

The Maximal Probability that k -wise Independent Bits are All 1

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Abstract

A k -wise independent distribution on n bits is a joint distribution of the bits such that each k of them are independent. In this paper we consider k -wise independent distributions with identical marginals, each bit has probability p to be 1. We address the following question: how high can the probability that all the bits are 1 be, for such a distribution? For a wide range of the parameters n, k and p we find an explicit lower bound for this probability which matches an upper bound given by Benjamini et al., up to multiplicative factors of lower order. In particular, for fixed k , we obtain the sharp asymptotic behavior. The question we investigate can be viewed as a relaxation of a major open problem in error-correcting codes theory, namely, how large can a linear error correcting code with given parameters be?

The question is a type of discrete moment problem, and our approach is based on showing that bounds obtained from the theory of the classical moment problem provide good approximations for it. The main tool we use is a bound controlling the change in the expectation of a polynomial after small perturbation of its zeros.

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1 Introduction

The problem of generalized inclusion-exclusion inequalities has been considered by many authors [B1854, B37, DS67, K75, P88, BP89, GX90, LN90]. In this problem one has n events A_1, \dots, A_n and the probabilities of intersections $\cap_{i \in S} A_i$ for all S with $|S| \leq k$. Given this information the goal is to bound the probability of $\cup_{i=1}^n A_i$ from above and from below. The classical Bonferroni inequalities state that the odd and even partial sums of the inclusion-exclusion formula provide such upper and lower bounds, respectively. But in many cases these bounds are far from being sharp, in the sense that much tighter bounds may be deduced from the same information.

In this paper we address a special case of this question. In our setting the events all have equal probability $\mathbb{P}(A_i) = p$ and are k -wise independent; that is, $\mathbb{P}(\cap_{i \in S} A_i) = p^{|S|}$ whenever $|S| \leq k$. When referring to this case we shall use a slightly different terminology and refer to the events A_1, \dots, A_n as n bits. For convenience, we consider the intersection of events instead of the union, which is equivalent by de Morgan's rules. With this terminology we are interested in estimating the probability of the AND of the bits given that their joint distribution is k -wise independent with identical marginals p . Besides the simplification arising from considering a particular case, this case is of special interest from several points of view.

First, k -wise independent distributions play a key role in the computer science literature where they are used for derandomization (there are many references, e.g. the survey [LW95]). Here is an example for the use of k -wise independence in this context: Assume that a given efficient *probabilistic* algorithm A works, even when the algorithm uses pairwise independent bits instead of truly independent random bits. Since there are pairwise independent distributions with small support, this implies that the algorithm can be converted to an *efficient deterministic* algorithm. In order to prove that A indeed works with access to only pairwise independent bits, one needs to show that the probabilities of certain events (that depend on A) do not change significantly when “moving” to a pairwise independent distribution.

Second, there is a strong connection between linear error correcting codes and k -wise independent distributions (when $p = \frac{1}{q}$ for a prime power q). Given a linear error-correcting code over $(GF(q))^n$ with minimal distance d , one may obtain a k -wise independent distribution with $k = d - 1$ and $p = \frac{1}{q}$.

by sampling uniformly at random from the dual of the code and replacing the resulting codeword by the indicator word of its zeros. Although by this construction one gets only distributions with a certain structure, this is by far the most common way to construct k -wise independent distributions. It gives a simple connection between the size of the code C , and the probability of getting the all 1's vector:

$$\mathbb{P}[(1, \dots, 1)] = \frac{1}{|C^\perp|} = \frac{|C|}{q^n},$$

where the probability is over the k -wise independent distribution constructed from C , and C^\perp is the dual of C . A very basic and open question in the theory of error correcting codes is how large can a linear error-correcting code be, for given n, d, q ([MS77], see also [DY04]). A large code immediately implies a large probability for the AND of the bits, hence investigating the maximal probability that the AND event can achieve for a given triplet n, k, p can be thought of as a relaxation of the error correcting codes question. However, in general, these two questions turn out not to be equivalent, even asymptotically in n , as an example from [BGP] shows:

- (i) For every 3-wise independent distribution μ on n bits with marginal probabilities $1/3$ that is obtained from a linear code as (roughly) described above, $\mu[(1, \dots, 1)] = O(\frac{1}{n \log n})$ (this is a version of Roth's theorem on 3-term arithmetic progressions for $(GF(3))^n$, see [M95].)
- (ii) There exists a 3-wise independent distribution μ' on n bits with marginal probabilities $1/3$ such that $\mu'[(1, \dots, 1)] = \Omega(\frac{1}{n})$.

An important property of the code-based constructions of k -wise independent distributions is that such distributions have small support. The support size is important for derandomization, as discussed above. In this paper we show existence of k -wise independent distributions that assign large probability to $(1, \dots, 1)$, but we do not show that they have small support.

Third, the question has intrinsic mathematical beauty. From an analytic perspective, when attempting its solution one is naturally led to discrete analogues of classical moment problems (classical quadrature formulas). Although some investigation of such discrete moment problems exists in the literature [KN77, Chap. VIII],[P88, BP89], they are much less understood

than their classical counterparts. Still, the classical theory sheds light on our problem and enables us to make progress on it and obtain quite precise answers. From a more geometric standpoint, the set of k -wise independent distributions is an interesting convex body, the structure of which we understand quite poorly. In this work we try to at least understand the projection of this body in one specific direction.

Finally, in the case $p = \frac{1}{2}$, the maximal probability of the AND event is also the maximal probability for any fixed string of bits (roughly, ‘translating’ a distribution by a constant vector, does not ‘affect’ the k -wise independence). In other words, for $p = \frac{1}{2}$ this maximal probability corresponds to the minimal min-entropy possible for a k -wise independent distribution, which seems a very basic property.

This work continues a previous work [BGP] in which an (explicit) upper bound for the AND event was found (as well as some lower bounds). The upper bound was derived as a solution to a relaxed maximization problem (see Section 3) which appears quite similar to the original problem. The similarity makes it natural to expect that the upper bound be quite close to the true maximal probability. Indeed, in this work we affirm this expectation in a large regime of the parameters.

1.1 Results

Denote by $M(n, k, p)$ the maximal probability of the AND event for a k -wise independent distribution on n bits with marginals p . For odd k it is shown in [BGP] that

$$M(n, k, p) = pM(n - 1, k - 1, p) \quad (k \text{ odd}), \quad (1.1)$$

hence it is enough to consider the case of even k . It is also shown there that

$$M(n, k, p) \leq \tilde{M}(n, k, p) \quad (1.2)$$

where $\tilde{M}(n, k, p)$ is the solution to a certain maximization problem (see Section 3) and satisfies for even k ,

$$\tilde{M}(n, k, p) = \frac{p^n}{\mathbb{P}(\text{Bin}(n, 1 - p) \leq \frac{k}{2})}. \quad (1.3)$$

Our main result is a lower bound for $M(n, k, p)$ matching the bound given by $\tilde{M}(n, k, p)$ up to multiplicative factors of lower order, in a large regime of the parameters. Specifically:

Theorem 1.1. *There exist constants $c_1, c_2, c_3 > 0$ such that the following holds. Let $n \in \mathbb{N}$, $k \in \mathbb{N}$ even, and $0 < p < 1$. Let $N = np(1 - p) - 1$. Assume*

$$k \leq c_1 \cdot N. \quad (1.4)$$

Then,

$$M(n, k, p) \geq \frac{c_3}{k} \exp\left(-c_2 \cdot \frac{k}{V(N/k)}\right) \tilde{M}(n, k, p), \quad (1.5)$$

where $V(a) = \exp\left(\sqrt{\log(a) \log \log(a)}\right)$.

The cases where (1.4) does not hold are not covered by Theorem 1.1. Some partial results on these cases were given in [BGP]. For the case $n(1 - p) \leq \frac{k}{2}$ the bound $p^n \leq M(n, k, p) \leq \tilde{M}(n, k, p) \leq 2p^n$ was shown, and for the case $(n - 1)p \leq 1$ it was shown that $M(n, k, p) = p^k$. The case $k = 2$ was also solved there.

