

Spectral Properties of the Threshold Network Model

Yusuke Ide ^{*}

*Faculty of Engineering, Kanagawa University,
Yokohama, 221-8686, Japan*

Norio Konno [†]

*Department of Applied Mathematics, Yokohama National University,
Yokohama, 240-8501, Japan*

Nobuaki Obata [‡]

*Graduate School of Information Sciences, Tohoku University,
Sendai, 980-8579, Japan*

Abstract

We study the spectral distribution of the threshold network model. The results contain an explicit description and its asymptotic behaviour.

1 Introduction

The *threshold network model* $\mathcal{G}_n(X, \theta)$, where X is a random variable, $n \geq 2$ is an integer and $\theta \in \mathbb{R}$ is a constant called a threshold, is a random graph on the vertex set $V = \{1, 2, \dots, n\}$ obtained as follows: let X_1, X_2, \dots, X_n be independent copies of X and draw an edge between two distinct vertices $i, j \in V$ if $X_i + X_j > \theta$. In other words, $\mathcal{G}_n(X, \theta)$ is specified by the random adjacency matrix $A = (A_{ij})$ defined by

$$A_{ij} = \begin{cases} I_{(\theta, \infty)}(X_i + X_j), & \text{if } i \neq j, \\ 0, & \text{otherwise,} \end{cases}$$

where I_B denotes the indicator function of a set B . As a small variant one may allow self-loops, see e.g., [4]. In this case the threshold network model is denoted by $\tilde{\mathcal{G}}_n(X, \theta)$,

^{*}E-mail: ide@kanagawa-u.ac.jp

[†]E-mail: konno@ynu.ac.jp

[‡]E-mail: obata@math.is.tohoku.ac.jp

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where two vertices $i, j \in V$ (possibly $i = j$) are connected if $X_i + X_j > \theta$. The adjacency matrix $\tilde{A} = (\tilde{A}_{ij})$ is given by

$$\tilde{A}_{ij} = I_{(\theta, \infty)}(X_i + X_j), \quad i, j \in V.$$

The threshold network model has been extensively studied as a reasonable candidate model of real world complex graphs (networks), which are often characterized by small diameters, high clustering, and power-law (scale-free) degree distributions [1, 2, 20]. In fact, the threshold network model belongs to the so-called hidden variable models [5, 22] and is known for being capable of generating scale-free networks. Their mean behavior [3, 5, 8, 9, 15, 21, 22] and limit theorems [7, 11–13] for the degree, the clustering coefficients, the number of subgraphs, and the average distance have been analyzed. See also [6, 11–14, 16, 17] for related works.

Spectral properties of the threshold network model are also of interest. As a simple case, the binary threshold model appears in [23]. The strong law of large numbers and central limit theorem for the rank of the adjacency matrix of the model with self-loops are given by [4]. Eigenvalues and eigenvectors of the Laplacian matrix of the model have been studied [18, 19]. For general results of spectral analysis of graphs see e.g. Hora–Obata [10]. The main purpose of this paper is to study the spectral distribution of the threshold network model. Our result covers the preceding study of the rank of the adjacency matrix.

This paper is organized as follows: In Section 2 we recall the hierarchical structure of the threshold network model and derive the spectral distribution of each sample graph (threshold graph). In Section 3 we obtain similar results for the threshold network model which admits self-loops. In Section 4 we derive some asymptotic behaviors of the spectral distributions and in Section 5 we give a simple example called the binary threshold model.

2 Spectra of threshold graphs

Each sample graph $G \in \mathcal{G}_n(X, \theta)$ has a hierarchical structure described by the so-called creation sequence, introduced by Hagberg–Schult–Swart [9]. Here we adopt a variant by Diaconis–Holmes–Janson [6]. Each G being determined by the values of random variables X_1, X_2, \dots, X_n , we arrange them in increasing order: $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$. If $X_{(1)} + X_{(n)} > \theta$, we have

$$\theta < X_{(1)} + X_{(n)} \leq X_{(2)} + X_{(n)} \leq \dots \leq X_{(n-1)} + X_{(n)},$$

which means that the vertex corresponding to $X_{(n)}$ is connected with the $n - 1$ other vertices. Otherwise, we have

$$\theta \geq X_{(1)} + X_{(n)} \geq \dots \geq X_{(1)} + X_{(3)} \geq X_{(1)} + X_{(2)},$$

which means that the vertex corresponding to $X_{(1)}$ is isolated. We set $s_n = 1$ or $s_n = 0$ according as the former case or the latter occurs. Then, according to the case we remove the random variable $X_{(n)}$ or $X_{(1)}$, we continue similar procedure to define s_{n-1}, \dots, s_2 . Finally, we set $s_1 = s_2$ and obtain a $\{0, 1\}$ -sequence $\{s_1, s_2, \dots, s_n\}$, which is called the *creation sequence* of G and is denoted by S_G .

