

## Parametrical Neural Network

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### Abstract

The storage capacity of the Hopfield model is about 15% of the network size. It can be increased significantly in the Potts-glass model of the associative memory only. In this model neurons can be in more than two different states. We show that even greater storage capacity can be achieved in *the parametrical neural network* (PNN) that is based on the parametrical four-wave mixing process that is well-known in nonlinear optics. We present a uniform formalism allowing us to describe both PNN and the Potts-glass associative memory. To estimate the storage capacity we use the Chebyshev-Chernov statistical technique.

Keywords: Associative Memory, Phase-Frequency Modulation, Optical Networks, Chebyshev-Chernov Method.

# 1 INTRODUCTION

In refs. [1],[2] a network based on the parametrical four-wave mixing process (FWM) [3] that is well-known in nonlinear optics was examined. Such a network is capable to hold and handle information that is encoded in the form of the phase-frequency modulation. In the network the signals propagate along interconnections in the form of quasi-monochromatic pulses at  $q$  different frequencies

$$\{\omega_l\}_1^q \equiv \{\omega_1, \omega_2, \dots, \omega_q\}. \quad (1)$$

The model is based on a parametrical neuron that is a cubic nonlinear element capable to transform and generate frequencies in the parametrical FWM-processes  $\omega_i - \omega_j + \omega_k \rightarrow \omega_r$ . Schematically this model of a neuron can be assumed as a device that is composed of a summator of input signals, a set of  $q$  ideal frequency filters  $\{\omega_l\}^q$ , a block comparing the amplitudes of the signals and  $q$  generators of quasi-monochromatic signals  $\{\omega_l\}^q$ .

Let  $\{K^{(\mu)}\}_1^p$  be a set of patterns each of which is a set of quasi-monochromatic pulses with frequencies defined by Eq.(1) and amplitudes equal to  $\pm 1$ :

$$K^{(\mu)} = (\kappa_1^{(\mu)}, \dots, \kappa_N^{(\mu)}), \text{ where } \kappa_i^{(\mu)} = \pm \exp(i\omega_{l_i^{(\mu)}} t), \quad \begin{cases} \mu = 1, \dots, p; \\ i = 1, \dots, N; \\ 1 \leq l_i^{(\mu)} \leq q. \end{cases} \quad (2)$$

The memory of the network is localized in interconnections  $T_{ij}$ ,  $i, j = 1, \dots, N$ , which accumulate the information about the states of  $i$ th and  $j$ th neurons in all the  $p$  patterns. We suppose that the interconnections are dynamic ones and that they are organized according to the Hebb rule:

$$T_{ij} = (1 - \delta_{ij}) \sum_{\mu=1}^p \kappa_i^{(\mu)} \kappa_j^{(\mu)*}, \quad i, j = 1, \dots, N. \quad (3)$$

The network operates as follows. A quasi-monochromatic pulse with a frequency  $\omega_{l_j}$  that is propagating along the  $(ij)$ -th interconnection from the  $j$ th neuron to the  $i$ th one, takes part in FWM-processes with the pulses stored in the interconnection,

$$\omega_{l_i^{(\mu)}} - \omega_{l_j^{(\mu)}} + \omega_{l_j} \rightarrow \{\omega_l\}_1^q.$$

The amplitudes  $\pm 1$  have to be multiplied. Summing up the results of these partial transformations over all patterns,  $\mu = 1, \dots, p$ , we obtain a packet of

quasi-monochromatic pulses, where all the frequencies from the set (1) are present. The amplitudes of the pulses are determined by the interconnection. This packet is the result of transformation of the pulse  $\omega_{l_j}$  by the interconnection  $T_{ij}$ , and it comes to the  $i$ th neuron. All such packets are summarized in this neuron. The summarized signal propagates through  $q$  parallel ideal frequency filters. The output signals from the filters are compared with respect to their amplitudes. The signal with the maximal amplitude activates the  $i$ -th neuron ('winner-take-all'). As a result it generates an output signal whose frequency and phase are the same as the frequency and the phase of the activating signal.

Generally, when three pulses interact, under a FWM-process always the fourth pulse appears. The frequency of this pulse is defined by the conservation laws only. However, in order that the abovementioned model works as a memory, an important condition must be add, which has to facilitate the propagation of the useful signal, and, in the same time, to suppress external noise. This condition is *the principle of incommensurability of frequencies* proponed in [1],[2]: *no combinations  $\omega_l - \omega_{l'} + \omega_{l''}$  can belong to the set (1), when all the frequencies are different*. Now we finished to describe the principle of the network operating. This network will be called *the parametrical neural network* (PNN).