To better understand the bound given in Theorem 1.1, we present some particular cases in the following

Corollary 1.2. *There exist $C, c > 0$ such that for all $n \in \mathbb{N}$, $k \in \mathbb{N}$ even, and $0 < p < 1$, letting $N = np(1 - p) - 1$ we have*

1. *For every $m > 0$, there exists $N_0 = N_0(m)$ such that if $N > N_0$ and $k \leq (\log N)^m$, then*

$$M(n, k, p) \geq \frac{c}{k} \tilde{M}(n, k, p).$$

2. *For every $0 < \beta < 1$, there exists $c(\beta) > 0$ and $N_0 = N_0(\beta)$ such that if $N > N_0$ and $k \leq N^\beta$, then*

$$M(n, k, p) \geq c \exp\left(-\frac{k}{\exp(c(\beta)) \sqrt{\log(k) \log \log(2k)}}\right) \tilde{M}(n, k, p).$$

3. *For any k satisfying $k \leq cN$,*

$$M(n, k, p) \geq ce^{-Ck} \tilde{M}(n, k, p).$$

By estimating $\tilde{M}(n, k, p)$ (using (1.3) and Claim 4.2 below) in the first two cases of the above corollary we obtain, using (1.2), explicit two-sided bounds on $M(n, k, p)$. They show that for a large range of the parameters, the leading order behavior of $M(n, k, p)$ is identified and for the case of constant k , the exact asymptotics is determined, as follows:

Corollary 1.3. *There exist $C, c > 0$ such that for all $n \in \mathbb{N}$, $k \in \mathbb{N}$ even, and $0 < p < 1$, letting $N = np(1 - p) - 1$ we have*

1. *For every $m > 0$, there exists $N_0 = N_0(m)$ such that if $N > N_0$ and $k \leq (\log N)^m$, then*

$$\frac{c}{\sqrt{k}} \left(\frac{pk}{2e(1-p)n} \right)^{k/2} \leq M(n, k, p) \leq C\sqrt{k} \left(\frac{pk}{2e(1-p)n} \right)^{k/2}. \quad (1.6)$$

2. *For every $0 < \beta < 1$, there exists $c(\beta) > 0$ and $N_0 = N_0(\beta)$ such that if $N > N_0$ and $k \leq N^\beta$, then*

$$\frac{1}{C} e^{-\frac{k}{U(k,\beta)}} \left(\frac{pk}{2e(1-p)n} \right)^{k/2} \leq M(n, k, p) \leq C\sqrt{k} e^{\frac{k^2}{2n}} \left(\frac{pk}{2e(1-p)n} \right)^{k/2}, \quad (1.7)$$

where $U(k, \beta) \stackrel{\text{def}}{=} \exp(c(\beta)\sqrt{\log(k) \log \log(2k)})$.

Let us compare this with known results, our novelty is in the lower bounds and so we only compare these. As far as the authors are aware, the best known lower bounds for $M(n, k, p)$ come from error-correcting codes and apply to the cases when $p = \frac{1}{q}$ for a prime power q . The most important case for applications is $p = \frac{1}{2}$. In this case it was known using BCH codes ([MS77],[AS00, Chapter 15]) that $M(n, k, \frac{1}{2}) \geq \left(\frac{c_1}{n}\right)^{\lfloor k/2 \rfloor}$ and also using the Gilbert-Varshamov bound [MS77] that $M(n, k, \frac{1}{2}) \geq c_2 \left(\frac{c_3(k-1)}{n}\right)^{k-1}$ for some constants $c_1, c_2, c_3 > 0$. In both cases our bound improves on the known asymptotic results for $k = o(n)$, but still growing to infinity with n .

Other cases where lower bounds were known are the cases in which $p = \frac{1}{q} \neq \frac{1}{2}$ for a prime power q . In these cases much less is known and even for the case of constant k and p , the best results we are aware of are of the form $M(n, k, p) \geq n^{-\alpha(k,p)(1+o(1))}$ where, except for a few cases, $\alpha(k, p)$ is strictly larger than $\lfloor \frac{k}{2} \rfloor$ (see [DY04] for a survey of such results). For example, in the

case $p = \frac{1}{3}$ and constant $k \geq 7$ it appears that the best known asymptotic result in n was $M(n, k, \frac{1}{3}) \geq cn^{-\lceil 2(k-1)/3 \rceil}$. Our results show that the correct asymptotic behavior for constant k and p is $M(n, k, p) = \Theta(n^{-\lfloor k/2 \rfloor})$.

Here is a high-level description of the proof of Theorem 1.1. We start by employing linear programming duality as in [BGP]. This duality shows that $M(n, k, p)$ is the minimum of the expectation $\mathbb{E}f(X)$, where $X \sim \text{Bin}(n, p)$, over all polynomials f from a certain class (see (2.5)). A similar duality shows that $\tilde{M}(n, k, p)$ is the minimum of the expectation $\mathbb{E}g(X)$, where $X \sim \text{Bin}(n, p)$, over all polynomials g from a strictly smaller class than that of the first minimization problem (see (3.2)). This latter minimization problem is exactly solvable using the methods of the classical moment problem. We continue by associating to each polynomial f from the class of the first problem, a polynomial g from the class of the second problem, obtained by perturbing the roots of f . It thus follows that

$$\frac{\tilde{M}(n, k, p)}{M(n, k, p)} \leq \max \left(\frac{\mathbb{E}g(X)}{\mathbb{E}f(X)} \right)$$

where the maximum ranges over all polynomials f from the class of the first problem and g is the polynomial associated to f . A bound for the RHS of the above inequality which yields Theorem 1.1 is then given by Theorem 4.1. Our methods can be used to bound the ‘change’ in expectation for other distributions as well (see Section 5 for more details). Such an argument can be applied to other problems where there is a classical moment problem analogue to discrete problems. It thus seems that Theorem 4.1 and its proof might be of independent interest.

Outline Section 2 gives a more precise description of the question we consider, and explains some useful facts about it, including the use of linear programming duality. Section 3 describes the relaxed version of the problem with emphasis on its similarity to the original problem. Our main result is explained in Section 4 where the result on polynomials and the reduction between them are described. We also do the computations needed to obtain Corollary 1.3 there. Finally, Section 5 proves the result on polynomials. Some open problems are presented in Section 6. For completeness, the appendix gives short proofs for the results of [BGP] that we use.

2 The problem and its dual

In this section we introduce notation for our problem and present it in more precise terms. We then continue to describe the dual of the problem, on which we shall concentrate in the following sections. Let $\mathcal{A}(n, k, p)$ be the set of all probability distributions on $\{0, 1\}^n$ which are k -wise independent and have identical marginals p . In other words, the distribution of (X_1, \dots, X_n) belongs to $\mathcal{A}(n, k, p)$ if $\mathbb{P}(\forall i \in S X_i = 1) = p^{|S|}$ for all S with $|S| \leq k$. Thinking of $\mathcal{A}(n, k, p)$ as a body in \mathbb{R}^{2^n} , it is convex. Hence, bounding the probability of the event $\text{AND} = \{\forall 1 \leq i \leq n X_i = 1\}$ under all probability distributions in $\mathcal{A}(n, k, p)$ is the same as finding

$$M(n, k, p) = \max_{\mathbb{Q} \in \mathcal{A}(n, k, p)} \mathbb{Q}(\text{AND}) \quad (2.1)$$

$$m(n, k, p) = \min_{\mathbb{Q} \in \mathcal{A}(n, k, p)} \mathbb{Q}(\text{AND}). \quad (2.2)$$

In [BGP] it was shown that for many choices of the parameters n, k, p we have $m(n, k, p) = 0$, making the bound in this direction perhaps less useful. In this work we concentrate on estimating M .

