the index $|\mathbf{Z}^3 : L_k|$ works out to be the desired value n.

For $k \equiv 0 \pmod 3$, the following vectors are in L_k:

$$\mathbf{v}_2 + \mathbf{v}_3 = (2a+1, -1, 0),$$
$$\mathbf{v}_1 + (2a-1)\mathbf{v}_2 + 2a\mathbf{v}_3 = (4a^2+2a+1, 0, -1).$$

Hence, we have $\mathbf{e}_2 = (2a+1)\mathbf{e}_1$ and $\mathbf{e}_3 = (4a^2+2a+1)\mathbf{e}_1$ in $\mathbf{Z}^3/L_k$, so $\mathbf{e}_1$ generates $\mathbf{Z}^3/L_k$. So $\mathbf{Z}^3/L_k$ is isomorphic to $\mathbf{Z}_n$, via an isomorphism taking $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $1, 2a+1, 4a^2+2a+1$. Similarly, for $k \equiv 1 \pmod 3$ we have

$$a\mathbf{v}_2 + (a-1)\mathbf{v}_3 = (2a^2-a+1, -1, 0),$$
$$(a+1)\mathbf{v}_2 + a\mathbf{v}_3 = (2a^2+a+1, 0, -1),$$

so $\mathbf{Z}^3/L_k$ is isomorphic to $\mathbf{Z}_n$ with generators $1, 2a^2-a+1, 2a^2+a+1$; and for $k \equiv 2 \pmod 3$ we have

$$\mathbf{v}_2 + \mathbf{v}_3 = (2a-1, -1, 0),$$
$$\mathbf{v}_1 + (2a-1)\mathbf{v}_2 + 2a\mathbf{v}_3 = (4a^2-2a+1, 0, -1),$$

so $\mathbf{Z}^3/L_k$ is isomorphic to $\mathbf{Z}_n$ with generators $1, 2a-1, 4a^2-2a+1$.

It remains to show that $S_k + L_k = \mathbf{Z}^3$. We will do only the case $k \equiv 1 \pmod 3$ here; the other two cases are handled by the same method, but with a few more subcases because of less symmetry.

For $k = 1$ one just has to show that $\mathbf{Z}_7$ with generators $1, 2, 4$ has diameter 1, and this is trivial to do directly; so we may assume $k > 1$ and hence $a > 1$.

Let $\mathbf{v}_4 = \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 = (-a, a-1, a+1)$. Then the vectors $\pm\mathbf{v}_i$ for $i = 1, 2, 3, 4$ give one member of L_k strictly within each of the eight octants of $\mathbf{Z}^3$, and all of the coordinates of these vectors have absolute value at most $a+1$.

We must show that each $\mathbf{x} \in \mathbf{Z}^3$ is in $S_k + L_k$. This is equivalent to showing that there is a member $\mathbf{w}$ of L_k such that $\mathbf{x} - \mathbf{w} \in S_k$, which in turn is equivalent to $\delta(\mathbf{x}, \mathbf{w}) \le k$, where δ is the l^1 (Manhattan) metric on $\mathbf{Z}^3$. Note that if $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are such that each coordinate of $\mathbf{y}$ is between (inclusively) the corresponding coordinates of $\mathbf{x}$ and $\mathbf{z}$, then $\delta(\mathbf{x}, \mathbf{y}) + \delta(\mathbf{y}, \mathbf{z}) = \delta(\mathbf{x}, \mathbf{z})$. From now on, we will state this situation more briefly as "$\mathbf{y}$ lies between $\mathbf{x}$ and $\mathbf{z}$."

Suppose we are given $\mathbf{x} \in \mathbf{Z}^3$. The idea is to repeatedly reduce $\mathbf{x}$ by adding members of L_k to it, until one reaches a vector which is within l^1-distance k of $\mathbf{0}$ or some other known member of L_k.

The first thing we will do is reduce $\mathbf{x}$ to a vector whose coordinates all have absolute value at most $a+1$. Suppose $\mathbf{x}$ does not already have this property. Let $\mathbf{v}$ be one of

the vectors $\pm\mathbf{v}_i$ $(i = 1, 2, 3, 4)$ such that the coordinates of $\mathbf{v}$ have the same signs as the corresponding coordinates of $\mathbf{x}$; if a coordinate of $\mathbf{x}$ is 0, then either sign is allowed for the corresponding coordinate of $\mathbf{v}$. Now look at $\mathbf{x}' = \mathbf{x} - \mathbf{v}$. If a coordinate of $\mathbf{x}$ has absolute value $\leq a+1$, then the corresponding coordinate of $\mathbf{x}'$ will also have absolute value $\leq a+1$, because of the sign matching and the fact that the coordinates of $\mathbf{v}$ have absolute value $\leq a+1$. If a coordinate of $\mathbf{x}$ has absolute value $> a+1$, then the corresponding coordinate of $\mathbf{x}'$ will be strictly smaller in absolute value. Therefore, repeating this procedure will lead after finitely many steps to a vector whose coordinates all have absolute value at most $a+1$.

If this new $\mathbf{x}$ lies between $\mathbf{0}$ and one of the vectors $\pm\mathbf{v}_i$, then we have $\delta(\mathbf{0}, \mathbf{x}) + \delta(\mathbf{x}, \pm\mathbf{v}_i) = \delta(\mathbf{0}, \pm\mathbf{v}_i)$. But all of the vectors $\pm\mathbf{v}_i$ satisfy $\delta(\mathbf{0}, \pm\mathbf{v}_i) = 2k+1$; since $\delta(\mathbf{0}, \mathbf{x})$ and $\delta(\mathbf{x}, \pm\mathbf{v}_i)$ are both integers, one of them must be at most k, so we are done with this $\mathbf{x}$.

We now break into cases depending on which octant the new $\mathbf{x}$ lies in. Since L_k is centrosymmetric, we only need to handle the octants containing $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$. Also, L_k is invariant under cyclic permutations of the three coordinates, since these leave $\mathbf{v}_1$ fixed and permute $\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$; hence, we may assume that the new $\mathbf{x}$ is in the octant of $\mathbf{v}_1$ or the octant of $\mathbf{v}_2$.

First suppose that $\mathbf{x}$ is now in the octant of $\mathbf{v}_1$ (all three coordinates nonnegative). If $\mathbf{x}$ is between $\mathbf{0}$ and $\mathbf{v}_1$, we are done. If two or more of the coordinates of $\mathbf{x}$ are equal to $a+1$, say (by cyclic symmetry of L_k) $\mathbf{x} = (a+1, a+1, r)$, then we have $\delta(\mathbf{x}, \mathbf{v}_1) \leq k$ unless $k = 4$ and $r = 0$, in which case $\delta(\mathbf{x}, \mathbf{v}_1 + \mathbf{v}_3) = 4 = k$.

If $\mathbf{x}$ has exactly one coordinate equal to $a+1$, say $\mathbf{x} = (a+1, r, s)$ with $0 \leq r, s \leq a$, then we can subtract $\mathbf{v}_1$ from $\mathbf{x}$ to get $\mathbf{x}' = (1, r-a, s-a)$, which is in the octant containing $-\mathbf{v}_4$. If $\mathbf{x}'$ lies between $\mathbf{0}$ and $-\mathbf{v}_4$, we are done. If not, then $r = 0$. Now let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_4 = (-a+1, -1, s+1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $s = a$, in which case $\mathbf{x} = (a+1, 0, a)$ and $\delta(\mathbf{x}, \mathbf{v}_2) = a+1 \leq k$.

The procedure when $\mathbf{x}$ is in the octant of $\mathbf{v}_2$ is similar. Either $\mathbf{x} = (r, s, t)$ lies between $\mathbf{0}$ and $\mathbf{v}_2$, or $s = -a-1$, or $t \geq a$. In the latter cases, let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_2$. When $s = -a-1$, we have $\mathbf{x}' = (r-a-1, -1, t-a+1)$; either this lies between $\mathbf{0}$ and one of the vectors $\pm\mathbf{v}_i$, or $\mathbf{x} + \mathbf{v}_3$ does. When $s \geq -a$ but $t \geq a$, try $\mathbf{x}' - \mathbf{v}_4$; either it lies between $\mathbf{0}$ and some $\pm\mathbf{v}_i$, or it is $(-1, 1, -a)$, $(a, 1, -a)$, or $(a, 1, -a+1)$. These last three lie within δ-distance k of $\mathbf{v}_3 - \mathbf{v}_1$, $\mathbf{v}_3$, and either $\mathbf{v}_3 + \mathbf{v}_2 - \mathbf{v}_1$ or $\mathbf{v}_3 + \mathbf{v}_2$, respectively. ∎

The authors conjecture that the graphs given by this theorem are actually the largest undirected Cayley graphs of Abelian groups on three generators for each diameter k.

This conjecture would imply that the lattice covering of $\mathbf{R}^3$ by $\bar{S}_{3/2}$ using the lattice L_{bcc} is optimal; that is, $8/9$ is the best possible efficiency for a lattice covering by regular octahedra. The latter statement seems quite plausible, but remains unproved at this point. We can prove a partial result, though, that a "small" adjustment to L_{bcc} cannot improve the covering:

Theorem 16. *Among those lattices L for which $\bar{S}_{3/2} + L = \mathbf{R}^3$, the lattice L_{bcc} is locally optimal; that is, for any other lattice L sufficiently near L_{bcc} such that $\bar{S}_{3/2} + L = \mathbf{R}^3$, the efficiency of the covering using L is less than $8/9$.*

Proof. Let us use the vectors $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, -1, 1)$, and $\mathbf{v}_3 = (1, 1, -1)$ as generating vectors for L_{bcc}; then a nearby lattice L will be generated by nearby vectors $\mathbf{v}'_1 = (a_1, b_1, c_1)$, $\mathbf{v}'_2 = (a_2, b_2, c_2)$, and $\mathbf{v}'_3 = (a_3, b_3, c_3)$. We can concatenate the three vectors $\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3$ to get a single vector $\mathbf{v}'$ in $\mathbf{R}^9$; similarly, let $\mathbf{v}$ be the concatenation $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Let $F(\mathbf{v}')$ be the determinant of the matrix with rows $\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3$. Note that $F(\mathbf{v}) = 4$; we must see that this point is a strict local maximum of $F(\mathbf{v}')$ for those points $\mathbf{v}'$ satisfying the constraint that $\bar{S}_{3/2} + L = \mathbf{R}^3$. We compute that the gradient of F at the point $\mathbf{v}$ is $\mathbf{g} = (0, 2, 2, 2, -2, 0, 2, 0, -2)$.