There are arguments going in for PNN. First of all, the frequency-phase modulation is more convenient for optical processing of signals. It allows us to back down an artificial adaptation of an optical network to amplitude modulated signals. Second, when signals with  $q$  different frequencies can propagate along one interconnection (this is an analog of the channel multiplexing), this, in fact, allows us to reduce the number of interconnections by a factor of  $q^2$ . Note, interconnections occupy nearly 98% of the area of neurochips. Third, the signal-noise analysis made with the aid of the Chebyshev-Chernov statistical method showed that the storage capacity of PNN was approximately  $q^2$  times as much as the storage capacity of the Hopfield model. Even if  $q \sim 10$ , the gain is two orders. For computer processing of colored images the standard value is  $q = 256$ . Consequently, comparing with the Hopfield model the gain is about five orders. Simultaneously with an increase of the storage capacity, the noise immunity of the network also increases. For example, we simulated PNN with the following parameters:  $N = 100$ ,  $q = 22$  and  $p = 200$ . This network recognized any 80% noisy pattern after 100 steps (in fact, in one pass over all the neurons). The same

network with the parameters  $N = 100$ ,  $q = 25$ ,  $p = 1000(!)$  recognized a 65% noisy pattern in 4-5 passes over all the neurons. We remind that some time ago the ability of the Hopfield model with  $N = 400$ ,  $p = 30$  to recognize a 30% noisy pattern was presented as a high mark in the patterns recognition [6].

In the present work we investigate the abilities of PNN. Here an important remark has to be done. Generally speaking, there are different parametrical FWM-processes complying with the principle of incommensurability of frequencies. For example in [1],[2] the parametrical FWM-process of the type

$$\omega_l - \omega_{l'} + \omega_{l''} = \begin{cases} \omega_{l''}, & \text{when } l' = l; \\ \omega_l, & \text{when } l' = l''; \\ \rightarrow 0 & \text{in other cases.} \end{cases}$$

was examined. The corresponding network will be called PNN-I. However, better results can be obtained for the parametrical FWM-process

$$\omega_l - \omega_{l'} + \omega_{l''} = \begin{cases} \omega_l, & \text{when } l'' = l'; \\ \rightarrow 0 & \text{in other cases.} \end{cases} \quad (4)$$

This network will be called PNN-II.

The organization of the paper is as follows. In Section 2 we introduce a vector formalism allowing us to formulate the problem in the general form. In this section the results for PNN-II are presented. In Section 3 the vector formalism is used to examine the Potts-glass neural network. We compare it with PNN-II. Some remarks are given in Conclusions. The details of calculations are in Appendix.

## 2 PNN-II

In fact, PNN is an associative memory of the Hopfield type with neurons, which can be in more than two different states. Such models of neural networks were examined previously (see, for example, [7]–[12]). Usually neurons are modeled with the aid of vectors, but not scalar quantities equal to  $\pm 1$  or  $0/1$ . The number of the *representative vectors* is equal to the number of different states of neurons. In the present Section for PNN-II we formulate the vector formalism and then estimate the storage capacity of such a network.

## 2.1 Vector Formalism

In order to describe the  $q$  different states (1) of neurons we use the set of basis vectors  $\mathbf{e}_l$  in the space  $R^q$ ,  $q \geq 1$ ,

$$\mathbf{e}_l = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad l = 1, \dots, q.$$

The state of the  $i$ th neuron is described by a vector  $\mathbf{x}_i$ ,

$$\mathbf{x}_i = x_i \mathbf{e}_{l_i}, \quad x_i = \pm 1, \quad \mathbf{e}_{l_i} \in R^q, \quad 1 \leq l_i \leq q, \quad i = 1, \dots, N.$$

The state of the network as a whole  $X$  is determined by a set of  $N$   $q$ -dimensional vectors  $\mathbf{x}_i$ :  $X = (\mathbf{x}_1, \dots, \mathbf{x}_N)$ . By analogy with Eq.(2) the  $p$  stored patterns are

$$X^{(\mu)} = (\mathbf{x}_1^{(\mu)}, \mathbf{x}_2^{(\mu)}, \dots, \mathbf{x}_N^{(\mu)}), \quad \mathbf{x}_i^{(\mu)} = x_i^{(\mu)} \mathbf{e}_{l_i^{(\mu)}}, \quad x_i^{(\mu)} = \pm 1, \quad \begin{cases} 1 \leq l_i^{(\mu)} \leq q, \\ \mu = 1, \dots, p. \end{cases}$$