A simplification of problems (2.1) and (2.2) is possible: Define the set

$$\mathcal{A}^s(n, k, p) \subseteq \mathcal{A}(n, k, p)$$

to be the set of symmetric distributions in $\mathcal{A}(n, k, p)$; that is, the joint distribution of (X_1, \dots, X_n) is in $\mathcal{A}^s(n, k, p)$ if it is in $\mathcal{A}(n, k, p)$ and (X_1, \dots, X_n) are exchangeable. Since the **AND** event is symmetric, one can show that

$$M(n, k, p) = \max_{\mathbb{Q} \in \mathcal{A}^s(n, k, p)} \mathbb{Q}(\text{AND}) \quad (2.3)$$

$$m(n, k, p) = \min_{\mathbb{Q} \in \mathcal{A}^s(n, k, p)} \mathbb{Q}(\text{AND}). \quad (2.4)$$

Note further that a distribution in $\mathcal{A}^s(n, k, p)$ may be identified with the integer random variable S which counts the number of bits that are 1. Note that such an S has the following properties:

- (I) S is supported on $\{0, 1, \dots, n\}$.
- (II) $\mathbb{E}S^i = \mathbb{E}X^i$ for $X \sim \text{Bin}(n, p)$ and $1 \leq i \leq k$.

The converse also holds (see [BGP]); that is,

Lemma 2.1. *For each random variable S satisfying (I) and (II), there exists $\mathbb{Q} \in \mathcal{A}^s(n, k, p)$ such that S has the distribution of the number of bits which are 1 under \mathbb{Q} .*

Relying on Lemma 2.1, we shall henceforth identify $\mathcal{A}^s(n, k, p)$ with distributions S satisfying (I) and (II) above. There is a short argument given below showing that the distribution S achieving the maximum in (2.3) is unique. Similar arguments are used in [KN77].

We can now think of problem (2.3) as a linear programming problem in $n+1$ variables, namely, find the maximum of $\mathbb{P}(S = n)$ under the constraints $\mathbb{P}(S = i) \geq 0$ for $0 \leq i \leq n$, $\sum_{i=0}^n \mathbb{P}(S = i) = 1$ and the linear conditions on $\mathbb{P}(S = i)$ given by (II) above. We shall estimate $M(n, k, p)$ using the dual linear programming problem [BGP]:

$$M(n, k, p) = \min_{P \in \mathcal{P}_k^d} \mathbb{E}_{\text{Bin}(n, p)} P(X), \quad (2.5)$$

where \mathcal{P}_k^d is the collection of polynomials $P : \mathbb{R} \rightarrow \mathbb{R}$ of degree at most k satisfying $P(i) \geq 0$ for $i \in \{0, 1, \dots, n-1\}$ and $P(n) \geq 1$ (the d in the notation stands for discrete). We shall bound $M(n, k, p)$ from below by showing that for each $P \in \mathcal{P}_k^d$, the above expectation is not too small.

Note that finding an optimal polynomial for the above problem gives more information than just $M(n, k, p)$. By the theorem of complementary slackness of linear programming, if Z is the set of zeros of an optimal polynomial in (2.5) then the support of the optimal distribution in (2.3) is contained in $Z \cup \{n\}$. This can also be seen probabilistically since if P is an optimal polynomial, then $\mathbb{E}P(S) = M(n, k, p)$ for any $S \in \mathcal{A}^s(n, k, p)$ (since P is of degree at most k). But for any $P \in \mathcal{P}_k^d$ we have $\mathbb{E}P(S) \geq \mathbb{P}(S = n)$, hence $\mathbb{P}(S = n) = M(n, k, p)$ only when $P(n) = 1$ and all the support of S besides $\{n\}$ is contained in the zero set of P . Of course once the support of the optimal S (or the zero set of an optimal polynomial) is known, the exact probabilities of S can be found by solving a system of linear equations. This system always has a unique solution (it has a Van der Monde coefficient matrix), which also proves the uniqueness of the distribution of S .

Prékopa in his work ([P88], see also [BP89]) considers in more generality the problem of estimating $\mathbb{P}(S = n)$ for the class of random variables S with

given first k moments (not necessarily those of the Binomial). He does not use probabilistic language and instead writes his work in linear programming terminology. Adapting one of his results to our situation, it reads

Theorem 2.2. (*Prékopa [P88, Theorem 9]*) *There exists an optimizing polynomial P for (2.5) of the following form. $P(n) = 1$ and P has k simple roots $z_1 < z_2 < \dots < z_k$, all contained in $\{0, 1, \dots, n - 1\}$. Furthermore*

1. *For even k , the roots come in pairs $z_{i+1} = z_i + 1$ for odd $1 \leq i \leq k - 1$.*
2. *For odd k , $z_1 = 0$ and the rest of the roots come in pairs $z_{i+1} = z_i + 1$ for even $2 \leq i \leq k - 1$.*

This result is also essentially contained in [KN77, Chap. VIII, sec. 3]. Figures 1 and 2 below present such optimizing polynomials for some choices of the parameters. The theorem is not so surprising when one recalls that we are trying to minimize the expectation of P under the positivity constraints of the class \mathcal{P}_k^d . The theorem is valid in the generality of Prékopa's work, i.e., the first k moments of S are given but they do not necessarily equal those of a Binomial random variable.

We remark that the case in which there is more than one optimizing polynomial is the case in which some degeneracy occurs in the problem, allowing the optimal distribution for (2.3) to be supported on less than $k + 1$ points.

3 The relaxed problem

As explained in the introduction, in [BGP] an upper bound for $M(n, k, p)$ was given. The bound was proven by considering a relaxed version of problems (2.3) and (2.5). In this section we describe this relaxed version (doing so, we follow the ideas presented in [BGP]). Problem (2.3) is replaced by

$$\tilde{M}(n, k, p) = \max_{S \in \mathcal{A}^c(n, k, p)} \mathbb{P}(S = n), \quad (3.1)$$

where $\mathcal{A}^c(n, k, p)$ (here the c stands for continuous) is the set of all real random variables S satisfying

- (I') S is supported on $[0, n]$.

(II') $\mathbb{E}S^i = \mathbb{E}X^i$ for $X \sim \text{Bin}(n, p)$ and $1 \leq i \leq k$.

Comparing conditions (I), (II) above to conditions (I'), (II') here we see that the only difference between the original and relaxed problems is that in the relaxed problem S may take non-integer values between 0 and n . Of course, inequality (1.2) follows trivially. In [BGP], the exact value of \tilde{M} was found, giving the formula (1.3) for even k . The reason that \tilde{M} is easier to handle than M is that the problem (3.1) is a special case of the Classical Moment Problem. Such problems have been solved, for example in the classical books [Ak65, Theorem 2.5.2], [KN77, Chap. III, sec. 3.2], and a great deal of theory has been developed around them.

We now consider the dual problem to (3.1), which is

$$\tilde{M}(n, k, p) = \min_{P \in \mathcal{P}_k^c} \mathbb{E}_{\text{Bin}(n, p)} P(X), \quad (3.2)$$

where \mathcal{P}_k^c is the collection of polynomials $P : \mathbb{R} \rightarrow \mathbb{R}$ of degree at most k satisfying $P(x) \geq 0$ for $x \in [0, n]$ and $P(n) \geq 1$ (the c stands for continuous). The optimizing polynomial is explicitly given in [Ak65], it equals 1 at n and for even k it has $\frac{k}{2}$ double roots in $[0, n]$ (for odd k it has one root at 0 and $\frac{k-1}{2}$ double roots in $(0, n)$). The location of the roots is given in terms of Krawtchouk polynomials, the orthogonal polynomials of the Binomial distribution. In Figures 3 and 4 we have drawn the optimizing polynomials for some specific parameters. Refer to [BGP] for more details on the optimizing polynomials.

It seems worth mentioning that for even k there is another problem which is equivalent to the relaxed dual problem (3.2). This other problem has been used by some authors to obtain similar upper bounds, sometimes without noting the equivalence to (3.2). This equivalence is also fundamental in the analysis of the Classical Moment Problem. The equivalent problem for even k is

$$\tilde{M}(n, k, p) = \min_{P \in \mathcal{P}_k^2} \mathbb{E}_{\text{Bin}(n, p)} P(X), \quad (3.3)$$

where \mathcal{P}_k^2 is the collection of polynomials $P : \mathbb{R} \rightarrow \mathbb{R}$ of the form $P = R^2$ where R is a polynomial of degree at most $\frac{k}{2}$ satisfying $|R(n)| \geq 1$. It is clear that $\mathcal{P}_k^2 \subseteq \mathcal{P}_k^c$ but in fact they are equal. This follows immediately from the Markov-Lukacs theorem (see for example [KN77, Chap. III, thm. 2.2])

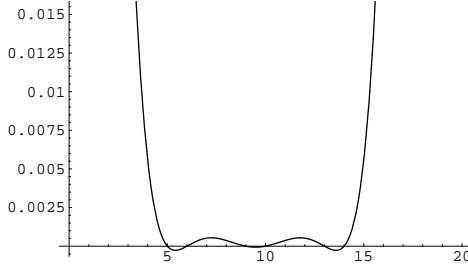


Figure 1: Optimizing polynomial in (2.5) for $n = 20, k = 6, p = \frac{1}{2}$.

