

**Eigenfunctions on the Finite Poincaré Plane**  
*by Jinghua Kuang \**

**1. Introduction.** Let  $F_q$  be a finite field with  $q$  elements ( $q$  odd.) Fix a non-square elemnet  $\delta \in F$ .

$$H_q = F_q(\sqrt{\delta}) - F_q = \{x + y\sqrt{\delta} \mid x, y \neq 0 \in F_q\}$$

is called the finite Poincaré plane. From [1], [2], [4], [7], we have the following facts:

1.  $G = \mathrm{GL}_2(F_q)$  acts on  $H_q$  by linear transformation:

$$gz = \frac{az + b}{cz + d}, \quad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \forall z \in H_q.$$

Let  $K = \left\{ \begin{pmatrix} a & b\delta \\ b & a \end{pmatrix} \right\}$ . One may identify  $G/K$  with  $H_q$  and with  $P = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \mid y \neq 0, x \in F_q \right\}$ .

2. There is a  $G$ -invariant pseudo-distance  $\Delta$  on  $H_q$ :

$$\Delta(z_1, z_2) = \frac{N(z_1 - z_2)}{\mathrm{Im}z_1 \cdot \mathrm{Im}z_2} \quad \forall z_1, z_2 \in H_q,$$

where  $N$  is the norm from  $F_q(\sqrt{\delta})$  to  $F_q$  and  $\mathrm{Im}z = y$  if  $z = x + y\sqrt{\delta}$ .

3. For  $a \in F_q^\times$  and  $a \neq 4\delta$ , a graph structure  $X_q(\delta, a)$  is defined on  $H_q$ :  $(z_1, z_2)$  is an edge iff  $\Delta(z_1, z_2) = a$ . Katz [3] and Li [5] proved that  $X_q(\delta, a)$  are Ramanujan graphs using different methods.

However, distributions of eigenvalues of the adjacency matrices have not been well understood. We will calculate the first and second moments of the asymptotic distribution of the eigenvalues of the adjacency matrices and provide evidence for Terras' conjecture in [7].

Let  $V$  be the space of all complex functions on  $H_q$ . Using character sums, Evans [2] constructed an orthogonal basis of  $V$  that diagonalizes all the adjacency matrices. Kuang [4], using group representation theory, also constructed such an orthogonal basis. This paper will also compare these two bases.

**2. Hecke Algebra.** Since  $K$  is the analogue of  $O(2)$  in the classical Poincaré upper half plane, it is natural to consider the Hecke algebra  $H(G, K)$ , which is defined as the algebra of all bi- $K$ -invariant complex functions on  $G$  under convolution. Since  $K$  is the isotropic subgroup of  $\sqrt{\delta}$ ,  $\Delta(g, 1) = \Delta(g^{-1}, 1)$  implies that  $KgK = Kg^{-1}K$ . Therefore,  $H(G, K)$  is commutative. Obviously,  $H(G, K)$  has dimension  $q$ . Hence there exist  $q$  idempotents in  $H(G, K)$ .

**3. Idempotents.** For an irreducible representation  $\pi$  of  $G$ , let  $\eta'_\pi(g) = \frac{1}{|K|} \sum_{k \in K} \mathrm{tr} \pi(kg)$ .  $\pi$  is spherical (i.e.  $\mathrm{Res}_K^G(\pi) \supset 1_K$ ) iff  $\eta'_\pi(g) \neq 0$ . From [4], we have

$$\eta'_{\pi_1} * \eta'_{\pi_2} = \begin{cases} 0, & \text{if } \pi_1 \not\simeq \pi_2; \\ \frac{|G|}{\dim(\pi)} \eta'_\pi, & \text{if } \pi_1 \simeq \pi_2 \simeq \pi. \end{cases}$$

Let  $\omega$  be a multiplicative character of  $F_q(\sqrt{\delta})$  of order  $q^2 - 1$ ,  $\chi$  be a multiplicative character of order  $q - 1$ , and  $s = \chi^{(q-1)/2}$ . There are  $q$  irreducible spherical representations of  $G$  (see [6]):

$$\begin{aligned} \pi &= 1_G; \\ \text{or} &= \rho(s, s) = q \text{ dimensional irreducible component of } \mathrm{Ind}_B^G(s, s); \\ \text{or} &= \rho(\chi^j, \bar{\chi}^j) = \mathrm{Ind}_B^G(\chi^j, \bar{\chi}^j), \quad j = 1, \dots, \frac{q-3}{2}; \\ \text{or} &= \rho_{\nu_j} = \text{cuspidal representation corresponding to } \nu_j = \omega^{j(q-1)}, \quad j = 1, \dots, \frac{q-1}{2}. \end{aligned}$$

