Generalization of a relation between the Riemann zeta function and Bernoulli numbers

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Abstract

A generalization of a well-known relation between the Riemann zeta function and Bernoulli numbers is obtained. The formula is a new representation of the Riemann zeta function in terms of a nested series of Bernoulli numbers.

1 A New Representation of the Riemann Zeta Function

Theorem 1

$$
\zeta(s) = -\frac{(2\pi)^s}{2}\, w^{s-1} \lim_{\hat{s}\to s} \left\{ \frac{\left(\frac{1}{2} + \sum_{n=1}^{\infty}(-1)^n \binom{\hat{s}-1}{n}\left[\frac{1}{2} + \sum_{m=1}^{n}\left(\frac{-1}{w}\right)^m \binom{n}{m}\frac{B_{m+1}}{(m+1)!}\right]\right)}{\cos\left(\frac{\pi\hat{s}}{2}\right)} \right\}
$$

(1)

for $\mathsf{Re}(s) > (1/w)$ where $s \in \mathsf{C}$, $w \in \mathsf{R}$, $w > 0$, the notation of binomial coefficient is extended such that

$$
\binom{s-1}{n} = \frac{1}{n!}\left[\prod_{k=0}^{n-1}(s-1-k)\right] = \frac{1}{n!}\frac{\Gamma(s)}{\Gamma(s-n)},
$$

B_m are the Bernoulli numbers with $B_1 = 1/2$, and the limit only needs to be taken when $s \in \{1, 3, 5, \ldots\}$ for which the denominator $\cos(\pi s/2)$ is 0.

The representation (1) can be seen as a generalization of the well-known relation

$$
\zeta(2n) = -\frac{(2\pi)^{2n}}{2}\frac{(-1)^n B_{2n}}{(2n)!} \quad (n \in \mathsf{Z}^+).
$$

This representation (1) of $\zeta(s)$ in terms of a nested series of B_n is distinct from the well-known Euler-Maclaurin summation representation [1, p.807, (23.2.3)] which also relates $\zeta(s)$ to B_n as follows:

$$\zeta(s) = \lim_{N\to\infty} \left[\begin{array}{l} \displaystyle\sum_{n=1}^{N} n^{-s} - \frac{1}{-s+1} N^{-s+1} - \frac{1}{2} N^{-s} \\ \displaystyle - \sum_{k=1}^{M} \frac{B_{2k}}{(2k)!} \frac{\partial^{2k-1}}{\partial x^{2k-1}} x^{-s}\Big|_{x=N} \\ + O(N^{-s-2M-1}) \end{array} \right] \tag{2}$$

$$(\mathsf{Re}(s) > -2M - 1) \,.$$

To prove Theorem 1, we shall have to introduce a binary tree and a set of operators for generating Bernoulli Numbers.

2 A Binary Tree for Generating Bernoulli Numbers

Definition 1 *Bernoulli numbers B_n are defined by [2, p.35, (1.13.1)]*

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \quad (|z| < 2\pi) \,. \tag{3}$$

Expanding the left hand side as a series and matching the coefficients on both sides give

$$B_1 = -1/2, \quad B_n \begin{cases} = 0 & , \quad \text{odd } n, \; n \neq 1 \\ \neq 0 & , \quad \text{even } n \end{cases} . \tag{4}$$

Now (3) can be rewritten as

$$\frac{z}{e^z - 1} + \frac{z}{2} = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} z^{2n} \,. \tag{5}$$

Alternatively, B_n can be defined as the solution of the recurrence relation

$$B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k \,, \quad B_0 = 1 \,. \tag{6}$$

A binary tree for generating Bernoulli numbers B_n can be constructed using two operators, O_L and O_R.

Definition 2 (The B_n-Generating Tree)

At each node of the binary tree sits a formal expression of the form $\dfrac{\pm 1}{a!\,b!\,\ldots}$.
The operators O_L and O_R are defined to act only on formal expressions of this form at the nodes of the tree as follows:

$$O_L \ : \ \frac{\pm 1}{a!\,b!\,\ldots} \longrightarrow \frac{\pm 1}{(a+1)!\,b!\,\ldots} \ ,$$

$$O_R \ : \ \frac{\pm 1}{a!\,b!\,\ldots} \longrightarrow \frac{\mp 1}{2!\,a!\,b!\,\ldots} \ .$$

Schematically,

- O_L acting on a node of the tree generates a branch downwards to the left (hence the subscript L in O_L) with a new node at the end of the branch.