Since in this model neurons are vectors, the local field  $\mathbf{h}_i$  affecting the  $i$ th neuron is a vector too. By analogy with the standard Hopfield model we write

$$\mathbf{h}_i = \sum_{j=1}^N \mathbf{T}_{ij} \mathbf{x}_j \quad (5)$$

The  $(q \times q)$ -matrix  $\mathbf{T}_{ij}$  describes the interconnection between the  $i$ th and the  $j$ th neurons. This matrix affects the vector  $\mathbf{x}_j \in R^q$ , converting it in a linear combination of basis vectors  $\mathbf{e}_l$ . This combination is an analog of the packet of quasi-monochromatic pulses that come from the  $j$ th neuron to the  $i$ th one after transformation in the interconnection (see Introduction). To satisfy the conditions (3) and (4), we need to take the matrices  $\mathbf{T}_{ij}$  as

$$\mathbf{T}_{ij} = (1 - \delta_{ij}) \sum_{\mu=1}^p \mathbf{x}_i^{(\mu)} \mathbf{x}_j^{(\mu)} {}^+, \quad i, j = 1, \dots, N, \quad (6)$$

where  $\delta_{ij}$  is the Kronecker symbol. The elements of these matrices are

$$T_{ij}^{(kl)} = (1 - \delta_{ij}) \sum_{\mu=1}^p (\mathbf{e}_k \mathbf{x}_i^{(\mu)}) (\mathbf{x}_j^{(\mu)} \mathbf{e}_l) \quad k, l = 1, \dots, q.$$

Let us define the dynamics of our  $q$ -dimensional neurons. Let  $X(t) = (\mathbf{x}_1(t), \dots, \mathbf{x}_N(t))$  be the state of the system at the time  $t$ . By definition the  $i$ th neuron at the time  $t + 1$  is oriented along a direction mostly close to the local field  $\mathbf{h}_i(t)$ . Let us clarify this definition. With the aid of (6) we write Eq.(5) in the form more convenient for analysis:

$$\mathbf{h}_i(t) = \sum_{l=1}^q A_l^{(i)} \mathbf{e}_l, \text{ where } A_l^{(i)} = \sum_{j(\neq i)}^N \sum_{\mu=1}^p (\mathbf{e}_l \mathbf{x}_i^{(\mu)}) (\mathbf{x}_j^{(\mu)} \mathbf{x}_j(t)). \quad (7)$$

Let  $k$  be the index relating to the amplitude that is maximal in modulus in the series (7):

$$|A_k^{(i)}| = \max_{1 \leq l \leq q} |A_l^{(i)}|.$$

Then according to our definition

$$\mathbf{x}_i(t+1) = \text{sgn}(A_k^{(i)}) \mathbf{e}_k. \quad (8)$$

The expression (8) is identical to the 'winner-take-all' rule of Introduction.

The evolution the system consists of consequent changes of orientations of vector-neurons according to the rule (8). We make the convention that if some of the amplitudes are maximal in modulus simultaneously, and the neuron is in one of these *unimprovable* states, its state does not change. Then it is easy to show that during the evolution of the network its *energy*  $H(t) = -1/2 \sum_{i=1}^N (\mathbf{h}_i(t) \mathbf{x}_i(t))$  decreases. In the end the system reaches a local energy minimum. In this state all the neurons  $\mathbf{x}_i$  are oriented in an unimprovable manner, and the evolution of the system come to its end. These states are the fixed points of the system. The necessary and sufficient conditions for a configuration  $X$  to be a fixed point is fulfillment of the set of inequalities:

$$(\mathbf{x}_i \mathbf{h}_i) \geq |(\mathbf{e}_l \mathbf{h}_i)|, \quad \forall l = 1, \dots, q; \quad \forall i = 1, \dots, N. \quad (9)$$