Using the lattice L_{bcc}, the point $(1/2, 1/2, 1/2)$, in the center of a face of $\bar{S}_{3/2}$, is covered by only two copies of $\bar{S}_{3/2}$, namely $\bar{S}_{3/2}$ itself and $\bar{S}_{3/2} + \mathbf{v}_1$, and it is on the boundary (a face) of each of these copies. If the lattice is altered slightly so that these two copies no longer touch, then the points in between will not be covered by any copy. In particular, if L is near L_{bcc} but $a_1 + b_1 + c_1 > 3$, then the point $(1/2, 1/2, 1/2 + \varepsilon)$ for small positive ε will not be in $\bar{S}_{3/2} + L$. So the constraint $\bar{S}_{3/2} + L = \mathbf{R}^3$ gives us the linear inequality $a_1 + b_1 + c_1 \le 3$. We will rewrite this as

$$\mathbf{u}_1 \cdot \mathbf{v}' \le 3, \qquad \text{where } \mathbf{u}_1 = (1, 1, 1, 0, 0, 0, 0, 0, 0).$$

The same argument for points on the other faces of the octahedron gives inequalities

$$\mathbf{u}_2 \cdot \mathbf{v}' \le 3, \qquad \text{where } \mathbf{u}_2 = (0, 0, 0, 1, -1, 1, 0, 0, 0),$$
$$\mathbf{u}_3 \cdot \mathbf{v}' \le 3, \qquad \text{where } \mathbf{u}_3 = (0, 0, 0, 0, 0, 0, 1, 1, -1),$$
$$\mathbf{u}_4 \cdot \mathbf{v}' \le 3, \qquad \text{where } \mathbf{u}_4 = (-1, 1, 1, 1, -1, -1, 1, -1, -1).$$

Next, consider the point $(1, 0, 1/2)$. This is in $\bar{S}_{3/2} + \mathbf{y}$ for four members $\mathbf{y}$ of L_{bcc}, namely $\mathbf{0}$, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_2 + \mathbf{v}_3$, and it is an edge point of each of these four copies. If the lattice is altered slightly, then a gap can open up near this point even if there are no gaps between octahedra adjacent at a face as above.

Specifically, if $\mathbf{v}'$ is near $\mathbf{v}$, ε is a very small positive number, and we define the point $\mathbf{x}$ by the linear equations

$$\mathbf{x} \cdot (1, -1, -1) = \mathbf{v}'_1 \cdot (1, -1, -1) + 3/2 + \varepsilon,$$
$$\mathbf{x} \cdot (-1, -1, 1) = (\mathbf{v}'_2 + \mathbf{v}'_3) \cdot (-1, -1, 1) + 3/2 + \varepsilon, \quad \text{and}$$
$$\mathbf{x} \cdot (-1, 1, -1) = \mathbf{v}'_2 \cdot (-1, 1, -1) + 3/2 + \varepsilon,$$

then $\mathbf{x}$ will be a point near $(1, 0, 1/2)$ which is not in $\bar{S}_{3/2} + \mathbf{y}$ for $\mathbf{y} \in \{\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_2 + \mathbf{v}'_3\}$. Adding up the three given equations yields

$$\mathbf{x} \cdot (-1, -1, -1) = \mathbf{v}'_1 \cdot (1, -1, -1) + \mathbf{v}'_2 \cdot (-2, 0, 0) + \mathbf{v}'_3 \cdot (-1, -1, 1) + 9/2 + 3\varepsilon.$$

If the right hand side of this equation is less than $-3/2$, then $\mathbf{x}$ will not be in $\bar{S}_{3/2}$ either, and hence will not be in $\bar{S}_{3/2} + L$. Since ε can be arbitrarily small, in order to have $\bar{S}_{3/2} + L = \mathbf{R}^3$, it is necessary to have

$$\mathbf{v}'_1 \cdot (1, -1, -1) + \mathbf{v}'_2 \cdot (-2, 0, 0) + \mathbf{v}'_3 \cdot (-1, -1, 1) \ge -6.$$

This can be rewritten as

$$\mathbf{u}_5 \cdot \mathbf{v}' \le 6, \qquad \text{where } \mathbf{u}_5 = (-1, 1, 1, 2, 0, 0, 1, 1, -1).$$

The same argument can be performed using the octahedra around $(1, 0, 1/2)$ in the opposite order, and there are 23 other points on the edges of $\bar{S}_{3/2}$ where the same configuration occurs. But one only gets six distinct inequalities from this; the other five are:

$$\mathbf{u}_6 \cdot \mathbf{v}' \le 6, \qquad \text{where } \mathbf{u}_6 = (0, 2, 0, 1, -1, 1, 1, -1, -1),$$
$$\mathbf{u}_7 \cdot \mathbf{v}' \le 6, \qquad \text{where } \mathbf{u}_7 = (-1, 1, 1, 1, -1, 1, 2, 0, 0),$$
$$\mathbf{u}_8 \cdot \mathbf{v}' \le 6, \qquad \text{where } \mathbf{u}_8 = (1, 1, 1, 0, -2, 0, 1, -1, -1),$$
$$\mathbf{u}_9 \cdot \mathbf{v}' \le 6, \qquad \text{where } \mathbf{u}_9 = (0, 0, 2, 1, -1, -1, 1, 1, -1),$$
$$\mathbf{u}_{10} \cdot \mathbf{v}' \le 6, \qquad \text{where } \mathbf{u}_{10} = (1, 1, 1, 1, -1, -1, 0, 0, -2).$$

Note that all ten of these inequalities are satisfied with equality when $\mathbf{v}' = \mathbf{v}$. Hence, they can be rewritten as $\mathbf{u}_i \cdot (\mathbf{v}' - \mathbf{v}) \le 0$ for $i = 1, 2, \ldots, 10$.

One can easily check that the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_7$ are linearly independent; their common null space (i.e., the set of $\mathbf{w}$ such that $\mathbf{u}_i \cdot \mathbf{w} = 0$ for all $i \le 7$) is generated by the independent vectors $\mathbf{w}_1 = (1, 0, -1, 1, 0, -1, 1, 0, 1)$ and $\mathbf{w}_2 = (-1, 1, 0, -1, -1, 0, -1, 1, 0)$. Also, we have

$$\mathbf{g} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_4 = \mathbf{u}_5 + \mathbf{u}_8 = \mathbf{u}_6 + \mathbf{u}_9 = \mathbf{u}_7 + \mathbf{u}_{10}.$$

Let C be the closed cone consisting of all vectors $\mathbf{t}$ in the subspace spanned by $\mathbf{u}_1, \ldots, \mathbf{u}_7$ such that $\mathbf{u}_i \cdot \mathbf{t} \le 0$ for all $i \le 10$. Then the above equations imply that $\mathbf{g} \cdot \mathbf{t} \le 0$ for all $\mathbf{t}$ in C, and equality can hold only when $\mathbf{t} = \mathbf{0}$. In particular, we have $\mathbf{g} \cdot \mathbf{t}_0 < 0$ for any unit vector $\mathbf{t}_0$ in C. The set of such $\mathbf{t}_0$ is closed and bounded, hence compact, so there is a positive number ε such that $\mathbf{g} \cdot \mathbf{t}_0 < -\varepsilon$ for all such $\mathbf{t}_0$. It follows that there is a neighborhood U of $\mathbf{g}$ such that, for any $\mathbf{g}'$ in U and any unit vector $\mathbf{t}_0$ in C, $\mathbf{g}' \cdot \mathbf{t}_0 < 0$. Since C is a cone, we have $\mathbf{g}' \cdot \mathbf{t} < 0$ for all $\mathbf{g}' \in U$ and all nonzero $\mathbf{t} \in C$.

We can compute that, for any real numbers r and s, the deteriminant for the lattice given by $\mathbf{v} + r\mathbf{w}_1 + s\mathbf{w}_2$ is

$$F(\mathbf{v} + r\mathbf{w}_1 + s\mathbf{w}_2) = 4(1 - r)(1 + s)(1 + r - s).$$

If $|r| + |s| < 1$, then the numbers $1 - r$, $1 + s$, and $1 + r - s$ are positive numbers with arithmetic mean 1, so their geometric mean is at most 1; this means that $F(\mathbf{v} + r\mathbf{w}_1 + s\mathbf{w}_2) \le 4$. Equality holds only when the above three numbers are equal, which is when $r = s = 0$.

Let U' be a convex neighborhood of $\mathbf{v}$ so small that $(\operatorname{grad} F)(\mathbf{v}') \in U$ for all $\mathbf{v}' \in U'$. Now, any vector $\mathbf{v}'$ sufficiently close to $\mathbf{v}$ can be expressed as $\mathbf{v} + \mathbf{t}_1 + \mathbf{t}_2$ where $\mathbf{t}_1$ is a (small) linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$, $\mathbf{t}_2$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_7$, and both $\mathbf{v} + \mathbf{t}_1$ and $\mathbf{v} + \mathbf{t}_1 + \mathbf{t}_2$ are in U'. If $\mathbf{v}'$ satisfies the condition $\bar{S}_{3/2} + L = \mathbf{R}^3$ and is near

to $\mathbf{v}$, then we must have $\mathbf{u}_i \cdot (\mathbf{v}' - \mathbf{v}) \leq 0$ for all $i \leq 10$, so $\mathbf{u}_i \cdot \mathbf{t}_2 \leq 0$ for all $i \leq 10$ (since $\mathbf{u}_i \cdot \mathbf{t}_1 = 0$), so $\mathbf{t}_2 \in C$. We have $F(\mathbf{v} + \mathbf{t}_1) \leq 4$, with equality holding only when $\mathbf{t}_1 = \mathbf{0}$. If $\mathbf{t}_2$ is nonzero, then for any $\mathbf{t}$ on the segment from $\mathbf{v} + \mathbf{t}_1$ to $\mathbf{v} + \mathbf{t}_1 + \mathbf{t}_2$ we have $\mathbf{t} \in U'$, so $(\operatorname{grad} F)(\mathbf{t}) \in U$, so $(\operatorname{grad} F)(\mathbf{t}) \cdot \mathbf{t}_2 < 0$; it follows that $F(\mathbf{v} + \mathbf{t}_1 + \mathbf{t}_2) < F(\mathbf{v} + \mathbf{t}_1)$. Therefore, $F(\mathbf{v}') \leq F(\mathbf{v})$, with equality holding only when $\mathbf{v}' = \mathbf{v}$. So $\mathbf{v}$ gives a local maximum of F, as desired.   ∎

It is still possible (though very unlikely) that a lattice quite different from L_{bcc} gives a more efficient covering. Theoretically, the search for an optimal lattice can be set up as a large optimization problem and solved once and for all, but this appears to be a formidable task.

One could begin this task by considering an arbitrary lattice L such that $\bar{S}_{3/2} + L = \mathbf{R}^3$ and this covering is reasonably efficient (at least as efficient as the covering from L_{bcc}). Such a lattice is generated by vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and we can carefully choose these generators so as to limit their lengths. In particular, we can choose $\mathbf{v}_1$ to be a nonzero member of L with minimal length. We can then choose $\mathbf{v}_2$ in L whose distance from the subspace of $\mathbf{R}^3$ spanned by $\mathbf{v}_1$ is as small as possible (but nonzero), and adjust $\mathbf{v}_2$ by subtracting off an integer multiple of $\mathbf{v}_1$ so as to ensure that the closest integer multiple of $\mathbf{v}_1$ to $\mathbf{v}_2$ is $\mathbf{0}$. One can similarly choose $\mathbf{v}_3$ to be as close as possible to (but not in) the subspace spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. These three chosen vectors will be a set of generating vectors for L. In order to have $\bar{S}_{3/2} + L = \mathbf{R}^3$, it is necessary that the length of $\mathbf{v}_1$ be no more than the diameter of $\bar{S}_{3/2}$; there are similar but slightly larger bounds on the lengths of $\mathbf{v}_2$ and $\mathbf{v}_3$. This limits our search for $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to a compact subset of nine-dimensional space. We must find the point in this subset which maximizes $\det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ subject to the constraint that $\bar{S}_{3/2} + L = \mathbf{R}^3$.