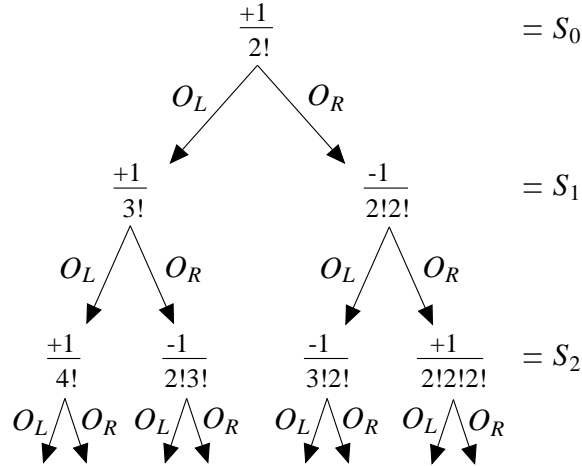- O_R acting on the same node generates a branch downwards to the right.

Figure 1: The binary tree that generates Bernoulli numbers.

The following finite series formed out of the two non-commuting operators

$$S_n = (O_L + O_R)^n \left(\frac{+1}{2!} \right) = \left(O_L^n + \sum_{k=0}^{n-1} O_L^{n-1-k} O_R\, O_L^k + \cdots + O_R^n \right) \left(\frac{+1}{2!} \right).$$

$$(7)$$

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is equivalent to the sum of terms on the n-th row of nodes across the tree.

Theorem 2 *Bernoulli numbers and the S_n series are related by*

$$\boxed{B_n = n!\, S_{n-1} \quad (n \geq 2)}\,.$$ (8)

For example,

$$
\begin{aligned}
B_3 &= 3!\, S_2 = 3!\, (O_L + O_R)^2 \left(\frac{+1}{2!}\right) = 3!\, (O_L + O_R)\, (O_L + O_R) \left(\frac{+1}{2!}\right) \\
&= 3!\, (O_L\, O_L + O_L\, O_R + O_R\, O_L + O_R\, O_R) \left(\frac{+1}{2!}\right) \\
&= 3! \left(\frac{+1}{4!} + \frac{-1}{2!\,3!} + \frac{-1}{3!\,2!} + \frac{+1}{2!\,2!\,2!}\right) = 0\,.
\end{aligned}
$$

Proof

The Riemann zeta function [3]

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad (\mathsf{Re}(s) > 1,\ s \in \mathsf{C})$$ (9)

can be analytically extended to the left-half of the complex plane $\mathsf{Re}(s) < 1$ by the Euler-Maclaurin Summation Formula (2).

Now, instead of adopting (2) directly, we choose to derive the analytic continuation of $\zeta(s)$ into the left-half of the complex plane step by step.

Consider the difference between (9) with its analogous integral.

$$
\begin{aligned}
&\lim_{N \to \infty} \left[\sum_{n=1}^{N} n^{-s} - \left(\int_1^N x^{-s} dx + \frac{1}{-s+1}\right)\right] \\
&= 1 - \frac{1}{-s+1} + \lim_{N \to \infty} \sum_{n=2}^{N} \left[n^{-s} - \int_{n-1}^{n} x^{-s} dx\right] \\
&= 1 - \frac{1}{-s+1} + \lim_{N \to \infty} \sum_{n=2}^{N} \left[n^{-s} - \frac{1}{-s+1} n^{-s+1} + \frac{1}{-s+1} (n-1)^{-s+1}\right]
\end{aligned}
$$

