

# Perfect dominating sets in the Cartesian products of prime cycles.

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## Abstract

We study the structure of a minimum dominating set of  $C_{2n+1}^n$ , the Cartesian product of  $n$  copies of the cycle of size  $2n + 1$ , where  $2n + 1$  is a prime.

KEYWORDS: PERFECT LEE CODES; DOMINATING SETS; DEFINING SETS.

## 1 Introduction

Let  $G$  and  $H$  be two graphs. The *Cartesian product* of  $G$  and  $H$  is a graph with vertices  $\{(x, y) : x \in G, y \in H\}$  where  $(x, y) \sim (x', y')$  if and only if  $x = x'$  and  $y \sim y'$ , or  $x \sim x'$  and  $y = y'$ . Let  $G^n$  denote the Cartesian product of  $n$  copies of  $G$ . This article deals with  $C_{2n+1}^n$  where  $C_{2n+1}$  is the cycle of size  $p := 2n + 1$  and  $p$  is a prime.

For our purpose, it is more convenient to view the vertices of  $C_{2n+1}^n$  as the elements of the group  $G := \mathbb{Z}_{2n+1}^n$ . Then  $x \sim y$  if and only if  $x - y = \pm e_i$  for some  $i \in [n]$ , where  $e_i = (0, \dots, 1, \dots, 0)$  is the unit vector with 1 at the  $i$ th coordinate. In other words,  $C_{2n+1}^n$  is the Cayley graph  $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$  over the group  $\mathbb{Z}_{2n+1}^n$  with the set of generators  $\mathcal{U} = \{\pm e_1, \dots, \pm e_n\}$ . From this point on, to emphasis the group structure of the graph we will use the Cayley graph notation  $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$  instead of the Cartesian product notation of  $C_{2n+1}^n$ .

Let  $u$  and  $v$  be two vertices of a graph  $G$ . We say that  $u$  dominates  $v$  if  $u = v$  or  $u \sim v$ . A subset  $S$  of the vertices of  $G$  is called a *dominating set*, if every vertex of  $G$  is dominated by at least one vertex of  $S$ . A dominating set is *perfect*, if no vertex is dominated by more than one vertex.

**Remark 1** Let  $G$  be a graph with  $m$  vertices. Every function  $f : V(G) \rightarrow \mathbb{C}$  can be viewed as a vector  $\vec{f} \in \mathbb{C}^m$ . Let  $A$  denote the adjacency matrix of  $G$ . Note that  $f : V(G) \rightarrow \{0, 1\}$  is the characteristic function of a perfect dominating set if and only if  $(A + I)\vec{f} = \vec{1}$ .

We are interested in perfect dominating sets of  $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ . Note that for an  $r$ -regular graph  $G = (V, E)$  a dominating set is perfect if and only if it is of size  $|V|/(r+1)$ . Since  $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$  is  $2n$ -regular and has  $(2n+1)^n$  vertices, a dominating set is perfect if and only if it is of size  $(2n+1)^{n-1}$ .

Fix an arbitrary  $(\epsilon_1, \dots, \epsilon_{n-1}) \in \{-1, 1\}^{n-1}$ , and a  $k \in \{0, \dots, 2n\}$ . The set

$$\{(x_1, \dots, x_{n-1}, k + \sum_{i=1}^{n-1} \epsilon_i(i+1)x_i) : x_i \in \mathbb{Z}_{2n+1} \ \forall i \in [n-1]\} \quad (1)$$

forms a perfect dominating set in  $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ , where the additions are in  $\mathbb{Z}_{2n+1}$ . To see this consider  $y = (y_1, \dots, y_n) \in \mathbb{Z}_{2n+1}^n$ . Let  $t = k + \sum_{i=1}^{n-1} \epsilon_i(i+1)y_i$ , and  $\Delta = t - y_n \pmod{2n+1}$  so that  $|\Delta| \leq n$ . If  $\Delta \in \{-1, 0, 1\}$  then  $y$  is dominated by  $(y_1, \dots, y_{n-1}, t)$ . If  $\Delta \notin \{-1, 0, 1\}$ , then with the notation  $j := |\Delta| - 1$ ,  $y$  is adjacent to  $(y_1, \dots, y_{j-1}, y_j - \epsilon_j \times \text{sgn}(\Delta), y_{j+1}, \dots, y_n)$ , which can easily be seen that is in the considered set.

