

Perfect dominating sets in the Cartesian products of prime cycles.

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Abstract

We study the structure of a minimum dominating set of C_{2n+1}^n , the Cartesian product of n copies of the cycle of size $2n+1$, where $2n+1$ is a prime.

KEYWORDS: PERFECT LEE CODES; DOMINATING SETS; DEFINING SETS.

1 Introduction

Let G and H be two graphs. The *Cartesian product* of G and H is a graph with vertices $\{(x, y) : x \in G, y \in H\}$ where $(x, y) \sim (x', y')$ if and only if $x = x'$ and $y \sim y'$, or $x \sim x'$ and $y = y'$. Let G^n denote the Cartesian product of n copies of G . This article deals with C_{2n+1}^n where C_{2n+1} is the cycle of size $p := 2n+1$ and p is a prime.

For our purpose, it is more convenient to view the vertices of C_{2n+1}^n as the elements of the group $G := \mathbb{Z}_{2n+1}^n$. Then $x \sim y$ if and only if $x - y = \pm e_i$ for some $i \in [n]$, where $e_i = (0, \dots, 1, \dots, 0)$ is the unit vector with 1 at the i th coordinate. In other words, C_{2n+1}^n is the Cayley graph $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ over the group \mathbb{Z}_{2n+1}^n with the set of generators $\mathcal{U} = \{\pm e_1, \dots, \pm e_n\}$. From this point on, to emphasize the group structure of the graph we will use the Cayley graph notation $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ instead of the Cartesian product notation of C_{2n+1}^n .

Let u and v be two vertices of a graph G . We say that u dominates v if $u = v$ or $u \sim v$. A subset S of the vertices of G is called a *dominating set*, if every vertex of G is dominated by at least one vertex of S . A dominating set is *perfect*, if no vertex is dominated by more than one vertex.

Remark 1 Let G be a graph with m vertices. Every function $f : V(G) \rightarrow \mathbb{C}$ can be viewed as a vector $\vec{f} \in \mathbb{C}^m$. Let A denote the adjacency matrix of G . Note that $f : V(G) \rightarrow \{0, 1\}$ is the characteristic function of a perfect dominating set if and only if $(A + I)\vec{f} = \vec{1}$.

We are interested in perfect dominating sets of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$. Note that for an r -regular graph $G = (V, E)$ a dominating set is perfect if and only if it is of size $|V|/(r+1)$. Since $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ is $2n$ -regular and has $(2n+1)^n$ vertices, a dominating set is perfect if and only if it is of size $(2n+1)^{n-1}$.

Fix an arbitrary $(\epsilon_1, \dots, \epsilon_{n-1}) \in \{-1, 1\}^{n-1}$, and a $k \in \{0, \dots, 2n\}$. The set

$$\{(x_1, \dots, x_{n-1}, k + \sum_{i=1}^{n-1} \epsilon_i(i+1)x_i) : x_i \in \mathbb{Z}_{2n+1} \ \forall i \in [n-1]\} \quad (1)$$

forms a perfect dominating set in $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$, where the additions are in \mathbb{Z}_{2n+1} . To see this consider $y = (y_1, \dots, y_n) \in \mathbb{Z}_{2n+1}^n$. Let $t = k + \sum_{i=1}^{n-1} \epsilon_i(i+1)y_i$, and $\Delta = t - y_n \pmod{2n+1}$ so that $|\Delta| \leq n$. If $\Delta \in \{-1, 0, 1\}$ then y is dominated by (y_1, \dots, y_{n-1}, t) . If $\Delta \notin \{-1, 0, 1\}$, then with the notation $j := |\Delta| - 1$, y is adjacent to $(y_1, \dots, y_{j-1}, y_j - \epsilon_j \times \text{sgn}(\Delta), y_{j+1}, \dots, y_n)$, which can easily be seen that is in the considered set.

There are many results in the direction of constructing perfect dominating sets in the Cartesian product of cycles (see [5] and its references). However the authors are unaware of any result in the direction of characterizing the structure of perfect dominating sets. We consider the simplest case $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ where $2n+1$ is a prime. Even in this simple case we are unable to characterize all the perfect dominating sets. However we prove the following theorem in this direction.