Given a creation sequence S_G let k_i and l_i denote the number of consecutive bits of 1's and 0's, respectively, as follows:

$$S_G = \{ \underbrace{1, \dots, 1}_{k_1}, \underbrace{0, \dots, 0}_{l_1}, \underbrace{1, \dots, 1}_{k_2}, \underbrace{0, \dots, 0}_{l_2}, \dots, \underbrace{1, \dots, 1}_{k_m}, \underbrace{0, \dots, 0}_{l_m} \}. \quad (1)$$

It may happen that $k_1 = 0$ or $l_m = 0$, but we have $k_2, \dots, k_m, l_1, \dots, l_{m-1} \geq 1$ and $m \geq 1$. Moreover, by definition we have two cases: (a) $k_1 = 0$ (equivalently $s_1 = 0$) and $l_1 \geq 2$; (b) $k_1 \geq 2$ (equivalently, $s_1 = 1$).

For example, if $S_G = \{1, 1, 0, 0, 1, 0, 1, 0\}$ then $k_1 = 2$, $l_1 = 2$, $k_2 = 1$, $l_2 = 1$, $k_3 = 1$, $l_3 = 1$ and Fig. 1 shows the shape of G .

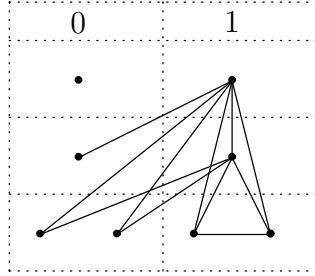


Figure 1: A threshold graph G corresponding to $S_G = \{1, 1, 0, 0, 1, 0, 1, 0\}$

The creation sequence S_G gives rise to a partition of the vertex set:

$$V = \bigcup_{i=1}^m V_i^{(1)} \cup \bigcup_{i=1}^m V_i^{(0)} \quad |V_i^{(1)}| = k_i, \quad |V_i^{(0)}| = l_i.$$

The subgraph induced by $V_i^{(1)}$ is the complete graph of k_i vertices, and that induced by $V_i^{(0)}$ is the null graph of l_i vertices. Moreover, every vertex in $V_i^{(1)}$ (resp. $V_i^{(0)}$) is connected (resp. disconnected) with all vertices in

$$V_1^{(1)} \cup \dots \cup V_i^{(1)} \cup V_1^{(0)} \cup \dots \cup V_{i-1}^{(0)}.$$

In general, a graph possessing the above hierarchical structure is called a *threshold graph* [14].

Theorem 1. Let G be a threshold graph with a creation sequence $S_G = \{s_1 = s_2, s_3, \dots, s_n\}$. Define k_i and l_i as in (1) and set

$$C_n(-1) = \sum_{i=1}^m k_i - (m-1) - I_{\{1\}}(s_1), \quad C_n(0) = \sum_{i=1}^m l_i - (m-1). \quad (2)$$

Then the spectral distribution of G is given by

$$\mu_n(G) = \frac{C_n(-1)}{n} \delta_{-1} + \frac{C_n(0)}{n} \delta_0 + \frac{1}{n} \sum_{j=1}^J \delta_{\lambda_j}, \quad J = 2(m-1) + I_{\{1\}}(s_1), \quad (3)$$

where $\{\lambda_j\}$ exhausts the eigenvalues of the matrix:

$$\begin{bmatrix} k_m - 1 & l_{m-1} & k_{m-1} & l_{m-2} & \dots & l_1 & k_1 \\ k_m & 0 & 0 & 0 & \dots & 0 & 0 \\ k_m & 0 & k_{m-1} - 1 & l_{m-2} & \dots & l_1 & k_1 \\ k_m & 0 & k_{m-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k_m & 0 & k_{m-1} & 0 & \dots & 0 & 0 \\ k_m & 0 & k_{m-1} & 0 & \dots & 0 & k_1 - 1 \end{bmatrix} \quad (4)$$

for $s_1 = 1$ (equivalently, $k_1 \geq 2$), or

$$\begin{bmatrix} k_m - 1 & l_{m-1} & k_{m-1} & l_{m-2} & \dots & k_2 & l_1 \\ k_m & 0 & 0 & 0 & \dots & 0 & 0 \\ k_m & 0 & k_{m-1} - 1 & l_{m-2} & \dots & k_2 & l_1 \\ k_m & 0 & k_{m-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k_m & 0 & k_{m-1} & 0 & \dots & k_2 - 1 & l_1 \\ k_m & 0 & k_{m-1} & 0 & \dots & k_2 & 0 \end{bmatrix} \quad (5)$$

for $s_1 = 0$ (equivalently, $k_1 = 0$). Moreover, any λ_j in (3) differs from 0 and -1 .

