Intrinsic metrics on graphs – A survey

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INTRINSIC METRICS ON GRAPHS - A SURVEY

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ABSTRACT. A few years ago various disparities for Laplacians on graphs and manifolds were discovered. The corresponding results are mostly related to volume growth in the context of unbounded geometry. Indeed, these disparities can now be resolved by using so called intrinsic metrics instead of the combinatorial graph distance. In this article we give an introduction to this topic and survey recent results in this direction. Specifically, we cover topics such as Liouville type theorems for harmonic functions, essential selfadjointness, stochastic completeness and upper escape rates. Furthermore, we determine the spectrum as a set, discuss upper and lower spectral bounds by isoperimetric constants and volume growth and study p-independence of spectra under a volume growth assumption.

CONTENTS

1. Introduction

There are many parallels in the analysis of Laplacians on graphs and manifolds. However, starting with examples discovered by Wojciechowski [82, 83, 84] various disparities surfaced that show very different behavior for certain phenomena on manifolds and graphs. These phenomena appeared in the context of unbounded geometry, i.e., when the corresponding graph Laplacian is an unbounded operator. In particular, these phenomena do not show for the normalized Laplacian. Furthermore, a common feature these phenomena share are that they deal with questions of volume growth and more specifically with distances.

Independently, Frank/Lenz/Wingert [22] developed a framework to study so called intrinsic metrics for general, in particular non-local, regular Dirichlet forms. Such metrics had already proven to be very effective in the context of strongly local regular Dirichlet forms, see e.g. the work of Sturm [78]. In [22] already various applications were given for general Dirichlet forms.

Indeed, the Laplace-Beltrami operator on a Riemannian manifold arises from a strongly local Dirichlet form and the corresponding intrinsic metric turns out to be the Riemannian metric. On the other hand, Laplacians on graphs arise from Dirichlet forms that are non-local and, thus, their forms do not fit in the framework of [78]. So, the

question arises what is a suitable metric for a given graph. The most immediate choice is the combinatorial graph distance which is given by the minimal number of edges needed to connect two vertices by a path. It turns out that the combinatorial graph distance is equivalent to an intrinsic metric in the sense of [22] if and only if the geometry is bounded. As a consequence many results known for Riemannian manifolds can be proven for graphs in the case of bounded geometry, but they fail for graphs with unbounded geometry if one considers the combinatorial graph distance.

Now, [22] provides a very general concept of intrinsic metrics which can be applied also to graph Laplacians. And, indeed, these metrics work as a remedy to recover the results of Riemannian manifolds for graphs (even with unbounded geometry) in very great generality. From this perspective, the case of bounded geometry and, in particular, the normalized Laplacian appears as a special case.

In this survey we review the concept of intrinsic metrics and present recent results in the context of weighted graphs. In the next two sections we introduce the basic concepts of weighted graphs, Laplacians and intrinsic metrics.

The sections after these introductional parts are structured in the following way: We first introduce the question and discuss the known results in the manifold setting. Then we examine the situation for graphs with bounded geometry and illustrate the problems that arise for unbounded geometry. Finally, we present results involving intrinsic metrics to recover the results for manifolds for general weighted graphs.

In Section 4 we study harmonic functions and discuss Liouville type theorems involving ℓ^p growth bounds, $p \in (1, \infty)$, which are classical results of Yau [85] and Karp [50] for manifolds. As a corollary, we obtain a criterion for recurrence. The recent results for graphs in this section are mainly based on results found in [36].

We use these results in Section 5 to draw consequences about the domain of the generators and, in particular, about essential selfadjointness for the case $p = 2$ which is a result of Gaffney [27] and Roelcke [71] for complete manifolds. For graphs, these are results based on [49, 36].

In Section 6 we consider stochastic completeness, a property which is equivalent to uniqueness of solutions of the heat equation and the Poisson equation in ℓ^∞. For this phenomena Wojciechowski [84] discovered examples of only little more than cubic volume growth with respect to the combinatorial graph distance that are not stochastically complete while the volume growth bound for manifolds is superexponential as proven by Grigor'yan, [29]. With the help of intrinsic metrics the result from the manifold setting can be recovered for locally finite graphs, see [20, 45]. After stochastic completeness, we take a look at upper escape rates of the related Markov process in Section 7. This is based on results of [48].

The ℓ^2-spectrum as a set can be determined by existence of subexponentially growing solutions in ℓ^2 which is a classical result by Shnol' [73] in $\mathbb{R}^n$ and was generalized by Boutet de Monvel/Lenz/Stollmann to strongly local Dirichlet forms, [7]. This is discussed in Section 8 and the references for graphs are found in [22, 51].

After this result on the spectrum as a set, we focus on the lower edge and put the focus on lower and upper spectral bounds via geometric invariants.

First, we consider lower bounds via isoperimetric constants in Section 9. The use of such a constant to estimate the bottom of the spectrum goes back to a classical theorem of Cheeger, [10]. For graphs a first result in this direction is due to Dodziuk [13] which is however only applicable for graphs with bounded vertex degree. This problem was pointed out in 1986 Dodziuk/Kendall, [16], and they gave a proof for the normalized Laplacian recovering the original result of Cheeger in this particular situation. Since then the general case remained open and a priori it is not even clear what role do metrics play in this problem at all. Recently, the problem was solved in [6] by the use of intrinsic metrics that now enter the definition of the boundary measure in the isoperimetric constant.

Upper bounds on the bottom of the spectrum can be given in terms of exponential volume growth of distance balls in Section 10 which is a result of Brooks [8] which is paralleled by result of Sturm [78]. Such a result fails for unbounded graph Laplacians and the combinatorial graph distance, since there are graphs with only little more than cubic volume growth that admit a spectral gap, [55]. Again, using intrinsic metrics the result that holds for Riemannian manifolds can be recovered for graphs and even for general Dirichlet forms, [38].

Finally, in Section 11.1 we study the spectrum of the Laplacian in dependence of the underlying ℓ^p space, $p \in [1, \infty]$. This question was answered for Schrödinger operators in $\mathbb{R}^n$ by Hempel/Voigt [39] in 1986. This question was later addressed by Sturm for manifolds [77] in terms of volume growth. It turns out that in the case of uniform subexponential volume growth the spectrum can be shown to be p-independent. For results on graphs the results are found in [5].

2. Graphs and Laplacians

In this section we introduce the basic notions for weighted symmetric graphs. We mainly follow the framework developed in [52].

2.1. **Graphs.** Let X be a discrete and countably infinite space and m a measure of full support on X, that is a function $m : X \to (0, \infty)$ which is extended additively to sets via $m(A) = \sum_{x \in A} m(x)$, $A \subseteq X$.

A graph (b, c) over X is a symmetric function $b : X \times X \to [0, \infty)$ with zero diagonal and

$$\sum_{y \in X} b(x, y) < \infty, \quad x \in X,$$

and $c : X \to [0, \infty)$. We say two vertices $x, y \in X$ are *neighbors* if $b(x, y) > 0$. In this case we write $x \sim y$. The function c can be thought to describe one-way-edges to a virtual point at infinity or as a potential or as a killing term. If $c \equiv 0$, then we speak of b as a graph over X. If we already fixed a measure, then we speak of graphs over (X, m).

We say a graph is *connected* if for all $x, y \in X$ there are $x = x_0 \sim \ldots \sim x_n = y$. If a graph is not connected one can restrict the attention to the connected components. Therefore, we assume that the graphs under consideration are connected.

Given a pair (b, c) an important special case of a measure m is the *normalizing measure*

$$n(x) = \sum_{y \in X} b(x, y) + c(x), \quad x \in X.$$

Another important special case is the *counting measure* $m \equiv 1$.

We say a graph is *locally finite* if every vertex has only finitely many neighbors, that is if the *combinatorial vertex degree* deg is finite at every vertex

$$\deg(x) = \#\{y \in X \mid x \sim y\} < \infty, \quad \text{for all } x \in X.$$

We speak of a graph with *standard weights* if $b : X \times X \to \{0, 1\}$ and $c \equiv 0$. In this case the normalizing measure n equals the combinatorial vertex degree deg. Obviously, by summability of b about vertices, graphs with standard weights are locally finite.

2.2. **Generalized forms and formal Laplacians.** We let $C(X)$ be the set of complex valued functions on X and $C_c(X)$ be the subspace of functions in $C(X)$ of finite support. In contrast to the standard theory of Dirichlet forms we deal here with complex valued functions as opposed to real valued functions. Nevertheless results from these two contexts can be transferred into each other by the virtue of standard arguments presented in [37, Appendix B].