The binomial expansion of the sum over n of the last term, with an

interchange of the order of summation, gives a set of series

$$
1-\frac{1}{-s+1}+\lim_{N\to\infty}\left[
\begin{aligned}
&\sum_{n=2}^{N}\left[n^{-s}-\frac{1}{-s+1}\,n^{-s+1}+\frac{1}{-s+1}\,n^{-s+1}+(-1)\,n^{-s}\right] &&(10)\\[2mm]
&+\ \frac{(-s)}{2!}\sum_{n=2}^{N}n^{-s-1} &&(11)\\[2mm]
&+\ \frac{(-1)(-s)(-s-1)}{3!}\sum_{n=2}^{N}n^{-s-2} &&(12)\\[2mm]
&\ \vdots\\[1mm]
&+\ \frac{(-1)^{m+1}(-s)(-s-1)\ldots(-s+1-m)}{(m+1)!}\sum_{n=2}^{N}n^{-s-m} &&(13)\\[2mm]
&\ \vdots
\end{aligned}
\right]
$$

$$
=\ 1-\frac{1}{-s+1}+\lim_{N\to\infty}\sum_{m=1}^{\infty}\left[\left(\prod_{j=1}^{m}(-s+1-j)\right)\frac{(-1)^{m+1}}{(m+1)!}\sum_{n=2}^{N}-s-m\right]
$$

$$
=\ 1-\frac{1}{-s+1}+\lim_{N\to\infty}\sum_{m=1}^{\infty}\left[\left(\prod_{j=1}^{m}(-s+1-j)\right)O_{L}^{m-1}\!\left(\frac{+1}{2!}\right)\sum_{n=2}^{N}n^{-s-m}\right],
$$

(note that the series (10) sums to zero). O_{L}^{m-1} represents a string of $(m-1)$ O_{L} operators, and $O_{L}^{m-1}(+1/2!)$ corresponds to the *first* term of $S_{m-1}, m\in\mathsf{Z}^{+}$ in (7), the *first* branch from the left in the $(m-1)$-th row of the tree. O_{L}^{0} is taken as the identity operator.

Now $\lim\limits_{N\to\infty}(1+\int_{1}^{N}kx^{-s}dx=\lim\limits_{N\to\infty}\left[kN^{-s+1}/(-s+1)\right]$ converges for $\mathsf{Re}(s)>1$, where arbitrary constants $k,p\in\mathsf{C}$. Therefore, $\lim\limits_{N\to\infty}\sum_{n=1}^{N}kn^{-s}$ converges for $\mathsf{Re}(s)>1$. Hence,

$$
\zeta(s)=\lim_{N\to\infty}\left[\sum_{n=1}^{N}n^{-s}-\left(\int_{1}^{N}x^{-s}dx+\frac{1}{-s+1}\right)\right]=\lim_{N\to\infty}\left[\sum_{n=1}^{N}n^{-s}-\frac{1}{-s+1}N^{-s+1}\right]
$$

defines an analytic continuation of $\zeta(s)$ for $\mathsf{Re}(s)>0$, since the series (11), (12) and (13) converge for $\mathsf{Re}(s)>0$, $\mathsf{Re}(s)>-1$, and $\mathsf{Re}(s)>-m+1$ respectively.

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Similarly, find the difference between the series (11) with its analogous integral.

$$\lim_{N\to\infty}\left[\sum_{n=2}^{N}\frac{(-s)}{2!}n^{-s-1}-\frac{(-s)}{2!}\left(\int_{2}^{N}x^{-s-1}dx+\frac{2^{-s}}{-s}\right)\right]$$

$$=\frac{(-s)\,2^{-s-1}}{2!}+\frac{2^{-s}}{-s}+\lim_{N\to\infty}\left[\begin{array}{l}\dfrac{(-s)(-s-1)}{2!\,2!}\displaystyle\sum_{n=3}^{N}n^{-s-2}\qquad\qquad\qquad(14)\\[2ex]\vdots\\[1ex]+\ \dfrac{(-1)^{q+1}(-s)(-s-1)\ldots(-s-q)}{2!\,(q+1)!}\displaystyle\sum_{n=3}^{N}n^{-s-1-q}\quad(15)\\[2ex]\vdots\end{array}\right]$$