There are many results in the direction of constructing perfect dominating sets in the Cartesian product of cycles (see [5] and its references). However the authors are unaware of any result in the direction of characterizing the structure of perfect dominating sets. We consider the simplest case  $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$  where  $2n+1$  is a prime. Even in this simple case we are unable to characterize all the perfect dominating sets. However we prove the following theorem in this direction.

**Theorem 1** *Let  $2n+1$  be a prime and  $S \subseteq \Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$  be a perfect dominating set. Then for every  $(x_1, \dots, x_n) \in \mathbb{Z}_{2n+1}^n$  and every  $i \in [n]$ ,*

$$|S \cap \{(y_1, \dots, y_n) : y_j = x_j \ \forall j \neq i\}| = 1.$$

Theorem 1 says that when  $2n+1$  is a prime, every parallel-axis line contains exactly one point from every perfect dominating set of  $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ . It is easy to construct examples to show that the condition of  $2n+1$  being a prime is necessary [4].

Let  $\mathcal{F}$  be a family of sets. For  $S \in \mathcal{F}$ , a set  $D \subseteq S$  is called a *defining set* for  $(S, \mathcal{F})$  (or for  $S$  when there is no ambiguity), if and only if  $S$  is the only superset of  $D$  in  $\mathcal{F}$ . The size of the minimum defining set for  $(S, \mathcal{F})$  is called its defining number. Defining sets are studied for various families of  $\mathcal{F}$  (See [3] for a survey on the topic). Let  $\mathcal{F}$  be the family of all minimum dominating sets of  $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ . Note that since  $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$  is regular and contains at least one perfect dominating

set, a set  $S \subseteq V(G)$  is a minimum dominating set if and only if it is a perfect dominating set. In [2] Chartrand et al. studied the size of defining sets of  $\mathcal{F}$  for  $n = 2$ . Based on this case they conjectured that the smallest defining set over all minimum dominating sets of  $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$  is of size exactly  $n$ . As it is noticed by Richard Bean [private communication], the conjecture fails for  $n = 3$ , as in this case there are perfect dominating sets with defining number 2 (See Remark 3). So far there is no nontrivial bound known for the defining numbers of minimum dominating sets of  $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ . We prove the following theorem.

**Theorem 2** *Let  $2n + 1$  be a prime and  $\mathcal{F}$  be the family of all minimum dominating sets of  $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ . Every  $S \in \mathcal{F}$  has a defining set of size at most  $n!2^n$ .*

The proof of Theorem 1 uses Fourier analysis on finite Abelian groups. In Section 2 we review Fourier analysis on  $\mathbb{Z}_p^n$ . Section 3 is devoted to the proof of Theorem 2. Section 4 contains further discussions about the defining sets of minimum dominating sets of  $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ .

## 2 Background

In this section we introduce some notations and review Fourier analysis on  $G = \mathbb{Z}_p^n$ . For a nice and more detailed, but yet brief introduction we refer the reader to [1]. See also [6] for a more comprehensive reference.

Aside from its group structure we will also think of  $G$  as a measure space with the uniform (product) measure, which we denote by  $\mu$ . For any function  $f : G \rightarrow \mathbb{C}$ , let

$$\int_G f(x) dx = \frac{1}{|G|} \sum_{x \in G} f(x).$$

The inner product between two functions  $f$  and  $g$  is  $\langle f, g \rangle = \int_G f(x) \overline{g(x)} dx$ . Let

$$\omega = e^{2\pi i/p},$$

where  $i$  is the imaginary number. For any  $x \in G$ , let  $\chi_x : G \rightarrow \mathbb{C}$  be defined as

$$\chi_x(y) = \omega^{\sum_{i=1}^n x_i y_i}.$$

It is easy to see that these functions form an orthonormal basis. So every function  $f : G \rightarrow \mathbb{C}$  has a unique expansion of the form  $f = \sum \hat{f}(x) \chi_x$ , where  $\hat{f}(x) = \langle f, \chi_x \rangle$  is a complex number.