For a graph (b, c) over X, the generalized quadratic form $\mathcal{Q} : C(X) \to [0, \infty]$ is given by

$$\mathcal{Q}(f) = \frac{1}{2} \sum_{x, y \in X} b(x, y) |f(x) - f(y)|^2 + \sum_{x \in X} c(x) |f(x)|^2$$

with generalized domain

$$\mathcal{D} = \{f \in C(X) \mid \mathcal{Q}(f) < \infty\}.$$

Since $\mathcal{Q}^{\frac{1}{2}}$ is a semi norm and satisfies the parallelogram identity, $\mathcal{Q}$ yields a semi scalar product on $\mathcal{D}$ by polarization via

$$\mathcal{Q}(f,g) = \frac{1}{2}\sum_{x,y\in X} b(x,y)(f(x)-f(y))\overline{(g(x)-g(y))} + \sum_{x\in X} c(x)f(x)\overline{g(x)}.$$

Moreover, for functions in

$$\mathcal{F} = \{f\in C(X) \mid \sum_{y\in X} b(x,y)|f(y)|^2 < \infty \text{ for all } x\in X\},$$

we define the generalized Laplacian $\mathcal{L}:\mathcal{F}\to C(X)$ by

$$\mathcal{L}f(x) = \frac{1}{m(x)}\sum_{y\in X} b(x,y)(f(x)-f(y)) + \frac{c(x)}{m(x)}f(x).$$

In [36, Lemma 4.7], a *Green's formula* is shown for functions $f\in\mathcal{F}$ and $\varphi\in C_c(X)$

$$\frac{1}{2}\sum_{x,y\in X} b(x,y)(f(x)-f(y))\overline{(\varphi(x)-\varphi(y))} + \sum_{x\in X} c(x)f(x)\overline{\varphi(x)}$$

$$= \sum_{x\in X}\mathcal{L}f(x)\overline{\varphi(x)}m(x) = \sum_{x\in X} f(x)\overline{\mathcal{L}\varphi(x)}m(x).$$

In the special case when the functions under consideration are in a certain Hilbert space the formula above might be expressed by scalar products, see Section 2.3.2.

We call $f\in\mathcal{F}$ a *solution* (respectively *subsolution* or *supersolution*) for $\lambda\in\mathbb{R}$ if $(\mathcal{L}-\lambda)f = 0$ (respectively $(\mathcal{L}-\lambda)f \leq 0$ or $(\mathcal{L}-\lambda)f \geq 0$). A solution (respectively subsolution or supersolution) for $\lambda = 0$ is called a *harmonic* (respectively *subharmonic* or *superharmonic*).

We say a function $f\in C(X)$ is *positive* if f is non-trivial and $f\geq 0$ and *strictly positive* if $f > 0$.

For two functions $f,g\in C(X)$ we denote the minimum of f and g by $f\wedge g$ and the maximum of f and g by $f\vee g$.

A *Riesz space* is a linear space equipped with a partial ordering which is consistent with addition, scalar multiplication and where the maximum and the minimum of two functions exist.

An important well known fact which is needed in the subsequent is that in order to study existence of non-constant (respectively non-zero) solutions for $\lambda \leq 0$ in a Riesz space, it suffices to study positive subharmonic functions.

Lemma 2.1. *Let (b,c) be a connected graph over (X,m) and $\mathcal{F}_0\subseteq\mathcal{F}$ be a Riesz space. If there are no non-constant positive subharmonic functions in $\mathcal{F}_0$, then there are no non-constant solutions for $\lambda \leq 0$ in $\mathcal{F}_0$. In particular, any constant solution for $\lambda < 0$ is zero.*

Proof. For a solution f for $\lambda \leq 0$ the positive part $f_+ = f \vee 0$, the negative part $f_- = -f \vee 0$ and the modulus $|f| = f_+ + f_-$ can directly be seen to be non-negative subharmonic functions. Thus, the statement follows from connectivity. The 'in particular' is obvious. $\qquad \square$

2.3. **Dirichlet forms and their generators.** The form and Laplacian introduced above are defined on spaces with very few properties. Next, we will consider these objects restricted to suitable Hilbert and Banach spaces.

Let $\ell^p(X, m)$ be the canonical complex-valued ℓ^p-spaces, $p \in [1, \infty]$, with norms

$$\|f\|_p = \Big(\sum_{x \in X} |f(x)|^p m(x) \Big)^{\frac{1}{p}}, \ p \in [1, \infty),$$

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

As $\ell^\infty(X, m)$ does not depend on m we also write $\ell^\infty(X)$. For $p = 2$, we have a Hilbert space $\ell^2(X, m)$ with scalar product

$$\langle f, g \rangle = \sum_{x \in X} f(x)\overline{g(x)}m(x), \qquad f, g \in \ell^2(X, m),$$

and we denote the norm $\| \cdot \| = \| \cdot \|_2$.

2.3.1. *Dirichlet forms.* Restricting the form $\mathcal{Q}$ to $\mathcal{D} \cap \ell^2(X, m)$, we see by Fatou's lemma that this restriction is lower semi-continuous and, thus, closed. Hence, the restriction of $\mathcal{Q}$ to $C_c(X)$ is closable.

Let $Q = Q_{b,c}$ be the quadratic form given by

$$D(Q) = \overline{C_c(X)}^{\|\cdot\|_\mathcal{Q}}, \ \text{ where } \| \cdot \|_\mathcal{Q} = \big(\mathcal{Q}(\cdot) + \| \cdot \|^2 \big)^{\frac{1}{2}}$$

$$Q(f) = \frac{1}{2} \sum_{x,y \in X} b(x, y)|f(x) - f(y)|^2 + \sum_{x \in X} c(x)|f(x)|^2, \qquad f \in D(Q).$$

It can be seen that Q is a *Dirichlet form* that is for any $f \in D(Q)$ we have $0 \vee f \wedge 1 \in D(Q)$ and

$$Q(0 \vee f \wedge 1) \leq Q(f).$$

(see [25, Theorem 3.1.1] for general theory and for a proof in the graph setting see [72, Proposition 2.10]). Obviously, Q is *regular*, that is $C_c(X) \cap D(Q)$ is dense in $D(Q)$ with respect to $\| \cdot \|_\mathcal{Q}$ and dense in $C_c(X)$ with respect to $\| \cdot \|_\infty$.

As it turns out, by [52, Theorem 7], all regular Dirichlet forms are given in this way. Indeed, this can be also derived directly from the Beurling-Deny representation formula [25, Theorem 3.2.1 and Theorem 5.2.1].

Theorem 2.2 (Theorem 7 in [52])**.** *If q is a regular Dirichlet on $\ell^2(X, m)$, then there is a graph (b, c) such that $q = Q_{b,c}$.*

2.3.2. *Markovian semigroups and their generators.* By general theory (see e.g. [80, Satz 4.14]), Q yields a positive selfadjoint operator L with domain $D(L)$ viz

$$Q(f,g) = \langle L^{\frac{1}{2}} f, L^{\frac{1}{2}} g \rangle, \qquad f, g \in D(Q).$$

By Green's formula it can be seen directly that L is a restriction of $\mathcal{L}$ which reads in the case when $f \in D(Q) \cap \mathcal{F}$ and $\varphi \in C_c(X)$ as

$$Q(f,\varphi) = \langle \mathcal{L}f, \varphi \rangle,$$

where the right hand side also equals $\langle f, \mathcal{L}\varphi \rangle$ whenever $\mathcal{L}\varphi \in \ell^2(X,m)$.

By the second Beurling-Deny criterion L gives rise to a Markovian semigroup e^{-tL}, $t > 0$, which extends consistently to all $\ell^p(X,m)$, $p \in [1,\infty]$, and is strongly continuous for $p \in [1,\infty)$. *Markovian* means that for functions $0 \le f \le 1$, one has $0 \le e^{-tL} f \le 1$.

We denote the generators of e^{-tL} on $\ell^p(X,m)$, $p \in [1,\infty)$, by L_p, that is

$$D(L_p) = \left\{ f \in \ell^p(X,m) \mid g = \lim_{t \to 0} \frac{1}{t}(I - e^{-tL})f \text{ exists in } \ell^p(X,m) \right\}$$

$$L_p f = g$$

and L_∞ is defined as the adjoint of L_1.