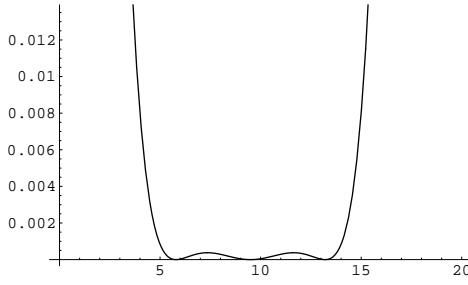


Figure 3: Optimizing polynomial in (3.2) for $n = 20, k = 6, p = \frac{1}{2}$.

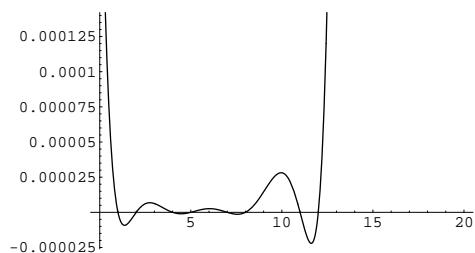


Figure 2: Optimizing polynomial in (2.5) for $n = 20, k = 8, p = \frac{3}{10}$.

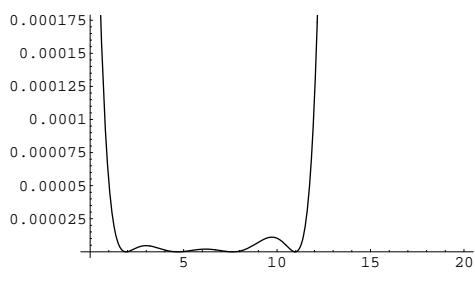


Figure 4: Optimizing polynomial in (3.2) for $n = 20, k = 8, p = \frac{3}{10}$.

Theorem 3.1. (*Markov-Lukacs*) *A polynomial P of even degree is non-negative on $[a, b]$ iff it is of the form*

$$P(x) = R^2(x) + (x - a)(b - x)Q^2(x) \quad (3.4)$$

for some polynomials Q and R .

4 Proof of main result

In this section we show how to reduce our main result, Theorem 1.1, to a result about polynomials. We also give the estimate on $\tilde{M}(n, k, p)$ required to deduce Corollary 1.3 from Corollary 1.2.

Theorem 1.1 is proved using the following general idea. Consider any polynomial $P \in \mathcal{P}_k^d$ of the form given in Prékopa's Theorem 2.2. Change the

location of its roots slightly to make each pair of adjacent roots into one double root. The new perturbed polynomial \tilde{P} is in \mathcal{P}_k^c . Show that the expectation under the $\text{Bin}(n, p)$ distribution of \tilde{P} is not much higher than that of P . Deduce that the expectation of the optimal polynomial in (2.5) is not much lower than the expectation of the optimal polynomial in (3.2).

The actual proof that the two expectations are close is somewhat complicated. A key ingredient is the use of discrete Chebyshev polynomials to bound the ratio of the value of P and \tilde{P} at certain points. The Chebyshev polynomials were previously used in a similar context; see, for example, [HLL97, S99].

The result we need about polynomials is the following. Let k be even and fix two polynomials

$$f(x) = \prod_{i=1}^{k/2} (x - a_i)(x - a_i - 1) \quad \text{and} \quad g(x) = \prod_{i=1}^{k/2} (x - a_i)^2, \quad (4.1)$$

with all $a_i \in \{0, 1, \dots, n-1\}$ and such that $a_i \neq a_j$ and $a_i \neq a_j + 1$ for $i \neq j$.

For a polynomial φ we denote $\mathbb{E}_{n,p}[\varphi] = \mathbb{E}[\varphi(X)]$ where $X \sim \text{Bin}(n, p)$ has Binomial distribution with parameters n and p .

Theorem 4.1. *There exist constants $c_1, c_2, c_3 > 0$ such that the following holds. Let $n \in \mathbb{N}$, $k \in \mathbb{N}$ even and $0 < p < 1$. Let $N = np(1-p) - 1$. Assume $k \leq c_1 \cdot N$. Then,*

$$\mathbb{E}_{n,p}[g] \leq c_3 k \cdot \exp\left(c_2 \cdot \frac{k}{V(N/k)}\right) \mathbb{E}_{n,p}[f],$$

where $V(a) = \exp\left(\sqrt{\log(a) \log \log(a)}\right)$.

The proof of Theorem 4.1 is given in Section 5 below.

Proof of Theorem 1.1. By Prékopa's Theorem 2.2, there exists f such that

$$M(n, k, p) = \frac{\mathbb{E}_{n,p}[f]}{f(n)},$$

f has the form given in (4.1), and $\frac{f}{f(n)} \in \mathcal{P}_k^d$. By Theorem 4.1,

$$\mathbb{E}_{n,p}[g] \leq c_3 k \cdot \exp\left(c_2 \cdot \frac{k}{V(N/k)}\right) \mathbb{E}_{n,p}[f],$$

for g as in (4.1). Since $\frac{g}{g(n)} \in \mathcal{P}_k^c$,

$$\tilde{M}(n, k, p) \leq \frac{\mathbb{E}_{n,p}[g]}{g(n)},$$

which completes the proof, as $g(n) \geq f(n)$. \square

Corollary 1.3 follows from Corollary 1.2 using the following bounds on $\tilde{M}(n, k, p)$.

Claim 4.2. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}$ even, and $0 < p < 1$. There exist $C, c > 0$ such that if $k \leq cn(1 - p)$ then*

$$c\sqrt{k} \left(\frac{pk}{2e(1-p)n} \right)^{k/2} \leq \tilde{M}(n, k, p) \leq C\sqrt{k} \left(\frac{pk}{2e(1-p)n} \right)^{k/2} e^{k^2/2n}.$$

Proof. We first recall that there exist $C_4, c_4 > 0$ such that $c_4\sqrt{k} \left(\frac{k}{2e} \right)^{k/2} \leq \left(\frac{k}{2} \right)! \leq C_4\sqrt{k} \left(\frac{k}{2e} \right)^{k/2}$. Hence

$$\binom{n}{k/2} \leq \frac{n^{k/2}}{(k/2)!} \leq \frac{1}{c_4\sqrt{k}} \left(\frac{2en}{k} \right)^{k/2}$$

and since $k \leq cn(1 - p)$ (for a small enough c), we have

$$\binom{n}{k/2} \geq \frac{(n - k/2)^{k/2}}{(k/2)!} \geq \frac{1}{C_4\sqrt{k}} \left(\frac{2en}{k} \right)^{k/2} e^{-k^2/2n}.$$

Hence

$$\begin{aligned} \mathbb{P} \left(\text{Bin}(n, 1 - p) \leq \frac{k}{2} \right) &\geq \binom{n}{k/2} p^{n-k/2} (1 - p)^{k/2} \geq \\ &\geq \frac{p^n}{C_4\sqrt{k}} \left(\frac{2e(1-p)n}{pk} \right)^{k/2} e^{-k^2/2n}. \end{aligned}$$

Similarly note that since $k \leq cn(1 - p)$, we have

$$\binom{n}{i} p^{n-i} (1 - p)^i \leq \frac{1}{2} \binom{n}{i+1} p^{n-i-1} (1 - p)^{i+1}$$

for $i \leq k/2$. Hence

$$\mathbb{P} \left(\text{Bin}(n, 1 - p) \leq \frac{k}{2} \right) \leq 2 \binom{n}{k/2} p^{n-k/2} (1 - p)^{k/2} \leq \frac{2p^n}{c_4\sqrt{k}} \left(\frac{2e(1-p)n}{pk} \right)^{k/2}.$$

The claim now follows by substituting the above estimates into (1.3). \square

5 Perturbing Roots of Polynomials

In this section we shall prove Theorem 4.1. For $n \in \mathbb{N}$, we denote $I_n = \{0, 1, \dots, n\}$. For two real numbers a and b , we denote $[a, b) = \{t : a \leq t < b\}$ and $[a, b] = \{t : a \leq t \leq b\}$. For given $n \in \mathbb{N}$ and $0 < p < 1$, we define $\mathbb{P}_{n,p}[x] = \binom{n}{x} p^x (1-p)^{n-x}$ for $x \in I_n$; i.e., the probability of x according to $\text{Bin}(n, p)$.