Theorem 1 Let $2n+1$ be a prime and $S \subseteq \Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ be a perfect dominating set. Then for every $(x_1, \dots, x_n) \in \mathbb{Z}_{2n+1}^n$ and every $i \in [n]$,

$$|S \cap \{(y_1, \dots, y_n) : y_j = x_j \ \forall j \neq i\}| = 1.$$

Theorem 1 says that when $2n+1$ is a prime, every parallel-axis line contains exactly one point from every perfect dominating set of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$. It is easy to construct examples to show that the condition of $2n+1$ being a prime is necessary [4].

Let \mathcal{F} be a family of sets. For $S \in \mathcal{F}$, a set $D \subseteq S$ is called a *defining set* for (S, \mathcal{F}) (or for S when there is no ambiguity), if and only if S is the only superset of D in \mathcal{F} . The size of the minimum defining set for (S, \mathcal{F}) is called its defining number. Defining sets are studied for various families of \mathcal{F} (See [3] for a survey on the topic). Let \mathcal{F} be the family of all minimum dominating sets of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$. Note that since $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ is regular and contains at least one perfect dominating

set, a set $S \subseteq V(G)$ is a minimum dominating set if and only if it is a perfect dominating set. In [2] Chartrand et al. studied the size of defining sets of \mathcal{F} for $n = 2$. Based on this case they conjectured that the smallest defining set over all minimum dominating sets of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ is of size exactly n . As it is noticed by Richard Bean [private communication], the conjecture fails for $n = 3$, as in this case there are perfect dominating sets with defining number 2 (See Remark 3). So far there is no nontrivial bound known for the defining numbers of minimum dominating sets of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$. We prove the following theorem.

Theorem 2 *Let $2n+1$ be a prime and \mathcal{F} be the family of all minimum dominating sets of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$. Every $S \in \mathcal{F}$ has a defining set of size at most $n!2^n$.*

The proof of Theorem 1 uses Fourier analysis on finite Abelian groups. In Section 2 we review Fourier analysis on \mathbb{Z}_p^n . Section 3 is devoted to the proof of Theorem 2. Section 4 contains further discussions about the defining sets of minimum dominating sets of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$.

2 Background

In this section we introduce some notations and review Fourier analysis on $G = \mathbb{Z}_p^n$. For a nice and more detailed, but yet brief introduction we refer the reader to [1]. See also [6] for a more comprehensive reference.

Aside from its group structure we will also think of G as a measure space with the uniform (product) measure, which we denote by μ . For any function $f : G \rightarrow \mathbb{C}$, let

$$\int_G f(x)dx = \frac{1}{|G|} \sum_{x \in G} f(x).$$

The inner product between two functions f and g is $\langle f, g \rangle = \int_G f(x)\overline{g(x)}dx$. Let

$$\omega = e^{2\pi i/p},$$

where i is the imaginary number. For any $x \in G$, let $\chi_x : G \rightarrow \mathbb{C}$ be defined as

$$\chi_x(y) = \omega^{\sum_{i=1}^n x_i y_i}.$$

It is easy to see that these functions form an orthonormal basis. So every function $f : G \rightarrow \mathbb{C}$ has a unique expansion of the form $f = \sum \widehat{f}(x)\chi_x$, where $\widehat{f}(x) = \langle f, \chi_x \rangle$ is a complex number.

3 Proof of Theorem 1

Let $\vec{0} = (0, \dots, 0)$, $\vec{1} = (1, \dots, 1)$, and $e_i = (0, \dots, 1, \dots, 0)$, the unit vector with 1 at the i -th coordinate. Let $p = 2n + 1$ be a prime and S be a perfect dominating set in G , and let f be the characteristic function of S , i.e. $f(x) = 1$ if $x \in S$ and $f(x) = 0$ otherwise. Let

$$D = \{\pm e_1, \pm e_2, \dots, \pm e_n\},$$

be the set of unit vectors and their negations. For every $\tau \in D$ define $f_\tau(x) = f(x + \tau)$. Note that

$$\widehat{f}_\tau(y) = \int f(x + \tau) \overline{\chi_y(x)} dx = \int f(x) \overline{\chi_y(x - \tau)} dx = \int f(x) \overline{\chi_y(x)} \chi_y(\tau) dx = \widehat{f}(y) \chi_y(\tau).$$