It can shown that L_p is a restriction of $\mathcal{L}$.

Theorem 2.3 (Theorem 5 in [52])**.** *Let (b,c) be a graph over (X,m) and $p \in [1,\infty]$. Then,*

$$L_p f = \mathcal{L}f, \qquad f \in D(L_p).$$

2.3.3. *Boundedness of the operators.* We next comment on the boundedness of the form Q and the operator L. The theorem below is taken from [37] and an earlier version can be found in [53, Theorem 11].

Theorem 2.4 (Theorem 9.3 in [37])**.** *Let (b,c) be a graph over (X,m). Then the following are equivalent:*

(i) $X \to [0,\infty)$, $x \mapsto \frac{1}{m(x)}\left(\sum_{y \in X} b(x,y) + c(x) \right)$ *is a bounded function.*
(ii) Q *is bounded on $\ell^2(X,m)$.*
(iii) L_p *is bounded for some $p \in [1,\infty]$.*
(iv) L_p *is bounded for all $p \in [1,\infty]$.*

Specifically, if the function in (i) is bounded by $D < \infty$, then $Q \le 2D$ and $\|L\|_p \le 2D$, $p \in [1,\infty]$.

2.3.4. *The compactly supported functions as a core.* It shall be observed that $C_c(X)$ is in general not included in $D(L)$. Indeed, one can give a characterization for this situation. The proof is rather immediate and we refer to [52, Proposition 3.3] or [35, Lemma 2.7.] for a reference.

Lemma 2.5. *Let (b, c) be a graph over (X, m). Then the following are equivalent:*

(i) $C_c(X) \subseteq D(L)$.
(ii) $\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$
(iii) *The functions $X \to [0, \infty)$, $y \mapsto \frac{1}{m(y)}b(x, y)$ are in $\ell^2(X, m)$ for all $x \in X$.*

In particular, the above assumptions are satisfied if the graph is locally finite or

$$\inf_{y \sim x} m(y) > 0, \qquad x \in X.$$

Moreover, the above assumptions imply $\ell^2(X, m) \subseteq \mathcal{F}$.

Proof. The equivalence of (ii) and (iii) follows from the abstract definition of the domain of L as $D(L) = \{f \in D(Q) \mid \text{there is } l \in \ell^2(X, m) \text{ such that } \langle l, \varphi \rangle = Q(f, \varphi) \text{ for all } \varphi \in D(Q)\}$. The equivalence of (i) and (ii) is a direct calculation, see [52, Proposition 3.3] and the 'in particular' statements are also immediate, see [52, 35]. $\square$

2.3.5. *Graphs with standard weights.* Two important special cases are graphs with standard weights, that is $b : X \times X \to \{0, 1\}$ and $c \equiv 0$.

For the counting measure $m \equiv 1$, we denote the operator L on $\ell^2(X) = \ell^2(X, 1)$ by Δ which operates as

$$\Delta f(x) = \sum_{y \sim x}(f(x) - f(y)), \quad f \in D(\Delta), x \in X.$$

By Lemma 2.4 the operator Δ is bounded if and only if deg is bounded, i.e., $\sup_{x \in X} \deg(x) < \infty$. In the unbounded case one has still $C_c(X) \subseteq D(\Delta)$ by Lemma 2.5 since standard weights imply local finiteness.

For the normalized measure $n = \deg$, we call the operator L on $\ell^2(X, \deg)$ the *normalized Laplacian* and denote it by Δ_n. The operator Δ_n acts as

$$\Delta_n f(x) = \frac{1}{\deg(x)} \sum_{y \sim x}(f(x) - f(y)), \quad f \in \ell^2(X, \deg), x \in X,$$

and, also by Lemma 2.4, Δ_n is always bounded by 2.

3. INTRINSIC METRICS

In this subsection we discuss the notion of intrinsic metrics for weighted graphs. As already discussed in the introduction such metrics have proven to be very effective in the context of strongly local Dirichlet

forms, see e.g. [78]. Recently, this concept was generalized to all regular Dirichlet forms and systematically studied by Frank/Lenz/Wingert in [22], (see also [60] for an earlier mentioning of the criterion for certain non-local forms). We will demonstrate in this article that these metrics can be used to obtain analogous results for graphs as in the case of manifolds.

3.1. **Definition of intrinsic metrics.** We call a symmetric map $\rho :$ $X \times X \to [0, \infty)$ with zero diagonal a *pseudo metric* if it satisfies the triangle inequality. By [22, Lemma 4.7, Theorem 7.3] it can be seen that the definition below coincides with the definition of an intrinsic metric for general regular Dirichlet forms, [22, Definition 4.1].

Definition 3.1. A pseudo metric ρ is called an *intrinsic metric* with respect to a graph (b, c) over (X, m) if for all $x \in X$

$$\sum_{y \in X} b(x, y) \rho^2(x, y) \leq m(x).$$

Similar definitions of such metrics were also introduced later in the context of graphs or jump processes under the name adapted metrics, see [20, 21, 34, 44, 48, 60].

3.2. **Examples and relations to other metrics.**

3.2.1. *The degree path metric.* A specific example of an intrinsic metric was introduced by Huang, [44, Definition 1.6.4] and it also appeared in [19]. Consider the pseudo metric $\rho_0 : X \times X \to [0, \infty)$ given by

$$\rho_0(x, y) = \inf_{x = x_0 \sim x_1 \sim \ldots \sim x_n = y} \sum_{i=1}^{n} \left(\mathrm{Deg}(x_{i-1}) \vee \mathrm{Deg}(x_i) \right)^{-\frac{1}{2}}, \qquad x \in X,$$

where $\mathrm{Deg} : X \mapsto (0, \infty)$ is the *weighted vertex degree* given by

$$\mathrm{Deg}(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y), \qquad x \in X.$$

We call such a metric that minimizes sums of weights over paths of edges a *path metric*.

It can be seen directly that ρ_0 defines an intrinsic metric for the graph (b, c) over (X, m)

$$\sum_{y \in X} b(x, y) \rho_0^2(x, y) \leq \sum_{y \in X} \frac{b(x, y)}{\mathrm{Deg}(x) \vee \mathrm{Deg}(y)} \leq \sum_{y \in X} \frac{b(x, y)}{\mathrm{Deg}(x)} = m(x).$$

There is the following intuition behind the definition of ρ_0. Consider the Markov process $(X_t)_{t \geq 0}$ associated to the semigroup e^{-tL} via

$$e^{-tL} f(x) = \mathbb{E}_x(f(X_t)), \qquad x \in X,$$

where $\mathbb{E}_x$ is the expected value conditioned on the process starting at x. The random walker modeled by this process jumps from a vertex

x to a neighbor y with probability $b(x,y)/\sum_z b(x,z)$. Moreover, the probability of not having left x at time t is given by

$$\mathbb{P}_x(X_s = x, 0 \leq s \leq t) = e^{-\mathrm{Deg}(x)t}.$$

Qualitatively, this indicates that the larger $\mathrm{Deg}(x)$, the faster the random walker leaves x. Looking at the definition of $\rho_0(x,y)$, the larger the degree of either x or y the closer are the two vertices. Combining these two observations, we see that the faster the random walker jumps along an edge the shorter the edge is with respect to ρ_0. Of course, the jumping time along an edge connecting x to y is not symmetric and depends on whether one jumps from x to y or from y to x as the degrees of x and y can be very different. In order to get a symmetric function, ρ_0 favors the vertex with the larger degree and the faster jumping time.

There is a direct analogy to the Riemannian setting in terms of mean exit times of small balls. Consider a small ball B_r of radius r on a d-dimensional Riemannian manifold, the first order term of the mean exit time of B_r is $r^2/2d$, [68]. Now, on a locally finite graph for a vertex x a 'small' ball with respect to ρ_0 can be thought to have radius $r = \inf_{y \sim x} \rho(x,y)/2$, namely this ball contains only the vertex itself. Now, computing the mean exit time of this ball gives $1/\mathrm{Deg}(x) \geq r^2$, where equality holds whenever $\mathrm{Deg}(x) = \max_{y \sim x} \mathrm{Deg}(y)$.

3.2.2. *The combinatorial graph distance.* We call the path metric defined by

$$d(x,y) =$$

$$\min \#\{n \in \mathbb{N}_0 \mid \text{there are } x_0, \ldots, x_n \text{ with } x = x_0 \sim \ldots \sim x_n = y\}$$

the *combinatorial graph distance*. The next lemma shows that the combinatorial graph distance is equivalent to an intrinsic metric if and only if the graphs has bounded geometry, i.e., Deg is bounded. This was already noted in [22, 54].

