

# Evolutionary Minority Games: the benefits of imitation

Richard Metzler

*Institut für Theoretische Physik und Astrophysik,  
Universität Würzburg, Am Hubland, D-97074 Würzburg, Germany*

Christian Horn

*Experimentalphysik III, Universität Kassel, Heinrich-Plett-Str.40, D-34132 Kassel, Germany*

In the original Evolutionary Minority Game, a segregation into two populations with opposing preferences is observed under many circumstances. We show that this segregation becomes more pronounced and more robust if the dynamics are changed slightly, such that strategies with above-average fitness become more frequent. Similar effects occur also for a generalization of the EMG to more than two choices, and for evolutionary dynamics of a different stochastic strategy for the Minority Game.

## I. INTRODUCTION

The Minority Game (MG) was introduced by Challet and Zhang in 1997 [1] as a model for the competition for limited resources. Although it has since been studied in more than 100 publications [2], and countless variations have been introduced, the basic scenario is still easy to explain: there is a population of  $N$  players who, at each time step  $t$ , have to make a decision  $\sigma_i^t \in \{-1, 1\}$ . Those who are in the minority win, the others lose (to avoid ambiguities,  $N$  is chosen to be odd). Direct communication and contracts among players are not allowed; however, the decision of the minority is public information, and players can base their decision on a finite number  $M$  of past decisions.

Global efficiency is measured by the standard deviation of the sum of individual decision,

$$\sigma^2 = \left\langle \left( \sum_{i=1}^N \sigma_i^t \right)^2 \right\rangle_t. \quad (1)$$

Random guessing by all players leads to  $\sigma^2 = N$ ; a smaller value indicates good coordination among players, a larger value is a sign of herd behavior.

One obvious feature of this game is that there is no unique best action for the players – if it existed, it would be the same for all players for symmetry reasons, and all players would lose.

From the point of view of economic game theory [3], the game has a large number of Nash equilibria – combinations of strategies where no player can improve his chances of winning by unilaterally changing his strategy. For example, if  $(N+1)/2$  players choose  $+1$  all the time, and  $(N-1)/2$  pick  $-1$ , the global loss is minimal ( $\sigma^2 = 1$ ); however, those who are on the losing side stay there forever, and a player who switches sides will cause the majority to flip, and continue losing. Simple combinatorics show that there are  $\binom{N}{(N-1)/2}$  such combinations.

Furthermore, there are even more Nash equilibria in mixed strategies: e.g., if all players pick  $+1$  with a probability of 0.5 and  $-1$  otherwise, no single player who

develops a preference for one option will have an advantage from this. However, if all players continue guessing, one gets  $\sigma^2 = N$ , as pointed out before; a better coordination would be desirable. A vast continuum of mixed strategies exists where no outcome is preferred – all of these are Nash equilibria.

In the absence of a unique best way to proceed, players have no choice but to adapt their strategy to their environment, i.e., the behavior of their co-players. The MG has become a testing ground for various forms of “bounded rationality”, i.e., more or less simplistic decision and learning algorithms for the agents, ranging from a choice between a small number of Boolean functions [1, 4, 5] and simple neural networks [6] to evolutionary algorithms [7].

This paper presents new aspects of the Evolutionary Minority Game (EMG) studied by Johnson et al. in [7], and introduces an evolutionary variation of the stochastic strategy described in [8]. Two central questions are: 1.) what are the consequences if a player who has “died” in the evolutionary process is replaced by a modified copy of a different player, rather than picking a strategy at random? 2.) Can the prescription be generalized to more than two choices –  $1, \dots, Q$  instead of  $\pm 1$ , as suggested in Ref. [9]? Let us start with a look at the original evolutionary MG.

## II. THE ORIGINAL EVOLUTIONARY MG

In its original formulation [7], the EMG works as follows: each player has access to a table which records, for each possible combination of  $M$  consecutive minority decisions, what the minority decision following the last occurrence of that combination was. Players have only two individual features, namely a score  $s_i$  and a probability  $p_i$ . With this probability  $p_i$ , they choose the action in the history table corresponding to the current history; otherwise they choose the opposite action.

Players who win gain a point on their score, whereas the others lose a point. If the score of a player drops below a certain threshold  $-d < 0$ , the player is replaced

by a new one with a reset score  $s_i = 0$  and a probability  $p_i$  that is either a modified copy of his predecessor's value or chosen entirely at random.

As was pointed out before [10], this scheme can be simplified by exploiting the fact that the entries in the history table are decoupled, and that there is complete symmetry between the actions +1 and -1. The simplest interpretation that gives the same stationary distribution  $P(p)$  would therefore be that each player picks +1 with probability  $p_i$  and -1 otherwise. This points to an analogy with classical game theory [11]: the Minority Game is an  $N$ -player negative-sum matrix game, and the  $p_i$  define the mixed strategies of each player.