We wish to bound the ratio between $\mathbb{E}_{n,p}[g]$ and $\mathbb{E}_{n,p}[f]$. We write

$$\frac{\mathbb{E}_{n,p}[g]}{\mathbb{E}_{n,p}[f]} = \frac{\sum_{x=0}^n \mathbb{P}_{n,p}[x]g(x)}{\sum_{x=0}^n \mathbb{P}_{n,p}[x]f(x)}. \quad (5.1)$$

The theorem then follows from the following two lemmas:

Lemma 5.1. *Let $x \in I_n$ be such that $f(x) \neq 0$. Then*

$$\frac{g(x)}{f(x)} \leq 2\sqrt{k}.$$

Lemma 5.2. *There exist universal constants $c_1, c_2, c_3 > 0$ such that the following holds. Let $N = np(1-p) - 1$. Assume $k \leq c_1 \cdot N$. Then, for every $x \in I_n$ there exists $w \in I_n$ satisfying*

$$\frac{\mathbb{P}_{n,p}[x]g(x)}{\mathbb{P}_{n,p}[w]f(w)} \leq c_3 \cdot \exp\left(c_2 \cdot \frac{k}{V(N/k)}\right), \quad (5.2)$$

where $V(a) = \exp\left(\sqrt{\log(a) \log \log(a)}\right)$.

The first lemma, whose proof is much simpler than the proof of the second lemma, is proved in Section 5.1. The second lemma addresses the case $f(x) = 0$ in which the first lemma does not apply, and is proved in Section 5.2. We note that the ‘simple’ ideas presented in the proof of the first lemma can yield a weaker version of the second lemma, with a bound of the form $c_3 \exp(c_2 k)$ on the RHS of (5.2). While significantly weaker, such a bound still yields the correct asymptotic behavior of $M(n, k, p)$ for constant k .

We now show how Theorem 4.1 follows from the two lemmas.

Proof of Theorem 4.1. Let

$$\text{Zeros}(f) = \{y \in I_n : f(y) = 0\}.$$

We shall denote the w that corresponds to x according to Lemma 5.2 by w_x . We write (5.1) using the above two lemmas and using the fact that for all $x \in I_n$, $\mathbb{P}_{n,p}[x] \geq 0$ and $f(x), g(x) \geq 0$ (due to the special structure (4.1) of the polynomials) as

$$\begin{aligned}\mathbb{E}_{n,p}[g] &\leq 2\sqrt{k} \sum_{x \in I_n \setminus \text{Zeros}(f)} \mathbb{P}_{n,p}[x] f(x) \\ &\quad + c_3 \cdot \exp\left(c_2 \cdot \frac{k}{V(N/k)}\right) \sum_{x \in \text{Zeros}(f)} \mathbb{P}_{n,p}[w_x] f(w_x) \\ &\leq (c_3 + 2)k \cdot \exp\left(c_2 \cdot \frac{k}{V(N/k)}\right) \mathbb{E}_{n,p}[f].\end{aligned}$$

□

5.1 Points that are not zeros of f

Proof of Lemma 5.1. Note that

$$\frac{g(x)}{f(x)} = \prod_{i=1}^{k/2} \frac{a_i - x}{a_i + 1 - x}. \quad (5.3)$$

Since $f(x) \neq 0$, we can partition the a_i 's into two sets:

$$S_1 = \{i : a_i < x\} \quad \text{and} \quad S_2 = \{i : a_i > x\}.$$

First, for each $i \in S_2$

$$0 \leq \frac{a_i - x}{a_i + 1 - x} \leq 1. \quad (5.4)$$

In addition,

$$\begin{aligned}0 \leq \prod_{i \in S_1} \frac{x - a_i}{x - a_i - 1} &\leq \prod_{i=1}^{k/2} \left(1 + \frac{1}{2i-1}\right) \leq 2 \exp\left(\sum_{i=2}^{k/2} \frac{1}{2i-1}\right) \\ &\leq 2 \exp\left(\frac{1}{2} \log(k-1)\right) \leq 2\sqrt{k}.\end{aligned} \quad (5.5)$$

The lemma follows by substituting (5.4) and (5.5) in (5.3). □

5.2 Points that are zeros of f

In this section we prove Lemma 5.2. We first describe a family of orthogonal polynomials, the discrete Chebyshev polynomials. Then we prove Claim 5.4 that uses these polynomials. Finally we use the Claim 5.4 to prove Lemma 5.2.

5.2.1 Orthogonal Polynomials

We now give some properties of a family of orthogonal polynomials studied by Chebyshev, sometimes called discrete Chebyshev polynomials. These properties are described and proved in [Sz75, Section 2.8]. We use these orthogonal polynomial to prove the following proposition.

Proposition 5.3. *Let $M \in \mathbb{N}$ and let G be a monic polynomial of degree d for $0 \leq d \leq \frac{M}{2}$, then*

$$\max_{i \in \{0, \dots, M-1\}} |G(i)| \geq \frac{M^d}{4^{d+1/2}} e^{-d^3/M^2}.$$

Proof. The family of polynomials $\{t_d\}_{d=0}^{M-1}$ defined below are orthogonal polynomials for the measure μ_M which assigns mass one to each integer $x \in \{0, 1, \dots, M-1\}$, see (5.7) for the chosen normalization. In other words for every $d, d' \in \{0, \dots, M-1\}$ such that $d \neq d'$,

$$\sum_{i=0}^{M-1} t_d(i) t_{d'}(i) = 0.$$

The polynomial t_d is

$$t_d(x) = d! \cdot \Delta^{(d)} \binom{x}{d} \binom{x-M}{d}, \quad (5.6)$$

where

$$\Delta G(x) \stackrel{\text{def}}{=} G(x+1) - G(x), \quad \Delta^{(d)} G \stackrel{\text{def}}{=} \Delta [\Delta^{(d-1)} G]$$

and

$$\binom{x}{d} \stackrel{\text{def}}{=} \frac{x(x-1)\cdots(x-d+1)}{d!}.$$

The normalization is chosen so that

$$\sum_{i=0}^{M-1} |t_d(i)|^2 = \frac{M(M^2 - 1^2)(M^2 - 2^2) \cdots (M^2 - d^2)}{2d + 1}. \quad (5.7)$$

The coefficient of x^{2d} in $d! \cdot \binom{x}{d} \binom{x-M}{d}$ is $\frac{1}{d!}$. Thus, by the linearity of Δ , and since for every $k \in \mathbb{N}$,

$$\Delta x^k = (x+1)^k - x^k = kx^{k-1} + \binom{k}{2} x^{k-2} + \cdots + 1,$$

the polynomial t_d has degree d , and the coefficient of x^d in t_d is $\binom{2d}{d}$. Thus, since every monic polynomial G of degree $d < M$ can be expanded as $G(x) = \binom{2d}{d}^{-1} t_d(x) + \sum_{i=0}^{d-1} a_i t_i(x)$, we have using (5.7)

$$\begin{aligned} \sum_{i=0}^{M-1} |G(i)|^2 &\geq \binom{2d}{d}^{-2} \sum_{i=0}^{M-1} |t_d(i)|^2 = \\ &= \binom{2d}{d}^{-2} \frac{M(M^2 - 1^2)(M^2 - 2^2) \cdots (M^2 - d^2)}{2d + 1}. \end{aligned} \quad (5.8)$$

Using the inequalities $1 - x \geq e^{-2x}$ ($0 \leq x \leq \frac{1}{4}$), $\binom{2d}{d} \leq 2 \frac{4^d}{\sqrt{\pi d}}$ ($d \geq 1$) and $\sum_{i=1}^d i^2 = \frac{d(d+1)(2d+1)}{6} \leq d^3$ ($d \geq 1$) we obtain for $1 \leq d \leq \frac{M}{2}$,

$$\sum_{i=0}^{M-1} |G(i)|^2 \geq \frac{\pi d M^{2d+1}}{4^{2d+1}(2d+1)} e^{-2 \sum_{i=1}^d i^2 / M^2} \geq \left(\frac{M}{4}\right)^{2d+1} e^{-2d^3/M^2}. \quad (5.9)$$

The proposition thus follows (the case $d = 0$ is straightforward). \square

5.2.2 A Segment With Few Zeros

In this section we prove an auxiliary claim, to be used in the next section as a main component in the proof of Lemma 5.2. The claim roughly states that given a segment with few zeros, we can find a point at which f obtains a ‘large’ value.