Lemma 3.2. *Let (b,c) be a graph over (X,m). The following are equivalent:*

(i) *The combinatorial graph distance d is equivalent to an intrinsic metric.*
(ii) Deg *is a bounded function.*
(iii) L *is a bounded operator.*

Proof. (i)$\Rightarrow$(ii): Let ρ be an intrinsic metric such that $C^{-1}\rho \leq d \leq C\rho$. Then,

$$\sum_{x \in X} b(x,y) = \sum_{x \in X} b(x,y)d^2(x,y) \leq C^2 \sum_{x \in X} b(x,y)\rho^2(x,y) \leq C^2 m(x),$$

for all $x \in X$. Hence, Deg $\leq C^2$.
(ii)$\Rightarrow$(i): Assume Deg $\leq C$ and consider the degree path metric ρ_0. Then, $\rho_1 = \rho_0 \wedge 1$ is an intrinsic metric as well. Clearly, $\rho_1 \leq d$. On

the other hand, by $\mathrm{Deg} \leq C$ we immediately get $\rho_1 \geq C^{-\frac{1}{2}}d$. The equivalence (ii)⇔(iii) follows from Theorem 2.4. $\qquad\square$

The theorem implies in particular that in the case of the normalizing measure $m = n$ the combinatorial graph distance is an intrinsic distance as $\mathrm{Deg} = n/m = 1$ in the case of $c \equiv 0$ and $\mathrm{Deg} \leq n/m = 1$ in general.

Furthermore, for a graph with standard weights and the counting measure associated to the Laplacian Δ, the combinatorial graph distance d is a multiple of an intrinsic metric if and only if the combinatorial vertex degree deg is bounded since $\mathrm{Deg} = \deg$.

3.2.3. *Comparison to the strongly local case.* An important difference to the case of strongly local Dirichlet forms is that in the graph case there is no maximal intrinsic metric. For example for a complete Riemannian manifold M the Riemannian distance d_M is the maximal C^1 metric ρ_M that satisfies

$$|\nabla \rho_M(o, \cdot)| \leq 1,$$

for all $o \in M$,where ∇_M is the Riemannian gradient. In fact, d_M can be recovered by the formula

$$d_M(x, y) = \sup\{f(x) - f(y) \mid f \in C_c^\infty(M)\ |\nabla_M f| \leq 1\}, \quad x, y \in X.$$

Now, for discrete spaces the maximum of two intrinsic metrics is not necessarily an intrinsic metric. In particular, consider the pseudo-metric σ

$$\sigma(x, y) = \sup\{f(x) - f(y) \mid f \in \mathcal{A}\}, \qquad x, y \in X,$$

where

$$\mathcal{A} = \Big\{f : X \to \mathbb{R} \mid \sum_{y \in X} b(x, y)|f(x) - f(y)|^2 \leq m(x) \text{ for all } x \in X\Big\}.$$

As discussed for the Riemannian case above, the strongly local analogue d_M of σ defines the maximal intrinsic metric in the strongly local case, but σ is in general not even equivalent to an intrinsic metric in the graph case.

A basic example where this can be seen immediately can be found in [22, Example 6.2]. More generally, this can be seen for arbitrary tree graphs with standard weights and the counting measure to which the operator Δ is associated. In this case $\sigma = \frac{1}{2}d$ and by discussion above we already know that the combinatorial graph distance d is typically not an intrinsic metric for Δ.

An abstract way to see that σ is in general not intrinsic is discussed in [54, Section 1]. Namely, the set of Lipschitz continuous functions Lip_ρ with respect to an intrinsic metric ρ is included in $\mathcal{A}$ and Lip_ρ is closed under taking suprema. On the other hand, $\mathcal{A}$ is in general not closed under taking suprema. Hence, in general Lip_ρ is not equal to

$\mathcal{A}$. This raises the question whether one can characterize the situation when these two spaces are different.

3.2.4. *Resistance metrics.* Another important metric appears in the context of resistance metrics. Let $r : X \times X \to [0, \infty)$ be given by

$$r(x, y) = \sup\{f(x) - f(y) \mid f \in \mathcal{D}, \; \mathcal{Q}(f) \leq 1\}, \qquad x, y \in X.$$

Indeed, r is the square root of the resistance metric as it appears e.g. in [58], see [28, Theorem 3.20]. In [28] this metric is related to intrinsic metrics.

Theorem 3.3 (Theorem 3.14 in [28])**.** *Let b be a connected graph over X. Then*

$$r = \sup\{\rho \mid \text{intrinsic metric for } b \text{ over } (X, m) \text{ with } m(X) = 1\}.$$

3.2.5. *Another path metric.* Colin de Verdière/Torki-Hamza/Truc [12] studied a path pseudo metric δ which is given as

$$\delta(x, y) = \inf_{x = x_0 \sim \ldots \sim x_n = y} \sum_{i=0}^{n-1} \left(\frac{m(x) \wedge m(y)}{b(x, y)} \right)^{\frac{1}{2}}, \qquad x, y \in X.$$

This metric is equivalent to the intrinsic metric ρ_0 if and only if the combinatorial vertex degree deg is bounded on the graph.

3.3. **A Hopf-Rinow theorem.** We shall stress that in general an intrinsic metric ρ (and in particular ρ_0) is not a metric and (X, ρ) might not even be locally compact, as can be seen from examples in [49, Example A.5]. However, for locally finite graphs and path metrics such as ρ_0 the situation is much tamer. For example one can show a Hopf-Rinow type theorem which in parts can also be found in [62]. Recall that for a path $\gamma = (x_n)$ the length with respect to a metric ρ is given by $l(\gamma) = \sum_{j \geq 0} \rho(x_j, x_{j+1})$ and, moreover, a path $\gamma = (x_n)$ is called a *geodesic* with respect to a metric ρ if $\rho(x_0, x_n) = l((x_0, \ldots, x_n))$.

Theorem 3.4 (Theorem A1 in [49])**.** *Let (b, c) be a locally finite connected graph over (X, m) and ρ be a path metric. Then, the following are equivalent:*
   (i) *(X, ρ) is complete as a metric space.*
   (ii) *(X, ρ) is geodesically complete, that is any infinite geodesic has infinite length.*
   (iii) *The distance balls in (X, ρ) are pre-compact (that is finite).*

3.4. **Some important conditions.** In the general situation, we will often make assumptions which are discussed next.

We say a pseudo metric ρ admits *finite balls* if (iii) in the theorem above is satisfied for ρ, i.e.,
   (B) The distance balls $B_r(x) = \{y \in X \mid \rho(x, y) \leq r\}$ are finite for all $x \in X$, $r \geq 0$.

A somewhat weaker assumption is that the weighted vertex *degree is bounded on balls*:

(D) The restriction of Deg to $B_r(x)$ is bounded for all $x \in X$, $r \geq 0$.

Clearly, (B) implies (D). Moreover, (D) is equivalent to the fact that $\mathcal{L}$ restricted to the ℓ^2 space of a distance ball is a bounded operator, confer Theorem 2.4.

The assumptions (B) and (D) can be understood as bounding ρ in a certain sense from below. Next, we come to an assumption which may be understood as an upper bound.

We say a pseudo-metric ρ has *finite jump size* if

(J) The jump size $s = \sup\{\rho(x,y) \mid x,y \in X,\ x \sim y\}$ is finite.

The assumptions (B) and (J) combined have strong geometric consequences.

Lemma 3.5. *Let (b,c) be a graph over (X,m) and ρ be an pseudo metric. If ρ satisfies (B) and (J), then the graph is locally finite.*

Proof. If there was a vertex with infinitely many neighbors, then there would be a distance ball containing all of them by finite jump size. However, this is impossible by (B). $\qquad\qquad\square$

3.5. **Construction of cut-off functions.** From an analyst's point of view a major interest in metrics is to construct cut-off functions with desirable properties. Let us illustrate in how intrinsic metrics serve this purpose.

Given a intrinsic metric ρ, a subset $A \subseteq X$ and $R > 0$, the most basic cut-off function is defined by $\eta = \eta_{A,R} : X \to [0,\infty)$

$$\eta(x) = \left(1 - \frac{\rho(x,A)}{R}\right) \wedge 0, \qquad x \in X.$$

Such test functions are often used to approximate a solution f by $\eta_{B_r,R} f$, where B_r is a ball with respect to an intrinsic metric ρ about some vertex.