Proof. Let $\mathbf{1}_{i,j}$ denote the $i \times j$ matrix consisting of only 1, $\mathbf{0}_{i,j}$ the $i \times j$ zero matrix, I_i the $i \times i$ identity matrix, and $\bar{\mathbf{1}}_{i,i} = \mathbf{1}_{i,i} - I_i$. By the hierarchical structure mentioned above, the adjacency matrix of G is represented in the form:

$$A_G = \begin{bmatrix} \mathbf{0}_{l_m, l_m} & \mathbf{0}_{l_m, k_m} & \mathbf{0}_{l_m, l_{m-1}} & \mathbf{0}_{l_m, k_{m-1}} & \mathbf{0}_{l_m, l_{m-2}} & \dots & \mathbf{0}_{l_m, l_1} & \mathbf{0}_{l_m, k_1} \\ \mathbf{0}_{k_m, l_m} & \bar{\mathbf{1}}_{k_m, k_m} & \mathbf{1}_{k_m, l_{m-1}} & \mathbf{1}_{k_m, k_{m-1}} & \mathbf{1}_{l_m, l_{m-2}} & \dots & \mathbf{1}_{k_m, l_1} & \mathbf{1}_{k_m, k_1} \\ \mathbf{0}_{l_{m-1}, l_m} & \mathbf{1}_{l_{m-1}, k_m} & \mathbf{0}_{l_{m-1}, l_{m-1}} & \mathbf{0}_{l_{m-1}, k_{m-1}} & \mathbf{0}_{l_{m-1}, l_{m-2}} & \dots & \mathbf{0}_{l_{m-1}, l_1} & \mathbf{0}_{l_{m-1}, k_1} \\ \mathbf{0}_{k_{m-1}, l_m} & \mathbf{1}_{k_{m-1}, k_m} & \mathbf{0}_{k_{m-1}, l_{m-1}} & \bar{\mathbf{1}}_{k_{m-1}, k_{m-1}} & \mathbf{1}_{k_{m-1}, l_{m-2}} & \dots & \mathbf{1}_{k_{m-1}, l_1} & \mathbf{1}_{k_{m-1}, k_1} \\ \mathbf{0}_{l_{m-2}, l_m} & \mathbf{1}_{l_{m-2}, l_m} & \mathbf{0}_{l_{m-2}, l_{m-1}} & \mathbf{1}_{l_{m-2}, k_{m-1}} & \mathbf{0}_{l_{m-2}, l_{m-2}} & \dots & \mathbf{0}_{l_{m-2}, l_1} & \mathbf{0}_{l_{m-2}, k_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{l_1, l_m} & \mathbf{1}_{l_1, k_m} & \mathbf{0}_{l_1, l_{m-1}} & \mathbf{1}_{l_1, k_{m-1}} & \mathbf{0}_{l_1, l_{m-2}} & \dots & \mathbf{0}_{l_1, l_1} & \mathbf{0}_{l_1, k_1} \\ \mathbf{0}_{k_1, l_m} & \mathbf{1}_{k_1, k_m} & \mathbf{0}_{k_1, l_{m-1}} & \mathbf{1}_{k_1, k_{m-1}} & \mathbf{0}_{k_1, l_{m-2}} & \dots & \mathbf{0}_{k_1, l_1} & \bar{\mathbf{1}}_{k_1, k_1} \end{bmatrix}.$$

The adjacency matrix A acts on \mathbb{C}^n from the left. We define subspaces of \mathbb{C}^n by

$$\begin{aligned} V_i(-1) &= \left\{ \begin{bmatrix} \mathbf{0}_{u_i+l_i} \\ \boldsymbol{\xi}_{k_i} \\ \mathbf{0}_{d_i} \end{bmatrix} : \xi_1 + \xi_2 + \dots + \xi_{k_i} = 0 \right\}, \quad 1 \leq i \leq m, \\ V_i(0) &= \left\{ \begin{bmatrix} \mathbf{0}_{u_i} \\ \boldsymbol{\eta}_{l_i} \\ \mathbf{0}_{k_i+d_i} \end{bmatrix} : \eta_1 + \eta_2 + \dots + \eta_{l_i} = 0 \right\}, \quad 1 \leq i \leq m-1, \\ V_m(0) &= \left\{ \begin{bmatrix} \boldsymbol{\eta}_{l_m} \\ \mathbf{0}_{k_m+d_m} \end{bmatrix} \right\}, \end{aligned}$$