## 2.2 Storage capacity of PNN-II

Let us estimate the storage capacity of the network in the limit  $N \gg 1$ . Suppose that the network starts from a distorted  $m$ th pattern

$$\tilde{X}^{(m)} = (a_1 \hat{b}_1 \mathbf{x}_1^{(m)}, a_2 \hat{b}_2 \mathbf{x}_2^{(m)}, \dots, a_N \hat{b}_N \mathbf{x}_N^{(m)}).$$

Here  $\{a_i\}_1^N$  and  $\{\hat{b}_i\}_1^N$  define a *phase noise* and a *frequency noise* respectively:  $a_i$  is a random value that is equal to  $-1$  or  $+1$  with the probabilities  $a$  and  $1-a$  respectively;  $b$  is the probability that the operator  $\hat{b}_i$  changes the state of the vector  $\mathbf{x}_i^{(m)} = x_i^{(\mu)} \mathbf{e}_{l_i^{(\mu)}}$ , and  $1-b$  is the probability that this vector remains unchanged.

Let us examine to what extent the neural network recognizes the pattern  $X^{(m)}$  correctly. The amplitudes  $A_l^{(i)}$  (7) have the form

$$A_l^{(i)} = \begin{cases} x_i^{(m)} \sum_{j=1}^{N-1} \xi_j + \sum_{r=1}^L \eta_r(l_i^{(m)}), & \text{when } l = l_i^{(m)}; \\ \sum_{r=1}^L \eta_r(l), & \text{when } l \neq l_i^{(m)}, \end{cases} \quad (10)$$

where  $\xi_j = a_j(\mathbf{x}_j^{(m)} \hat{b}_j \mathbf{x}_j^{(m)})$ ,  $\eta_r(l) \equiv \eta_j^{(\mu)}(l) = a_j(\mathbf{e}_l \mathbf{x}_i^{(\mu)})(\mathbf{x}_j^{(\mu)} \hat{b}_j \mathbf{x}_j^{(m)})$ ,  $j = 1, \dots, N$ ,  $j \neq i$ ,  $\mu = 1, \dots, p$ ,  $\mu \neq m$ . For simplicity, when writing the quantities  $\eta$  in place of the superscript  $\mu$  and the subscript  $j$  we use the subscript  $r$  which takes  $L = (N-1)(p-1)$  different values  $r = 1, \dots, L$ .

Let us note that when the patterns  $\{X^{(\mu)}\}_1^p$  are uncorrelated, the quantities  $\xi_j$  and  $\eta_r$  can be considered as independent random variables described by the probability distributions

$$\xi_j = \begin{cases} +1, & (1-a)(1-b) \\ 0, & b \\ -1, & (1-a)b \end{cases}, \quad \eta_r(l) = \begin{cases} +1, & 1/2q^2 \\ 0, & 1-1/q^2 \\ -1, & 1/2q^2 \end{cases}. \quad (11)$$

Since the distributions of the quantities  $\eta_r(l)$  are independent of  $l$ , in what follows we simply write  $\eta_r$ . According to the rule (9), the  $i$ th neuron finds itself in the state  $\mathbf{x}_i^{(m)}$  when two conditions for the amplitudes (10) are fulfilled:

$$\operatorname{sgn}(A_{l_i^{(m)}}^{(i)}) = x_i^{(m)}, \quad \sum_{j=1}^{N-1} \xi_j + x_i^{(m)} \sum_{r=1}^L \eta_r \geq |\sum_{r=1}^L \eta_r|.$$

Otherwise there will be an error in the recognition of the vector  $\mathbf{x}_i^{(m)}$ . Since the random variable  $x_i^{(m)} \eta_r$  has the same distribution as  $\eta_r$ , the probability

of this error is

$$\Pr_i = \Pr \left\{ \sum_{j=1}^{N-1} \xi_j + \sum_{r=1}^L \eta_r < 0 \right\}. \quad (12)$$

To estimate the value of  $\Pr_i$  we use the well-known Chebyshev-Chernov method [4],[5] (see Appendix). As a result we obtain the expression for the probability of the error in the recognition of the pattern  $X^{(m)}$ :

$$\Pr_{err} = N \exp \left( -\frac{N(1-2a)^2}{2p} \cdot q^2(1-b)^2 \right) \quad (13)$$

When  $N$  increases, this probability tends to zero, if  $p$  as function of  $N$  increases slower than

$$p_c = \frac{N(1-2a)^2}{2 \ln N} \cdot q^2(1-b)^2 \quad (14)$$

This allows us to use (14) as an asymptotically possible value of the storage capacity of PNN-II.