Let us illustrate an estimate which often occurs as a crucial step in the analysis. To this end, let $s \in [0,\infty]$ be the jump size of ρ. In the analysis one often arrives at a term such as below which is then further

estimated using the intrinsic metric property as illustrated below

$$\sum_{x \in X} |f(x)|^2 \sum_{y \in X} b(x, y)(\eta(x) - \eta(y))^2$$

$$= \sum_{x \in B_{R+s} \setminus B_{r-s}} |f(x)|^2 \sum_{y \in B_{R+s} \setminus B_{r-s}} b(x, y)(\eta(x) - \eta(y))^2$$

$$\leq \frac{1}{(R-r)^2} \sum_{x \in B_{R+s} \setminus B_{r-s}} |f(x)|^2 \sum_{y \in B_{R+s} \setminus B_{r-s}} b(x, y)(\rho(x, B_r) - \rho(y, B_r))^2$$

$$\leq \frac{1}{(R-r)^2} \sum_{x \in B_{R+s} \setminus B_{r-s}} |f(x)|^2 \sum_{y \in B_{R+s} \setminus B_{r-s}} b(x, y)\rho^2(x, y)$$

$$\leq \frac{1}{(R-r)^2} \sum_{x \in B_{R+s} \setminus B_{r-s}} |f(x)|^2 m(x)$$

$$= \frac{1}{(R-r)^2} \|f 1_{B_{R+s} \setminus B_{r-s}}\|^2.$$

For example consider a graph with standard weights and the counting measure. If ones constructs η with the combinatorial graph distance instead, this yields the vertex degree function $\deg(x)$ instead of $m(x) = 1$ in the fourth step. Hence, in order to do the final step one has to assume additionally that deg is bounded.

Finally, let us discuss the virtue of the assumptions of finite balls (B) and finite jump size (J) in light of these considerations:

If (B) holds, then $\eta_{B_r,R} f$ is in $C_c(X)$ which is for example useful to apply Green's formula.

Secondly, if (J) holds then $s < \infty$. So, at the end of the estimate above we only have the ℓ^2 norm of f on a ball rather than on the whole space.

4. LIOUVILLE TYPE THEOREMS

The classical Liouville theorem states that if a harmonic function is bounded from above, then the function is constant. Here, we look into boundedness assumptions such as ℓ^p growth bounds. First, we present Yau's L^p Liouville theorem and Karp's improved bound in the case of manifolds. Secondly, we discuss the case of the normalized Laplacian for graphs and the results that have been proven for this operator. Thirdly, we present theorems that recover Yau's and Karp's results for weighted graphs using intrinsic metrics. As a consequence, this yields a sufficient criterion for recurrence. Finally, we round off the section by a result which seems to hold exclusively for discrete spaces.

Throughout this section, keep in mind the fact that absence of non-constant positive subharmonic functions implies the absence of non-constant harmonic functions, Lemma 2.1.

4.1. **Yau's and Karp's theorem for manifolds.** We consider a connected Riemannian manifold M, together with its Laplace Beltrami operator Δ_M. A twice continuously differentiable function f on M is called harmonic (respectively subharmonic) if $\Delta_M f = 0$ (respectively $\Delta_M f \leq 0$).

In 1976 Yau [85] proved that on a complete Riemannian manifold M any harmonic function or positive subharmonic function in $L^p(M)$ is already constant.

This result was later strengthened by Karp in 1982, [50]. Namely, any harmonic function or positive subharmonic function f that satisfies

$$\inf_{r_0 > 0} \int_{r_0}^{\infty} \frac{1}{\|f 1_{B_r}\|_p^p} dr = \infty,$$

is already constant, where 1_{B_r} is the characteristic function of the geodesic ball B_r about some arbitrary point in the manifold. Karp's result has Yau's theorem as a direct consequence.

Later in 1994 Sturm generalized Karp's theorem to strongly local Dirichlet forms, where balls are taken with respect to the intrinsic metric. The underlying assumption on the metric is that it generates the original topology and all balls are relatively compact.

4.2. **Liouville theorems for normalized graph Laplacians.** For graphs b over (X, m) the first results in this direction were obtained for the normalizing measure $m = n$. In this case, the operator L is bounded, see Section 2.4, and the combinatorial graph distance d is an intrinsic metric, see Section 3.2.2. (Of course, the fact that a function is harmonic depends only on the graph b and not on the measure m, but being in an ℓ^p space does.)

Starting 1997 with Holopainen/Soardi [40], Rigoli/Salvatori/Vignati [70], Masamune [59], eventually in 2013 Hua/Jost [42] showed that if a harmonic or positive subharmonic function f satisfies

$$\liminf_{r \to \infty} \frac{1}{r^2} \|f 1_{B_r(x)}\|_p^p < \infty,$$

for some $p \in (1, \infty)$ and $x \in X$, then f must be constant. Here, the balls are taken with respect to the natural graph distance. This directly implies Yau's theorem for $p \in (1, \infty)$. Moreover, Hua/Jost [42] also show Yau's theorem for $p = 1$.

4.3. **Liouville theorems involving intrinsic metrics.** For graphs b over a general discrete measure space one can not expect such results to hold without any further conditions: Namely, being harmonic does not depend on the measure, so for any harmonic function f there is a measure m such that f is in $\ell^p(X, m)$, $p \in (1, \infty)$.

The following theorem for $p \in (1, \infty)$ is found in [43, Corollary 1.2] with the additional assumption of finite jump size (J). Below we sketch how to omit (J) based on an idea which was communicated by Huang

[47]. For $p = 2$, the theorem (without the assumption (J)) is found in [35] based on ideas developed in [43, 49] and [62], see also [63, 64].

Theorem 4.1 (Corollary 1.2 in [43] and [47]). *Let (b, c) be a connected graph over (X, m) and ρ be an intrinsic metric with bounded degree on balls (D). If $f \in \ell^p(X, m)$, $p \in (1, \infty)$, is a positive subharmonic function then f is constant.*

Sketch of the proof. The key element of the proof is a Caccioppoli inequality stating that there is $C > 0$ such that such that for any positive subharmonic function f and $0 < r < R$

$$\sum_{x,y \in B_r} b(x, y)(f(x) \vee f(y))^{p-2}|f(x) - f(y)|^2 \le \frac{C}{(R - r)^2}\|f\|_p^p.$$

Letting first $R \to \infty$ and afterwards $r \to \infty$ readily yields constancy of f.

The assumption (J) was used in [43, Lemma 3.1] only for Green's formula applied to $f \in \mathcal{F}$ such that $f_r = f 1_{B_r} \in \ell^p(X, m)$ and $g \in \ell^q(X, m)$, $\operatorname{supp} g \subseteq B_r$ with $1/p + 1/q = 1$. It was pointed out by Huang [47] that (D) is sufficient to show

$$\sum_{x \in X}(\mathcal{L}f_r)(x)g(x)m(x) = \frac{1}{2}\sum_{x,y \in B_r} b(x, y)(f_r(x) - f_r(y))\overline{(g(x) - g(y))},$$

since the term including vertices outside of B_r can be seen to be bounded. $\square$

The case ℓ^1 is more subtle in the general situation. In [43, Theorem 1.7] it was shown that for the stochastic complete graphs Yau's Liouville theorem remains true for $p = 1$ (see Section 6 for definition of stochastic completeness). Otherwise, there are counterexamples, see [43, Section 4].

Refining the Caccioppoli inequality from the proof of Theorem 4.1 above, [43, Lemma 3.1], and applying it recursively, we obtain a discrete version of Karp's theorem. However, for the recursive scheme we need the jump size to be finite (J).

Theorem 4.2 (Theorem 1.1 in [43]). *Let b be a connected graph over (X, m) and ρ be an intrinsic metric with bounded degree on balls (D) and finite jump size (J). If f is a positive subharmonic function such that for some $p \in (1, \infty)$ and $x \in X$*

$$\inf_{r_0 > 0} \int_{r_0}^{\infty} \frac{1}{\|f 1_{B_r(x)}\|_p^p} dr = \infty,$$

then f is constant, where 1_B is the characteristic function of a set $B \subseteq X$.