( $B$  is the Borel subgroup of  $G$ .) Let  $\eta_\pi = \frac{\dim(\pi)}{|G|} \eta'_\pi$ . Let  $\eta_0 = \eta_{1_G}$ ,  $\eta_j = \eta_{\rho(\chi^j, \bar{\chi}^j)}$ ,  $j = 1, \dots, (q-3)/2$ ,  $\eta_{\frac{q-1}{2}} = \eta_{\rho(s, s)}$  and  $\eta_{\frac{q-1}{2}+j} = \eta_{\rho_{\nu_j}}$ ,  $j = 1, \dots, (q-1)/2$ . Then

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**Theorem 1 (Kuang [4]).**  $\eta_0, \eta_1, \dots, \eta_{q-1}$  are the  $q$  idempotents in  $H(G, K)$ .

**4. Representation of  $H(G, K)$ .**  $H(G, K)$  has a natural representation on  $L^2(H_q) = L^2(P)$ , the space of all complex functions on  $H_q = P$ :

$$T_\varphi(f)(p) = \frac{1}{|K|} f * \varphi(p) = \frac{1}{|K|} \sum_{h \in G} f(h) \varphi(h^{-1}p) = \sum_{p_1 \in P} f(p_1) \varphi(p_1^{-1}p),$$

for  $\varphi \in H(G, K)$ ,  $f \in L^2(P)$  and  $p \in P$ .

For the usual inner product  $\langle \cdot, \cdot \rangle$  on  $L^2(P)$ , we have  $\langle T_\varphi(f_1), f_2 \rangle = \langle f_1, T_{\hat{\varphi}}(f_2) \rangle$ , where  $\hat{\varphi}(h) = \overline{\varphi(h^{-1})}$ . That is,  $H(G, K)$  acts on  $L^2(P)$  self-adjointly. Hence, there exists a basis of simultaneous eigenfunctions of  $H(G, K)$ .

**5. Construction of the Bases.** Let  $\psi$  be a fixed non-trivial additive character of  $F_q$ . Denote  $\psi_a(x) = \psi(ax)$ . Define

$$\chi_i(g) = \chi^i(y) \text{ if } gK = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} K, i = 1, \dots, q-1.$$

$$\Psi_a(g) = \delta(y)\psi_a(x) \text{ if } gK = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} K, a \in F_q^\times,$$

where  $\delta(y) = 1$  if  $y = 1$ ;  $= 0$  otherwise. Then

**Theorem 2 (Kuang [4]).**  $\chi_i, i = 1, \dots, q-1$  and  $\Psi_a * \eta_i, i = 1, \dots, q-1, a \in F_q^\times$  are simultaneous eigenfunctions of  $H(G, K)$  and make up an orthogonal basis of  $L^2(P)$ . Moreover,  $T_{\eta_j}(\chi_i) = \chi_i$  if  $\eta_j = \eta_{\rho(\chi^{\pm i}, \bar{\chi}^{\pm i})}$ ;  $= 0$  otherwise.  $T_{\eta_j}(\Psi_a * \eta_i) = \Psi_a * \eta_i$  if  $i = j$ ;  $= 0$  otherwise.

Terras [8] constructed the following functions:

$$K_{i,\psi_a}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \psi_a(-x)\chi^i(y) \sum_{u \in F_q} \psi_a(u)\bar{\chi}^i(u^2 - \delta y^2).$$

Evans [2], based on Velasquez's work according to [9], constructed the following functions:

$$H_{t,i,\psi_a}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \frac{\psi_a(-x)}{q+1} \sum_{u \in F_q} {}^*\psi_a(u) \sum_{\alpha \in U_1} \omega^i(\alpha) s\left(\alpha + \frac{1}{\alpha} + \frac{\delta y}{t} + \frac{t}{\delta y} - \frac{u^2}{ty}\right),$$

where  $U_1 = \{\alpha \in F_q^\times \mid N(\alpha) = 1\}$  and the asterisk indicates that when  $y = \pm t$ , the  $u = 0$  term is multiplied by  $q+1$ .

**Theorem 3 (Terras, Velasquez, Evans [2]).**  $K_{i,\psi_a}, i = 1, \dots, q-1, a \in F_q$  and  $H_{t,i,\psi_a}, t \in F_q^\times, i = 1, \dots, q-1, a \in F_q$  are simultaneous eigenfunctions of  $H(G, K)$ . There exist a  $t_0$  for any fixed  $i$  and  $a$  such that  $K_{i,\psi_0}, i = 1, \dots, q-1, K_{i,\psi_a}, i = 1, \dots, (q-1)/2, a \in F_q$  and  $H_{t_0,i,\psi_a}, i = 1, \dots, (q-1)/2, a \in F_q^\times$  make up an orthogonal basis of  $L^2(P)$ .