This constraint looks infinitary, but it can actually be reduced to finitely many sets of linear inequalities. To see this, note that, using the above upper bounds on the lengths of the vectors $\mathbf{v}_i$ along with the assumed lower bound on the lattice determinant (the covering must be at least as efficient as that from L_{bcc}), we can get lower bounds on the lengths of the $\mathbf{v}_i$, the angles between them, and associated quantities such as the distance from $\mathbf{v}_3$ to the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. These will allow us to get upper bounds on the absolute values of integers a_1, a_2, a_3 such that $\bar{S}_{3/2} + a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$ overlaps or almost touches $\bar{S}_{3/2}$. (In other words, we get an upper bound on the number M from the proof of Proposition 7.) So we only have to consider finitely many of the lattice translates of $\bar{S}_{3/2}$ when trying to cover the space near $\bar{S}_{3/2}$ (which is all that is needed, by Proposition 6).

There are only finitely many configurations (specifications of arrangements and overlaps) for these finitely many translates of $\bar{S}_{3/2}$. For each such configuration, the assertion that there are no 'gaps' in the coverage of the space near $\bar{S}_{3/2}$ becomes a list of linear inequalities like the inequalities $\mathbf{u}_i \cdot \mathbf{v}' \leq b$ from the proof of Theorem 16. So we need to optimize a cubic function (the lattice determinant) subject to a list of linear inequalities in order to find the optimal version of each configuration, and then compare the resulting values to find the best configuration.

Unfortunately, there is a very large number of possible configurations (for an example

of the possibilities for complicated configurations, see Figure 4 later in this paper), so this finite computation appears to be beyond reach at present. Of course, a different approach to the problem might lead to a more feasible computation.

One might hope to be able to use the arguments of Proposition 7 and Theorem 8 in reverse, to get an upper bound on the efficiency of lattice coverings of $\mathbf{R}^3$ by the octahedron $\bar{S}_1$ by showing that any extremely efficient real lattice covering would lead to integer lattice coverings more efficient than what the computation actually found. To do this, one would fix a value for the distance ρ from Proposition 7, and then use the method described above to get an upper bound on the number M from that Proposition. If there is actually a lattice covering of $\mathbf{R}^3$ by $\bar{S}_1$ using the lattice L generated by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ having a specified large determinant (equivalently, a specified large efficiency), then we can round the coordinates of these vectors to the nearest multiples of $1/k$ to get vectors $\mathbf{v}_1', \mathbf{v}_2', \mathbf{v}_3'$ generating a lattice L'. By Proposition 7, if $1/(2k)$ is less than $\eta = \rho/M$, then we will have $\bar{S}_{1+3\rho} + L' = \mathbf{R}^3$, so $\bar{S}_{k+3k\rho} + kL' = \mathbf{R}^3$. But kL' is an integer lattice; if n is its determinant, then this lattice covering will yield an Abelian Cayley graph on three generators with size n and diameter at most $k + 3k\rho$. The fact that L' is close to L means that we can get a lower bound on n from the determinant of L. If the actual computational search showed that there is no Abelian Cayley graph of such a size for this diameter, then our original assumption that there was a lattice L giving a covering of that efficiency must have been false.

Unfortunately, the constants involved are such that even the large computation done so far does not suffice to get a bound less than 1 for the efficiency of L (even if we are optimistic enough to assume that M is as small as 3 or 4). It probably requires searches for values of k larger than 500 in order to get actual results from this method; such searches are completely out of range at the moment.

Three generators, directed. For the directed case of three generators, we want to study lattice coverings of $\mathbf{R}^3$ by the trirectangular tetrahedron $\bar{S}_k'$. (Since lattice covering efficiency is affine invariant, it makes no difference which particular tetrahedron we consider.) One hopes that one can discretize these coverings to get good lattice coverings of $\mathbf{Z}^3$ by S_k', and hence good directed Cayley graphs.

The best three-dimensional directed toroidal mesh with diameter k is $\mathbf{Z}_{b_0+1} \times \mathbf{Z}_{b_1+1} \times \mathbf{Z}_{b_2+1}$, where $b_i = \lfloor (k+i)/3 \rfloor$; this has about $(1/27)k^3$ vertices for large k. This corresponds to the covering of $\mathbf{R}^3$ by $\bar{S}_1'$ using the cubic lattice $(1/3)\mathbf{Z}^3$; this covering has efficiency $2/9$.

It is more difficult to find a candidate for a good covering lattice (or, equivalently, a large subset which gives a lattice tiling) for the tetrahedron than it was for the octahedron. One possible method is to try to find the three-dimensional analogue of the L-tromino used for the triangle; this leads one to consider the tetracube shown on the left of Figure 3. In order for the shape to fit into $\bar{S}_1'$, the edge-length of the subcubes should be $1/4$. It is easy to see that this shape does indeed tile space, using the lattice generated by $(1/2, 0, 0)$, $(0, 1/2, 0)$, and $(1/4, 1/4, -1/4)$ (this is just $(1/4)L_{\text{bcc}}$); since the shape has volume $1/16$ while $\bar{S}_1'$ has volume $1/6$, we get a lattice covering of $\mathbf{R}^3$ by $\bar{S}_1'$ with efficiency $3/8$.

The discrete form of this shape, scaled by a factor s_i in the i'th dimension, is a subset
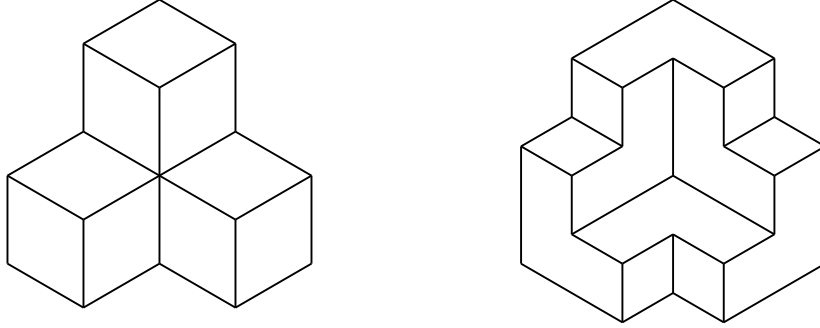
FIGURE 3. Two subsets of the tetrahedron $\bar{S}'_1$ which tile space.

of $\mathbf{Z}^3$ of size $4s_1s_2s_3$ which gives a lattice tiling of $\mathbf{Z}^3$; this subset is included in S'_k, where $k = s_1 + s_2 + s_3 + \max(s_1, s_2, s_3) - 3$. Optimizing this for a given $k \geq 1$ gives a subset of S'_k which tiles and has size $4a_3a_4a_5$, where $a_i = \lfloor (k+i)/4 \rfloor$.

One can obtain another lattice covering of $\mathbf{R}^3$ by tetrahedra as follows. If one cuts off the four corners of a regular tetrahedron at planes passing through the midpoints of the edges (so one removes four half-size regular tetrahedra), then what is left is a regular octahedron with volume $1/2$ that of the tetrahedron. We have a lattice giving a covering of $\mathbf{R}^3$ by this octahedron with efficiency $8/9$; the same lattice therefore gives a covering of $\mathbf{R}^3$ by the original tetrahedron with efficiency $4/9$.

If one uses an affine transformation to change the regular tetrahedron to the tetrahedron $\bar{S}'_1$, then the corresponding lattice will be generated by $(1/6, 1/6, 1/6)$, $(1/6, -1/2, 1/6)$, and $(1/6, 1/6, -1/2)$. One fundamental region for this lattice is an affinely distorted truncated octahedron. Another one can be obtained by the method of Lemma 3, using an ordering $\prec$ which orders vectors primarily by the sum of their coordinates; the resulting region is shown at the right of Figure 3. This shape consists of 16 cubes of edge-length $1/6$, for a total volume of $2/27$, which, as expected, is $4/9$ of $\mathrm{vol}(\bar{S}'_1) = 1/6$.

Discretizing this new shape with scale factors $s_1 \leq s_2 \leq s_3$ gives a subset of $\mathbf{Z}^3$ of size $16s_1s_2s_3$ which gives a lattice tiling of $\mathbf{Z}^3$; this subset is included in S'_k, where $k = s_1 + 2s_2 + 3s_3 - 3$. Another simple optimization shows that, for any given $k \geq 3$, we get a subset of S'_k which tiles and has size $16\hat{a}_3\hat{a}_4\hat{a}_6$, where $\hat{a}_i = \lfloor (k+i)/6 \rfloor$. For large k (in fact, for all $k \geq 30$), this new lattice gives a better covering of $\mathbf{Z}^3$ by S'_k than the preceding one did, but for smaller k the preceding one sometimes does better.

Aguiló, Fiol, and Garcia [1] also work with this shape, but discretize it in a rotationally symmetric way rather than in each dimension separately; the Cayley graphs they obtain are slightly larger than the graphs of size $16\hat{a}_3\hat{a}_4\hat{a}_6$ given above, but still of the form $(2/27)k^3 + O(k^2)$.

In order to see whether these lattice coverings give close-to-optimal Cayley graphs, the authors performed a computer search for the best (smallest-diameter) directed Abelian Cayley graphs on three generators. This extends similar computations performed by Aguiló, Fiol, and Garcia [1] and by Fiduccia, Forcade, and Zito [14]. The latter paper also contains a useful upper bound: an Abelian Cayley digraph on three generators with

diameter k has size at most $3(k+3)^3/25$. This improves the obvious upper bound $|S_k|$ when $k > 7$.

Comparing the above figures with the output from the authors' computations gives a slight surprise: the best cyclic groups do substantially better than the groups from the above coverings. The data are shown in Table 2; here 'FFZ' is the Fiduccia-Forcade-Zito upper bound, 'Impr.' refers to the larger of the sizes obtained from the two improved constructions above, 'AFG' is the size attained by the Aguiló-Fiol-Garcia construction, and the remaining columns are analogous to those of Table 1. The computations were run on Abelian groups of sizes up to and including 4871; this means that the entries marked with an asterisk in the n_c' column (for which the FFZ bound is greater than 4871) have not been completely proven optimal, but it is extremely likely that they are.

Note that in three cases, $k = 7, 31, 33$, the best cyclic Cayley graph was not achieved using 1 as one of the generators. If one is required to use 1 as a generator (which may be useful when actually building the corresponding loop network), then the best one can do is size 78 for $k = 7$ (with generators $1, 6, 49$), size 3178 for $k = 31$ (with generators $1, 386, 1295$), and size 3794 for $k = 33$ (with generators $1, 469, 2094$).

There is one other difference between this case and the undirected case: here there are values of k for which one can do better with general Abelian groups than with cyclic groups. The improved values obtained from non-cyclic groups are shown in Table 3. A number of these optimal graphs are actually obtained by applying Proposition 5(b) to smaller Cayley graphs; for instance, the Abelian graph for $k = 17$ is obtained this way from the cyclic graph for $k = 7$, which is the reason that these two graphs give exactly the same real-covering efficiency (.504).

The values in the n_c' column of Table 2 are so much larger than those in the preceding two columns that it is clear that the real lattices used for the preceding columns were not optimal. This is made explicit in the last column of the table, which gives the efficiency of the real lattice covering obtained from the computed integer lattice covering via Proposition 4(b). For $k = 1$ and $k = 3$ these coverings are just (scaled versions of) the two coverings we explicitly constructed above; but later coverings obviously do substantially better.

The best real covering obtained from these computations is that for $k = 7$, with efficiency .504. As in the undirected case, we can reconstruct generators for the lattice from the given generating set $2, 9, 35$ for $\mathbf{Z}_{84}$; after simplification, the resulting generating vectors are $(-2, 2, 2)$, $(3, -3, 3)$, and $(4, 3, -1)$. We now have a computer-assisted proof that the lattice generated by these vectors gives a lattice covering of $\mathbf{R}^3$ by $\bar{S}_{10}'$; but one can obtain useful extra information (as well as, perhaps, more satisfaction) by proving this directly.