Let

$$g = f + \sum_{\tau \in D} f_\tau.$$

We have

$$g = \left(\sum_{y \in G} \widehat{f}(y) \chi_y \right) + \sum_{\tau \in D} \sum_{y \in G} \widehat{f}_\tau(y) \chi_y = \sum_{y \in G} \widehat{f}(y) \left(\sum_{\tau \in D \cup \{\vec{0}\}} \chi_y(\tau) \right) \chi_y. \quad (2)$$

Since f is the characteristic function of a perfect dominating set, we have $g(x) = 1$, for every $x \in G$. So $g = \chi_{\vec{0}}$. By uniqueness of Fourier expansion, for every $y \neq \vec{0}$,

$$0 = \widehat{g}(y) = \widehat{f}(y) \sum_{\tau \in D \cup \{\vec{0}\}} \chi_y(\tau) = \widehat{f}(y) \left(1 + \sum_{i=1}^n \omega^{y_i} + \sum_{i=1}^n \omega^{-y_i} \right). \quad (3)$$

Now we turn to the key step of the proof. Since $2n + 1$ is a prime, (3) implies that whenever $\widehat{f}(y) \neq 0$, we have

$$\{y_1, \dots, y_n\} \cup \{-y_1, \dots, -y_n\} = \{1, \dots, 2n\}. \quad (4)$$

Denote the set of all y satisfying (4) by \mathcal{Y} . For $1 \leq i \leq n$, let

$$D_i = \{k e_i : 0 \leq k \leq 2n\}.$$

Define $g_i = \sum_{\tau \in D_i} f_\tau$. Similar to (2), we get

$$g_i = \sum \widehat{f}(y) \left(\sum_{\tau \in D_i} \chi_y(\tau) \right) \chi_y.$$

When $y \in \mathcal{Y}$, since $y_i \neq 0$, we have

$$\sum_{\tau \in D_i} \chi_y(\tau) = \sum_{k=0}^{2n} \omega^{ky_i} = 0.$$

When $y \notin \mathcal{Y}$ and $y \neq \vec{0}$, $\widehat{f}(y) = 0$. So

$$g_i = \left(\widehat{f}(0) \sum_{\tau \in D_i} \chi_{\vec{0}}(\tau) \right) \chi_{\vec{0}} = \chi_{\vec{0}} = 1. \quad (5)$$

Note that $g_i(x)$ counts the number of elements in $S \cap \{(y_1, \dots, y_n) : y_j = x_j \forall j \neq i\}$. This completes the proof.

Remark 2 The above proof can be translated to the language of linear algebra (However in the linear algebra language the key observation (4) becomes less obvious). Indeed, let $m = (2n+1)^n$ denote the number of vertices. From Remark 1 we know that $f : \mathbb{Z}_{2n+1}^n \rightarrow \mathbb{C}$ is the characteristic function of a perfect dominating set if and only if $(A + I)\vec{f} = \vec{1}$, where A is the adjacency matrix of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$. The reader may notice that in the proof of Theorem 1, $\vec{g} = (A + I)\vec{f}$, and thus (2) shows that $\vec{\chi}_y$ is a family of orthonormal eigenvectors of $A + I$. Moreover, among these eigenvectors, the ones that correspond to the 0 eigenvalue are exactly $\vec{\chi}_y$ with $y \in \mathcal{Y}$. Hence the rank of $A + I$ is $m - |\mathcal{Y}| = (2n+1)^n - n!2^n$. We will use this fact in the proof of Theorem 2.