where

$$\boldsymbol{\xi}_k = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_k \end{bmatrix}, \quad \boldsymbol{\eta}_l = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_l \end{bmatrix}, \quad \mathbf{1}_j = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \mathbf{0}_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and

$$u_i = \sum_{j=i+1}^m (l_j + k_j), \quad d_i = \sum_{j=1}^{i-1} (l_j + k_j).$$

Since A_G acts on $V_i(-1)$ as the scalar operator with -1 , it possesses the eigenvalues -1 with multiplicity at least

$$\sum_{i=1}^m \dim V_i(-1) = \sum_{i=1}^m (k_i - 1) = \sum_{i=1}^m k_i - m$$

if $k_1 \geq 2$ (i.e., $s_1 = 1$), and

$$\sum_{i=2}^m \dim V_i(-1) = \sum_{i=2}^m (k_i - 1) = \sum_{i=2}^m k_i - (m - 1)$$

if $k_1 = 0$ (i.e., $s_1 = 0$). In any case, the multiplicity is at least $C_n(-1)$ defined in (2). Similarly, acting on $V_i(0)$ as a scalar operator with 0 , A_G possesses the eigenvalues 0 with multiplicity at least $C_n(0)$.

Let W be the orthogonal complement to $\bigoplus_{i=1}^m (V_i(-1) \oplus V_i(0))$. The matrix representation of A_G on W with respect to the basis

$$\mathbf{v}_i = \begin{bmatrix} \mathbf{0}_{u_i+l_i} \\ \mathbf{1}_{k_i} \\ \mathbf{0}_{d_i} \end{bmatrix}, \quad 1 \leq i \leq m, \quad \text{and} \quad \mathbf{w}_i = \begin{bmatrix} \mathbf{0}_{u_i} \\ \mathbf{1}_{l_i} \\ \mathbf{0}_{k_i+d_i} \end{bmatrix}, \quad 1 \leq i \leq m-1.$$

is given by (4) or by (5) according as $k_1 \geq 2$ or $k_1 = 0$. Then, one may verify easily the eigenvalues of the matrices (4) and (5) are different from -1 nor 0 . \square

Remark After simple calculation we see that the eigenvalues $\lambda_1, \dots, \lambda_J$ in (3) are obtained from the characteristic equations

$$M(\lambda) = 0,$$

where

$$M(\lambda) = \det \begin{bmatrix} k_m - 1 - \lambda & l_{m-1} & k_{m-1} & l_{m-2} & \dots & l_1 & k_1 \\ k_m & -\lambda & 0 & 0 & \dots & 0 & 0 \\ k_m & 0 & k_{m-1} - 1 - \lambda & l_{m-2} & \dots & l_1 & k_1 \\ k_m & 0 & k_{m-1} & -\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k_m & 0 & k_{m-1} & 0 & \dots & -\lambda & 0 \\ k_m & 0 & k_{m-1} & 0 & \dots & 0 & k_1 - 1 - \lambda \end{bmatrix}$$

if $k_1 \geq 2$ (i.e., $s_1 = 1$), and

$$M(\lambda) = \det \begin{bmatrix} k_m - 1 - \lambda & l_{m-1} & k_{m-1} & l_{m-2} & \dots & k_2 & l_1 \\ k_m & -\lambda & 0 & 0 & \dots & 0 & 0 \\ k_m & 0 & k_{m-1} - 1 - \lambda & l_{m-2} & \dots & k_2 & l_1 \\ k_m & 0 & k_{m-1} & -\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k_m & 0 & k_{m-1} & 0 & \dots & k_2 - 1 - \lambda & l_1 \\ k_m & 0 & k_{m-1} & 0 & \dots & k_2 & -\lambda \end{bmatrix}$$

if $k_1 = 0$ (i.e., $s_1 = 0$). Simple calculation shows that

$$M(-1) = \begin{cases} k_1 \cdots k_m \cdot l_1 \cdots l_{m-1}, & \text{if } k_1 \geq 2 \text{ (i.e., } s_1 = 1\text{),} \\ k_2 \cdots k_m \cdot (l_1 - 1) \cdot l_2 \cdots l_{m-1}, & \text{otherwise,} \end{cases}$$

and

$$M(0) = \begin{cases} (k_1 - 1) \cdot k_2 \cdots k_m \cdot l_1 \cdots l_{m-1}, & \text{if } k_1 \geq 2 \text{ (i.e., } s_1 = 1\text{),} \\ k_2 \cdots k_m \cdot l_1 \cdots l_{m-1}, & \text{otherwise,} \end{cases}$$

from which we see also that $\{\lambda_j\}$ do not contain -1 or 0 .