When  $q = 1$ , Eqs.(13)-(14) transform into well-known results for the standard Hopfield model (in this case there is no frequency noise,  $b = 0$ ). When  $q$  increases, the probability of the error (13) decreases exponentially, i.e. the noise immunity of PNN increases noticeably. In the same time the storage capacity of the network increases proportionally to  $q^2$ . In contrast to the Hopfield model the number of the patterns  $p$  can be much greater than the number of neurons.

For example, let us set a constant value  $\Pr_{err} = 0.01$ . In the Hopfield model, with this probability of the error we can recognize any of  $p = N/10$  patterns, each of which is less than 30% noisy. In the same time, PNN-II with  $q = 64$  allows us to recognize any of  $p = 5N$  patterns with 90% noise, or any of  $p = 50N$  patterns with 65% noise.

In Fig.1 we give an example of the restoration of 90% distorted pattern ( $a = 0$ ,  $b = 0.9$ ). Here the parameters of the network are  $N = 100$ ,  $p = 200$ ,  $q = 32$ . The pattern is a picture of a dog. The gray squares are noisy pixels. The states of the network after 50 and 100 steps are shown.

In Fig.2 for different values of  $q$  we show the dependence of the probability of a pattern recognition  $\mathbf{P}_{rec} = 1 - \Pr_{err}$  as function of the frequency noise  $\mathbf{b} = b \cdot 100\%$ ,  $b \in [0, 1]$ , when  $\alpha = p/N = 2$  (solid line); the phase noise is equal to zero,  $a = 0$ . We see that if  $q = 20$ , we can recognize correctly any

pattern when the noise is less than 70%, and if  $q = 30$ , any pattern when the noise is less than 85%. Generally, if the noise is less than a critical value  $b_c$ ,

$$b_c = 1 - \frac{2}{q} \sqrt{\frac{p}{N}}, \quad (15)$$

PNN can recognize a noisy pattern for sure, and if  $b > b_c$ , the probability of recognition tends to zero. Our computer simulations confirm these results.

### 3 Potts-glass neural networks

The models of associative memory with neurons that can be in more than two different states have been investigated by a lot of authors [7]–[12]. All these models are related with the Potts model of magnetic. The last generalizes the Ising model for the case of the spin variable that takes  $q > 2$  different values [13],[14]. In all these works the authors used the same well-known approach relating the Ising model with the Hopfield model (see, for example, [15]). Namely, in place of the short-range interaction between two nearest spins the Hebb type interconnections between all vector-neurons were used. As a result, long-range interactions appeared. Then in the mean-field approximation it was possible to calculate the statistical sum and, consequently, to construct the phase diagram. Different regions of the phase diagram were interpreted in the terms of the ability of the network to recognize noisy patterns.

Among all the models of  $q$ -state associative memory, characteristics of the *anisotropic Potts-glass neural network* (APGNN) [9], [11],[12] are most close to PNN-II. In other models the storage capacity is less than even for the Hopfield model. Below we describe APGNN in terms of our vector formalism and compare it with PNN-II.

APGNN consists of  $N$  neurons each of which can be in  $q$  different states. Now to describe the states of the neurons in place of the basis vectors  $\mathbf{e}_l \in \mathbb{R}^q$  (see Subsection 2.1)  $q$ -dimensional vectors of a special type are used. Namely,

the  $l$ th state of a neuron is described by a column-vector  $\mathbf{d}_l \in \mathbb{R}^q$ ,

$$\mathbf{d}_l = \frac{1}{q} \begin{pmatrix} -1 \\ \vdots \\ q-1 \\ \vdots \\ -1 \end{pmatrix}, \quad l = 1, \dots, q.$$

The state of the  $i$ -th neuron is described by a vector  $\mathbf{x}_i = \mathbf{d}_{l_i}$ ,  $1 \leq l_i \leq q$ ,  $i = 1, \dots, N$ . The state of the network as a whole  $X$  is determined by a set of  $N$   $q$ -dimensional vectors  $\mathbf{x}_i$ :  $X = (\mathbf{x}_1, \dots, \mathbf{x}_N)$ . The  $p$  stored patterns are

$$X^{(\mu)} = (\mathbf{x}_1^{(\mu)}, \dots, \mathbf{x}_N^{(\mu)}), \quad \mathbf{x}_i^{(\mu)} = \mathbf{d}_{l_i^{(\mu)}}, \quad 1 \leq l_i^{(\mu)} \leq q, \quad \mu = 1, 2, \dots, p.$$