In the original EMG, a deceased player is replaced by a new player whose strategy  $p_i$  is a modified version of his predecessor's value: a random number  $\Delta$  with a given variance  $V$  is added to the previous value, with reflecting or cyclic boundary conditions at  $p = 0$  and  $p = 1$ . It turns out that neither the exact value of the threshold  $d$  nor the number  $N$  of players play a significant role, and that the typical size of mutations changes the results quantitatively, but not qualitatively: the stationary probability distribution  $P(p)$  develops two peaks at  $p = 0$  and  $p = 1$ , while there is still a significant probability for intermediate values of  $p$ .

First attempts to calculate the probability distribution analytically were only moderately accurate [12, 13]: they assumed that the reason for the self-organized segregation was only in the self-interaction of agents, and none of the two choices was systematically preferred. As newer studies [14, 15] have shown, this is not true: most of the time, there is a significant preference for one of the two options, and players who prefer this option have higher losses and a higher chance to be replaced. The preference for one side undergoes rather regular oscillations, with accompanying oscillations of the scores of players with one or the other preference. (The presence of these oscillations also means that the distribution  $P(p)$  is time-dependent, and becomes stationary only when averaged over a long time compared to the oscillation period. Whenever we speak of a stationary distribution from now on, we mean it in that sense.)

These publications also reported that, if a winning player receives  $R < 1$  points rather than 1 point, the segregation into extreme opinions vanishes below a certain point  $R_c$  and is replaced by a preference for undecided agents with  $p \approx 0.5$ . This is a rather remarkable result: after all, the aim of the game is unaffected by the modified payoff  $R$  – it is still advantageous to be in the minority, the chances of winning still only depend on the set of  $\{p_i\}$ , and the optimal configuration still has  $(N-1)/2$  players on one side and  $(N+1)/2$  on the other. What has changed is the dynamics of the game, and it is these dynamics that prevent the system from finding a more advantageous state.

The crucial point about the evolutionary dynamics that have been considered so far is this: *they do not systematically favor mutations with a higher fitness* – they

are not *compatible* in the sense of Ref. [16]: “...for any dynamic compatible with a properly defined fitness function fitter strategies should increase relative to less fit strategies.” We will demonstrate this in the limit of infinitely large mutations, which amounts to the same as choosing a new strategy completely at random. Simulations indicate that the results apply to small mutations as well.

Let us take the fitness  $f(p)$  associated with a strategy  $p$  to be the negative of the probability of a player using  $p$  of dying in a given round (in previous studies, this was assumed to be the average gain divided by the threshold). The strategies are distributed following a probability density  $P(p)$ , with a cumulative probability distribution

$$C(p, t) = \int_0^p P(p') dp'. \quad (2)$$

The expected number of players with  $p_i < p$  who mutate in a given round is  $\int_0^p -f(p')P(p')dp'$ . Out of the replacements for these players, a fraction of  $p$  has a strategy  $p_i < p$ . The updated probability function is therefore

$$\begin{aligned} C(p, t+1) = & C(p, t) + \int_0^p f(p')P(p')dp' \\ & - p \int_0^1 f(p')P(p')dp'. \end{aligned} \quad (3)$$

Going to continuous time and differentiating with respect to  $p$ , the integro-differential equation for  $P(p)$  looks as follows:

$$\frac{dP(p)}{dt} = P(p)f(p) - \int_0^1 P(p')f(p')dp' = P(p)f(p) - \bar{f}. \quad (4)$$

Keeping in mind that the fitness always takes negative values, Eq. (4) means that if  $P(p)$  is small enough, it increases even if the fitness associated with it is below average – this is clearly not what is desired.

The problem can be remedied (in principle) by a small change in the dynamics: a player is replaced not by a copy of himself, but by a copy of another player. This makes sense in various interpretations: in an economic situation, “dying” could have the meaning of “going broke”, and a player who tries a new start wants to imitate one of his (apparently more successful) competitors. In a biological setting, an organism literally dies, and an offspring of another organism takes its place. With this new mechanism, Eq. (3) takes the form

$$\begin{aligned} C(p, t+1) = & C(p, t) + \int_0^p f(p')P(p')dp' - \\ & C(p) \int_0^1 f(p')P(p')dp', \end{aligned} \quad (5)$$

which leads to the dynamic

$$\frac{\dot{P}(p)}{P(p)} = f(p) - \bar{f}. \quad (6)$$

This describes a so-called Malthus process, where the frequency of strategies with above-average fitness increases exponentially. The problem with this dynamic is that once a strategy becomes extinct, there is no way of reviving it. In the limit of infinitely many players that was tacitly assumed above, this is not a problem, since the probabilities for a strategy never go to 0 in finite time. However, with a finite number of players imitating each other, this would eventually lead to a small number of sub-populations, each of which exclusively plays one of the mixed strategies that happened to survive the initial stage (this scenario resembles a variation of the Backgammon model [17]).