Claim 5.4. *Let $x \in I_n$, let $R, m \in \mathbb{N}$ be such that $m \geq 2R$, let L be a non-negative integer, and let $\tau > 4$. If*

$$|\text{Zeros}(f) \cap [x+m, x+\tau m]| \leq R, \quad (5.10)$$

and

$$|\text{Zeros}(f) \cap (x, x + m/\tau)| \geq L, \quad (5.11)$$

then there exists $w \in \mathbb{N} \cap [x + 2m, x + 3m]$ such that

$$\frac{g(x)}{f(w)} \leq 8 \cdot \exp\left(\frac{12k}{\tau} + 6R - L \log \tau\right).$$

Similarly, if instead of (5.10) and (5.11) we have

$$|\text{Zeros}(f) \cap (x - \tau m, x - m)| \leq R,$$

and

$$|\text{Zeros}(f) \cap (x - m/\tau, x)| \geq L,$$

then there exists $w \in \mathbb{N} \cap [x - 3m, x - 2m]$ such that

$$\frac{g(x)}{f(w)} \leq 8 \cdot \exp\left(\frac{12k}{\tau} + 6R - L \log \tau\right).$$

Proof. Assume without loss of generality that (5.10) and (5.11) hold (a similar argument holds for the second case). Let $x + 2m \leq w \leq x + 3m$ be such that $f(w) \neq 0$. Write

$$\frac{g(x)}{f(w)} = \prod_{i=1}^{k/2} \frac{(x - a_i)^2}{(w - a_i)(w - a_i - 1)}. \quad (5.12)$$

We partition the a_i 's into six subsets S_1, S_2, \dots, S_6 according to the definitions below, and bound (5.12) over each subset separately. The partition is

$$\begin{aligned} S_1 &= \{i \in \{1, \dots, k/2\} : a_i < x\}, \\ S_2 &= \{i \in \{1, \dots, k/2\} : x \leq a_i < x + m/\tau\}, \\ S_3 &= \{i \in \{1, \dots, k/2\} : x + m/\tau \leq a_i < x + m\}, \\ S_4 &= \{i \in \{1, \dots, k/2\} : x + m \leq a_i < x + 4m\}, \\ S_5 &= \{i \in \{1, \dots, k/2\} : x + 4m \leq a_i < x + \tau m\}, \text{ and} \\ S_6 &= \{i \in \{1, \dots, k/2\} : x + \tau m \leq a_i\}. \end{aligned}$$

For every $i \in S_1 \cup S_3$,

$$0 \leq \left| \frac{x - a_i}{w - a_i - 1} \right| \leq 1,$$

which implies

$$0 \leq \prod_{i \in S_1 \cup S_3} \frac{(a_i - x)^2}{(w - a_i)(w - a_i - 1)} \leq 1. \quad (5.13)$$

For every $i \in S_2$,

$$0 \leq \frac{a_i - x}{w - a_i - 1} \leq \frac{1}{\tau},$$

which implies

$$0 \leq \prod_{i \in S_2} \frac{(a_i - x)^2}{(w - a_i)(w - a_i - 1)} \leq \tau^{-L}. \quad (5.14)$$

To argue about S_4 , define the polynomial

$$F(\xi) \stackrel{\text{def}}{=} \prod_{i \in S_4} (\xi - a_i)(\xi - a_i - 1).$$

Denote $d = 2|S_4|$, the degree of F . Since $2|S_4| \leq R \leq \frac{m}{2}$, we deduce from Proposition 5.3 that there exists $w_0 \in \mathbb{N} \cap [x + 2m, x + 3m]$ such that

$$|F(w_0)| \geq \frac{m^d}{4^{d+1/2}} e^{-d^3/m^2}.$$

Hence, since $\prod_{i \in S_4} (a_i - x)^2 \leq (4m)^d$,

$$0 \leq \prod_{i \in S_4} \frac{(a_i - x)^2}{(w_0 - a_i)(w_0 - a_i - 1)} \leq 2 \cdot e^{3R+R^3/m^2} \leq 2 \cdot e^{4R}. \quad (5.15)$$

For every $i \in S_5$,

$$0 \leq \frac{a_i - x}{a_i - w} \leq 4.$$

Hence, since $|S_5| \leq \frac{R+1}{2}$,

$$0 \leq \prod_{i \in S_5} \frac{(a_i - x)^2}{(a_i - w)(a_i + 1 - w)} \leq 4 \cdot 4^R. \quad (5.16)$$

Similarly, since $2|S_6| \leq k$,

$$0 \leq \prod_{i \in S_6} \frac{(a_i - x)^2}{(a_i - w)(a_i + 1 - w)} \leq \left(\frac{\tau}{\tau - 3} \right)^{2|S_6|} \leq \exp \left(\frac{12k}{\tau} \right). \quad (5.17)$$

Therefore, plugging (5.14), (5.13), (5.16), (5.17) and (5.15) into (5.12) with $w = w_0$,

$$\frac{g(x)}{f(w_0)} \leq 8 \cdot \exp \left(\frac{12k}{\tau} + 6R - L \log \tau \right).$$

□

5.2.3 Finding good w

The following claim shows that there exists a w that is ‘close’ to x on which f obtains a ‘large’ value.

Claim 5.5. *Let $x \in I_n$, $k \geq Z \in \mathbb{N}$, and $\tau \geq e^{12}$. Let K be the smallest integer such that*

$$\left\lfloor \frac{\log \tau}{6} \right\rfloor^{\frac{K-1}{2}} \geq \frac{k}{Z}.$$

Then, there exist integers $w_1 > x$ and $w_2 < x$ such that for each $w \in \{w_1, w_2\}$,

$$3Z \leq |w - x| \leq 9Z\tau^K$$

and

$$\frac{g(x)}{f(w)} \leq 8 \cdot \exp \left(\frac{12k}{\tau} + 6Z \right).$$

Proof. We show the existence of w_1 , the existence of w_2 can be shown similarly.

Let $Z_0 = Z_1 = Z$. Let $m_0 = 3Z_0$ and $m_1 = \tau m_0$. If either

$$|\text{Zeros}(f) \cap [x + m_0, x + m_1]| \leq Z_0 \quad (5.18)$$

or

$$|\text{Zeros}(f) \cap [x + m_1, x + m_2]| \leq Z_1, \quad (5.19)$$

then by Claim 5.4, with $L = 0$, $R = Z$, $m = m_0 \geq 2Z$ for (5.18) and $m = m_1 \geq 2Z$ for (5.19), there exists $w_1 \in \mathbb{N}$ such that

$$3Z \leq w_1 - x \leq 9Z\tau$$

and

$$\frac{g(x)}{f(w_1)} \leq 8 \cdot \exp\left(\frac{12k}{\tau} + 6Z\right).$$

Thus, assume that both (5.18) and (5.19) do not hold. Define Z_i and m_i for $i \geq 2$ as

$$Z_i = \left\lfloor \frac{\log \tau}{6} \right\rfloor Z_{i-2} \quad \text{and} \quad m_i = \tau m_{i-1}.$$