**6. Comparison of the Bases.** We now prove that the two bases constructed above are essentially same. Let us denote by  $\pi_i$  the spherical representation that give rise to  $\eta_i$ .

**Theorem 4.** The following identities hold.

$$\begin{aligned} \overline{C_i(0)} \cdot \chi_i &= K_{i,1}, i = 1, \dots, q-1; \\ \Psi_a * \eta_i &= \frac{\dim(\pi_i)}{|G|(q+1)} C_i(a) K_{i,\psi_{-a}}, i = 1, \dots, (q-1)/2; \\ \Psi_a * \eta_{i+(q-1)/2} &= \frac{\dim(\pi_{i+(q-1)/2})}{|G|} H_{-\delta,i,\psi_{-a}}, i = 1, \dots, (q-1)/2, \end{aligned}$$

where  $C_i(a) = \sum_{x \in F_q} \psi_a(x) \chi^i(x^2 - \delta)$ .

**Proof.**  $K_{i,1} = \overline{C_i(0)} \cdot \chi_i$  is obvious. Since for any pair of  $a, b \in F_q$ ,  $K_{i,\psi_b}$  and  $\Psi_a * \eta_i$  are eigenfunctions of  $H(G, K)$  of the same eigenvalues ( $i = 1, \dots, (q-1)/2$ ),  $\Psi_a * \eta_i$  is a complex linear combination of  $K_{i,\psi_b}$ , ( $b \in F_q$ ). Now,

$$\begin{aligned} \langle K_{i,\psi_b}, \Psi_a * \eta_i \rangle &= \sum_{p \in P} K_{i,\psi_b}(p) \overline{\Psi_a * \eta_i}(p) \\ &= \sum_{p \in P} \sum_{u \in F_q} \psi_b(u-x) \chi^i(y) \overline{\chi^i(u^2 - \delta y^2)} \overline{\eta_i}(p) \sum_{v \in F_q} \psi(-(a+b)v) \\ &= 0 \quad \text{if } a+b \neq 0 \end{aligned}$$

Hence  $\Psi_a * \eta_i$  is a multiple of  $K_{i,\psi_{-a}}$ . Using the equation (2.17) in [2], we calculate

$$\begin{aligned} \Psi_a * \eta_i(1) &= \sum_{v \in F_q} \psi_a(v) \eta_i\left(\begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}\right) \\ &= \frac{\dim(\pi_i)}{|G|(q+1)} C_i(a) \overline{C_i(a)}. \end{aligned}$$

And  $K_{i,\psi_{-a}}(1) = \overline{C_i(a)}$ . Therefore,  $\Psi_a * \eta_i = \frac{\dim(\pi_i)}{|G|(q+1)} C_i(a) K_{i,\psi_{-a}}$ .

For the third identity, we use the equation (2.16) in [2]. Let  $|S(z)|$  be the cardinality of the  $K$ -orbit of  $z$  in  $H_q$ . Let  $\hat{i} = i + (q-1)/2$ . Then,

$$\begin{aligned} \Psi_a * \eta_{\hat{i}}(p) &= \sum_{u \in F_q} \psi_a(u) \eta_{\hat{i}}\left(\begin{pmatrix} y & x-u \\ 0 & 1 \end{pmatrix}\right) \\ &= \sum_{u \in F_q} \psi_a(x-u) \eta_{\hat{i}}\left(\begin{pmatrix} y & u \\ 0 & 1 \end{pmatrix}\right) \\ &= \psi_a(x) \sum_{u \in F_q} \psi_a(-u) \frac{\dim(\pi_{\hat{i}})}{|G|} \frac{1}{|S(u+y\sqrt{\delta})|} \sum_{\alpha \in U_1} \omega^i(\alpha) s\left(\alpha + \frac{1}{\alpha} - 2 + \frac{\Delta(u+y\sqrt{\delta}, \sqrt{\delta})}{\delta}\right) \\ &= \psi_a(x) \sum_{u \in F_q} \psi_a(-u) \frac{\dim(\pi_{\hat{i}})}{|G|} \frac{1}{|S(u+y\sqrt{\delta})|} \sum_{\alpha \in U_1} \omega^i(\alpha) s\left(\alpha + \frac{1}{\alpha} + \frac{u^2}{y\delta} - y - \frac{1}{y}\right) \\ &= \frac{\dim(\pi_{\hat{i}})}{|G|} H_{-\delta, i, \psi_{-a}}(p) \end{aligned}$$

**7. Comments and Questions.** For any function  $f$  on  $G$ , define the Fourier coefficient

$$F(g; a, f) = \sum_{x \in F_q} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi_a(x).$$