In particular, the theorem above implies the result of Hua/Jost [42] and even implies that a harmonic function f satisfying

$$\limsup_{r\to\infty} \frac{1}{r^2 \log r} \|f 1_{B_r(x)}\|_p^p < \infty,$$

for some $p \in (1, \infty)$, is constant.

4.4. **Recurrence.** As a direct consequence of Karp's theorem we get a sufficient criterion for recurrence of a graph. A connected graph b over X is called *recurrent* if for all m and some (all) $x, y \in X$, we have

$$\int_0^\infty e^{-tL} 1_{\{x\}}(y) dt = \infty$$

which is equivalent to absence of non-constant bounded subharmonic functions. For a collection of various equivalent statements of recurrence, see [43, Proposition 3.3] and references therein.

Similar analogous results to the criterion below are due to [50, Theorem 3.5] and [78, Theorem 3] which generalizes for example [16, Theorem 2.2], [70, Corollary B], [81, Lemma 3.12], [32, Corollary 1.4], [61, Theorem 1.2] on graphs.

Theorem 4.3 (Corollary 1.6 in [43])**.** *Let b be a connected graph over (X, m) and ρ be an intrinsic metric with bounded degree on balls (D) and finite jump size (J). If for some $x \in X$*

$$\int_1^\infty \frac{r}{m(B_r(x))} dr = \infty,$$

then the graph is recurrent.

4.5. **Graphs with measure bounded from below.** Above we discussed existence of certain solutions in terms of metric properties of the underlying graph. There is a result for graphs of a completely different flavor which seems to have no analogue in the non-discrete setting. The condition (A) below is about (X, m) only as a measure space and the combinatorial structure of the graph.

(A) $\sum_{n=1}^\infty m(x_n) = \infty$ for all infinite paths (x_n).

In particular, (A) is satisfied if

$$\inf_{x \in X} m(x) > 0$$

which for example holds if m is constant as in the case of the counting measure or if $m = \deg$.

This condition yields a result for absence of certain solutions in $\ell^p(X, m)$ which in contrast to the metric result above includes the case $p = 1$. Since the proof is rather short we include it here.

Theorem 4.4 (Lemma 3.2 in [52])**.** *Let (b, c) be a graph over (X, m) such that every infinite path has infinite measure (A). Then there are no positive subharmonic function in $\ell^p(X, m)$, $p \in [1, \infty)$.*

Proof. Let f be non-negative and subharmonic. Then, $\mathcal{L}f(x) \leq 0$ evaluated at some x gives, using $f \geq 0$,

$$f(x) \leq \frac{1}{\sum_{y \in X} b(x,y)} \sum_{y \in X} b(x,y)f(y)$$

Thus, whenever there is $x' \sim x$ with $f(x') < f(x)$ there must be $y \sim x$ such that $f(x) < f(y)$. Such x' and x exist whenever f is non-constant. Letting $x_0 = x$, $x_1 = y$ and proceeding inductively there is a sequence (x_n) of vertices such that $0 < f(x) < f(x_n) < f(x_{n+1})$, $n \geq 0$. Now, (A) implies that f is not in $\ell^p(X, m)$. On the other hand, if f is constant, then (A) again implies $f \equiv 0$. $\square$

It is an open problem to unify Theorem 4.1 and Theorem 4.4.

5. Domain of the generators and essential selfadjointness

In this section we address the question of identifying the domain of the generators L_p. Classically, the special case $p = 2$ received particular attention. Going back to investigations of Friedrichs and von Neumann a classical question is whether a symmetric operator on a Hilbert space has a unique selfadjoint extension. This property is often studied under the name essential selfadjointness.

We say a symmetric operator on a dense subspace of a Hilbert space is *essentially selfadjoint* if it has a unique selfadjoint extension. Moreover, we say the operator has a *unique Markovian extension* if there is a unique selfadjoint extension such that the corresponding semigroup is Markovian. Clearly, essential selfadjointness implies uniqueness of Markovian extensions.

A sufficient criterion for essential selfadjointness for positive symmetric operators on ℓ^2 is the absence of solutions for $\lambda < 0$ in ℓ^2 to the equation $\mathcal{L}u = \lambda u$, see [52, Proof of Theorem 6].

The connection of essential selfadjointness to metric completeness for the Laplace Beltrami operator on Riemannian manifolds is that if there exists a boundary one might have to choose suitable 'boundary conditions' in order to obtain a selfadjoint operator.

We first discuss the manifold case which is often referred to Gaffney's theorem. Secondly, we consider weighted graph and recover Gaffney's theorem by the virtue of intrinsic metrics. Furthermore, we determine the domain of the generators on ℓ^p.

5.1. **Gaffney's theorem for manifolds.** We first discuss the question for the Laplace Beltrami operator on a Riemannian manifold.

A result going back to the work of Gaffney [26, 27] essentially states that on a geodesically complete manifold the so called Gaffney Laplacian is essentially selfadjoint which is equivalent to the uniqueness of

Markovian extensions of the minimal Laplacian. Independently, essential selfadjointness of the Laplace Beltrami operator the compactly supported infinitely often differentiable functions was shown by Roelcke, [71]. For later results in this direction see also [11, 76].

5.2. **Domain of the generators for graphs and intrinsic metrics.** The first results connecting metric completeness and uniqueness of selfadjoint extensions were obtained by Colin de Verdière/Torki-Hamza/Truc [12] and Milatovic [62, 63]. These results were proven for (magnetic) Schrödinger operators on graphs with bounded combinatorial vertex degree and the metric δ discussed in Section 3.2.5. As discussed there, δ is equivalent to an intrinsic metric if and only if the combinatorial vertex degree is bounded. Here, we consider intrinsic metrics to deal with the general case of unbounded vertex degree.

In Lemma 2.5 we demonstrated that for a graph we may not have $\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$. Hence, in general $\mathcal{L}$ is not a symmetric operator on the subspace $C_c(X)$ of $\ell^2(X, m)$. Nevertheless, we can still identify the domain of the generator of the form. The following result is found in [49] for graph Laplacians and in [35] for magnetic Schrödinger operators. We only sketch the idea of the proof which follows essentially from Theorem 4.1 above and standard arguments found in [52, Proof of Theorem 5 and 6]. One may find a version of the theorem below in [43, Corollary 1.4].

Theorem 5.1. *Let (b, c) be a graph over (X, m) and ρ be an intrinsic metric with bounded degree on balls (D). Then,*

$$D(L_p) = \{f \in \ell^p(X, m) \mid \mathcal{L}f \in \ell^p(X, m)\}, \quad \text{for all } p \in (1, \infty).$$

If additionally $\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$, then $\mathcal{L}|_{C_c(X)}$ is essentially selfadjoint on $\ell^2(X, m)$.

Combining this with the Hopf-Rinow type theorem, Theorem 3.4, we obtain an analogue to a classical result in Riemannian geometry which is often referred to as Gaffney's theorem.

Corollary 5.2 (Theorem 2 in [49])**.** *Let b be a locally finite graph over (X, m) and ρ be an intrinsic path metric. If (X, ρ) is metrically complete, then $\mathcal{L}|_{C_c(X)}$ is essentially selfadjoint on $\ell^2(X, m)$.*

5.3. **Graphs with measure bounded from below.** Viewing the graph as a measure space we obtain the following result as consequence of Theorem 4.4.

Theorem 5.3 (Theorem 5 in [52])**.** *Let (b, c) be a graph over (X, m) such that every infinite path has infinite measure (A). Then,*

$$D(L_p) = \{f \in \ell^p(X, m) \mid \mathcal{L}f \in \ell^p(X, m)\} \quad \text{for all } p \in [1, \infty).$$

If additionally $\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$, then $\mathcal{L}|_{C_c(X)}$ is essentially selfadjoint on $\ell^2(X, m)$.

6. Stochastic completeness

In the sections above we excluded the case of solutions in ℓ^∞. Indeed, this case is more subtle and the topic of this section. Specifically, we study is stochastic completeness, a property which is also referred to as conservativeness, honesty or non-explosion.

There is a huge body of literature from various mathematical fields and for references we restrict ourselves a very small selection. For stochastic completeness in the context of discrete Markov processes there is work by Feller [17, 18] and [69] in the late 50's, for manifolds there is work going back to Azencott [3] and Grigor'yan [29, 30]. For positive contraction semigroups we mention Arlotti/Banasiak [1] and Mokhtar-Kharroubi/Voigt [65] and for strongly local Dirichlet forms there is for example work by Sturm [78].