3 Spectra of threshold graphs with self-loops

The idea of a creation sequence in Section 2 can be applied to the threshold network model which allows self-loops. With each $G \in \tilde{\mathcal{G}}_n(X, \theta)$ we associate a creation sequence $\tilde{S}_G = \{\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n\}$ as follows: if $X_{(1)} + X_{(n)} > \theta$, we have

$$\theta < X_{(1)} + X_{(n)} \leq X_{(2)} + X_{(n)} \leq \cdots \leq X_{(n-1)} + X_{(n)} \leq X_{(n)} + X_{(n)},$$

which implies that the vertex corresponding to $X_{(n)}$ is connected with the $n - 1$ other vertices and has a self-loop. Otherwise,

$$\theta \geq X_{(1)} + X_{(n)} \geq \cdots \geq X_{(1)} + X_{(3)} \geq X_{(1)} + X_{(2)} \geq X_{(1)} + X_{(1)},$$

which means that the vertex corresponding to $X_{(1)}$ is isolated and has no self-loops. We set $\tilde{s}_n = 1$ or $\tilde{s}_n = 0$ according as the former case or the latter occurs. Then, according to the case we remove the random variable $X_{(n)}$ or $X_{(1)}$, we continue similar procedure to define $\tilde{s}_{n-1}, \dots, \tilde{s}_2$. Finally, letting $X_{(*)}$ be the last remained random variable, set $\tilde{s}_1 = 1$ if $X_{(*)} > \theta/2$ and $\tilde{s}_1 = 0$ otherwise. In this case G is called a *threshold graph with self-loops* associated with a creation sequence $\tilde{S} = \{\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n\}$. We note that if $\tilde{s}_j = 1$ the corresponding vertex has a self-loop and otherwise no self-loop.

Given a creation sequence $\tilde{S} = \{\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n\}$ we define k_j and l_j as in (1). It may happen that $k_1 = 0$ and $l_m = 0$, but $k_2, \dots, k_m, l_1, \dots, l_{m-1} \geq 1$ and $m \geq 1$. The adjacency

matrix of G is of the form:

$$\tilde{A}_G = \begin{bmatrix} \mathbf{0}_{l_m, l_m} & \mathbf{0}_{l_m, k_m} & \mathbf{0}_{l_m, l_{m-1}} & \mathbf{0}_{l_m, k_{m-1}} & \mathbf{0}_{l_m, l_{m-2}} & \dots & \mathbf{0}_{l_m, l_1} & \mathbf{0}_{l_m, k_1} \\ \mathbf{0}_{k_m, l_m} & \mathbf{1}_{k_m, k_m} & \mathbf{1}_{k_m, l_{m-1}} & \mathbf{1}_{k_m, k_{m-1}} & \mathbf{1}_{l_m, l_{m-2}} & \dots & \mathbf{1}_{k_m, l_1} & \mathbf{1}_{k_m, k_1} \\ \mathbf{0}_{l_{m-1}, l_m} & \mathbf{1}_{l_{m-1}, k_m} & \mathbf{0}_{l_{m-1}, l_{m-1}} & \mathbf{0}_{l_{m-1}, k_{m-1}} & \mathbf{0}_{l_{m-1}, l_{m-2}} & \dots & \mathbf{0}_{l_{m-1}, l_1} & \mathbf{0}_{l_{m-1}, k_1} \\ \mathbf{0}_{k_{m-1}, l_m} & \mathbf{1}_{k_{m-1}, k_m} & \mathbf{0}_{k_{m-1}, l_{m-1}} & \mathbf{1}_{k_{m-1}, k_{m-1}} & \mathbf{1}_{k_{m-1}, l_{m-2}} & \dots & \mathbf{1}_{k_{m-1}, l_1} & \mathbf{1}_{k_{m-1}, k_1} \\ \mathbf{0}_{l_{m-2}, l_m} & \mathbf{1}_{l_{m-2}, l_m} & \mathbf{0}_{l_{m-2}, l_{m-1}} & \mathbf{1}_{l_{m-2}, k_{m-1}} & \mathbf{0}_{l_{m-2}, l_{m-2}} & \dots & \mathbf{0}_{l_{m-2}, l_1} & \mathbf{0}_{l_{m-2}, k_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{l_1, l_m} & \mathbf{1}_{l_1, k_m} & \mathbf{0}_{l_1, l_{m-1}} & \mathbf{1}_{l_1, k_{m-1}} & \mathbf{0}_{l_1, l_{m-2}} & \dots & \mathbf{0}_{l_1, l_1} & \mathbf{0}_{l_1, k_1} \\ \mathbf{0}_{k_1, l_m} & \mathbf{1}_{k_1, k_m} & \mathbf{0}_{k_1, l_{m-1}} & \mathbf{1}_{k_1, k_{m-1}} & \mathbf{0}_{k_1, l_{m-2}} & \dots & \mathbf{0}_{k_1, l_1} & \mathbf{1}_{k_1, k_1} \end{bmatrix}. \quad (6)$$