From [4], we see  $\langle \Psi_a * \eta_i, \Psi_a * \eta_i \rangle = |K| \cdot q \cdot F(1; a, \eta_i)$ . So  $F(1; a, \eta_i) \neq 0$ . But can one evaluate  $F(1; a, \eta_i)$ ? We have  $F(1; a, \eta_i) = \Psi_{-a} * \eta_i(1)$ . Hence, for  $i = 1, \dots, (q-1)/2$ ,

$$\begin{aligned} F(1; a, \eta_i) &= \frac{\dim(\pi_i)}{|G|(q+1)} |C_i(-a)|^2 = \frac{\dim(\pi_i)}{|G|(q+1)} |C_i(a)|^2 \\ F(1; a, \eta_i) &= \frac{\dim(\pi_{\hat{i}})}{|G|} \sum_{u \in F_q} \psi_a(u) \frac{1}{|S(u+\sqrt{\delta})|} \sum_{\alpha \in U_1} \omega^i(\alpha) s\left(\alpha + \frac{1}{\alpha} + \frac{u^2}{\delta} - 2\right) \end{aligned}$$

However, we don't know much about  $C_i(a)$  and about the last set of sums. We note that  $C_i(0)$  is essentially a Jacobi sum, in fact,  $C_i(0) = -\chi^i(-\delta) J(\chi^i, s)$ .

**8. Connection to  $X_q(\delta, a)$ .** There are  $q$  double cosets:  $D_a = KgK$ , where  $\Delta(g, 1) = a \in F_q$ . Let  $\varphi_a$  be the characteristic function of the set  $D_a$ . Then  $A_a = \frac{1}{|K|}T_{\varphi_a}$  acting on  $L^2(H_q)$  is the adjacency matrix of  $X_q(\delta, a)$ . Therefore each basis constructed in Section 5 diagonalizes all the adjacency matrices. Let  $S_a = D_a \cap P$ . Then

$$\lambda_i(a) = \frac{|G|}{\dim(\pi_i)} |S_a| \eta_i(D_a), \quad i = 0, 1, \dots, q-1$$

are all the eigenvalues of  $A_a$ . Katz [3], Li [5] proved that  $|\lambda_i(a)| \leq 2\sqrt{q}$  for  $i = 1, \dots, q-1$ , which confirms that  $X_a(\delta, a)$  is Ramanujan.

Fix  $a \neq 4\delta \in F^\times$ , Terras [7] conjectured that  $\{\lambda_i(a)/\sqrt{q} \mid i = 1, \dots, q-1\}$  asymptotically has Sato-Tate distribution, i.e. for  $E \subset [-2, 2]$ ,

$$\lim_{q \rightarrow \infty} \frac{1}{q-1} |\{\lambda_i(a) / \lambda_i(a)/\sqrt{q} \in E\}| = \frac{1}{2\pi} \int_E \sqrt{4-x^2} dx.$$

**9. Moments.** Define a  $q \times q$  matrix

$$M = \left( \sqrt{\frac{|S_a|}{\dim(\pi_i)}} \eta_i(S_a) \right)_{i=0,1,\dots,q-1, a \in F_q}.$$

The idempotent property of  $\eta_i$  ( $i = 0, \dots, q-1$ ) implies that  $MM' = \frac{1}{|K||G|}I_q$ . ( $M'$  is the transpose of  $M$ .) Hence,  $M'M = \frac{1}{|K||G|}I_q$ , that gives

$$\begin{aligned} \frac{1}{q-1} \sum_{i=0}^{q-1} \frac{\dim(\pi_i)}{q} \cdot \frac{\lambda_i(a)}{\sqrt{q}} &= 0 \\ \frac{1}{q-1} \sum_{i=0}^{q-1} \frac{\dim(\pi_i)}{q} \cdot \left( \frac{\lambda_i(a)}{\sqrt{q}} \right)^2 &= \frac{|G||S_a|}{|K|q^2(q-1)} \end{aligned}$$

That, in turn, implies

$$\begin{aligned} \lim_{q \rightarrow \infty} \frac{1}{q-1} \sum_{i=1}^{q-1} \frac{\lambda_i(a)}{\sqrt{q}} &= 0 \\ \lim_{q \rightarrow \infty} \frac{1}{q-1} \sum_{i=1}^{q-1} \left( \frac{\lambda_i(a)}{\sqrt{q}} \right)^2 &= 1. \end{aligned}$$

Therefore, the first and second moments of  $\{\lambda_i(a)/\sqrt{q} \mid i = 1, \dots, q-1\}$  asymptotically match with those of the Sato-Tate distribution.

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