4 Proof of Theorem 2

As it is observed in Remark 1, every perfect dominating set of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ corresponds to a zero-one vector $\vec{f} \in \mathbb{C}^m$ that satisfies $(A + I)\vec{f} = \vec{1}$. Let

$$V := \text{span}\{\vec{f} : f \in \mathcal{F}\}.$$

Trivially

$$\dim V \leq 1 + (m - \text{rank}(A + I)) = 1 + n!2^n.$$

Also for a subset D of vertices of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$, define

$$V_D := \text{span}\{\vec{f} : f \in \mathcal{F} \text{ and } \forall x \in D, f(x) = 1\},$$

Note that $V = V_\emptyset$.

To prove Theorem 2 we start from $D = \emptyset$. At every step, if D does not extend uniquely to S , then there exists a vertex $v \in S$ such that $\dim V_{D \cup \{v\}} < \dim V_D$; we add v to D . Since $\dim V_\emptyset \leq 1 + n!2^n$, we can obtain a set D of size at most $n!2^n$ such that the dimension of V_D is at most 1. This completes the proof as there is at most one non-zero, zero-one vector in a vector space of dimension 1.

5 Future directions

We ask the following question:

Question 1 For a prime $2n + 1$, are there examples of perfect dominating sets in $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ that are not of the form (1)?

If the answer to Question 1 turns out to be negative, then we can improve the bound of Theorem 2:

Proposition 1 Let $p = 2n + 1$ be a prime, and let \mathcal{T} denote the set of perfect dominating sets of the form (1). Every (S, \mathcal{T}) where $S \in \mathcal{T}$ has a defining set of size $1 + \lceil \frac{n-1}{\lfloor \log_2 p \rfloor} \rceil$.

Proof. Suppose that $S \in \mathcal{T}$. Then S is of the form:

$$\{(x_1, \dots, x_{n-1}, k + \sum_{i=1}^{n-1} \epsilon_i(i+1)x_i) : x_i \in \mathbb{Z}_p \forall i \in [n-1]\}.$$

Let $m = \lfloor \log_2 p \rfloor$. We will use the easy fact that for any $c \in \mathbb{Z}_p$, the equation $\sum_{i=0}^{m-1} \epsilon_i 2^i =_p c$ has at most one solution $(\epsilon_0, \epsilon_1, \dots, \epsilon_{m-1}) \in \{-1, +1\}^m$. For $i, j \geq 0$, define $\alpha_{i,j} \in \mathbb{Z}_p$ to be the solution to $(i+j+1)\alpha_{i,j} =_p 2^j$.

Let $u = (0, 0, \dots, 0, b)$ be the unique vertex in S with the first $n-1$ coordinates equal to 0, and for every $1 \leq i \leq n-1$ consider the unique vector

$$u_i = (0, \dots, 0, \alpha_{i,0}, \alpha_{i,1}, \dots, \alpha_{i,k_i}, 0, \dots, 0, b_i) \in S,$$

where $\alpha_{i,0}$ is in the i th coordinate and $k_i = \min(m-1, n-i-1)$. We claim that the set $D = \{u, u_0, u_m, \dots, u_{m(\lceil \frac{n-1}{m} \rceil - 1)}\}$ is a defining set for (S, \mathcal{T}) . Since S is of form (1), clearly $k = b$, and for every $0 \leq i \leq \lceil \frac{n-1}{m} \rceil - 1$, we have:

$$b_{mi} - b = \sum_{j=0}^{k_{mi}} \epsilon_{mi+j} (mi + j + 1) \alpha_{mi,j} = \sum_{j=0}^{k_{mi}} \epsilon_{mi+j} 2^j.$$

The above equation has only one solution $(\epsilon_{mi}, \epsilon_{mi+1}, \dots, \epsilon_{mi+k_{mi}}) \in \{-1, +1\}^{k_{mi}+1}$. Considering this for all $u_{mi} \in D$ determines $(\epsilon_1, \epsilon_2, \dots, \epsilon_{n-1})$. Thus the set D is a defining set for (S, \mathcal{T}) . ■

Remark 3 For $n = 2, 3$ the answer to Question 1 is negative. Thus when $n = 3$, Proposition 1 implies that there is a defining set of size 2 for a perfect dominating set. This disproves the conjecture of [2] which is already observed by Richard Bean [private communication].

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