Proposition 17. *Let L_7' be the lattice in $\mathbf{R}^3$ generated by the vectors $(-2, 2, 2)$, $(3, -3, 3)$, and $(4, 3, -1)$; then $\bar{S}_{10}' + L_7' = \mathbf{R}^3$.*

k	$\lvert S'_k \rvert$	FFZ	Toroidal	Impr.	AFG	n'_c	Generators	$n'_c/\lvert S'_k \rvert$	$n'_c/\operatorname{vol}(\bar{S}'_{k+3})$
0	1		1		1	1		1	.222222
1	4		2	4	4	4	$1, 2, 3$	1	.375000
2	10		4	4	7	9	$1, 3, 4$	.900000	.432000
3	20		8	16	16	16	$1, 4, 5$	.800000	.444444
4	35		12	16	19	27	$1, 4, 17$	.771428	.472303
5	56		18	32	31	40	$1, 6, 15$	.714286	.468750
6	84		27	32	50	57	$1, 13, 33$	.678571	.469136
7	120	120	36	48	56	84	$2, 9, 35$	.700000	.504000
8	165	159	48	72	86	111	$1, 31, 69$	.672727	.500376
9	220	207	64	128	128	138	$1, 11, 78$	.627273	.479167
10	286	263	80	128	134	176	$1, 17, 56$	.615385	.480655
11	364	329	100	144	182	217	$1, 13, 119$	.596154	.474490
12	455	405	125	192	243	273	$1, 14, 153$	.600000	.485333
13	560	491	150	256	252	340	$1, 90, 191$	.607143	.498047
14	680	589	180	288	333	395	$1, 35, 271$	.580882	.482394
15	816	699	216	432	432	462	$1, 29, 97$	.566176	.475309
16	969	823	252	432	441	560	$1, 215, 326$	.577915	.489867
17	1140	960	294	500	549	648	$1, 76, 237$	.568421	.486000
18	1330	1111	343	576	676	748	$1, 41, 147$	.562406	.484613
19	1540	1277	392	600	688	861	$1, 27, 463$	.559091	.485162
20	1771	1460	448	768	844	979	$1, 22, 351$	.552795	.482781
21	2024	1658	512	1024	1024	1140	$1, 45, 196$	.563241	.494792
22	2300	1875	576	1024	1036	1305	$1, 246, 1030$	.567391	.501120
23	2600	2109	648	1024	1228	1440	$1, 126, 415$	.553846	.491579
24	2925	2361	729	1280	1445	1616	$1, 56, 257$	.552479	.492608
25	3276	2634	810	1372	1460	1788	$1, 154, 1452$	.545788	.488703
26	3654	2926	900	1600	1715	1963	$1, 90, 780$	.537219	.482923
27	4060	3240	1000	2000	2000	2224	$1, 425, 704$	.547783	.494222
28	4495	3574	1100	2000	2015	2442	$1, 964, 1372$	.543270	.491826
29	4960	3932	1210	2048	2315	2693	$1, 39, 942$	.542944	.493103
30	5456	4312	1331	2400	2646	2920	$1, 540, 831$	.535191	.487520
31	5984	4716	1452	2400	2664	3220	$7, 30, 2277$	.538102	.491553
32	6545	5145	1584	2880	3042	3591*	$1, 1519, 2031$	.548663	.502531
33	7140	5598	1728	3456	3456	3850*	$2, 475, 1177$	.539216	.495113
34	7770	6078	1872	3456	3474	4191*	$1, 748, 2652$	.539382	.496437
35	8436	6584	2028	3456	3906	4468*	$1, 353, 2789$	.529635	.488555

TABLE 2. Best directed Cayley graphs of cyclic groups, three generators.

k	n'_a	Group	Generators	$n'_a/|S'_k|$	$n'_a/\operatorname{vol}(\bar{S}'_{k+3})$
12	279	$\mathbf{Z}_{93} \times \mathbf{Z}_3$	$(1,0), (9,1), (10,2)$	.613187	.496000
17	672	$\mathbf{Z}_{168} \times \mathbf{Z}_2 \times \mathbf{Z}_2$	$(2,1,0), (9,0,0), (35,0,1)$	.589474	.504000
18	752	$\mathbf{Z}_{188} \times \mathbf{Z}_4$	$(1,0), (13,2), (14,1)$	.565414	.487204
19	888	$\mathbf{Z}_{222} \times \mathbf{Z}_2 \times \mathbf{Z}_2$	$(1,0,0), (142,1,0), (180,0,1)$	.576623	.500376
26	1980	$\mathbf{Z}_{330} \times \mathbf{Z}_6$	$(1,0), (123,2), (234,3)$	.541872	.487105
27	2268	$\mathbf{Z}_{252} \times \mathbf{Z}_3 \times \mathbf{Z}_3$	$(2,0,0), (9,1,0), (35,0,1)$	.558621	.504000
28	2448	$\mathbf{Z}_{816} \times \mathbf{Z}_3$	$(1,0), (427,0), (564,1)$	.544605	.493035
29	2720	$\mathbf{Z}_{680} \times \mathbf{Z}_2 \times \mathbf{Z}_2$	$(1,0,0), (191,1,0), (90,0,1)$	.548387	.498047
30	2997	$\mathbf{Z}_{333} \times \mathbf{Z}_3 \times \mathbf{Z}_3$	$(1,0,0), (31,1,0), (180,0,1)$	.549304	.500376
35	4500*	$\mathbf{Z}_{300} \times \mathbf{Z}_{15}$	$(1,0), (3,1), (214,7)$	.533428	.492054

TABLE 3. Best directed Cayley graphs of Abelian groups, three generators.

Proof. First, note that the following vectors are in L'_7:

$$
\begin{aligned}
&\mathbf{v}_1 = (-2,2,2) && \mathbf{v}_8 = (1,-1,5) && = \mathbf{v}_1 + \mathbf{v}_2 \\
&\mathbf{v}_2 = (3,-3,3) && \mathbf{v}_9 = (-1,1,7) && = 2\mathbf{v}_1 + \mathbf{v}_2 \\
&\mathbf{v}_3 = (4,3,-1) && \mathbf{v}_{10} = (3,4,-6) && = \mathbf{v}_3 - \mathbf{v}_1 - \mathbf{v}_2 \\
&\mathbf{v}_4 = (6,1,-3) = \mathbf{v}_3 - \mathbf{v}_1 && \mathbf{v}_{11} = (-7,7,1) && = 2\mathbf{v}_1 - \mathbf{v}_2 \\
&\mathbf{v}_5 = (5,-5,1) = \mathbf{v}_2 - \mathbf{v}_1 && \mathbf{v}_{12} = (-5,-2,8) = 2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 \\
&\mathbf{v}_6 = (1,6,-4) = \mathbf{v}_3 - \mathbf{v}_2 && \mathbf{v}_{13} = (-1,8,-2) = \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 \\
&\mathbf{v}_7 = (2,5,1) \quad = \mathbf{v}_1 + \mathbf{v}_3 && \mathbf{v}_{14} = (8,-1,-5) = \mathbf{v}_3 - 2\mathbf{v}_1
\end{aligned}
$$

For each vector $\mathbf{v}_i = (r,s,t)$, we have $1 \le r+s+t \le 8$; hence, the translated tetrahedron $\bar{S}'_{10} + \mathbf{v}_i$ intersects the plane $x+y+z = 10$. In fact, the intersection is a triangle whose vertices have coordinates $(10-s-t, s, t)$, $(r, 10-r-t, t)$, and $(r, s, 10-r-s)$.

Figure 4 shows the upper face of $\bar{S}'_{10}$. For each $i \le 14$, it indicates which part of this face is covered by the translate $\bar{S}'_{10} + \mathbf{v}_i$. (The face is divided up into unit triangles, each of which is labeled by the value(s) of i for which $\bar{S}'_{10} + \mathbf{v}_i$ covers that triangle.) Clearly each unit triangle is labeled, so the translates $\bar{S}'_{10} + \mathbf{v}_i$ cover the entire upper face of $\bar{S}'_{10}$.

In fact, since each $\mathbf{v}_i$ has coordinates summing to at least 1, the smaller translates $\bar{S}'_9 + \mathbf{v}_i$, $i \le 14$, cover the upper face of $\bar{S}'_{10}$. For any $\mathbf{x}$ in this upper face, there is an i such that $\mathbf{x} \in \bar{S}'_9 + \mathbf{v}_i$, so $\bar{S}'_1 + \mathbf{x} \subseteq \bar{S}'_1 + \bar{S}'_9 + \mathbf{v}_i = \bar{S}'_{10} + \mathbf{v}_i$. Since $\bar{S}'_{11}$ is the union of $\bar{S}'_{10}$ and the sets $\bar{S}'_1 + \mathbf{x}$ for $\mathbf{x}$ in the upper face of $\bar{S}'_{10}$, we have $\bar{S}'_{11} \subseteq \bar{S}'_{10} + L'_7$, and hence $\bar{S}'_{11} + L'_7 \subseteq \bar{S}'_{10} + L'_7$.

We now prove by induction that, for all integers $k \ge 10$, $\bar{S}'_k + L'_7 \subseteq \bar{S}'_{10} + L'_7$. The case $k = 10$ is trivial. If it is true for k, then

$$
\bar{S}'_{k+1} + L'_7 = \bar{S}'_1 + \bar{S}'_k + L'_7 \subseteq \bar{S}'_1 + \bar{S}'_{10} + L'_7 = \bar{S}'_{11} + L'_7 \subseteq \bar{S}'_{10} + L'_7,
$$

so it is true for $k+1$.
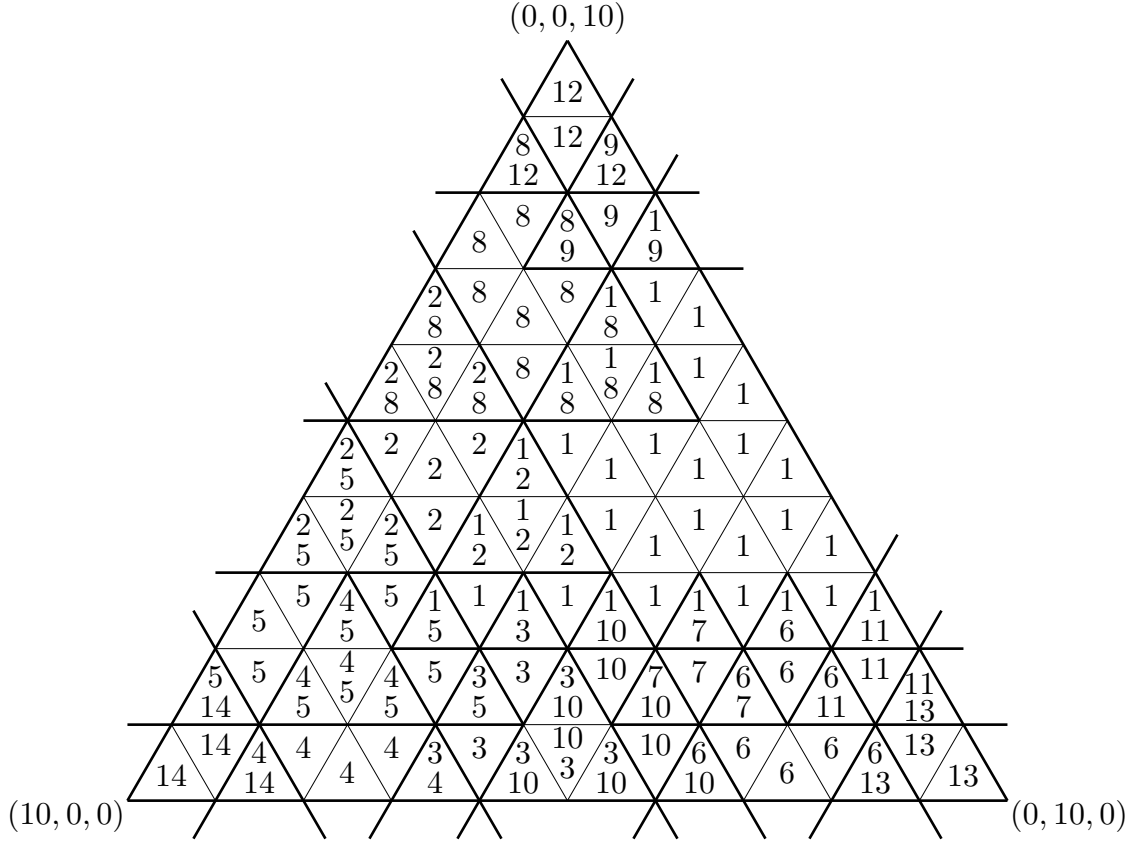
FIGURE 4. Coverage of one face of $\bar{S}'_{10}$ under L'_7.