$$=\frac{(-s)\,2^{-s-1}}{2!}+\frac{2^{-s}}{-s}+\lim_{N\to\infty}\sum_{m=1}^{\infty}\left[\left(\prod_{j=1}^{m+1}(-s+1-j)\right)\frac{(-1)^{m+1}}{2!\,(m+1)!}\sum_{n=3}^{N}n^{-s-m-1}\right]$$

$$=\frac{(-s)\,2^{-s-1}}{2!}+\frac{2^{-s}}{-s}+\lim_{N\to\infty}\sum_{m=1}^{\infty}\left[\left(\prod_{j=1}^{m+1}(-s+1-j)\right)O_R\,O_L^{m-1}\!\left(\frac{+1}{2!}\right)\sum_{n=3}^{N}n^{-s-m-1}\right]$$

where $O_R\,O_L^{m-1}(+1/2!)$ corresponds to the *second* term of S_m, $m\in\mathsf{Z}^+$, the *second* branch from the left in the m-th row of the tree, and $q\in\mathsf{Z}^+$, $\mathsf{q}>2$. Hence,

$$\begin{aligned}\zeta(s)&=\lim_{N\to\infty}\left[\sum_{n=1}^{N}n^{-s}-\left(\int_{1}^{N}x^{-s}+\frac{1}{-s+1}dx\right)-\frac{(-s)}{2!}\left(\int_{2}^{N}x^{-s-1}+\frac{2^{-s}}{-s}dx\right)\right]\\[1ex]&=\lim_{N\to\infty}\left[\sum_{n=1}^{N}n^{-s}-\frac{1}{-s+1}N^{-s+1}-\frac{1}{2!}N^{-s}\right]\\[1ex]&=\lim_{N\to\infty}\left[\sum_{n=1}^{N}n^{-s}-\frac{1}{-s+1}N^{-s+1}-S_0\,N^{-s}\right]\end{aligned}$$

defines an analytic continuation of $\zeta(s)$ for $\mathsf{Re}(s)>-1$, since the series (14) and (15) converge for $\mathsf{Re}(s)>-1$ and $\mathsf{Re}(s)>-q$ respectively. Note that $(+1/2!)$, the coefficient of N^{-s}, corresponds to S_0, the starting node (zeroth row) of the tree.

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To analytically extend $\zeta(s)$ to $\mathsf{Re}(s) > -2$, we again subtract in a similar way the set of all the series of n^{-s-2} (e.g., (11) and (12)), which diverge for $\mathsf{Re}(s) \leq -1$, to get sets of series of higher orders that converge for $\mathsf{Re}(s) > -2$.

Similarly, the difference between the series (11) with its analogous integral, and the corresponding one for series (12) are

$$\frac{(-s)(-s-1)\,3^{-s-1}}{2!\,2!} + \frac{3^{-s-1}}{-s-1} + \lim_{N\to\infty}\sum_{m=1}^{\infty}\left[\left(\prod_{j=1}^{m+3}(-s+1-j)\right)O_L\,O_R\,O_L^{m-1}\!\left(\frac{+1}{2!}\right)\sum_{n=4}^{N}n^{-s-m-3}\right]$$

and

$$\frac{(-s)(-s-1)\,3^{-s-1}}{2!\,2!} + \frac{3^{-s-1}}{-s-1} + \lim_{N\to\infty}\sum_{m=1}^{\infty}\left[\left(\prod_{j=1}^{m+3}(-s+1-j)\right)O_R^2\,O_L^{m-1}\!\left(\frac{+1}{2!}\right)\sum_{n=4}^{N}n^{-s-m-3}\right]$$

respectively, where $O_L\,O_R\,O_L^{m-1}(+1/2!)$ and $O_R^2\,O_L^{m-1}(+1/2!)$ correspond to the *third* and *fourth* terms respectively of $S_{m+1}, m \in \mathsf{Z}^+$, the *third* and *fourth* branch from the left in the $(m+1)$-th row of the tree. Hence,