Since the intervals

$$[x + m_0, x + m_1), \dots, [x + m_K, x + m_{K+1})$$

are disjoint, since the number of zeros of f is k , and since $Z_K \geq k$, let i be the smallest integer so that

$$|\text{Zeros}(f) \cap [x + m_i, x + m_{i+1})| \leq Z_i$$

and

$$|\text{Zeros}(f) \cap (x, x + m_{i-1})| \geq Z_{i-2}.$$

Since $m_i \geq 2Z_i$, by Claim 5.4 with $L = Z_{i-2}$, $R = Z_i$ and $m = m_i$, there exists $w_1 \in \mathbb{N}$ such that

$$3Z \leq w_1 - x \leq 9Z\tau^K$$

and

$$\frac{g(x)}{f(w_1)} \leq 8 \cdot \exp\left(\frac{12k}{\tau} + 6Z_i - Z_{i-2} \log \tau\right) \leq 8 \cdot \exp\left(\frac{12k}{\tau}\right).$$

□

5.2.4 Probability Estimates

Claim 5.6. Let $\ell \in \{1, \dots, n\}$, $0 < p < 1$, and let $N = np(1 - p) - 1$. Assume $N > 0$. Set $\mu = \lfloor pn \rfloor$. If $\ell \leq (n - \mu)/2$, then

$$\mathbb{P}_{n,p}[\mu + \ell] \geq \exp\left(-\frac{3\ell^2}{2N}\right) \mathbb{P}_{n,p}[\mu].$$

In addition, if $\ell \leq \mu/2$, then

$$\mathbb{P}_{n,p}[\mu - \ell] \geq \exp\left(-\frac{8\ell^2}{N}\right) \mathbb{P}_{n,p}[\mu].$$

Proof. Assume that $\ell - 1 \leq (n - \mu)/2$. Then,

$$\begin{aligned} \frac{\mathbb{P}_{n,p}[\mu]}{\mathbb{P}_{n,p}[\mu + \ell]} &= \left(\frac{1-p}{p}\right)^\ell \cdot \frac{\mu^\ell \prod_{i=1}^\ell (1+i/\mu)}{(n-\mu)^\ell \prod_{i=0}^{\ell-1} (1-i/(n-\mu))} \\ &\leq \left(\frac{1-p}{p}\right)^\ell \cdot \frac{\mu^\ell}{(n-\mu)^\ell} \exp\left(\sum_{i=1}^\ell \frac{i}{\mu} + \frac{2(i-1)}{n-\mu}\right) \\ &= \left(\frac{1-p}{p}\right)^\ell \cdot \frac{\mu^\ell}{(n-\mu)^\ell} \exp\left(\ell \cdot \left(\frac{(\ell+1)(n-\mu) + 2\mu(\ell-1)}{2\mu(n-\mu)}\right)\right) \\ &\leq \exp\left(\ell \cdot \left(\frac{\ell(n+\mu) + n}{2\mu(n-\mu)}\right)\right) \\ &\leq \exp\left(\frac{3\ell^2}{2n} \cdot \left(\frac{1}{p(1-p) - 1/n}\right)\right). \end{aligned}$$

This proves the first assertion. For the second assertion note that

$$\ell \leq \frac{\mu}{2} \leq \frac{\lfloor pn \rfloor}{2} = \frac{n - \lfloor (1-p)n \rfloor}{2}.$$

Recall that the binomial measure decreases as the distance from its expectation increases. Thus,

$$\mathbb{P}_{n,p}[\mu - \ell] = \mathbb{P}_{n,(1-p)}[n - \mu + \ell] \geq \mathbb{P}_{n,(1-p)}[\lfloor (1-p)n \rfloor + 1 + \ell].$$

In addition, since $(\ell + 1) - 1 \leq \frac{n - \lfloor (1-p)n \rfloor}{2}$, the proof of the first assertion implies

$$\begin{aligned}
\mathbb{P}_{n,p}[\mu] &\leq \exp\left(\frac{3}{2N}\right) \cdot \mathbb{P}_{n,p}[\mu + 1] \\
&\leq \exp\left(\frac{3}{2N}\right) \cdot \mathbb{P}_{n,p}[n - \lfloor (1-p)n \rfloor] \\
&= \exp\left(\frac{3}{2N}\right) \cdot \mathbb{P}_{n,(1-p)}[\lfloor (1-p)n \rfloor] \\
&\leq \exp\left(\frac{3}{2N}\right) \cdot \exp\left(\frac{3(\ell+1)^2}{2N}\right) \cdot \mathbb{P}_{n,(1-p)}[\lfloor (1-p)n \rfloor + 1 + \ell] \\
&\leq \exp\left(\frac{3((\ell+1)^2 + 1)}{2N}\right) \cdot \mathbb{P}_{n,p}[\mu - \ell],
\end{aligned}$$

which completes the proof since $\ell \geq 1$. \square

5.2.5 Proof of Lemma 5.2

Proof. Let $x \in I_n$ and $\mu = \lfloor pn \rfloor$. Let $k \geq Z \in \mathbb{N}$ and $\tau \geq e^{12}$, to be determined. Let $K = K(\tau, Z)$ be the smallest integer such that

$$\left\lfloor \frac{\log \tau}{6} \right\rfloor^{\frac{K-1}{2}} \geq \frac{k}{Z}.$$

First assume $x < \mu$. We use Claim 5.5 to find $w > x$ such that $w \leq x + 9Z\tau^K$ and

$$\frac{g(x)}{f(w)} \leq 8 \cdot \exp\left(\frac{12k}{\tau} + 6Z\right).$$

Now if $w < \mu$ we certainly have $\mathbb{P}_{n,p}[x] \leq \mathbb{P}_{n,p}[w]$. If $w > \mu$ then we can use Claim 5.6 with $\ell = w - \mu \leq w - x \leq 9Z\tau^K$, provided that $\ell \leq \frac{n-\mu}{2}$, to obtain

$$\frac{\mathbb{P}_{n,p}[x]}{\mathbb{P}_{n,p}[w]} \leq \exp\left(\frac{8(9Z\tau^K)^2}{N}\right).$$

Thus,

$$\frac{\mathbb{P}_{n,p}[x]g(x)}{\mathbb{P}_{n,p}[w]f(w)} \leq 8 \cdot \exp\left(\frac{8(9Z\tau^K)^2}{N} + \frac{12k}{\tau} + 6Z\right).$$

This also holds for $x \geq \mu$, by using Claim 5.5 to find $w < x$, and the estimate in Claim 5.6 involving $\mathbb{P}_{n,p}[\mu - \ell]$, provided that $\ell \leq \frac{\mu}{2}$.

Set

$$\tau = \frac{1}{100} \exp(\sqrt{\log(N/k) \log \log(N/k)}) \text{ and } Z = \left\lceil \frac{k}{\tau} \right\rceil.$$

Since c_1 is small enough (recall that $k \leq c_1 \cdot N$), $\tau \geq e^{12}$ and a short calculation shows that

$$K \leq \frac{1}{4} \sqrt{\frac{\log(N/k)}{\log \log(N/k)}}.$$

Since $\ell \leq 9Z\tau^K$, this implies that $\ell \leq \frac{N}{2} \leq \min\left\{\frac{\mu}{2}, \frac{n-\mu}{2}\right\}$ (for small enough c_1). Thus,

$$\frac{\mathbb{P}_{n,p}[x]g(x)}{\mathbb{P}_{n,p}[w]f(w)} \leq 8 \cdot \exp\left(c_4 \left(\frac{k}{\tau} + 1\right)\right),$$

for a constant $c_4 > 0$, since $\frac{Z\tau^{2K}}{N} \leq \sqrt{\frac{k}{N}} \leq 1$ and $Z \leq \frac{k}{\tau} + 1$. The lemma follows. \square

6 Open Problems

1. What is the value of $M(n, k, p)$ in the range of the parameters not treated by our theorem, namely $k \geq Cnp(1-p)$?
2. What is the actual ratio of $M(n, k, p)$ and $\tilde{M}(n, k, p)$?
3. Is there also a similarity between the optimal distributions of our original and relaxed problems (problems (2.3) and (3.1))? As explained in Section 2, this is related to whether the optimizing polynomials in the dual problems are similar. As hinted by Figures 1-4, calculations in particular cases seem to indicate this to be the case. The similarity seems especially strong in the case $p = \frac{1}{2}$.
4. In the setting of Theorem 4.1, What is the best ratio between $\mathbb{E}_{n,p}[g]$ and $\mathbb{E}_{n,p}[f]$? I.e., the best bound on the change in the expectation of the polynomial after small perturbation of its zeros.