Finally, for any $\mathbf{y} \in \mathbf{R}^3$, there is a member $\mathbf{w}$ of L'_7 such that the coordinates of $\mathbf{y} - \mathbf{w}$ are all positive (e.g., let $\mathbf{w}$ be a large multiple of $-\mathbf{v}_7$). Then $\mathbf{y} - \mathbf{w} \in \bar{S}'_k$ for some k, so $\mathbf{y} \in \bar{S}'_k + L'_7$ and hence $\mathbf{y} \in \bar{S}'_{10} + L'_7$. Therefore, $\bar{S}'_{10} + L'_7 = \mathbf{R}^3$. ∎

This covering has efficiency $\det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) / \mathrm{vol}(\bar{S}'_{10}) = .504$. Hence, Corollary 10(b) gives:

Corollary 18. *For all k, there is a directed Cayley graph of an Abelian group on three generators which has diameter k and size at least $0.084k^3 + O(k^2)$.* ∎

We can now use the method of (the real version of) Lemma 3 to get a fundamental region $T'_7 \subseteq \bar{S}'_{10}$ for the lattice L'_7. (Recall that any such region must have volume $\det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 84$.) To do this, just start with $\bar{S}'_{10}$, look at each of the vectors $\mathbf{v}_i$ ($i \leq 14$) defined above, and delete those points of $\bar{S}'_{10}$ which lie in $\bar{S}'_{10} + \mathbf{v}_i$. (We have $\mathbf{v}_i \succ \mathbf{0}$ for all i if $\prec$ orders vectors primarily by the sum of coefficients.) What is left is the set shown in Figure 5; since this is the union of 84 unit cubes, we know that there is no need to subtract further translates $\bar{S}'_{10} + \mathbf{w}$. This set was obtained independently by Fiduccia, Forcade, and Zito [14].
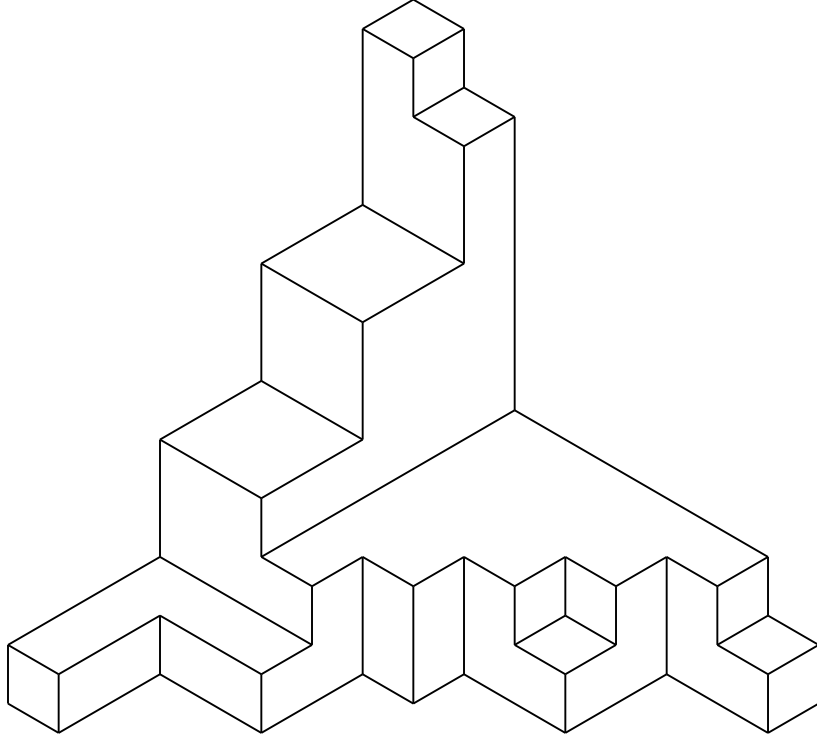
FIGURE 5. A subset of the tetrahedron $\bar{S}'_{10}$ which tiles space.

So this set T'_7 gives a lattice tiling of $\mathbf{R}^3$ using L'_7. This tiling is quite unusual; the translates of T'_7 fit together in a peculiar way, seeming to wind around each other. One interesting fact is that each translate of T'_7 is adjacent to (i.e., shares a boundary segment of positive area with) 28 other translates, a surprisingly high number. (T'_7 itself is adjacent to $T'_7 + \mathbf{v}_i$ and $T'_7 - \mathbf{v}_i$ for $i \leq 14$.)

In many of the tilings we constructed explicitly, there was a polycube fundamental region like T'_7, but there was also an alternate fundamental region which was convex; for instance, for the optimal covering of $\mathbf{R}^2$ by right triangles, one could use either an L-tromino or a hexagon as the fundamental region. Clearly L'_7 has convex fundamental regions (e.g., its Voronoi regions), but it turns out that they are unsuitable for the current problem:

Proposition 19. *There is no convex fundamental region for the lattice L'_7 included within the tetrahedron $\bar{S}'_{10}$.*

Proof. Since T'_7 gives a lattice tiling of $\mathbf{R}^3$ by L'_7, every point of $\mathbf{R}^3$, except for those lying on boundaries of the tiling, can be translated by a vector in L'_7 to a unique point of T'_7. In particular, if we look at the part of $\bar{S}'_{10}$ lying outside T'_7, then we can break it up into finitely many parts (in fact, just cut it along the integer translates of the three coordinate planes) which can be translated in a unique way by members of L'_7 so as to lie within T'_7.

If one does this, one finds that there are parts of T'_7 which do not get covered by translates of parts of $\bar{S}'_{10} \setminus T'_7$. Most of these uncovered parts look like inverted copies of $\bar{S}'_1$ (i.e., translates of $-\bar{S}'_1$), although there are some larger ones. In particular, the sets

$(1, 1, 1) - \bar{S}'_1$, $(8, 1, 1) - \bar{S}'_1$, $(1, 8, 1) - \bar{S}'_1$, and $(1, 1, 8) - \bar{S}'_1$ are not covered by such translates. This implies that each of those four sets is disjoint (except for boundaries) from all of the translates $\bar{S}'_{10} + \mathbf{w}$ for $\mathbf{w} \in L'_7 \setminus \{\mathbf{0}\}$. It follows that any fundamental region for L'_7 included within $\bar{S}'_{10}$ must include all four of these sets.

If the fundamental region is also convex, then it must contain any convex combinations of points in those four sets; in particular, it must include the sets $(3, 1, 1) - \bar{S}'_1$ and $(1, 3, 3) - \bar{S}'_1$. But $(1, 3, 3) = (3, 1, 1) + \mathbf{v}_1$, so the $\mathbf{v}_1$-translate of the region overlaps the region itself in a set of positive volume, which is impossible for a fundamental region of L'_7. Therefore, no fundamental region of L'_7 within $\bar{S}'_{10}$ can be convex. ∎

There is no obvious reason why the lattice L'_7 should be exactly optimal for a lattice covering of $\mathbf{R}^3$ by the tetrahedron $\bar{S}'_{10}$. In the case of undirected graphs on three generators, the lattices obtained for each k were not optimal, but were closer and closer approximations to the lattice L_{bcc}, which does appear to be optimal. One would expect something similar to occur in the directed case, but it does not; the real lattice efficiencies in the last columns of Tables 2 and 3 go up and down irregularly and do not (so far) exceed the value .504 attained by L'_7.

Given this, it seems reasonable to examine L'_7 and try to adjust it slightly in order to improve its efficiency; there should be some locally optimal lattice which L'_7 is an approximation to, and we would like to find it. Quite surprisingly, it turns out that no adjustment is necessary. Just before submitting the present paper, the authors found a recent paper of Forcade and Lamoreaux [15] proving this same result by a method slightly different from that presented here.

Theorem 20 (Forcade and Lamoreaux [15])**.** *Among those lattices L for which $\bar{S}'_{10} + L = \mathbf{R}^3$, the lattice L'_7 is locally optimal.*

Proof. We use the same methods as for Theorem 16. Recall the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{14}$ from Proposition 18. The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ generate L'_7; a nearby lattice L will be generated by nearby vectors $\mathbf{v}'_1 = (a_1, b_1, c_1)$, $\mathbf{v}'_2 = (a_2, b_2, c_2)$, and $\mathbf{v}'_3 = (a_3, b_3, c_3)$. Again concatenate $\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to get $\mathbf{v}'$ and $\mathbf{v}$ in $\mathbf{R}^9$. Let $F(\mathbf{v}')$ be the determinant of the matrix with rows $\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3$; then we have $F(\mathbf{v}) = 84$, and we want to see that $F(\mathbf{v}') < 84$ for any other $\mathbf{v}'$ near $\mathbf{v}$ for which the corresponding lattice L satisfies $\bar{S}'_{10} + L = \mathbf{R}^3$. We compute that the gradient of F at the point $\mathbf{v}$ is $\mathbf{g} = (-6, 15, 21, 8, -6, 14, 12, 12, 0)$.