To get a well-defined final state independent of initial conditions, it is therefore necessary to add a small mutation to the copied strategy to explore unoccupied areas in strategy space – for example by adding a Gaussian of variance  $V \ll 1$  to  $p$ , with reflecting boundary conditions. If this is done, a stationary probability distribution emerges that is strongly peaked at 0 and 1 and vanishes for intermediate  $p$ , as seen in Fig. 1. Global losses are reduced dramatically: for example, for  $V = 10^{-4}$ , one gets  $\sigma^2 \approx 0.021N$  instead of  $\sigma^2 \approx 0.31N$  for the original EMG for large  $d$ . Coordination can be improved even more by decreasing  $V$ , at the cost of a longer equilibration time. Furthermore, the results of decreasing the reward  $R$  for winning, as suggested in Ref. [14], are less dramatic: decreasing  $R$  does not destroy segregation; however, the peaks at extreme strategies become wider, and global efficiency decreases somewhat.

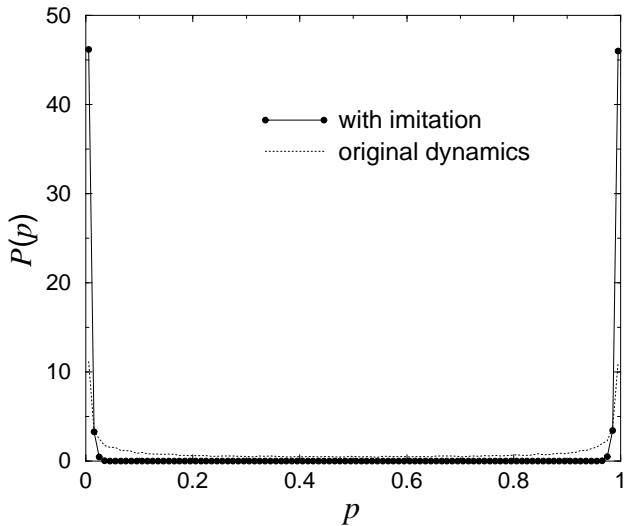


FIG. 1: Stationary distribution  $P(p)$  of the EMG with imitation, compared to the EMG without imitation with large mutations. Parameters were  $N = 501$ ,  $d = 10$ ,  $R = 1$ ,  $V = 10^{-4}$ .

### III. MULTI-CHOICE EMG

Generalizations of the MG to more than two options were studied in [9] (with agents using neural networks) and [18, 19] (with agents using a set of decision tables). The basic idea is simple:

- each player now picks an action (or “room”)  $\sigma_i^t \in 1, \dots, Q$  out of  $Q$  options;
- the number of players  $N_q$  who chose each option is determined:  $N_q = \sum_i^N \delta_{\sigma_i^t, q}$ ;
- the option chosen by the fewest players (the “least crowded room”, with occupation  $N_{min}$ ) is declared winner (a coin toss decides in case of a tie);
- the players who chose the winning room receive an award (let us say, one point), whereas the others lose  $1/(Q - 1)$  points.

Global efficiency can be measured by taking the analog of Eq. (1) either for the occupation of the winning room:

$$\sigma_{min}^2 = \left\langle \left( N_{min} - \frac{N}{Q} \right)^2 \right\rangle_t \quad (7)$$

or the occupation of all rooms:

$$\sigma_Q^2 = \frac{1}{Q} \left\langle \sum_q^Q \left( N_q - \frac{N}{Q} \right)^2 \right\rangle_t . \quad (8)$$

In many cases, the latter quantity differs from  $\sigma_{min}^2$  only by a constant factor and is easier to calculate [9]. For the reference case of random guessing,  $\sigma_Q^2$  takes the value of  $N/Q$ .

The generalization of the evolutionary MG to multiple choices leaves several options. We choose the one that yields a standard multi-player matrix game: each player is equipped with a strategy vector  $\mathbf{p}_i$ , with entries  $p_{i,q} \geq 0$  that give the probability of player  $i$  choosing room  $q$ . These vectors obey the normalization constraint  $\sum_q p_{i,q} = 1$ .

How strongly a player specializes in one option can be measured using the self-overlap of his strategy vector:

$$O_i = \sum_q p_{i,q}^2 . \quad (9)$$

This quantity varies from  $1/Q$  for a completely undecided player to 1 for a player who chooses one option exclusively. The average over the population,  $O = \sum_i O_i / N$ , is therefore a good measure of the degree of specialization among players.

If a player’s score drops below the threshold  $-d$ , he is replaced by a player with a different strategy vector. Again, two different paths suggest themselves: either, as in Ref. [7], the player is replaced by an altered copy of himself (or, in the extreme case of large mutations, a

randomly chosen new player); or the gap is filled by the mutated offspring of another, randomly chosen player. We ran simulations starting with random initial vectors uniformly distributed on the simplex (see the Appendix). The same picture emerges as in the binary-choice case:

If deceased players are replaced by copies of themselves or players with random strategies, very little specialization occurs. The stationary distribution of  $O_i$  gets slightly more contributions from larger values; however, the mean value shifts only little. E.g., for  $Q = 3$  and  $d = 10$ , the average  $O$  changes from  $1/2$  for no adaption to  $\approx 0.540$  for replacing deceased players with randomly chosen strategies. Correspondingly, global efficiency increases only slightly: in this case, from  $\sigma_Q^2 = 1/3$  to  $\approx 0.242$ .