### 3 Proof of Theorem 1

Let  $\vec{0} = (0, \dots, 0)$ ,  $\vec{1} = (1, \dots, 1)$ , and  $e_i = (0, \dots, 1, \dots, 0)$ , the unit vector with 1 at the  $i$ -th coordinate. Let  $p = 2n + 1$  be a prime and  $S$  be a perfect dominating set in  $G$ , and let  $f$  be the characteristic function of  $S$ , i.e.  $f(x) = 1$  if  $x \in S$  and  $f(x) = 0$  otherwise. Let

$$D = \{\pm e_1, \pm e_2, \dots, \pm e_n\},$$

be the set of unit vectors and their negations. For every  $\tau \in D$  define  $f_\tau(x) = f(x + \tau)$ . Note that

$$\hat{f}_\tau(y) = \int f(x + \tau) \overline{\chi_y(x)} dx = \int f(x) \overline{\chi_y(x - \tau)} dx = \int f(x) \overline{\chi_y(x)} \chi_y(\tau) dx = \hat{f}(y) \chi_y(\tau).$$

Let

$$g = f + \sum_{\tau \in D} f_\tau.$$

We have

$$g = \left( \sum_{y \in G} \hat{f}(y) \chi_y \right) + \sum_{\tau \in D} \sum_{y \in G} \hat{f}_\tau(y) \chi_y = \sum_{y \in G} \hat{f}(y) \left( \sum_{\tau \in D \cup \{\vec{0}\}} \chi_y(\tau) \right) \chi_y. \quad (2)$$

Since  $f$  is the characteristic function of a perfect dominating set, we have  $g(x) = 1$ , for every  $x \in G$ . So  $g = \chi_{\vec{0}}$ . By uniqueness of Fourier expansion, for every  $y \neq \vec{0}$ ,

$$0 = \hat{g}(y) = \hat{f}(y) \sum_{\tau \in D \cup \{\vec{0}\}} \chi_y(\tau) = \hat{f}(y) \left( 1 + \sum_{i=1}^n \omega^{y_i} + \sum_{i=1}^n \omega^{-y_i} \right). \quad (3)$$

Now we turn to the key step of the proof. Since  $2n + 1$  is a prime, (3) implies that whenever  $\hat{f}(y) \neq 0$ , we have

$$\{y_1, \dots, y_n\} \cup \{-y_1, \dots, -y_n\} = \{1, \dots, 2n\}. \quad (4)$$

Denote the set of all  $y$  satisfying (4) by  $\mathcal{Y}$ . For  $1 \leq i \leq n$ , let

$$D_i = \{ke_i : 0 \leq k \leq 2n\}.$$

Define  $g_i = \sum_{\tau \in D_i} f_\tau$ . Similar to (2), we get

$$g_i = \sum \hat{f}(y) \left( \sum_{\tau \in D_i} \chi_y(\tau) \right) \chi_y.$$

When  $y \in \mathcal{Y}$ , since  $y_i \neq 0$ , we have

$$\sum_{\tau \in D_i} \chi_y(\tau) = \sum_{k=0}^{2n} \omega^{ky_i} = 0.$$

When  $y \notin \mathcal{Y}$  and  $y \neq \vec{0}$ ,  $\hat{f}(y) = 0$ . So

$$g_i = \left( \hat{f}(0) \sum_{\tau \in D_i} \chi_{\vec{0}}(\tau) \right) \chi_{\vec{0}} = \chi_{\vec{0}} = 1. \quad (5)$$

Note that  $g_i(x)$  counts the number of elements in  $S \cap \{(y_1, \dots, y_n) : y_j = x_j \ \forall j \neq i\}$ . This completes the proof.

**Remark 2** The above proof can be translated to the language of linear algebra (However in the linear algebra language the key observation (4) becomes less obvious). Indeed, let  $m = (2n+1)^n$  denote the number of vertices. From Remark 1 we know that  $f : \mathbb{Z}_{2n+1}^n \rightarrow \mathbb{C}$  is the characteristic function of a perfect dominating set if and only if  $(A + I)\vec{f} = \vec{1}$ , where  $A$  is the adjacency matrix of  $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ . The reader may notice that in the proof of Theorem 1,  $\vec{g} = (A + I)\vec{f}$ , and thus (2) shows that  $\vec{\chi}_y$  is a family of orthonormal eigenvectors of  $A + I$ . Moreover, among these eigenvectors, the ones that correspond to the 0 eigenvalue are exactly  $\vec{\chi}_y$  with  $y \in \mathcal{Y}$ . Hence the rank of  $A + I$  is  $m - |\mathcal{Y}| = (2n+1)^n - n!2^n$ . We will use this fact in the proof of Theorem 2.