The local field  $\mathbf{h}_i$  affecting the  $i$ -th neuron is the vector  $\mathbf{h}_i = \sum_{j=1}^N \mathbf{T}_{ij} \mathbf{x}_j$ , where  $(q \times q)$ -matrices  $\mathbf{T}_{ij}$  describe the interconnections between the  $i$ -th and the  $j$ -th neurons. These matrices are

$$\mathbf{T}_{ij} = (1 - \delta_{ij}) \sum_{\mu=1}^p \mathbf{x}_i^{(\mu)} \mathbf{x}_j^{(\mu)^\perp}, \quad i, j = 1, \dots, N.$$

The same as in Subsection 2.1 the dynamics of APGNN is defined as follows: the  $i$ -th neuron at the next time step  $t + 1$  is oriented along a direction mostly close to the local field  $\mathbf{h}_i(t)$  at the time  $t$ . During the evolution of the network the energy  $H(t) = -1/2 \sum_{i=1}^N (\mathbf{h}_i(t) \mathbf{x}_i(t))$  decreases. The necessary and sufficient conditions for a configuration  $X$  to be a fixed point is fulfillment of the set of inequalities:

$$(\mathbf{x}_i \mathbf{h}_i) \geq (\mathbf{d}_l \mathbf{h}_i), \quad \forall l = 1, \dots, q; \quad \forall i = 1, \dots, N.$$

We see that PNN-II and APGNN are much alike. The difference between these models is that, first, in APGNN the vectors  $\mathbf{d}_l$  are nonorthogonal, and, second, in APGNN there are no amplitudes  $\pm 1$  relating with the vectors  $\mathbf{d}_l$ .

When  $q = 2$ , APGNN is the same as the standard Hopfield model[9]. Repeating the argumentation of Subsection 2.2. we can estimate the storage capacity of APGNN for  $N \gg 1$ . We must only take into account that there is no phase noise  $\{a_i\}_1^N$  in this model. The distorted  $m$ th pattern has a form

$$\tilde{X}^{(m)} = (\hat{b}_1 \mathbf{x}_1^{(m)}, \hat{b}_2 \mathbf{x}_2^{(m)}, \dots, \hat{b}_N \mathbf{x}_N^{(m)}).$$

As above, the random operator  $\hat{b}_i$  with the probability  $b$  changes the state of the vector  $\mathbf{x}_i^{(m)}$ , and with the probability  $1-b$  this vector remains unchanged. Now, the probability of the error in the recognition of the vector  $\mathbf{x}_i^{(m)}$  is

$$\Pr_i = \Pr \left\{ \sum_{j=1}^{N-1} \xi_j + \sum_{r=1}^L \eta_r(l_I^{(m)}) < \sum_{r=1}^L \eta_r(l) \right\},$$

where the independent random variables  $\xi_j = (\mathbf{x}_j^{(m)} \hat{b}_j \mathbf{x}_j^{(m)})$ ,  $\eta_r = (\mathbf{d}_l \mathbf{x}_i^{(\mu)}) (\mathbf{x}_j^{(\mu)} \hat{b}_j \mathbf{x}_j^{(m)})$  are distributed

$$\xi_j = \begin{cases} (q-1)/q, & 1-b \\ -1/q, & b \end{cases}, \quad \eta_r = \begin{cases} 1/q^2, & (q-1)/q^2 \\ 0, & (q-2)/q \\ -1/q, & (q-1)/q^2 \\ -(q-1)/q, & 1/q^2 \end{cases}$$

Naturally, it is true for the randomized patterns  $\{X^{(\mu)}\}_1^p$ , only.

Similarly to calculations of Appendix we obtain the expression for the probability of the error in the recognition of the pattern  $X^{(m)}$ ,

$$\Pr_{err} = N \exp \left( -\frac{N}{2p} \frac{q(q-1)}{2} (1-\bar{b})^2 \right), \quad \bar{b} = \frac{q}{q-1} b. \quad (16)$$

Then the asymptotically possible value of the storage capacity of APGNN is

$$p_c = \frac{N}{2 \ln N} \frac{q(q-1)}{2} (1-\bar{b})^2. \quad (17)$$

When  $q = 2$ , these expressions give the known estimates for the Hopfield model. For  $q > 2$  the storage capacity of APGNN is  $q(q-1)/2$  times as large as the storage capacity of the Hopfield model. In [9] the same factor was obtained by fitting the results of numerical calculations. Our approach allows us to obtain the same result rigorously.