However, copying another player (with small mutations) gives excellent coordination: in the stationary state, all players specialize strongly in one of the choices – the self-overlap of strategies is close to 1 (see Fig. 2). The width of the probability distribution of  $O_i$  again depends on the magnitude of mutations: the smaller the mutations, the narrower the peak. As before, eliminating mutations altogether prevents coordination.

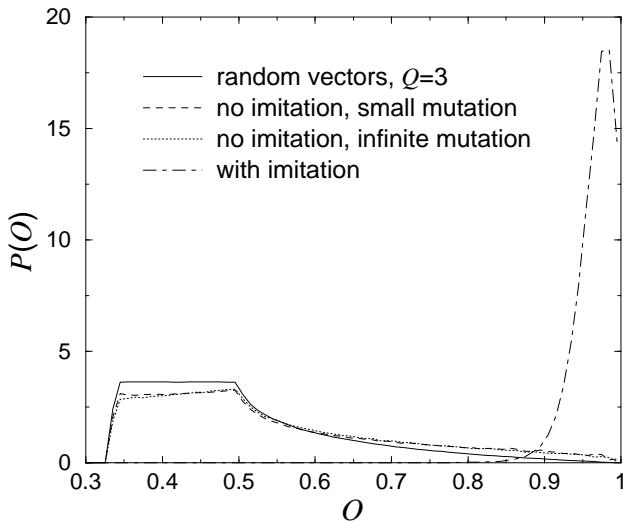


FIG. 2: Stationary distribution  $P(O)$  of the self-overlap of strategy vectors for  $Q = 3$  options. The solid line give the initial state of vectors chosen uniformly on the simplex. The dashed and dotted line give the result of the original dynamics: it matters little if replaced players undergo small mutations or are chosen completely at random. The dot-dashed line shows the effect of imitation with small mutation. The same parameters as in Fig. 1 were used.

#### IV. THE STOCHASTIC MG

In Ref. [8], a strategy was presented that looks similar to the EMG described above: again, each agent is equipped with a probability  $p_i$  that characterizes his be-

havior. The meaning of  $p$  is different, though: if a player wins in a given round, he is content and repeats his choice  $\sigma_i^t$  in the following round. If he loses, however, the agent may rethink his game plan and switch to the opposite action with probability  $p_i$ . This prescription amounts to a one-step Markov process which can be solved analytically in some regimes if all players use the same  $p$  [8, 20].

For large  $p$  (of order 1), a finite fraction of the population switches at every time step, resulting in large global losses ( $\sigma^2 = \mathcal{O}(N^2)$ ). Furthermore, the majority flips very frequently. The stationary probability distribution of  $A = \sum_i \sigma_i^2$  takes the shape of two roughly Gaussian peaks centered at  $\pm Np/(2 - p)$ .  
Pine finished – Clo

However, if  $p$  scales with  $p = 2x/N$ ,  $x = \mathcal{O}(1)$ , there is very good coordination ( $\sigma^2 = 1 + 4x + 4x^2/3$  as  $N \rightarrow \infty$ ), and the minority does not switch at every time step. The stationary probability distribution is centered at  $A = 0$ , with a width of roughly  $2x$ .

The relative simplicity of the mathematical description breaks down if each player is allowed to have an individual  $p_i$ : if players are distinguishable, it is no longer enough to state how many of them are on one side or the other to completely characterize the system. However, with a few approximations, even then some insight can be gained if complications like an evolutionary dynamics are introduced.

We start with a scenario analogous to that described in Sec. II, which we will call SEMG (stochastic evolutionary Minority Game): each of the  $N$  players has an individual probability  $p_i$  of switching, which is initially a uniform random number. In the case of a loss, the player loses a point; otherwise he wins one. If his score drops below  $-d$ , his probability is replaced, and his score reset to 0. One would expect that players with smaller  $p$  have an advantage over those with large  $p$ , and one would hope that players organize themselves to a stationary state with  $p_i \propto 1/N$ .

If mutations of players are performed analogous to the original EMG (new players are modified copies of the deceased ones), a stationary distribution  $P(p)$  emerges in which small  $p$  are more likely than large ones, but there is still a significant tail towards large  $p$  (see Fig. 3). Simulations show that contrary to the original EMG, the details of the mutation process (size of the mutation, reflecting/cyclical boundary conditions etc.) have no impact at all on this distribution. Neither does the threshold  $d$ . The number of players  $N$  has only a small effect – the shape of the distribution for larger  $p$  does not change significantly, but  $P(p)$  increases for very small  $p$  as  $N$  increases. This effect is explained later; however, it is much too small to achieve a mean  $p$  that scales with  $1/N$ .