## 4 Proof of Theorem 2

As it is observed in Remark 1, every perfect dominating set of  $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$  corresponds to a zero-one vector  $\vec{f} \in \mathbb{C}^m$  that satisfies  $(A + I)\vec{f} = \vec{1}$ . Let

$$V := \text{span}\{\vec{f} : f \in \mathcal{F}\}.$$

Trivially

$$\dim V \leq 1 + (m - \text{rank}(A + I)) = 1 + n!2^n.$$

Also for a subset  $D$  of vertices of  $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ , define

$$V_D := \text{span}\{\vec{f} : f \in \mathcal{F} \text{ and } \forall x \in D, f(x) = 1\},$$

Note that  $V = V_\emptyset$ .

To prove Theorem 2 we start from  $D = \emptyset$ . At every step, if  $D$  does not extend uniquely to  $S$ , then there exists a vertex  $v \in S$  such that  $\dim V_{D \cup \{v\}} < \dim V_D$ ; we add  $v$  to  $D$ . Since  $\dim V_\emptyset \leq 1 + n!2^n$ , we can obtain a set  $D$  of size at most  $n!2^n$  such that the dimension of  $V_D$  is at most 1. This completes the proof as there is at most one non-zero, zero-one vector in a vector space of dimension 1.

## 5 Future directions

We ask the following question:

**Question 1** *For a prime  $2n + 1$ , are there examples of perfect dominating sets in  $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$  that are not of the form (1)?*

If the answer to Question 1 turns out to be negative, then we can improve the bound of Theorem 2:

**Proposition 1** *Let  $p = 2n + 1$  be a prime, and let  $\mathcal{T}$  denote the set of perfect dominating sets of the form (1). Every  $(S, \mathcal{T})$  where  $S \in \mathcal{T}$  has a defining set of size  $1 + \lceil \frac{n-1}{\lceil \log_2 p \rceil} \rceil$ .*

**Proof.** Suppose that  $S \in \mathcal{T}$ . Then  $S$  is of the form:

$$\{(x_1, \dots, x_{n-1}, k + \sum_{i=1}^{n-1} \epsilon_i(i+1)x_i) : x_i \in \mathbb{Z}_p \ \forall i \in [n-1]\}.$$

Let  $m = \lceil \log_2 p \rceil$ . We will use the easy fact that for any  $c \in \mathbb{Z}_p$ , the equation  $\sum_{i=0}^{m-1} \epsilon_i 2^i =_p c$  has at most one solution  $(\epsilon_0, \epsilon_1, \dots, \epsilon_{m-1}) \in \{-1, +1\}^m$ . For  $i, j \geq 0$ , define  $\alpha_{i,j} \in \mathbb{Z}_p$  to be the solution to  $(i+j+1)\alpha_{i,j} =_p 2^j$ .

Let  $u = (0, 0, \dots, 0, b)$  be the unique vertex in  $S$  with the first  $n-1$  coordinates equal to 0, and for every  $1 \leq i \leq n-1$  consider the unique vector

$$u_i = (0, \dots, 0, \alpha_{i,0}, \alpha_{i,1}, \dots, \alpha_{i,k_i}, 0, \dots, 0, b_i) \in S,$$

where  $\alpha_{i,0}$  is in the  $i$ th coordinate and  $k_i = \min(m-1, n-i-1)$ . We claim that the set  $D = \{u, u_0, u_m, \dots, u_{m(\lceil \frac{n-1}{m} \rceil - 1)}\}$  is a defining set for  $(S, \mathcal{T})$ . Since  $S$  is of form (1), clearly  $k = b$ , and for every  $0 \leq i \leq \lceil \frac{n-1}{m} \rceil - 1$ , we have:

$$b_{mi} - b = \sum_{j=0}^{k_{mi}} \epsilon_{mi+j}(mi+j+1)\alpha_{mi,j} = \sum_{j=0}^{k_{mi}} \epsilon_{mi+j} 2^j.$$

The above equation has only one solution  $(\epsilon_{mi}, \epsilon_{mi+1}, \dots, \epsilon_{mi+k_{mi}}) \in \{-1, +1\}^{k_{mi}+1}$ . Considering this for all  $u_{mi} \in D$  determines  $(\epsilon_1, \epsilon_2, \dots, \epsilon_{n-1})$ . Thus the set  $D$  is a defining set for  $(S, \mathcal{T})$ . ■

**Remark 3** For  $n = 2, 3$  the answer to Question 1 is negative. Thus when  $n = 3$ , Proposition 1 implies that there is a defining set of size 2 for a perfect dominating set. This disproves the conjecture of [2] which is already observed by Richard Bean [private communication].

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