Referring back to Figure 4, we see that the point $(1, 1, 8)$ is on the boundary of $\bar{S}'_{10} + \mathbf{v}_i$ for $i = 8, 9, 12$, as well as on the boundary of $\bar{S}'_{10}$ itself; one can check that no other L'_7-translate of $\bar{S}'_{10}$ is near this point. The nearby lattice L contains points $\mathbf{0}$, $\mathbf{v}'_8 = \mathbf{v}'_1 + \mathbf{v}'_2$, $\mathbf{v}'_9 = 2\mathbf{v}'_1 + \mathbf{v}'_2$, and $\mathbf{v}'_{12} = 2\mathbf{v}'_1 + \mathbf{v}'_2 - \mathbf{v}'_3$. For any small positive ε, the point $(a_1 + a_2 - \varepsilon, 2b_1 + b_2 - \varepsilon, 2c_1 + c_2 - c_3 - \varepsilon)$, which is near $(1, 1, 8)$, will not be in $\mathbf{v}'_8 + \bar{S}'_{10}$ because its first coordinate is smaller that that of $\mathbf{v}'_8$. By looking at second and third coordinates respectively, we see that this point is not in $\mathbf{v}'_9 + \bar{S}'_{10}$ or $\mathbf{v}'_{12} + \bar{S}'_{10}$ either. Hence, in order to have $\bar{S}'_{10} + L = \mathbf{R}^3$, this point must be in $\bar{S}'_{10}$ itself, so we must have

$$a_1 + a_2 + 2b_1 + b_2 + 2c_1 + c_2 - c_3 - 3\varepsilon \leq 10.$$

Since ε can be arbitrarily small, we need

$$a_1 + a_2 + 2b_1 + b_2 + 2c_1 + c_2 - c_3 \leq 10$$

in order to have $\bar{S}'_{10} + L = \mathbf{R}^3$. So we have the constraint

$$\mathbf{u}_1 \cdot \mathbf{v}' \leq 10, \qquad \text{where } \mathbf{u}_1 = (1, 2, 2, 1, 1, 1, 0, 0, -1).$$

The same reasoning applied at the points $(1, 2, 7)$, $(3, 2, 5)$, $(5, 2, 3)$, $(5, 3, 2)$, $(4, 4, 2)$, $(3, 5, 2)$, $(2, 6, 2)$, $(1, 7, 2)$, $(8, 1, 1)$, $(6, 3, 1)$, $(3, 6, 1)$, and $(1, 8, 1)$ gives the constraints

$$\mathbf{u}_2 \cdot \mathbf{v}' \leq 10, \qquad \text{where } \mathbf{u}_2 = (1, 1, 2, 1, 0, 1, 0, 0, 0),$$
$$\mathbf{u}_3 \cdot \mathbf{v}' \leq 10, \qquad \text{where } \mathbf{u}_3 = (0, 1, 1, 1, 0, 1, 0, 0, 0),$$
$$\mathbf{u}_4 \cdot \mathbf{v}' \leq 10, \qquad \text{where } \mathbf{u}_4 = (-1, 1, 0, 1, 0, 1, 0, 0, 0),$$
$$\mathbf{u}_5 \cdot \mathbf{v}' \leq 10, \qquad \text{where } \mathbf{u}_5 = (-1, 0, 1, 1, 0, 0, 0, 1, 0),$$
$$\mathbf{u}_6 \cdot \mathbf{v}' \leq 10, \qquad \text{where } \mathbf{u}_6 = (0, -1, 1, 0, -1, 0, 1, 1, 0),$$
$$\mathbf{u}_7 \cdot \mathbf{v}' \leq 10, \qquad \text{where } \mathbf{u}_7 = (-1, 1, 1, -1, 0, 0, 1, 1, 0),$$
$$\mathbf{u}_8 \cdot \mathbf{v}' \leq 10, \qquad \text{where } \mathbf{u}_8 = (1, 0, 1, 0, -1, 0, 1, 1, 0),$$
$$\mathbf{u}_9 \cdot \mathbf{v}' \leq 10, \qquad \text{where } \mathbf{u}_9 = (0, 2, 1, -1, -1, 0, 1, 0, 0),$$
$$\mathbf{u}_{10} \cdot \mathbf{v}' \leq 10, \qquad \text{where } \mathbf{u}_{10} = (-2, -1, -1, 0, 0, 1, 1, 1, 0),$$
$$\mathbf{u}_{11} \cdot \mathbf{v}' \leq 10, \qquad \text{where } \mathbf{u}_{11} = (-1, 0, -1, 0, 0, 1, 1, 1, 0),$$
$$\mathbf{u}_{12} \cdot \mathbf{v}' \leq 10, \qquad \text{where } \mathbf{u}_{12} = (-1, 0, 1, -1, -1, 0, 1, 1, 1),$$
$$\mathbf{u}_{13} \cdot \mathbf{v}' \leq 10, \qquad \text{where } \mathbf{u}_{13} = (0, 1, 2, -1, -1, -1, 1, 1, 0).$$

Again note that all thirteen of these inequalities are satisfied with equality when $\mathbf{v}' = \mathbf{v}$. Hence, they can be rewritten as $\mathbf{u}_i \cdot (\mathbf{v}' - \mathbf{v}) \leq 0$ for $i = 1, 2, \ldots, 13$.

One can easily check that the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6, \mathbf{u}_7, \mathbf{u}_8, \mathbf{u}_{10}$ are linearly independent; their common null space is generated by the vector $\mathbf{w} = (0, 0, 0, 1, 1, -1, 2, -1, 1)$. (The other five vectors $\mathbf{u}_i$ are also orthogonal to $\mathbf{w}$, so they are linear combinations of the eight listed above.) Also, we have

$$\mathbf{g} = \mathbf{u}_1 + 4.8\mathbf{u}_2 + 6.4\mathbf{u}_4 + \mathbf{u}_5 + 1.6\mathbf{u}_6 + 3.2\mathbf{u}_7 + 3.4\mathbf{u}_8 + \mathbf{u}_9 + 1.8\mathbf{u}_{10} + \mathbf{u}_{12}.$$

Let C be the closed cone consisting of all vectors $\mathbf{t}$ in the subspace spanned by $\mathbf{u}_1, \ldots, \mathbf{u}_{13}$ such that $\mathbf{u}_i \cdot \mathbf{t} \leq 0$ for all $i \leq 13$. Then the above equations imply that $\mathbf{g} \cdot \mathbf{t} \leq 0$ for all $\mathbf{t}$ in C, and equality can hold only when $\mathbf{t} = \mathbf{0}$. Hence, as in Theorem 16, there is a neighborhood U of $\mathbf{g}$ such that, for any $\mathbf{g}'$ in U and any nonzero $\mathbf{t}$ in C, $\mathbf{g}' \cdot \mathbf{t} < 0$.

We can compute that, for any real number r, the determinant for the lattice given by $\mathbf{v} + r\mathbf{w}$ is $F(\mathbf{v} + r\mathbf{w}) = 84 - 12r^2$. Clearly this is at most 84, with equality only when $r = 0$.

Now, any vector $\mathbf{v}'$ close to $\mathbf{v}$ can be expressed as $\mathbf{v}+\mathbf{t}_1+\mathbf{t}_2$ where $\mathbf{t}_1$ is a small multiple of $\mathbf{w}$ and $\mathbf{t}_2$ is a small linear combination of the vectors $\mathbf{u}_1,\ldots,\mathbf{u}_{13}$. The reasoning from Theorem 16 shows that $\mathbf{t}_2$ must be in C if $\bar{S}'_{10}+L=\mathbf{R}^3$. Also as in that Theorem, we find that $F(\mathbf{v}+\mathbf{t}_1)\leq F(\mathbf{v})$ with equality only when $\mathbf{t}_1=\mathbf{0}$, and $F(\mathbf{v}+\mathbf{t}_1+\mathbf{t}_2)\leq F(\mathbf{v}+\mathbf{t}_1)$ with equality only when $\mathbf{t}_2=\mathbf{0}$. Therefore, $F(\mathbf{v}')\leq F(\mathbf{v})$, with equality holding only when $\mathbf{v}'=\mathbf{v}$. So $\mathbf{v}$ gives a local maximum of F, as desired. ∎

This and the computational evidence make it plausible that L'_7 actually gives an optimal lattice covering of $\mathbf{R}^3$ by $\bar{S}'_{10}$, and hence that the asymptotic formula in Corollary 18 is optimal.

More than three generators. In higher dimensions, analogues of many of the preceding constructions exist, but they do not produce lattice coverings as efficient as one would hope for.

For lattice coverings with the d-dimensional dual cube, one can use the d-dimensional body-centered cubic lattice (the set of vectors in $\mathbf{Z}^d$ whose coordinates are all odd or all even). By the same argument as for the three-dimensional case, this lattice gives a lattice covering of $\mathbf{R}^d$ by $\bar{S}_{d/2}$. The efficiency of this covering is $2^{d-1}/\operatorname{vol}(\bar{S}_{d/2})=2^{d-1}d!/d^d$, which is 2^{d-1} times the efficiency of the covering using the ordinary cubic lattice $\mathbf{Z}^d$.

As usual, the Cayley graph corresponding to this lattice is a twisted toroidal mesh. For a given number m, one can connect the elements of $\mathbf{Z}_{2m}^{d-1}\times\mathbf{Z}_m$ as in an ordinary toroidal mesh, except that the wraparound connections for the last coordinate are twisted along all of the other coordinates: $(x_1,\ldots,x_{d-1},m-1)$ is connected to $(x_1\pm m,\ldots,x_{d-1}\pm m,0)$. This gives a graph of diameter $\lfloor dm/2\rfloor$ and size $2^{d-1}m^d$, which is about 2^{d-1} times as large as the best ordinary toroidal mesh of this diameter.

One can optimize this slightly. Given the dimension d and the desired diameter k, let q and r be the quotient and remainder when $2k+1$ is divided by d; we assume k is large enough that $q>0$. Then a good lattice L to use is the body-centered cubic lattice above scaled up by a factor $q+1$ in each of the first r coordinates and a factor q in the remaining $d-r$ coordinates. The resulting $\mathbf{Z}^d/L$ is isomorphic to $(\mathbf{Z}_{2q+2}^r\times\mathbf{Z}_{2q}^{d-r})/H$ with the canonical generators, where H is the two-element subgroup $\{\mathbf{0},(q+1,\ldots,q+1,q,\ldots,q)\}$ (there are r $q+1$'s); it can be laid out as a twisted toroidal mesh on $\mathbf{Z}_{2q+2}^r\times\mathbf{Z}_{2q}^{d-r-1}\times\mathbf{Z}_q$ or on $\mathbf{Z}_{2q+2}^{r-1}\times\mathbf{Z}_{2q}^{d-r}\times\mathbf{Z}_{q+1}$. If q is even and $r>0$, this Cayley graph is isomorphic to that of $\mathbf{Z}_{q+1}\times\mathbf{Z}_{2q+2}^{r-1}\times\mathbf{Z}_{2q}^{d-r}$ with the generators $\mathbf{e}_2,\ldots,\mathbf{e}_d$ and $(1,\ldots,1,q,\ldots,q)$ with r 1's; if q is odd or $r=0$, then it is isomorphic to the Cayley graph of $\mathbf{Z}_{2q+2}^r\times\mathbf{Z}_{2q}^{d-r-1}\times\mathbf{Z}_q$ with generators $\mathbf{e}_1,\ldots,\mathbf{e}_{d-1}$ and $(q+1,\ldots,q+1,1,\ldots,1)$ with r $q+1$'s. The size of this graph is slightly larger than the size of the cyclic Cayley graph constructed by Chen and Jia [5], but the ratio of the two sizes tends to 1 for large k.

For the directed case, we must consider lattice coverings by d-simplices; as usual, by affine invariance, it doesn't matter which simplex is used. One can show that a lattice for covering with a given d-simplex is given by the following generating vectors: for each face of the simplex, take a vector which is twice the vector from the centroid of the simplex to the centroid of that face. (This gives $d+1$ vectors, but they sum to $\mathbf{0}$, so just take d of

them.) The efficiency of this covering works out to be $d!2^d/(d^d(d+1))$.

Unfortunately, in both cases, the efficiency decreases exponentially with d: by Stirling's formula,

$$\frac{2^{d-1}d!}{d^d} \sim \sqrt{\frac{\pi d}{2}}\left(\frac{2}{e}\right)^d$$

and

$$\frac{d!2^d}{d^d(d+1)} \sim \sqrt{\frac{2\pi}{d}}\left(\frac{2}{e}\right)^d.$$

This seems to be the case for all known explicitly constructed lattice coverings by these shapes (and by spheres).