## 4 Conclusions

For  $q \gg 1$  the storage capacity of APGNN is two times less than the storage capacity of PNN-II (compare Eq.(17) with Eq.(14) for  $a \sim 0$ ). When calculating the probability of the error in the recognition, the additional factor

two appears in the exponent (see Eqs.(13),(16)). This leads to a significant decrease of a noise immunity of APGNN comparing with PNN-II. This is well seen in Fig.2, where for APGNN the dashed line shows the dependence of the pattern recognition  $\mathbf{P}_{\text{rec}}$  on the value of the frequency noise  $\mathbf{b}$  under the same conditions as for PNN-II (the solid line). The superiority of PNN-II is easily seen, especially in the region of not so large values of  $q \sim 10$ . For APGNN the critical value of the noise  $b_c$  (15) is less than the analogous characteristic of PNN-II by a quantity  $\frac{0.8}{q} \sqrt{\frac{p}{N}}$ .

In conclusion, we would like to note that our approach allows us to describe not only the optical neural networks of the parametrical type, but also neural networks in which information is encoded in the form of phase delays of pulses in interconnections. It is much more easy to realize such a network in form of a device.

## 5 Appendix

The following equation is true for the probability  $\Pr_i$  (12):

$$\Pr_i \leq \Pr \left\{ \sum_{j=1}^{N-1} \xi_j + \sum_{r=1}^L \eta_r \leq 0 \right\} = \Pr \left\{ - \sum_{j=1}^{N-1} \xi_j - \sum_{r=1}^L \eta_r \geq 0 \right\}$$

Using the known approach of exponential estimates of the Chebyshev type, for any positive  $z > 0$  we obtain:

$$\Pr_i \leq \overline{\exp \left( z \left( - \sum_{j=1}^{N-1} \xi_j - \sum_{r=1}^L \eta_r \right) \right)} = \left( \overline{\exp(-z\xi_j)} \left( \overline{\exp(-z\eta_r)} \right)^{p-1} \right)^{N-1}$$

The over-line means an averaging over all possible realizations, and the last equality follows from independence of the random variables  $\xi_j$  and  $\eta_r$ .

Taking into account the distributions (11), it is easy to obtain the averages

$$\overline{\exp(-z\xi_j)} = (1-a)(1-b)e^{-z} + b + a(1-b)e^z, \quad \overline{\exp(-z\eta_r)} = e^{-z}/2q^2 + 1 - 1/q^2 + e^z/2q^2.$$

Changing the variables  $e^z = x$  and introducing functions  $f_1(x)$  and  $f_2(x)$ ,

$$f_1(x) = a(1-b)x + b + \frac{(1-a)(1-b)}{x}, \quad f_2(x) = \frac{1}{2q^2} \left( x + \frac{1}{x} \right) + 1 - \frac{1}{q^2},$$

we obtain that for any positive  $x$  the following estimate is valid:

$$\Pr_i \leq \left( f_1(x) f_2^{p-1}(x) \right)^{N-1}. \quad (18)$$

To obtain the minimal possible value of the probability  $\Pr_i$ , we need to find the value of the variable  $x$  minimizing the right-hand side of Eq.(18). This leads us to the equation

$$(p-1)(x^2 - 1) + \frac{a(1-b)x^2 - (1-a)(1-b)}{a(1-b)x^2 + bx + (1-a)(1-b)}(x^2 + 2(q^2 - 1)x + 1) = 0.$$

When  $p \gg 1$ , the proper root of this equation up to the terms of the order of  $1/p$  is equal to  $x_1 = 1 + q^2(1-2a)(1-b)/(p-1)$ . Substituting this value of  $x$  in Eq.(18), we obtain

$$\Pr_i \leq \left( 1 - \frac{q^2(1-2a)^2(1-b)^2}{2(p-1)} \right)^{N-1} \cong \exp \left( -\frac{N(1-2a)^2}{2p} \cdot q^2(1-b)^2 \right).$$

This inequality gives the estimate (13) for the probability of the error in the recognition of the pattern  $X^{(m)}$ .