Below we discuss a volume growth bound for manifolds that implies stochastic completeness due to Grigor'yan [29, 30]. Afterwards, we discuss stochastic completeness for graphs and show how Grigor'yan's bound fails when one considers the combinatorial graph metric. Finally, we present results that recover the result for manifolds for weighted graphs with intrinsic metrics due to Folz [20] and Huang [45].

6.1. **Grigor'yan's theorem for manifolds.** For a Riemannian manifold M stochastic completeness is defined by the property that the diffusion semigroup of the Laplace-Beltrami operator leaves the constant function 1 invariant. This can be seen to be equivalent to uniqueness of bounded solutions for the heat equation, [30, Theorem 6.2]. In 1986 Grigor'yan [29] proved a theorem which gives a sufficient condition for stochastic completeness in terms of volume growth for connected Riemannian manifolds. In particular, this theorem states that if

$$\inf_{r_0>0} \int_{r_0}^\infty \frac{r}{\log(\mathrm{vol}(B_r(x)))} dr = \infty,$$

for some $x \in M$, then the manifold is stochastically complete. In particular, this implies stochastic completeness for manifolds whose volume growth is less than e^{r^2}. Grigor'yan's theorem was later generalized to strongly local regular Dirichlet forms by Sturm [78].

6.2. **Characterization of stochastic completeness for graphs.** Next, we discuss the notion of stochastic completeness for graphs in more detail. We start with a characterization linking stochastic completeness to solutions in ℓ^∞.

Variants of the next proposition can be found in [17, 18, 69] in the context of discrete Markov processes, in [82] for graphs with standard weights and in [52, Theorem 1] for weighted graphs (which is given there even for non vanishing killing term c).

Proposition 6.1. *Let b be a connected graph over (X, m). The follow-ing are equivalent:*

(i) *$e^{-tL}1 = 1$ for some (all) $t > 0$.*

(ii) *For any (some) $\lambda < 0$, there is no non-trivial $u \in \ell^\infty(X)$ such that*

$$\mathcal{L}u = \lambda u.$$

(iii) *For any (some) $f \in \ell^\infty(X)$ there is a unique solution $[0, \infty) \to \ell^\infty(X)$, $t \mapsto u_t$ to*

$$-\mathcal{L}u = \partial_t u, \quad u_0 = f.$$

We call a graph b over (X, m) *stochastically complete* if one of the equivalences of the proposition above is satisfied.

There is a physical interpretation to (i) of the above theorem. This concerns the question whether heat leaves the graph in finite time. Assume the graph is not stochastically complete, i.e., $e^{-tL}1 < 1$ for some $t > 0$. Let $0 \le f \in \ell^1(X, m)$ model the distribution of heat in the graph at time $t = 0$. Then, the distribution of heat at time $t > 0$ is given by $e^{-tL}f$ and the amount of heat in the graph at time $t > 0$ is given by

$$\sum_{x \in X} e^{-tL} f(x) m(x) = \langle e^{-tL} f, 1 \rangle = \langle f, e^{-tL} 1 \rangle < \langle f, 1 \rangle = \sum_{x \in X} f(x) m(x),$$

where the right hand side is the amount of heat in the graph at time $t = 0$. Hence, the amount of heat in the graph decreases in time, if the graph is not stochastically complete.

6.3. The combinatorial graph distance and polynomial growth.
Consider the combinatorial graph distance on a graph with standard weights with the counting measure. Wojciechowski discovered in his PhD Thesis [82] that spherically symmetric trees, whose branching numbers grow more than linearly, are not stochastically complete. This counts for a volume growth of $r! \sim e^{r \log r}$ with respect to the com-binatorial graph distance. Later, Wojciechowski [84, Example 4.11] gave even examples of stochastically complete graphs with polynomial volume growth. These considerations were generalized for weighted weakly spherically symmetric graphs in [55].

The examples Wojciechowski considered are so called *anti-trees*. Specif-ically, anti-trees are highly connected graphs that are characterized by the property that a vertex in a sphere (with respect to a root vertex) is connected to every neighbor in the succeeding sphere, see Figure 1 below for an example.

So, for an anti-tree let s_r be the number of vertices with combinatorial graph distance r to a root vertex and let $v_r = s_0 + \ldots + s_r$, $r \ge 0$.
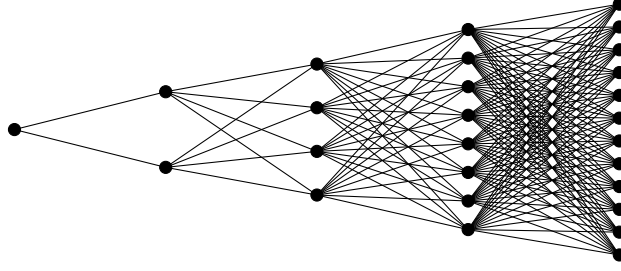
FIGURE 1. An anti-tree with $s_{r+1} = 2^r$

Then, Wojciechowski shows that the anti-tree with standard weights is stochastically complete if and only if

$$\sum_{r=0}^{\infty} \frac{v_r}{s_r s_{r+1}} = \infty$$

which results in a threshold of $v_r \sim r^3$. Hence, there are graphs with only little more than cubic volume growth that are not stochastically complete. This stands in clear contrast to Grigor'yan's volume growth bound for manifolds which was about e^{r^2}. Thus, the example clearly shows that the combinatorial graph distance is the wrong candidate to obtain an analogue of Grigor'yan's result.

Moreover, it was shown later in [34] that the threshold of r^3 is sharp for the combinatorial graph distance.

6.4. **Grigor'yan's result for graphs with intrinsic metrics.** The following theorem was proven by Folz [20] and later a simplified proof was given by Huang [45]. Before, a first step in this direction was taken by Masamune/Uemura [60] and Grigor'yan/Huang/Masamune [34].

Theorem 6.2 (Theorem 1 of [20]). *Let b be a locally finite, connected graph over (X, m) and ρ be an intrinsic metric with finite balls (B) and finite jump size (J). If for some $x \in X$*

$$\inf_{r_0 > 0} \int_{r_0}^{\infty} \frac{r}{\log(m(B_r(x)))} dr = \infty,$$

then the graph is stochastically complete.

The assumption of finite jump size and the assumption that the metric ρ is intrinsic were weakened by Huang [45] to so called weakly adapted metrics that is a metric ρ for which there is $c > 0$ such that

$$\sum_{y \in X} b(x, y)(\rho(x, y) \wedge c)^2 \leq m(x), \quad x \in X.$$

The approach of the proof uses quantum graphs. These are graphs were the edges are real intervals which are glued at the vertices. For some background see [56] and also Barlow/Bass [4] who already used a similar idea considering cable systems. The corresponding Laplace

operator operates as $-\frac{d^2}{dx^2}$ on the intervals and certain boundary conditions are imposed such that it becomes a selfadjoint operator associated to a Dirichlet form. This Dirichlet form is strongly local and, thus, Grigor'yan's result holds by the virtue of [78]. In order to compare the diffusion on the quantum graph and the discrete graphs one can add a certain number of loops to each vertex [20]. Alternatively, one can model the time change by appropriate Dirac measures on the vertices [45].

7. Upper escape rate

Khintchine's law of the iterated logarithm states for the Brownian motion $(B_t)_{t \geq 0}$ on $\mathbb{R}$,

$$\limsup_{t \to \infty} \frac{|B_t|}{\sqrt{2t \log \log t}} = 1,$$

almost surely. In particular, for the probability conditioned on the Brownian motion starting at zero, we have

$$\mathbb{P}_0\Big(|B_t| \leq R(t) = \sqrt{(2 + \varepsilon)t \log \log t} \text{ for all sufficiently large } t\Big) = 1.$$

for arbitrary $\varepsilon > 0$. Such a function R is called an *upper escape rate function*. Below we discuss first results on manifolds in this direction by Grigor'yan, Hsu and Qin before we get to weighted graphs and intrinsic metrics.