$$\zeta(s) = \lim_{N\to\infty}\left[\begin{array}{l}\sum_{n=1}^{N}n^{-s} - \left(\int_1^N x^{-s}dx + \frac{1}{-s+1}\right)\\[2mm] - \frac{(-s)}{2!}\left(\int_2^N x^{-s-1}dx + \frac{2^{-s}}{-s}\right)\\[2mm] - (-s)(-s-1)\,O_L\!\left(\frac{+1}{2!}\right)\left(\int_3^N x^{-s-2}dx + \frac{3^{-s-1}}{-s-1}\right)\\[2mm] - (-s)(-s-1)\,O_R\!\left(\frac{+1}{2!}\right)\left(\int_3^N x^{-s-2}dx + \frac{3^{-s-1}}{-s-1}\right)\end{array}\right]$$

$$= \lim_{N\to\infty}\left[\sum_{n=1}^{N}n^{-s} - \frac{1}{-s+1}N^{-s+1} - S_0\,N^{-s} - S_1\,(-s)\,N^{-s-1}\right] \quad (16)$$

defines an analytic continuation of $\zeta(s)$ for $\mathsf{Re}(s) > -2$, where $S_0 = (+1/2!)$ and $S_1 = (O_L + O_R)(+1/2!)$.

Comparing series (16) with the Euler-Maclaurin Summation Formula (2) for $M = 1$,

$$\zeta(s) = \lim_{N\to\infty}\left[\sum_{n=1}^{N}n^{-s} - \frac{1}{-s+1}N^{-s+1} - \frac{1}{2}N^{-s} - \frac{B_2}{2!}(-s)\,N^{-s-1}\right],$$

we find that $B_2 = 2!\,S_1$.

At this point, we observe that every time we find the difference between a divergent series and its analogous integral, the resulting binomial expansion has the effect of inserting additional operators $O_R\,O_L^{m-1}$ immediately before $(+1/2!)$. When we write down these sequences, the tree appears.

To obtain further analytic continuation of $\zeta(s)$, we make use of the tree and write

$$
\zeta(s) \;=\; \lim_{N\to\infty}
\left[
\begin{array}{l}
\displaystyle\sum_{n=1}^{N} n^{-s} - \frac{1}{-s+1}\,N^{-s+1} - S_0\,N^{-s} \\[2ex]
\displaystyle - \sum_{m=1}^{M}\left[\left(\prod_{j=1}^{m}(-s+1-j)\right) S_m\,N^{-s-m}\right]
\end{array}
\right]
$$

where $\qquad S_m = (O_L + O_R)^m\left(\dfrac{+1}{2!}\right).$

Comparing the above with the Euler-Maclaurin Summation Formula (2), and noting that B_n vanishes for odd $n \ge 3$, and so the coefficients of N^{-s-n+1} also vanish for odd $n \ge 3$, we get (8).

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

By observation, the sum-across-the-tree representation of S_n in (7) can also be seen to be equivalent to the following determinant known to generate B_n.

$$
S_n = (-1)^n
\begin{vmatrix}
\dfrac{1}{2!} & 1 & 0 & 0 & 0 & \cdots & 0 \\[2ex]
\dfrac{1}{3!} & \dfrac{1}{2!} & 1 & 0 & 0 & \cdots & 0 \\[2ex]
\dfrac{1}{4!} & \dfrac{1}{3!} & \dfrac{1}{2!} & 1 & 0 & \cdots & 0 \\[1ex]
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\[1ex]
\dfrac{1}{(n-2)!} & \ddots & \ddots & \dfrac{1}{3!} & \dfrac{1}{2!} & 1 & 0 \\[2ex]
\dfrac{1}{(n-1)!} & \dfrac{1}{(n-2)!} & \ddots & \ddots & \dfrac{1}{3!} & \dfrac{1}{2!} & 1 \\[2ex]
\dfrac{1}{n!} & \dfrac{1}{(n-1)!} & \dfrac{1}{(n-2)!} & \ddots & \ddots & \dfrac{1}{3!} & \dfrac{1}{2!}
\end{vmatrix}.
$$

3 The Tree-Generating Operator and Bernoulli Function

We can now expand the tree-generating operator $(O_L + O_R)$ raised to the complex power $(s-1)$ acting on $\left(\dfrac{+1}{2!}\right)$ as follows.