### List of Figures

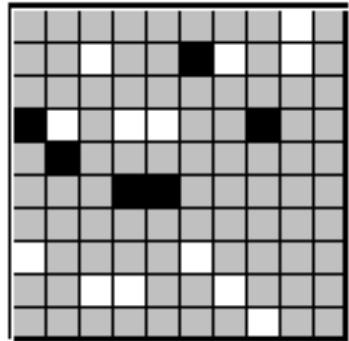
**Fig.1.** The restoration of the pattern with 90% frequency noise ( $b = 0.9$ ), when  $N = 100$ ,  $p = 200$  and  $q = 32$ . The pattern is a picture of a dog. The gray squares are noisy pixels. The states of the network after 50 and 100 steps are shown.

**Fig.2.** The probability of the pattern recognition  $\mathbf{P}_{\text{rec}} = 1 - \Pr_{\text{err}}$  versus frequency noise  $\mathbf{b} = b \cdot 100\%$ ,  $b \in [0, 1]$  for different values of  $q$  and  $\alpha = p/N = 2$  for PNN-II (solid line) and for the Potts-glass neural network (dashed line).

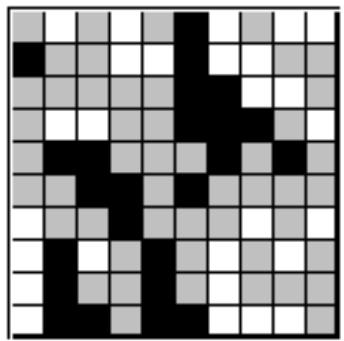
## References

- [1] Kryzhanovsky, B.V. & Mikaelian, A.L. (2002). On recognition ability of neuronet based on neurons with parametrical frequencies conversion. *Dokladi RAS (math.-phys.)*, **383**(3), 1-4.
- [2] Fonarev, A., Kryzhanovsky, B.V. et al. (2001). Parametric dynamic neural network recognition power. *Optical Memory & Neural Networks*, **10**(4), 31-48.
- [3] Bloembergen, N. (1966). Nonlinear optics.
- [4] Chernov, N. (1952). A mesure of asymptotic efficiency for tests of hypothesis based on the sum of observations. *Ann. Math. Statistics*, **23**, 493-507.
- [5] Kryzhanovsky, B.V., Mikaelyan, A.L., Koshelev, V.N. & Fonarev, A. (2000). On recognition error bound for associative Hopfield memory. *Optical Memory & Neural Networks*, **9**(4), 267-276.
- [6] Kinzel, W. (1987). Spin glasses and memory. *Physica Scripta*, **3**, 398-401.
- [7] Noest, J. (1988). Discrete-state phasor neural networks. *Physical Review A*, **38**, 2196-2199.
- [8] Cook, J. (1989). The mean-field theory of a Q-state neural network model. *Journal of Physics A*, **22**, 2000-2012.
- [9] Kanter, I. (1988). Potts-glass models of neural networks. *Physical Review A*, **37**, 2739-2742.
- [10] Vogt, H., Zippelius, A. (1992). Invariant recognition in Potts glass neural networks. *Journal of Physics A*, **25**, 2209-2226.
- [11] Bolle, D., Dupont, P. & van Mourik, J. (1991). Stability properties of Potts neural networks with biased patterns and low loading. *Journal of Physics A*, **24**, 1065-1081.

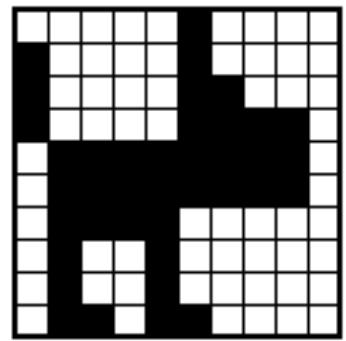
- [12] Bolle, D., Dupont, P. & Huyghebaert, J. (1992). Thermodynamics properties of the q-state Potts-glass neural network. *Phys. Rev. A*, **45**, 4194-4197.
- [13] Wu, F.Y. (1982). The Potts model. *Review of Modern Physics*, **54**, 235-268.
- [14] Baxter, R. (1982). Exactly Solved Models in Statistical Mechanics. London: Academic Press.
- [15] Hertz, J., Krogh, A. & Palmer, R. (1991). Introduction to the Theory of Neural Computation. Addison-Wesley.



$t=0$



$t=50$



$t=100$