Repeating a similar argument as in Theorem 1, we come to the following

Theorem 2. Let G be a threshold graph with self-loops associated with a creation sequence $\tilde{S} = \{\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n\}$ and its adjacency matrix given as in (6). Set

$$\tilde{C}_n(0) = n - 2(m-1) - I_{\{1\}}(\tilde{s}_1). \quad (7)$$

Then the spectral distribution of G is given by

$$\tilde{\mu}_n(G) = \frac{\tilde{C}_n(0)}{n} \delta_0 + \frac{1}{n} \sum_{j=1}^J \delta_{\lambda_j}, \quad J = 2(m-1) + I_{\{1\}}(\tilde{s}_1) \quad (8)$$

where $\{\lambda_j\}$ exhaust the eigenvalues of

$$\begin{bmatrix} k_m & l_{m-1} & k_{m-1} & l_{m-2} & \dots & l_1 & k_1 \\ k_m & 0 & 0 & 0 & \dots & 0 & 0 \\ k_m & 0 & k_{m-1} & l_{m-2} & \dots & l_1 & k_1 \\ k_m & 0 & k_{m-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k_m & 0 & k_{m-1} & 0 & \dots & 0 & 0 \\ k_m & 0 & k_{m-1} & 0 & \dots & 0 & k_1 \end{bmatrix}$$

for $\tilde{s}_1 = 1$ (i.e., $k_1 \geq 1$), or

$$\begin{bmatrix} k_m & l_{m-1} & k_{m-1} & l_{m-2} & \dots & k_2 & l_1 \\ k_m & 0 & 0 & 0 & \dots & 0 & 0 \\ k_m & 0 & k_{m-1} & l_{m-2} & \dots & k_2 & l_1 \\ k_m & 0 & k_{m-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k_m & 0 & k_{m-1} & 0 & \dots & k_2 & l_1 \\ k_m & 0 & k_{m-1} & 0 & \dots & k_2 & 0 \end{bmatrix}$$

for $\tilde{s}_1 = 0$ (i.e., $k_1 = 0$). Moreover, any λ_j in (8) differs from 0.

Remark The eigenvalues $\lambda_1, \dots, \lambda_J$ in (8) are obtained from the characteristic equations:

$$\det \begin{bmatrix} -\lambda & 0 & \dots & 0 & 0 & 0 & \dots & 0 & k_m \\ \lambda & -\lambda & \dots & 0 & 0 & 0 & \dots & k_{m-1} & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \ddots & -\lambda & 0 & k_2 & \dots & 0 & 0 \\ 0 & 0 & \dots & \lambda & k_1 - \lambda & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & l_1 & \lambda & -\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & l_{m-2} & \dots & 0 & 0 & 0 & \ddots & -\lambda & 0 \\ l_{m-1} & 0 & \dots & 0 & 0 & 0 & \dots & \lambda & -\lambda \end{bmatrix} = 0$$

for $s_1 = 1$ (i.e., $k_1 \geq 1$), or

$$\det \begin{bmatrix} -\lambda & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & k_m \\ \lambda & -\lambda & \dots & 0 & 0 & 0 & 0 & \dots & k_{m-1} & 0 \\ 0 & \lambda & \ddots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda & -\lambda & k_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & l_1 + \lambda & -\lambda & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & l_2 & 0 & \lambda & -\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & l_{m-2} & \dots & 0 & 0 & 0 & 0 & \ddots & -\lambda & 0 \\ l_{m-1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \lambda & -\lambda \end{bmatrix} = 0$$

for $s_1 = 0$ (i.e., $k_1 = 0$).

4 Limit theorems

In this section we discuss asymptotic behaviors of the spectral distributions obtained in the previous sections.