As mentioned before,  $p = \mathcal{O}(1)$  means that the minority changes sides at practically every time step. Assuming that this is true, it is possible to calculate the average gain of a player with a given  $p$ , and hence his expected lifetime. From this, the stationary distribution can be calculated. Let us start with a more general approach, which assumes very simple dynamics of the global deci-

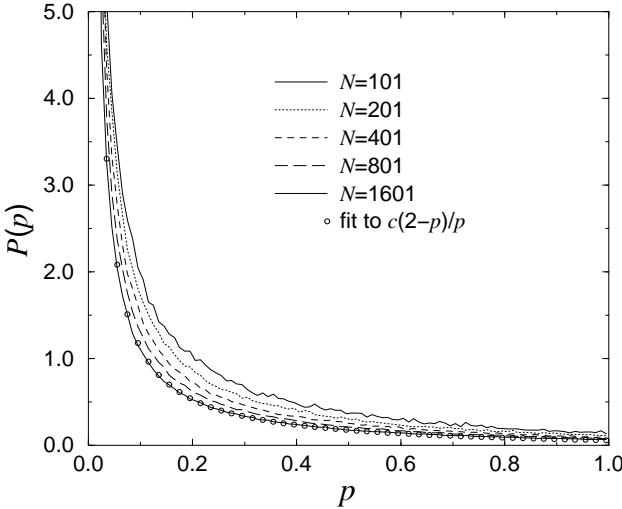


FIG. 3: Stationary distribution  $P(p)$  in the Stochastic EMG, for different values of  $N$ . New players use randomly chosen values of  $p$ ; the threshold is  $d = 10$ .

sion and neglects the impact of a single player on that decision. If the global minority stays the same in two consecutive time steps with a given probability  $\mu$ , the average gain of a player can be determined as follows: with a probability  $w(p)$ , the player wins in a given round. Consequently, he does not switch sides, and wins again in the next round with probability  $\mu$ . However, if he lost in the first round, he can win by changing sides if the minority stays on the same side (probability  $\mu p$ ) or if he insists on his opinion, and the minority switches (probability  $(1 - \mu)(1 - p)$ ). For a given  $\mu$ , the probability of winning is therefore

$$w(p) = \mu w(p) + (\mu p + (1 - \mu)(1 - p))(1 - w(p)). \quad (10)$$

For the average gain  $g(p) = 2w(p) - 1$ , this gives

$$g(p) = \frac{-p(1 - 2\mu)}{1 + (1 - p)(1 - 2\mu)}. \quad (11)$$

As long as  $g(p)$  is systematically negative and does not undergo fluctuations on time scales comparable to the lifetime of a player, it is safe to assume that the mean lifetime  $L(p)$  is  $d/g(p)$  – the score is a random walk with a negative drift which outweighs the diffusive motion for sufficiently large  $d$ .

Assuming that the average  $p$  is large enough to ensure that the majority will flip at every time step – i.e.,  $\mu = 0$ , and  $g(p) = p/(p - 2)$  – one can now identify  $-1/L(p) = p/d(2 - p)$  with the fitness  $f(p)$  and use Eq. (4) to derive, for the stationary state,

$$P(p) \propto \frac{2 - p}{p}. \quad (12)$$

This probability distribution has the unpleasant property of diverging at  $p = 0$ , since agents with  $p = 0$  are assigned

an infinite lifetime. In practice, three effects come into play: first, agents never have exactly  $p = 0$ . Second, even for agents with  $p = 0$ , their impact on the decision will give them a very small negative average gain and a long, but not infinitely long life. Third,  $\mu$  is very small, but never strictly equal to 0; if the probability of large values of  $p$  becomes too small, the simplistic assumptions about the dynamics no longer hold, and  $\mu$  increases. Together, these effects are responsible that for any given set of parameters, a stationary distribution emerges. Eq. (12) allows for a good fit to these stationary distributions measured in simulations, as seen in Fig. 3.

The average  $p$  that emerges from these simulations is of order 1 (to be more specific, around 0.10, with the precise value depending on  $N$  and  $d$ ). This means that the solution is self-consistent: the value of  $p$  that results from the dynamics is large enough to justify the assumptions that went into estimating it.

Analogous to Sec. II, the evolutionary dynamic of replacing a player with a random player or a copy of the old one does not always favor strategies with higher fitness. However, the same step can be taken to improve coordination: if deceased players are replaced with a copy of another player chosen at random, the relative growth of  $P(p)$  is proportional to  $f(p) - \bar{f}$ , just as in Eq. (6).

Unfortunately, Eq. (11) is not applicable for very small  $p$  (since it neglects the impact of the considered agent), and there is no simple equation that gives the fitness as a function of the strategy for all regimes. Nevertheless, there seems to be no situation where having a higher  $p$  gives better results. Hence, the evolutionary dynamic should lead to a state with minimal  $p$  for all agents. A similar problem as in Sec. II comes into play here, although it does not have quite as troubling effects: with a finite number of players, the best possible coordination is for all players to adapt the smallest value of  $p$  that survived the initial stage. However, this value is usually of order  $1/N$  – if initial values are chosen at random, they have an average distance of  $1/N$ .

Just as in Sec. II, the sensitivity to the initial state can be removed by adding a small random number to  $p_i$  when a new player is created. As seen in Fig. 4, results are similar: a peak centered at  $p = 0$  emerges, whose width depends on the size of the mutations. With sufficiently small mutations,  $p = \mathcal{O}(1/N)$  and  $\sigma^2 = \mathcal{O}(1)$  can easily be achieved.