On the other hand, in 1959 Rogers [18] gave a nonconstructive proof that there exist much more efficient lattice coverings by these shapes (or by any convex body) in high dimensions; and he gave an even better result for the case of spheres. More recently Gritzmann [16] extended the latter result to apply to any convex body with a sufficient number of mutually orthogonal hyperplanes of symmetry. (The number required is quite small: only $\lfloor\log_2\ln d\rfloor + 5$.) Gritzmann's result states that there is a constant c (not depending on d or on the convex body) such that, for any convex body K in $\mathbf{R}^d$ with the above number of mutually orthogonal planes of symmetry, there is a lattice covering of $\mathbf{R}^d$ by K with density at most $cd(\ln d)^{1+\log_2 e}$.

The regular dual d-cube and the regular d-simplex do have the required symmetry planes for large enough d. This is clear for the dual d-cube; it has the same d orthogonal planes of symmetry as the d-cube it is dual to. For the regular d-simplex, note that the perpendicular bisector of an edge is a hyperplane of symmetry, and that edges which do not share a vertex have orthogonal directions (the easiest way to see this is to look at the regular d-simplex in $\mathbf{R}^{d+1}$ whose vertices are $\mathbf{e}_1,\ldots,\mathbf{e}_{d+1}$, and take dot products), so one can find $\lceil d/2\rceil$ mutually orthogonal hyperplanes of symmetry. Therefore, we get lattice coverings of the specified density for large d, and by adjusting the constant c we can make the bound apply for all d (for these two particular shapes). Therefore, letting $\bar{c} = c^{-1}$, we can use Corollary 10 to get:

Theorem 21. *There is a constant $\bar{c} > 0$ (not depending on d or k) such that, for any fixed $d > 1$ and for all k, there exist undirected Cayley graphs of Abelian groups on d generators having diameter $\leq k$ and size at least*

$$\frac{2^d\bar{c}}{d!d(\ln d)^{1+\log_2 e}}k^d + O(k^{d-1}),$$

and there exist directed Cayley graphs of Abelian groups on d generators having diameter $\leq k$ and size at least

$$\frac{\bar{c}}{d!d(\ln d)^{1+\log_2 e}}k^d + O(k^{d-1}).$$

■

The coverings produced by this method are probably fairly strange. We seem to have run into this already in three dimensions, for the directed case; for the undirected case it apparently happens later.

Layouts with short wires. The obvious way to lay out a toroidal mesh is as a rectangular array with mesh connections between adjacent nodes in the array and with long wires connecting opposite ends of the array; these long wires may cause communications delays. However, there is a standard trick for rearranging the layout so as to remove the need for long wires. In the one-dimensional case, instead of placing the nodes in the order $1, 2, 3, \ldots, n$ (where i is connected to $i + 1$ for $i = 1, 2, \ldots, n-1$ and n is connected to 1), one can place them in the order $1, n, 2, n-1, 3, n-2, \ldots$; then the maximum required wire length is only twice the mesh spacing. In higher dimensions, one can apply the same trick to each dimension separately, and again the required wire length is twice the mesh spacing.

It is not immediately obvious that this interleaving trick can be applied to twisted toroidal meshes; a simple interleaving in each dimension would not make the twisted cross-connections short. But it is possible to get short-wire layouts for the twisted meshes in a similar way. One approach is to perform the interleaving *twice* on the long dimensions of the mesh; for instance, if the mesh has length 16 in one of the long dimensions, then the nodes would be arranged in the order

$$1, 9, 16, 8, 2, 10, 15, 7, 3, 11, 14, 6, 4, 12, 13, 5.$$

Then wires in this dimension would have length at most 4 times the mesh spacing. Now, when one does a single interleaving on the short dimension, the twisted cross-connections become short as well.
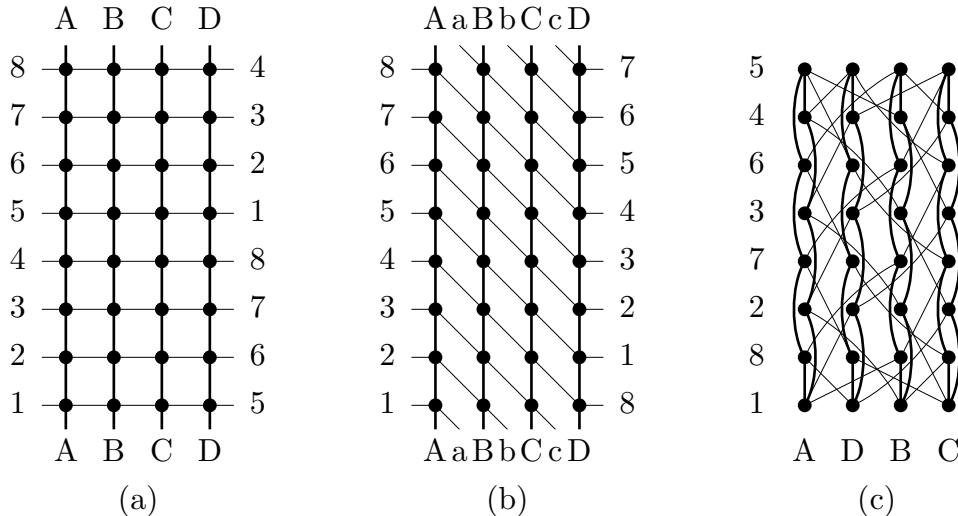

FIGURE 6. Short-wire layout for a twisted toroidal mesh.

Another method is shown in Figure 6. Here the idea is to modify the original arrangement (a) by shearing the mesh (rotating the i'th level in the short dimension by $i-1$ units

in each of the long dimensions), as shown in (b), so that the twisted cross-connections become (almost) straight. Then one can do an ordinary interleaving in each dimension to get the result shown in (c). This gives a layout in which the maximum wire length is $2\sqrt{d}$ times the mesh spacing.

Some of the other Abelian Cayley graphs we have considered can be treated similarly, especially the ones which differ only slightly from twisted toroidal meshes. For the optimal two-generator undirected Abelian Cayley graph, if one starts with the almost-rectangular layout shown in Figure 1 (a $(k+1) \times (k+1)$ square next to a $k \times k$ square), and performs a shear as in Figure 6, then the result is a $2k \times (k+1)$ rectangle with one node left over; this can then be interleaved to get a short-wire layout. A more difficult case is the two-generator directed graph from Theorem 13 and Corollary 14; here one can start with the natural L-shaped layout and perform shears on separate parts to obtain a rectangle with dimensions $(a + 2b) \times a$ (made up of three subrectangles with different shear patterns) where the necessary cross-connections are almost straight across, and hence interleaving will give a good layout.

Generators of order 2. The undirected Cayley graphs produced so far all have even degree (twice the number of generators). If one is interested in undirected Cayley graphs of odd degree, one will have to use a generator of order 2.

Using d unrestricted generators plus one order-2 generator, one can get an undirected Abelian Cayley graph of a given diameter which is about twice as large as one can get using d unrestricted generators alone. More precisely, if $n_a(d, k)$ is the size of the largest Abelian Cayley graph of diameter k using d generators and $n_a^+(d, k)$ is the size of the largest such graph using d generators plus one order-2 generator, then

$$2n_a(d, k - 1) \leq n_a^+(d, k) \leq 2n_a(d, k).$$

To see this, first let G be generated by $g_1, \ldots, g_d$ and ρ where ρ has order 2. If G has diameter at most k using these generators, and H is the subgroup of size 2 generated by ρ, then G/H is generated by the images $g_i + H$ for $1 \leq i \leq d$ with diameter at most k, so $|G/H| \leq n_a(d, k)$, so $|G| \leq 2n_a(d, k)$; hence, $n_a^+(d, k) \leq 2n_a(d, k)$. On the other hand, if G is generated by $g_1, \ldots, g_d$ with diameter at most $k - 1$, then $G \times \mathbf{Z}_2$ is generated by $(g_i, 0)$ for $i = 1, \ldots, d$ and $(0, 1)$, and has diameter at most k using these generators; this shows that $2n_a(d, k - 1) \leq n_a^+(d, k)$.

We can also study $n_a^+(d, k)$ using the same methods that were used for $n_a(d, k)$. The appropriate free (universal) group to use here is the group $\mathbf{Z}^d \times \mathbf{Z}_2$, with the canonical generators $(\mathbf{e}_i, 0)$ for $i = 1, \ldots, d$ and $(\mathbf{0}, 1)$. The set of elements of this group which can be written as a word of length at most k in the generators is precisely $W_k = (S_k \times 0) \cup (S_{k-1} \times 1)$. For any Abelian group G with generators $g_1, \ldots, g_d$ and ρ (ρ of order 2), there is a unique homomorphism from $\mathbf{Z}^d \times \mathbf{Z}_2$ to G taking $(\mathbf{e}_i, 0)$ to g_i and $(\mathbf{0}, 1)$ to ρ; the Cayley graph of G using these generators has diameter at most k if and only if the homomorphism maps W_k onto G. So the obvious upper limit for the size of G is $|W_k| = |S_k| + |S_{k-1}|$.

We are now led to study quotient groups $(\mathbf{Z}^d \times \mathbf{Z}_2)/N$ where N is a (normal) subgroup of $\mathbf{Z}^d \times \mathbf{Z}_2$ of finite index; we want such an N of largest possible index such that $W_k + N = \mathbf{Z}^d \times \mathbf{Z}_2$.

One simple possibility is that $N \subseteq \mathbf{Z}^d \times \{0\}$; in this case the resulting group is just $(\mathbf{Z}^d/N_0) \times \mathbf{Z}_2$ where $N = N_0 \times \{0\}$. It is easy to see that the diameter of this group is precisely one more than the diameter of $\mathbf{Z}^d/N_0$ using the canonical d generators.

Note that N_0 is a d-dimensional lattice; let $\mathbf{v}_1, \ldots, \mathbf{v}_d$ be a list of generators for this lattice. Now let N' be the subgroup of $\mathbf{Z}^d \times \mathbf{Z}_2$ generated by $(\mathbf{v}_i, 1)$ for $i = 1, \ldots, d$. Then we have

$$|\mathbf{Z}^d \times \mathbf{Z}_2 : N'| = 2\,|\mathbf{Z}^d : N_0| = |\mathbf{Z}^d \times \mathbf{Z}_2 : N|\,.$$

Furthermore, the diameter of $(\mathbf{Z}^d \times \mathbf{Z}_2)/N'$ is at most one more than the diameter of $\mathbf{Z}^d/N_0$, which means that it is no larger than the diameter of $(\mathbf{Z}^d \times \mathbf{Z}_2)/N$.

This shows that, when trying to determine $n_a^+(d, k)$, we may restrict ourselves to studying subgroups N of $\mathbf{Z}^d \times \mathbf{Z}_2$ of finite index which are *not* included in $\mathbf{Z}^d \times \{0\}$.

So choose $\mathbf{g} \in \mathbf{Z}^d$ such that $(\mathbf{g}, 1) \in N$. The subgroup $N \cap (\mathbf{Z}^d \times \{0\})$ is of index 2 in N and hence of finite index in $\mathbf{Z}^d \times \mathbf{Z}_2$. So we have $N \cap (\mathbf{Z}^d \times \{0\}) = L \times \{0\}$ for some d-dimensional lattice L. Note that $(2\mathbf{g}, 0) = 2(\mathbf{g}, 1) \in N$, so $2\mathbf{g} \in L$. (Normally $\mathbf{g}$ will not be in L; if $\mathbf{g} \in L$, then $(\mathbf{g}, 0) \in N$, so $(\mathbf{0}, 1) \in N$, so the order-2 generator collapses to the identity in the quotient group.) Also, we have $|\mathbf{Z}^d \times \mathbf{Z}_2 : N| = |\mathbf{Z}^d : L|$.