Lemma 1

$$(17)$$

$$
\begin{aligned}
(O_L + O_R)^{s-1}&\left(\frac{+1}{2!}\right) \\
&= \left(w\mathbf{1} - [w\mathbf{1} - (O_L + O_R)]\right)^{s-1}\left(\frac{+1}{2!}\right) \\
&= w^{s-1}\left(\mathbf{1} - \left[\mathbf{1} - \frac{1}{w}(O_L + O_R)\right]\right)^{s-1}\left(\frac{+1}{2!}\right) \\
&= w^{s-1}\left(\mathbf{1} + \sum_{n=1}^{\infty}(-1)^n\binom{s-1}{n}\left[\mathbf{1} - \frac{1}{w}(O_L + O_R)\right]^n\right)\left(\frac{+1}{2!}\right) \\
&= w^{s-1}\left(\mathbf{1} + \sum_{n=1}^{\infty}(-1)^n\binom{s-1}{n}\left[\mathbf{1} + \sum_{m=1}^{n}\left(\frac{-1}{w}\right)^m\binom{n}{m}(O_L + O_R)^m\right]\right)\left(\frac{+1}{2!}\right) \\
&= w^{s-1}\left(\frac{1}{2} + \sum_{n=1}^{\infty}(-1)^n\binom{s-1}{n}\left[\frac{1}{2} + \sum_{m=1}^{n}\left(\frac{-1}{w}\right)^m\binom{n}{m}\frac{B_{m+1}}{(m+1)!}\right]\right)
\end{aligned}
$$

which converges for $\mathsf{Re}(s) > (1/w)$ *where* $s \in \mathsf{C}$, $\mathsf{w} \in \mathsf{R}$, $\mathsf{w} > 0$, *and* $\mathbf{1}$ *is the identity operator.*

Theorem 3 (Bernoulli Function)

$$
\begin{aligned}
B(s) &= \Gamma(1+s)\,(O_L + O_R)^{s-1}\left(\frac{+1}{2!}\right) &(18)\\
&= w^{s-1}\,\Gamma(1+s)\left(\frac{1}{2} + \sum_{n=1}^{\infty}(-1)^n\binom{s-1}{n}\left[\frac{1}{2} + \sum_{m=1}^{n}\left(\frac{-1}{w}\right)^m\binom{n}{m}\frac{B_{m+1}}{(m+1)!}\right]\right)
\end{aligned}
$$

which converges for $\mathsf{Re}(s) > (1/w)$ *where* $s \in \mathsf{C}$, $\mathsf{w} \in \mathsf{R}$, $\mathsf{w} > 0$.

Proof

From (7) and (8), we have

$$
B_n = n!\,(O_L + O_R)^{n-1}\left(\frac{+1}{2!}\right) = \Gamma(1+n)\,(O_L + O_R)^{n-1}\left(\frac{+1}{2!}\right) \quad (n \geq 2)\,.
$$

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Analytically extending the tree-generating operator $(O_L + O_R)$ with (17) in Lemma 1 effectively turns the sequence of B_n into a function $B(s)$ as the analytic continuation of B_n.
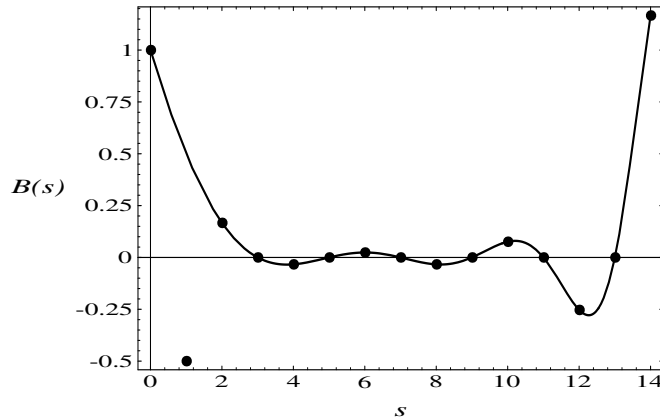
□

Figure 2: The curve $B(s)$ runs through the points of all (n, B_n) except $(1, B_1)$.