One of the drawbacks of introducing small mutations is that their size is a new parameter that has to be adjusted to get a  $\sigma^2$  of order 1. One of the conceptual flaws of the SMG was that players had to be aware of the size of the population to justify an adequate choice of  $p$ . One might have hoped that in an evolutionary scheme, the correct scaling for  $p$  would emerge naturally. With the dynamic of copying with mutations, it does not. Maybe an evolutionary mechanism that mutates the size of mutations would solve this.

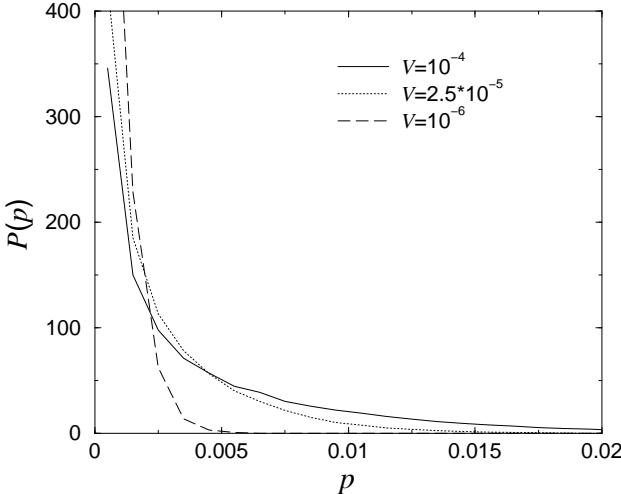


FIG. 4: Stationary distribution  $P(p)$  in the Stochastic EMG with copying. The variance  $V$  of mutations influences the width of the distribution. The simulation included  $N = 2000$  players, with a threshold  $d = 10$ .

## V. MULTI-CHOICE STOCHASTIC MG

The stochastic MG can be generalized to multiple choices in several ways, none of which is quite as natural as the generalization of the EMG in Sec. III. One of the most intuitive ways is this: again, players have the choice between  $Q$  different actions, or ‘‘rooms’’. Those who chose the least-crowded room are content and choose it again, whereas all others decide, with probability  $p_i$ , to look for alternatives. If they are not informed about which room won, they randomly pick one of the room that they did not choose the last time. Another plausible scenario is that they know which room won, and choose it with probability  $p$  the next time.

In both cases, the system can still be considered a one-step Markov process. However, the state of the system must now be characterized by  $Q - 1$  values  $N_1, \dots, N_{Q-1}$ , which give the number of players that chose each action (the remaining value  $N_Q$  can be calculated from normalization constraints:  $\sum_q N_q = N$ ), and the joint probability distribution is a  $Q - 1$ -dimensional tensor, or a function living in  $\mathbb{R}^{Q-1}$  if one wants to go to continuous variables. Transition probabilities look even worse, taking the form of  $2(Q - 1)$ -dimensional tensors or integral kernels. Put briefly, this problem is only accessible to simulations and very crude approximations.

In the limit of  $p = \mathcal{O}(1)$ ,  $N \rightarrow \infty$ , the behavior is analogous to that detailed in Sec. IV: finite fractions of players move from room to room, and the minority option changes at every time step. Suitable variables are  $n_q = N_q/N$ , the fractions of players who chose option  $q$ . Occupation probabilities  $P(n_q)$  turn out to be a superposition of  $Q$  Gaussian peaks whose widths decrease with  $N$ . Self-consistent values for the centers of the peaks can

be found analytically, as the following example for  $Q = 3$  will show:

At any given step, there are three occupation numbers, which we order  $n_1 < n_2 < n_3$ . Room 1 will now receive players from rooms 2 and 3, whereas rooms 2 and 3 gain players from the respective other room and lose players to all other rooms. Neglecting fluctuations, the rate equations for the occupations  $n_i^+$  at the next time step look like this:

$$\begin{aligned} n_1^+ &= n_1 + (p/2)(n_2 + n_3); \\ n_2^+ &= n_2 - pn_2 + (p/2)n_3; \\ n_3^+ &= n_3 - pn_3 + (p/2)n_2. \end{aligned} \quad (13)$$

If one can find a permutation of  $n_i$  such that each  $n_i$  is equal to  $n_j^+$  with some  $j \neq i$ , one has a solution. In the present case, the solution for Eq. (13) is  $n_1^+ = n_3$ ,  $n_2^+ = n_1$  and  $n_3^+ = n_2$  for  $0 < p \leq 2/3$ , and  $n_1^+ = n_3$ ,  $n_2^+ = n_2$  and  $n_3^+ = n_1$  for  $2/3 \leq p < 1$ . The corresponding equations are

$$\begin{aligned} n_1 &= \frac{4 - 6p + 3p^2}{3(2 - p)^2} \text{ for } p \leq 2/3, 1 - \frac{6}{10 - 3p} \text{ for } p > 2/3; \\ n_2 &= \frac{4(1 - p)}{3(2 - p)^2} \text{ for } p \leq 2/3, \frac{2}{10 - 3p} \text{ for } p > 2/3; \\ n_3 &= \frac{2}{3(2 - p)} \text{ for } p \leq 2/3, \frac{4}{10 - 3p} \text{ for } p > 2/3. \end{aligned} \quad (14)$$

This solution agrees well with behavior observed in simulations, as Fig. 5 shows.