It is now easy to see that

$$(W_k + N) \cap (\mathbf{Z}^d \times \{0\}) = ((S_k + L) \cup (S_{k-1} + \mathbf{g} + L)) \times \{0\}.$$

Hence, in order to have $W_k + N = \mathbf{Z}^d \times \mathbf{Z}_2$, it is necessary to have

$$(*) \qquad\qquad (S_k + L) \cup (S_{k-1} + \mathbf{g} + L) = \mathbf{Z}^d.$$

This necessary condition is also sufficient, because

$$(W_k + N) \cap (\mathbf{Z}^d \times \{1\}) = ((S_{k-1} + L) \cup (S_k + \mathbf{g} + L)) \times \{1\}$$
$$= (((S_k + L) \cup (S_{k-1} + \mathbf{g} + L)) + \mathbf{g}) \times \{1\}.$$

So our goal is to find such a lattice L and extra generator $\mathbf{g}$ (with $2\mathbf{g} \in L$) so that $(*)$ is satisfied and $|\mathbf{Z}^d : L|$ is as large as possible.

We are now ready to consider specific values of d. As usual, the case $d = 1$ is easy. The maximal possible value of $|\mathbf{Z} : L|$ is $|W_k| = 4k$, and this value is attained by letting L be generated by the element $4k$, with $\mathbf{g} = 2k$. This leads to the cyclic Cayley graph on the group $\mathbf{Z}_{4k}$ with unrestricted generator 1 and order-2 generator $2k$.

For $d = 2$ we have a situation very similar to that in Figure 1 (lattice coverings with Aztec diamonds), but not identical because we must use two different shapes. The upper bound on $|\mathbf{Z}^2 : L|$ is $|W_k| = 4k^2 + 2$. For $k = 1$ this bound is actually attainable; it leads to the Cayley graph from $\mathbf{Z}_6$ with unrestricted generators 1 and 2 and order-2 generator 3. But for $k > 1$ the pieces S_k and S_{k-1} do not fit together well enough to give a perfect
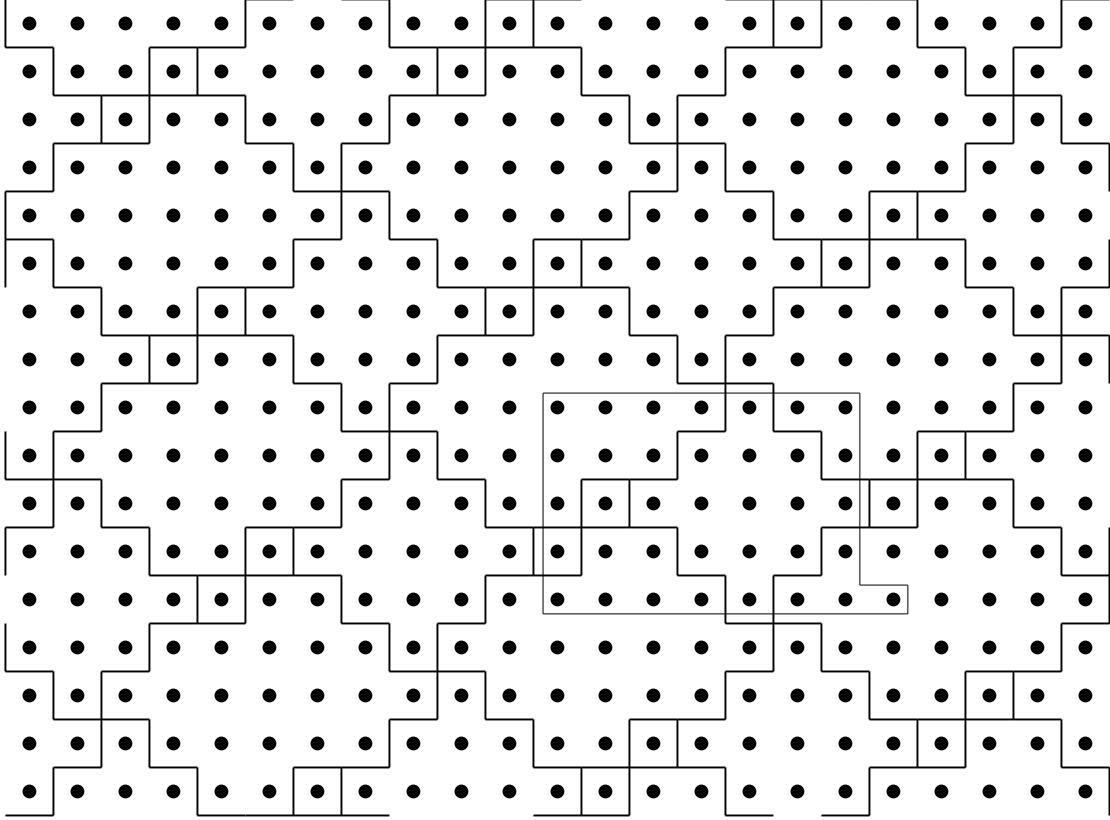
FIGURE 7. Lattice covering of $\mathbf{Z}^2$ using S_k and S_{k-1} (shown for $k = 3$).

tiling. The best one can do is the lattice L generated by $(2k + 1, 1)$ and $(-1, 2k - 1)$, with the extra generator $\mathbf{g} = (k, k)$, as shown in Figure 7.

The graph of diameter k resulting from this covering is the Cayley graph of the cyclic group $\mathbf{Z}_{4k^2}$ with unrestricted generators 1 and $2k - 1$ and order-2 generator $2k^2$. One can get another Cayley graph of this size by using the lattice generated by $(2k, 0)$ and $(0, 2k)$, but this graph will not be cyclic; it comes from the group $\mathbf{Z}_{2k} \times \mathbf{Z}_{2k}$.

The outlined shape in Figure 7 (a $(2k + 1) \times (2k - 1)$ rectangle with one extra point) is a fundamental region which is convenient for an actual layout of nodes in a network. Note that a $2k \times 2k$ rectangle (or for that matter, any rectangle with both sides greater than 1) will not work as a fundamental region for this lattice. The alternative lattice in the previous paragraph does allow one to use a layout which is a $2k \times 2k$ rectangle; in fact, this is just a toroidal mesh. However, in either case one will have to make the extra connections specified by the order-2 generator.

When one moves to $d = 3$, it becomes harder to get optimal results, so again the authors resorted to a computational search. For $k = 1$ the best graph is the Cayley graph of $\mathbf{Z}_8$ with unrestricted generators $1, 2, 3$ and order-2 generator 4; for $k = 2$ the best is $\mathbf{Z}_{26}$ with unrestricted generators $1, 2, 8$ and order-2 generator 13. For $3 \leq k \leq 10$ the optimal results, like those for three generators alone, form a pattern of period 3, as shown

in Table 4. (Again the best graphs are all cyclic. This time the parameter a is defined to be the integer nearest $2k/3$.)

$k \bmod 3$	0	1	2
a	$2k/3$	$(2k+1)/3$	$(2k-1)/3$
Lattice generators	$(2a, 1, -1)$ $(-1, 2a, -1)$ $(1, 1, 2a)$	$(2a-1, -1, 0)$ $(1, 2a, -1)$ $(0, 1, 2a-1)$	$(2a+1, -1, 0)$ $(1, 2a, -1)$ $(0, 1, 2a+1)$
Extra generator	$(a, a+1, a-1)$	$(a, a, a-1)$	$(a+1, a, a)$
Cyclic group size	$\dfrac{64k^3 + 108k}{27}$	$\dfrac{64k^3 + 60k - 16}{27}$	$\dfrac{64k^3 + 60k + 16}{27}$
Unrestricted generators	1 $4a^3 - 2a^2 + 2a - 1$ $4a^3 - 2a^2 + 4a - 1$	1 $2a - 1$ $4a^2 - 2a + 1$	1 $2a + 1$ $4a^2 + 2a + 1$
Order-2 generator	$4a^3 + 3a$	$4a^3 - 4a^2 + 3a - 1$	$4a^3 + 4a^2 + 3a + 1$

TABLE 4. Best undirected Cayley graphs of cyclic groups of diameter k using three generators plus one order-2 generator ($3 \le k \le 10$).

We can now apply the methods in the proof of Theorem 15 to show:

Theorem 22. *For each $k \ge 3$, the cyclic undirected Cayley graph using the group and generators specified in Table 4 has diameter k.* ∎

The authors again conjecture that these are actually the optimal such Abelian Cayley graphs for all $k \ge 3$, not just for $3 \le k \le 10$.

For $d > 3$ one can get reasonably good results by letting L be approximately a cubic lattice, with $\mathbf{g}$ near the center of one of the cubes; this makes $L \cup (\mathbf{g} + L)$ approximately a body-centered cubic lattice. Again, though, the efficiency of this covering decreases exponentially with d; one can do much better using the results of Gritzmann [16].

One can also consider the possibility of using more than one generator of order 2. For instance, one could look at Cayley graphs of degree $2d+2$ obtained by using d unrestricted generators and two generators of order 2.

However, this is not going to he helpful if one wants to construct large undirected Cayley graphs of a given degree and diameter, at least if the diameter is substantially larger than the degree. For instance, suppose that the degree is fixed as $2d + 2$. If one uses d unrestricted generators and two order-2 generators, then the size of the resulting undirected Abelian Cayley graph of diameter k is at most 4 times the number of points in

the d-dimensional shape S_k. (More precisely, by the methods used above for one order-2 generator, one gets a limit of $|S_k| + 2|S_{k-1}| + |S_{k-2}|$.) This limit is $O(k^d)$, which is less than the $O(k^{d+1})$ one gets by using $d + 1$ unrestricted generators. The same argument shows that using more than two order-2 generators cannot be useful for large k; one gets larger graphs by replacing two order-2 generators with one unrestricted generator.

If one is interested in the small-diameter case, though (especially when the diameter is less than or equal to the degree), then order-2 generators must be considered. The most extreme version of this would be to make *all* of the generators have order 2. This then becomes precisely the covering radius problem for binary linear codes; see the surveys by Cohen et al. [8,9] for more on this problem. The resulting graphs would be hypercubes with additional diagonal connections to reduce the diameter.

Conclusions. We have shown that one can construct Cayley graphs of Abelian groups which have substantially more vertices than traditional toroidal meshes, but retain certain desirable features. In particular, routing on the twisted toroidal meshes is easily described in almost the same manner as on toroidal meshes, and the twisted toroidal meshes host the discrete non-periodic orthogonal grids used in numerical calculations in exactly the same way that toroidal meshes do. In addition, we have shown how our d-dimensional meshes can be constructed with physical wire lengths that remain constant with increasing diameter (and increasing number of vertices) just as the corresponding toroidal meshes can. We have given results which are provably optimal in 2 dimensions and probably optimal in 3 dimensions—the physically interesting cases.

In the sequel to this paper, we will show that our methods can be extended to cover certain types of nilpotent groups. These groups yield graphs with cardinalities which still increase polynomially with diameter for a given degree, but with an exponent which is larger than in the Abelian case. One class of groups for which we obtain optimal results includes the groups discussed in Draper and Faber [12]. In particular, we show that the particular groups analyzed in that paper are not optimal for large diameters.

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