Figure 2 shows a plot of $B(s)$ for real s. All the Bernoulli numbers B_n agree with Bernoulli function $B(n)$ except at $n = 1$, i.e., $B(n) = B_n$ for $n = 0$ or $n \geq 2$,

$$B(1) = 1/2 \quad \text{but} \quad B_1 = -1/2 . \tag{19}$$

We shall now address the surprising discrepancy between $B(1)$ and B_1.

4   Fixing of the Arbitrary Sign Convention of B_1

$B_1 = -1/2$ has largely been adopted as the standard sign convention partly due to elegance in notation and partly due to its widespread usage although there had been suggestions for favoring the sign convention $B_1 = 1/2$.

Looking back at (3) to (6) in Definition 1, we see that the sign convention of B_1 was arbitrary. Figure 2 show that the analytic continuation of B_n actually fixes the arbitrary sign convention of B_1. A mathematical fact should precede notational elegance or personal preference.

Definition 3 (Redefinition of Bernoulli Numbers)

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To have consistency between Bernoulli numbers B_n and their analytic continuation $B(s)$, we should redefine B_n as

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} (-1)^n \frac{B_n}{n!} z^n \quad (|z| < 2\pi) , \tag{20}$$

or

$$B_n = \frac{(-1)^{n+1}}{n+1} \sum_{k=0}^{n-1} (-1)^k \binom{n+1}{k} B_k , \quad B_0 = 1 . \tag{21}$$

The factor $(-1)^n$ introduced in (20) and (21) only changes the sign in the conventional definition of the only non-zero odd Bernoulli numbers, B_1, from $B_1 = -1/2$ to the redefined $B_1 = B(1) = 1/2$.

5 Proof of the New Representation of $\zeta(s)$

We now have covered sufficient concepts to prove Theorem 1.

Proof

B_n are related to the Riemann zeta function $\zeta(s)$ as [2, p.34]

$$\begin{aligned}
\zeta(-n) &= -\frac{B_{n+1}}{n+1} & (n \in \mathsf{Z}^+) , \\
\zeta(2n) &= \frac{(-1)^{n+1}(2\pi)^{2n}}{2 \, (2n)!} B_{2n} & (n \in \mathsf{Z}^+) .
\end{aligned}$$

Hence, $\zeta(1-s) = -B(s)/s \ (s \in \mathsf{C})$. Replacing $B(s)$ with the series in (18) and noting that $\Gamma(1+s)/s = \Gamma(s)$ gives

$$\zeta(1{-}s) = -w^{s-1}\, \Gamma(s) \left(\frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n \binom{s-1}{n} \left[\frac{1}{2} + \sum_{m=1}^{n} \left(\frac{-1}{w} \right)^m \binom{n}{m} \frac{B_{m+1}}{(m+1)!} \right] \right) \tag{22}$$

which converges for $\mathsf{Re}(1{-}s) < 1 - (1/w)$ where $s \in \mathsf{C}$, $\mathsf{w} \in \mathsf{R}$, $\mathsf{w} > 0$.

The functional equation of the Riemann zeta function [4] relates $\zeta(1{-}s)$ to $\zeta(s)$ as

$$\zeta(1{-}s) = 2 \, (2\pi)^{-s} \, \Gamma(s) \, \cos\left(\frac{\pi s}{2} \right) \zeta(s) . \tag{23}$$

Applying the functional equation (23) to (22) yields

$$\cos\left(\frac{\pi \hat{s}}{2} \right) \zeta(s) = -\frac{(2\pi)^s}{2} (w)^{s-1} \left(\frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n \binom{s-1}{n} \left[\frac{1}{2} + \sum_{m=1}^{n} \left(\frac{-1}{w} \right)^m \binom{n}{m} \frac{B_{m+1}}{(m+1)!} \right] \right) \tag{24}$$

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(24) in the limit form gives the Theorem.

$\square$

References

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1970).

[2] H. Bateman, *Higher Transcendental Functions, Vol 1.* (McGraw-Hill, New York, 1953).

[3] E.C. Titchmarsh, *The Theory of the Riemann zeta-function*, (OUP, Oxford, 1986).

[4] L.V. Ahlfors, *Complex Analysis 2nd ed.* (McGraw-Hill, New York, 1966).