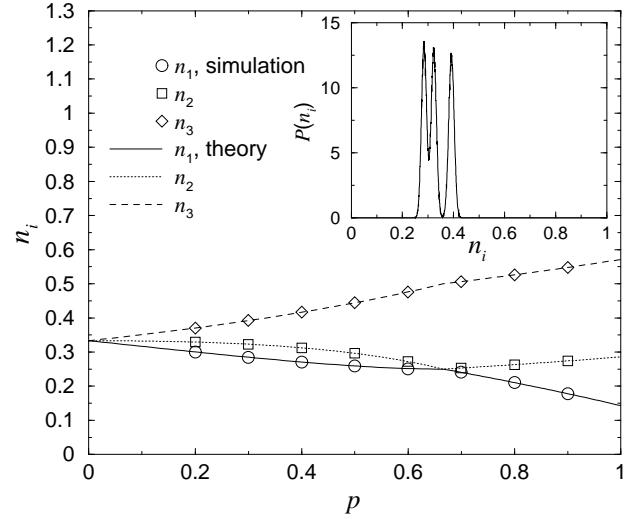


FIG. 5: Centers of the peaks of  $\pi(n)$  in the stochastic MG with  $Q = 3$ . Simulations agree well with Eqs. (14). The inset shows the probability distribution for  $p = 0.3$  and  $N = 2000$ .

For small uniform  $p$  of order  $Q/N$ , analytical results are hard to find, for the mentioned reasons. Evidence from simulations shows that the system organizes itself into a probability distribution close to optimal coordination. The details of the distribution depend, even for

large  $N$ , on  $N \bmod Q$ . An example of such a distribution is shown in Fig. 6.

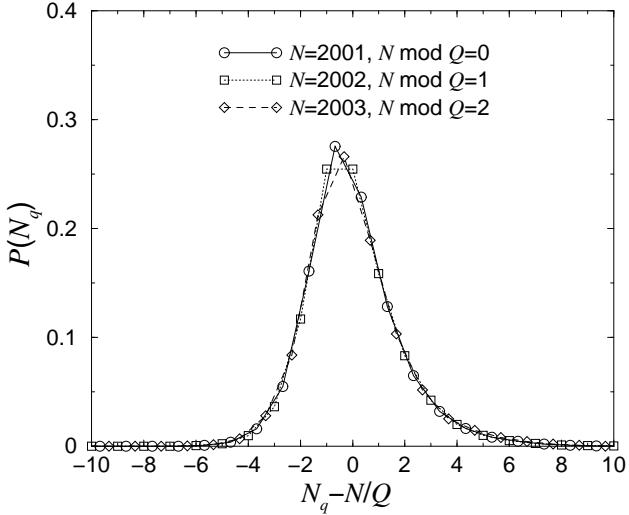


FIG. 6: Stationary probability distribution of occupation numbers  $N_q$  in the multi-choice SMG, with uniform  $p = 1/2000 \approx 1/N$ , for  $Q = 3$ .

Evolutionary dynamics can be introduced exactly analogous to the previous sections, and very similar results are observed: choosing a new player at random and replacing a player by a mutated copy of himself yields the same stationary probability distribution, which has a long tail towards larger  $p$ . Average values for  $p$  are around 0.2, the probability distribution of occupation numbers  $N_q$  shows multiple peaks.

The alternative dynamic of copying another player with mutation gives a sharp peak around  $p = 0$ , with  $\sigma_Q^2$  on the order of 1 for sufficiently small mutations.

## VI. CONCLUDING REMARKS

We have shown that the self-organized segregation observed in the evolutionary Minority Game is not only much more pronounced, but also more robust to modifications of the payoff scheme if a suitable dynamic is used – one that allows strategies with above-average fitness to grow, rather than keeping sub-par strategies alive. Copying another player’s strategy is a suitable way of doing this. The copy has to be modified by a small mutation to eliminate sensitivity to initial conditions.

We have also introduced a natural generalization of the EMG to multiple choices, evolutionary dynamics for the Stochastic MG, and an extension of the latter to multiple choices. The results in all cases have striking similarities: properly chosen evolutionary dynamics lead to near-optimal coordination and drastic suppression of losses, compared to random guessing.

It is somewhat ironic that the key ingredient to these dynamics is copying the strategy of another player – in

a game where the goal is to take different actions than the majority. One could argue that in the regime of the EMG where players have very strong preferences for either side, the EMG is not too different from the Stochastic MG: players stick to their opinion if possible; if they lose too often, they either copy a player from their own side (which changes little about the situation), or they copy a player from the other side – they switch their output. The presence of accumulated scores makes the situation more complicated than that, yielding preference oscillations whose wavelength depends on the threshold.