5. Find upper and lower bounds for the maximal probability that all the bits are 1, for the class of *almost k*-wise independent distributions. Similarly to k -wise independent distributions, such distributions have also proven quite useful for the derandomization of algorithms in computer science.

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7 Appendix

We provide here short proofs for the results of [BGP] that we use.

Proof of (1.1). Fix $n \in \mathbb{N}, 0 < p < 1$ and an odd $k \in \mathbb{N}$. Let P be the optimal polynomial for the problem (2.5) for these n, k and p . By the second part of Theorem (2.2) we know that

$$P(z) = \frac{z \prod_{i=1}^{(k-1)/2} (z - z_{2i})(z - z_{2i+1})}{n \prod_{i=1}^{(k-1)/2} (n - z_{2i})(n - z_{2i+1})}. \quad (7.1)$$

Now note that

$$\begin{aligned} & \mathbb{E}_{\text{Bin}(n,p)} P(Z) \\ &= \sum_{z=0}^n \frac{z \prod_{i=1}^{(k-1)/2} (z - z_{2i})(z - z_{2i+1})}{n \prod_{i=1}^{(k-1)/2} (n - z_{2i})(n - z_{2i+1})} \binom{n}{z} p^z (1-p)^{n-z} \\ &= \sum_{z=1}^n \frac{\prod_{i=1}^{(k-1)/2} (z - z_{2i})(z - z_{2i+1})}{\prod_{i=1}^{(k-1)/2} (n - z_{2i})(n - z_{2i+1})} \binom{n-1}{z-1} p^z (1-p)^{n-z} \\ &= p \sum_{z=0}^{n-1} \frac{\prod_{i=1}^{(k-1)/2} (z - (z_{2i} - 1))(z - (z_{2i+1} - 1))}{\prod_{i=1}^{(k-1)/2} (n - z_{2i})(n - z_{2i+1})} \binom{n-1}{z} p^z (1-p)^{n-1-z} \\ &= p \cdot \mathbb{E}_{\text{Bin}(n-1,p)} Q(Z), \end{aligned} \quad (7.2)$$

where

$$Q(z) = \frac{\prod_{i=1}^{(k-1)/2} (z - (z_{2i} - 1))(z - (z_{2i+1} - 1))}{\prod_{i=1}^{(k-1)/2} (n - 1 - (z_{2i} - 1))(n - 1 - (z_{2i+1} - 1))}. \quad (7.3)$$

Note that Q is of degree $k - 1$ and satisfies $Q(i) \geq 0$ for $i \in \{0, 1, \dots, n - 2\}$, and $Q(n - 1) = 1$. Hence,

$$M(n, k, p) \geq pM(n - 1, k - 1, p).$$

To prove that equality holds, we can carry the above reasoning in the reverse direction by starting with the optimal polynomial Q to problem (2.5), which, by Theorem (2.2), is of the form (7.3). Then noting that (7.2) still holds for a polynomial P of the form (7.1), which is of degree k and satisfies $P(i) \geq 0$ for $i \in \{0, 1, \dots, n - 1\}$, and $P(n) = 1$. \square

Proof of Lemma 2.1. Define a distribution \mathbb{Q}_S on $\{0, 1\}^n$ by

$$\mathbb{Q}_S(\{x\}) = \mathbb{P}(S = |x|) \cdot \binom{n}{|x|}^{-1}$$

for $x \in \{0, 1\}^n$, where $|x|$ is the number of 1's in x . By definition, \mathbb{Q}_S is symmetric and S has the distribution of the number of 1's in \mathbb{Q}_S . It remains to verify that each bit has marginal probability p , and the k -wise independence property. Let \tilde{S} be a random variable with the $\text{Bin}(n, p)$ distribution; i.e., $\mathbb{P}(\tilde{S} = i) = \binom{n}{i} p^i (1 - p)^{n-i}$. It is straight-forward to verify that $\mathbb{Q}_{\tilde{S}}$ is the distribution of n independent $\text{Bernoulli}(p)$ random variables. Fix $1 \leq i_1 < i_2 < \dots < i_k \leq n$ and $y_1, \dots, y_k \in \{0, 1\}$. Let j be the number of 1's in (y_1, \dots, y_k) . Note that

$$\begin{aligned} \mathbb{Q}_S(\{x \in \{0, 1\}^n : x_{i_1} = y_1, \dots, x_{i_k} = y_k\}) &= \sum_{i=j}^{n-k+j} \binom{n-k}{i-j} \frac{\mathbb{P}(S = i)}{\binom{n}{i}} \\ &= \frac{(n-k)!}{n!} \sum_{i=j}^{n-k+j} \mathbb{P}(S = i) \prod_{m=0}^{j-1} (i-m) \prod_{m=0}^{k-j-1} (n-m-i) = \mathbb{E}P_j(S), \end{aligned}$$

where P_j is defined by

$$P_j(z) = \frac{(n-k)!}{n!} \prod_{m=0}^{j-1} (z-m) \prod_{m=0}^{k-j-1} (n-m-z).$$

Since P_j is a polynomial of degree k and since S has the same first k moments as \tilde{S} ,

$$\begin{aligned} \mathbb{Q}_S(\{x \in \{0, 1\}^n : x_{i_1} = y_1, \dots, x_{i_k} = y_k\}) &= \mathbb{E}P_j(S) \\ &= \mathbb{E}P_j(\tilde{S}) = \mathbb{Q}_{\tilde{S}}(\{x \in \{0, 1\}^n : x_{i_1} = y_1, \dots, x_{i_k} = y_k\}), \end{aligned}$$

as required. \square

Proof of (1.3). Following the methods of the classical moment problem, we use the equivalence of (3.2) and (3.3) and solve the latter problem. Fix $n \in \mathbb{N}, 0 < p < 1$ and an even $k \in \mathbb{N}$. Let $P = R^2$ for a polynomial R of degree at most $k/2$ satisfying $|R(n)| \geq 1$. Let $\{K_i\}_{i=0}^n$ be the (n, p) -Krawtchouk polynomials; i.e., the orthogonal polynomials corresponding to the $\text{Bin}(n, p)$ distribution normalized so that $\mathbb{E}_{\text{Bin}(n, p)} K_i^2(Z) = 1$. Write

$$R(z) = \sum_{i=0}^{k/2} a_i K_i(z).$$

Note that

$$\mathbb{E}_{\text{Bin}(n, p)} P(Z) = \mathbb{E}_{\text{Bin}(n, p)} R^2(Z) = \sum_{i=0}^{k/2} a_i^2.$$

Hence the problem (3.3) reduces to minimizing $\sum_{i=0}^{k/2} a_i^2$ under the constraint that $|R(n)| = |\sum_{i=0}^{k/2} a_i K_i(n)| \geq 1$. By Cauchy-Schwarz,

$$\sum_{i=0}^{k/2} a_i^2 \sum_{i=0}^{k/2} K_i^2(n) \geq \left(\sum_{i=0}^{k/2} a_i K_i(n) \right)^2 \geq 1.$$

Hence the optimal value of the problem (3.3) is $\frac{1}{\sum_{i=0}^{k/2} K_i^2(n)}$ and the optimal polynomial is (up to multiplication by (-1))

$$R(z) = \frac{1}{\sum_{i=0}^{k/2} K_i^2(n)} \sum_{i=0}^{k/2} K_i(n) K_i(z).$$

Since the Krawtchouk polynomials equal [Sz75]

$$K_i(x) = \binom{n}{i}^{-\frac{1}{2}} (p(1-p))^{-\frac{i}{2}} \sum_{j=0}^i (-1)^{i-j} \binom{n-x}{i-j} \binom{x}{j} p^{i-j} (1-p)^j,$$

and in particular

$$K_i(n) = \binom{n}{i}^{\frac{1}{2}} \left(\frac{1-p}{p} \right)^{\frac{i}{2}},$$

we deduce (1.3). \square