We first consider the case where the distribution of X is discrete and given by

$$\mathbb{P}(X = i) = p_i, \quad i = 0, 1, \dots, \quad \sum_{i=0}^{\infty} p_i = 1.$$

Let $m \geq 1$ be a fixed integer. Take a particular threshold $\theta = 2m - 1$ and assume that $p_i > 0$ for $i = 0, 1, \dots, 2m - 1$. It follows from the strong law of large numbers that

$$\begin{aligned} l_i &= \#\{j : X_j = m - i\}, \quad i = 1, \dots, m, \\ k_i &= \#\{j : X_j = m - 1 + i\}, \quad i = 1, \dots, m - 1, \\ k_m &= \#\{j : X_j \geq 2m - 1\}, \\ l_i &= k_i = 0, \quad i \geq m + 1, \end{aligned}$$

for large n almost surely. Moreover, denoting by F the distribution function of X , we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^m l_i = F(m-1) \quad \text{a.s.} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^m k_i = 1 - F(m-1) \quad \text{a.s.}$$

With these observation we easily obtain the following

Theorem 3. *Notations and assumptions being as above, the spectral distributions of $\mathcal{G}_n(X, 2m-1)$ verifies*

$$\lim_{n \rightarrow \infty} \mu_n(G) = (1 - F(m-1)) \cdot \delta_{-1} + F(m-1) \cdot \delta_0 \quad \text{a.s.}$$

Similarly, the spectral distributions of $\tilde{\mathcal{G}}_n(X, 2m-1)$ verifies

$$\lim_{n \rightarrow \infty} \tilde{\mu}_n(G) = \delta_0 \quad \text{a.s.}$$

Remark Similar results hold when the distribution of X is discrete and F has only finite number of jumps in $(-\infty, \theta/2]$ or $(\theta/2, \infty)$. But no simple description is known for a general case.

Next we consider the case where the distribution of X is continuous. As is stated by Bose–Sen [4] implicitly, if the distribution of X is continuous and symmetric around 0, then the distribution of zero and one entries in the creation sequence \tilde{S} of each graph generated by $\tilde{\mathcal{G}}_n(X, 0)$ is the same as the distribution of a sequence of i.i.d. Bernoulli random variables $\{\tilde{Y}_i\}_{i=1,2,\dots,n}$ with success probability 1/2, that is,

$$\mathbb{P}(\tilde{Y}_i = 0) = \mathbb{P}(\tilde{Y}_i = 1) = 1/2, \quad \text{for } i = 1, 2, \dots, n.$$

This means that $\tilde{S} = \{\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n\} \stackrel{d}{=} \{\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_n\}$. Recall that the creation sequence S of each graph generated by $\mathcal{G}_n(X, 0)$ is always satisfied with $s_1 = s_2$. Then we observe that $S = \{s_1 = s_2, s_3, \dots, s_n\} \stackrel{d}{=} \{Y_2, Y_3, \dots, Y_n\}$, where $\{Y_i\}_{i=2,3,\dots,n}$ be the sequence of i.i.d. Bernoulli random variables with success probability 1/2, similarly.

Taking the above consideration into account, we obtain asymptotic behaviors of coefficients of point measures on -1 and 0 appearing in $\mu_n(G)$ and $\tilde{\mu}_n(G)$.

Theorem 4. *Assume that the distribution of X is continuous and symmetric around 0. Define $C_n(-1)$, $C_n(0)$ and $\tilde{C}_n(0)$ as in (2) and (7). Then we have*

$$(1) \quad \lim_{n \rightarrow \infty} C_n(-1)/n = \lim_{n \rightarrow \infty} C_n(0)/n = 1/4 \quad \text{a.s.}$$

$$(2) \quad \sqrt{n}(C_n(-1)/n - 1/4) \Rightarrow N(0, 1/4) \text{ and } \sqrt{n}(C_n(0)/n - 1/4) \Rightarrow N(0, 1/4) \text{ as } n \rightarrow \infty.$$

$$(3) \quad \lim_{n \rightarrow \infty} \tilde{C}_n(0)/n = 1/2 \quad \text{a.s.}$$

$$(4) \quad \sqrt{n}(\tilde{C}_n(0)/n - 1/2) \Rightarrow N(0, 1/4) \text{ as } n \rightarrow \infty.$$