Despite the increased efficiency and robustness that the imitation mechanism has brought, the dynamics are still complex, and a thorough analytical treatment has not been found yet. The same holds for the SEMS, where the interplay between the probability distributions of outputs and of strategies is difficult to handle analytically. Maybe future research will fill in the missing details.

## Acknowledgments

R. M. acknowledges financial support by the GIF. We thank Georg Reents for his enthusiasm about the Stochastic MG, for ideas, helpful discussions and supervision. We also thank Wolfgang Kinzel for his support and supervision.

## APPENDIX A: CONSTRUCTING UNIFORMLY DISTRIBUTED VECTORS ON A SIMPLEX

There are many conceivable methods of finding  $Q$ -dimensional random vectors  $\mathbf{p}$  that obey the constraints of probabilities,  $p_i \geq 0$  and  $\sum_i p_i = 1$ . However, the easiest ones do not give a uniform distribution on the simplex of allowed vectors: for example, forcing a set of uniform random numbers between 0 and 1 to obey the constraints by dividing them by their sum emphasizes vectors in the center of the simplex due to projection effects.

The following method generates uniformly distributed probability vectors from uniformly generated random numbers. We include it because it may be useful to the reader for other applications.

The space of allowed vectors  $\mathbf{p}$  is spanned by linear combinations

$$\mathbf{p} = \sum_q a_q \mathbf{b}_q \quad (\text{A1})$$

of a set of  $Q$  basis vectors  $\mathbf{b}_q$

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ \vdots \end{pmatrix}, \dots \quad (\text{A2})$$

with coefficients  $1 = a_1 \geq a_2 \geq \dots \geq a_Q \geq 0$ . The components must be chosen with a suitably weighted probability distribution to account for the fact that a larger coefficient  $a_2$  allows for more combinations of  $a_3, a_4$  etc. Since the volume of the sub-simplex limited by  $a_q$  is proportional to  $a_q^{Q-q}$ , the appropriate distribution is

$$\text{Prob}(a_q) = \begin{cases} \frac{Q-q+1}{a_{q-1}} a_q^{Q-q} & \text{for } 0 \leq a_q \leq a_{q-1} \\ 0 & \text{else} \end{cases}. \quad (\text{A3})$$

Consequently, a set of coefficients  $\{a_2, \dots, a_Q\}$  can be

calculated from a set of random numbers  $\{r_2, \dots, r_Q\}$  uniformly distributed between 0 and 1 by a simple transformation [21]:

$$a_1 = 1; \quad a_q = a_{q-1} r_q^{1/(Q-q+1)}. \quad (\text{A4})$$

Eq. (A1) then gives the desired vector on the simplex.

- [1] D. Challet and Y.-C. Zhang, *Physica A*, 246:407, 1997.
- [2] Minority game homepage (with extensive bibliography), <http://www.unifr.ch/econophysics/minority>.
- [3] Manfred J. Holler and Gerhard Illig, *Einführung in die Spieltheorie*. Springer-Verlag, Berlin, 1991.
- [4] D. Challet and M. Marsili, *Phys. Rev. E*, 60(6A):R6271–R6274, 1999.
- [5] Matteo Marsili, Damien Challet, and Ricardo Zecchina, cond-mat/9908480, 1999.
- [6] R. Metzler, W. Kinzel, and I. Kanter, *Phys. Rev. E*, 62(2):2555, 2000.
- [7] N. F. Johnson, P. M. Hui., R. Jonson, and T. S. Lo, *Phys. Rev. Lett.*, 82(16):3360, 1999.
- [8] G. Reents, R. Metzler, and W. Kinzel, *Physica A*, 299:253–261, 2001.
- [9] Liat Ein-Dor, Richard Metzler, Ido Kanter, and Wolfgang Kinzel, *Phys. Rev. E*, 63:066103, 2001.
- [10] E. Burgos and Horacio Ceva, cond-mat/0003179, 2000.
- [11] John von Neumann and Oskar Morgenstern, *Theory of Games and Economic Behaviour*. Princeton University Press, Princeton, N.J., 1944.
- [12] T. S. Lo, P. M. Hui, and N. F. Johnson, *Phys. Rev. E*, 62:4393, 2000.
- [13] R. D'holst and G. J. Rodgers, *Physica A*, 270:514–525, 1999.
- [14] Shahar Hod and Ehud Nakar, *Phys. Rev. Lett.*, 88:238702, 2002.
- [15] Ehud Nakar and Shahar Hod, cond-mat/0206056, 2002.
- [16] Daniel Friedman, *Econometrica*, 59(3):637–666, 1991.
- [17] C. Godrèche, J. P. Bouchaud, and M. Mezard, *J. Phys. A*, 28:603–611, 1995.
- [18] F. K. Chow and H. F. Chau, cond-mat/0109166, 2001.
- [19] F. K. Chow and H. F. Chau, cond-mat/0210608, 2002.
- [20] Christian Horn, Diplomarbeit, Universität Würzburg, 2001.
- [21] William H. Press, Saul A. Teukolsky, William T. Vetterling, and Brian P. Flannery, *Numerical Recipes in C*. Cambridge University Press, Cambridge, 1988.