Proof. Note the following relations:

$$\begin{aligned}
C_n(-1) &= \sum_{i=1}^m k_i - (m-1) - I_{\{1\}}(s_1) \\
&\stackrel{d}{=} \left(Y_2 + \sum_{i=2}^n Y_i \right) - \sum_{i=2}^{n-1} (1-Y_i)Y_{i+1} - Y_2 = Y_2 + \sum_{i=2}^{n-1} Y_i Y_{i+1}, \\
C_n(0) &= \sum_{i=1}^m l_i - (m-1) \\
&\stackrel{d}{=} \left\{ (1-Y_2) + \sum_{i=2}^n (1-Y_i) \right\} - \sum_{i=2}^{n-1} (1-Y_i)Y_{i+1} \\
&= 2 - Y_2 - Y_n + \sum_{i=2}^{n-1} (1-Y_i)(1-Y_{i+1}), \\
\tilde{C}_n(0) &= n - 2(m-1) - I_{\{1\}}(\tilde{s}_1) \\
&\stackrel{d}{=} n - 2 \sum_{i=1}^{n-1} (1-\tilde{Y}_i)\tilde{Y}_{i+1} - \tilde{Y}_1 = 1 - \tilde{Y}_n + \sum_{i=1}^{n-1} (1-\tilde{Y}_i + \tilde{Y}_{i+1})(1+\tilde{Y}_i - \tilde{Y}_{i+1}).
\end{aligned}$$

We then easily check that

$$\mathbb{E}[C_n(-1)] = \mathbb{E}[C_n(0)] - \frac{1}{2} = \frac{n}{4}$$

and

$$\mathbb{E}[\tilde{C}_n(0)] = \frac{n}{2}.$$

Applying a similar argument as in [4, Theorem 1], we have the assertion. \square

When the distribution of X is continuous and symmetric around $\theta/2$, we can obtain similar results for $\mathcal{G}_n(X, \theta)$ and $\tilde{\mathcal{G}}_n(X, \theta)$ by straightforward modification. Study covering a more general situation is now in progress.

5 Binary threshold model

In this section we give a simple example. The threshold network model defined by Bernoulli trials X_1, X_2, \dots, X_n with success probability p , i.e., $0 < P(X_i = 1) = p < 1$, and a threshold $0 \leq \theta < 1$ is called the *binary threshold model* and is denoted by $\mathcal{G}_n(p)$. For $G \in \mathcal{G}_n(p)$ the partition of the vertex set V is given by

$$V = V^{(1)} \cup V^{(0)}, \quad V^{(1)} = \{i ; X_i = 1\}, \quad V^{(0)} = \{i ; X_i = 0\}.$$

Theorem 5. *For $G \in \mathcal{G}_n(p)$ we set $|V^{(1)}| = k$ and $|V^{(0)}| = l$. Then the spectral distribution of G is given by*

$$\mu_{k,l} = \frac{k-1}{n} \delta_{-1} + \frac{l-1}{n} \delta_0 + \frac{1}{n} \delta_{\lambda_+} + \frac{1}{n} \delta_{\lambda_-},$$

where

$$\lambda_{\pm} = \frac{k-1 \pm \sqrt{(k-1)^2 + 4kl}}{2}. \quad (9)$$

Proof. We need only to apply Theorem 1 with $l_1 = l$, $l_2 = k_1 = 0$, $k_2 = k$ and $m = 2$. In this case, (5) becomes

$$\begin{bmatrix} k-1 & l \\ k & 0 \end{bmatrix},$$

of which the eigenvalues are λ_{\pm} in (9). \square

Corollary 1. *Let $\mu_n(G)$ be the spectral distribution of $G \in \mathcal{G}_n(p)$. Then we have*

$$\lim_{n \rightarrow \infty} \mu_n(G) = p \cdot \delta_{-1} + (1-p) \cdot \delta_0 \quad a.s.$$

Proof. By the strong law of large numbers, see also Theorem 3. \square

As for the mean spectral distribution we have

Theorem 6. *The mean spectral distribution of the binary threshold model $\mathcal{G}_n(p)$ is given by*

$$\begin{aligned} \mu_n &= \left(p - \frac{1}{n} \right) \delta_{-1} + \left(1 - p - \frac{1}{n} \right) \delta_0 \\ &\quad + \frac{1}{n} \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} (\delta_{\lambda_{-(k)}} + \delta_{\lambda_{+(k)}}), \end{aligned} \quad (10)$$

where

$$\lambda_{\pm}(k) = \frac{k-1 \pm \sqrt{(k-1)^2 + 4k(n-k)}}{2}, \quad k = 0, 1, \dots, n.$$

Proof. Since

$$P(|V^{(1)}| = k, |V^{(0)}| = l) = \binom{n}{k} p^k (1-p)^l, \quad k+l=n,$$

the mean spectral distribution is given by

$$\mu = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^l \mu_{k,l}.$$

Then (10) follows from Theorem 5 by direct computation. \square

Corollary 2. *Let μ_n be mean spectral distribution of the binary threshold model $\mathcal{G}_n(p)$. Then we have*

$$\lim_{n \rightarrow \infty} \mu_n = p \cdot \delta_{-1} + (1-p) \cdot \delta_0.$$

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