7.1. **Upper escape rate functions for manifolds.** For a complete connected Riemannian manifold M which satisfies

$$\inf_{r_0 > 0} \int_{r_0}^{\infty} \frac{r}{\log(\mathrm{vol}(B_r(x)))} dr = \infty,$$

for some $x \in M$, we have seen in the previous section that M is stochastically complete. Imposing this assumption the result of Hsu/Qin [41] gives for the Brownian motion $(B_t)_{t \geq 0}$ on M and the probability $\mathbb{P}_x$ conditioned on the process to start in $x \in M$,

$$\mathbb{P}_x\Big(|B_t| \leq R(t) \text{ for all sufficiently large } t\Big) = 1,$$

where the upper escape rate function R is a multiple of the inverse function of

$$t \mapsto \int_6^t \frac{r}{\log \mathrm{vol}(B_r) + \log \log r} dr.$$

Earlier results in this direction are found in [31] and [33]. For strongly local Dirichlet forms this result was recently generalized by Ouyang, [67].

7.2. **Upper escape rate functions for weighted graphs.** For a graph b over (X, m), let (X_t) be Markov process with generator L. The process is characterized by the formula

$$e^{-tL} f(x) = \mathbb{E}_x(f(X_t)), \qquad t > 0, \ f \in C_c(X).$$

The following theorem on the upper escape rate is found in [48] which improves the results of [46].

Theorem 7.1 (Theorem 1.7 of [48]). *Let b be a locally finite connected graph over (X, m) such that $\inf_{x \in X} m(x) > 0$ and ρ be an intrinsic metric with finite balls (B) and finite jump size (J). If for some $x \in X$*

$$\inf_{r_0 > 0} \int_{r_0}^{\infty} \frac{r}{\log(m(B_r(x)))} dr = \infty,$$

then

$$\mathbb{P}_x\Big(|X_t| \leq R(t) \text{ for all sufficiently large } t \Big) = 1,$$

where the upper escape rate function R is a multiple of the inverse function of

$$t \mapsto \int_{t_0}^{t} \frac{r}{\log \operatorname{vol}(B_r) + \log \log r} dr.$$

for some given t_0.

Note that the proof found in [48] also gives an alternative argument to show stochastic completeness under the volume growth condition above.

It can be shown that the escape rate is sharp for anti-trees. Moreover, for weakly spherically symmetric graphs there are even more precise results, confer [46, Section 6].

8. Subexponentially bounded solutions and the spectrum

So far, we considered solutions for $\lambda < 0$. In this section we turn to solutions for $\lambda \geq 0$ to study the spectrum $\sigma(L)$ of the operator L which is included in $[0, \infty)$ since L is positive.

The meta-theorem behind this is often referred to as a Shnol' type theorem and it states that λ is in the spectrum if there is a subexponentially bounded solution for λ. The classical Shnol' theorem, see [73], deals with the Schrödinger equation in $\mathbb{R}^d$. It was rediscovered by Simon in [74, 75] and it was proven in [7] for strongly local Dirichlet forms by Boutet de Monvel/Lenz/Stollmann.

A function $f \in C(X)$ on a connected graph is said to be *subexponentially bounded* with respect to an intrinsic metric ρ if for all $\alpha > 0$ and some (all) $x \in X$

$$e^{-\alpha \rho(x, \cdot)} f \in \ell^2(X, m).$$

For bounded Laplacians on graphs a Shnol' type theorem was proven in [36]. Using intrinsic metrics we get a general version of this result. Such a result can be derived from a Shnol' inequality [22, Corollary 12.2] combined with a one-dimensional estimate. As it is stated here the theorem can be found in [51, Theorem 19].

Theorem 8.1 (Corollary 12.2 in [22]). *Let (b, c) be a connected graph over (X, m) and ρ be an intrinsic metric with finite balls (B) and finite jump size (J). If for some $\lambda \in \mathbb{R}$ there is a non-trivial subexponentially bounded solution, then $\lambda \in \sigma(L)$.*

The proof is based again on a Caccioppoli type inequality.

A basic corollary which we improve later in Section 10 is the following.

Corollary 8.2. *Let b be a connected graph over (X, m) and ρ be an intrinsic metric with finite balls (B) and finite jump size (J). If*

$$\limsup_{r \to \infty} \frac{1}{r} \log m(B_r(x)) \leq 0,$$

for some $x \in X$, then $\inf \sigma(L) = 0$.

Proof. Under the assumption above the constant functions which are solutions for $\lambda = 0$ are subexponentially bounded. Hence, the statement follows from the theorem above. $\square$

9. Isoperimetric constants and lower spectral bounds

In this section we aim for lower bounds on the bottom of the spectrum

$$\lambda_0(L) = \inf \sigma(L)$$

via so called isoperimetric estimates. Such estimates are often referred to as Cheeger's inequality.

We first discuss the result on manifolds going back to Cheeger from 1960. Then, we discuss how an analogous result was proven in the 80's for the normalized Laplacian by Dodziuk/Kendall and what problems occur for the operator Δ. Finally, we examine how intrinsic metrics can be used to establish this inequality for general graph Laplacians.

9.1. **Cheeger estimates for manifolds.** For a non-compact Riemannian manifold M the isoperimetric constant or Cheeger constant is defined as

$$h_M = \inf_S \frac{\text{Area}(\partial S)}{\text{vol(int}(S))},$$

where S runs over all hypersurfaces cutting M into a precompact piece int(S) and an unbounded piece. Denote by $\lambda_0(\Delta_M)$ the bottom of the

spectrum of the Laplace-Beltrami. The well known Cheeger inequality reads as

$$\lambda_0(\Delta_M) \geq \frac{h_M^2}{4}.$$

See [10] for Cheeger's original work on the compact case and [9] for a discussion of the non-compact case.

9.2. **Cheeger estimates for graphs with standard weights.** There is an enormous amount of literature on isoperimetric inequalities especially for finite graphs. Here, we restrict ourselves to infinite graphs and only mention [2] for finite graphs as one of the first papers.

The *boundary* of a set $W \subseteq X$ is defined as the set of edges emanating from W, i.e.,

$$\partial W = \{(x, y) \in W \times X \setminus W \mid x \sim y\}$$

In 1984 Dodziuk, [13], considered graphs with standard weights and the counting measure. The isoperimetric constant he studied is closely related to

$$h_1 = \inf_{W \subseteq X \text{ finite}} \frac{|\partial W|}{|W|}$$

and Dodziuk's proof yields

$$\lambda_0(\Delta) \geq \frac{h_1^2}{2D},$$

with $D = \sup_{x \in X} \deg(x)$. This analogue of Cheeger's inequality is effective for graphs with bounded vertex degree. However, for unbounded vertex degree the bound becomes trivial. The following example illustrates how this bound can be seen to be non optimal.

Example 9.1. Let T_k, $k \geq 2$, be the k-regular rooted tree with standard weights. We glue T_k to T_{k^4} at the root and denote the resulting graph by G_k. We find that the infimum of the bottom of the spectrum of Δ_{G_k} on G_k is assumed at $\lambda_0(\Delta_{G_k}) = \lambda_0(\Delta_{T_k}) = ((k+1) - 2\sqrt{k})$. In particular, $\lambda_0(\Delta_{G_k}) \to \infty$ for $k \to \infty$. The constant $h_1(G_k)$ is assumed by considering larger and larger balls on T_k, i.e., $h_1(G_k) = h_1(T_k) = k$. Finally, $D = k^4$. Hence, the bound in Dodziuk's inequality is $k^2/2(k+1)^4 \to 0$, $k \to \infty$. In summary, $\lambda_0 \to \infty$ while $h_1^2/2D \to 0$ as $k \to \infty$.

Two years later Dodziuk and Kendall [16] proposed a solution to this issue by considering graphs with standard weights and the normalizing measure $n = \deg$ instead. The corresponding isoperimetric constant is

$$h_n = \inf_{W \subseteq X \text{ finite}} \frac{|\partial W|}{\deg(W)}$$

and they proved in [16] for the normalized Laplacian Δ_n

$$\lambda_0(\Delta_n) \geq \frac{h_n^2}{2}.$$

This analogue of Cheeger's inequality does not have the disadvantage illustrated in the example above. It seems that from this point on the operator Δ was rather neglected in the spectral geometry of graphs and the normalized Laplacian Δ_n gained momentum.

This estimate can be generalized to arbitrary edge weights b. Define the area of the boundary of a finite set $W \subseteq X$ by

$$b(\partial W) = \sum_{(x,y) \in \partial W} b(x,y)$$

and for the normalizing measure n with $n(W) = 2 \sum_{x,y \in W} b(x,y)$ the normalized Cheeger constant is given by

$$\alpha_n = \inf_{W \subseteq X \text{ finite}} \frac{b(\partial W)}{n(W)}.$$

With further improvements to Dodziuk/Kendall's bound found in [24, 53] one has for the operator L associated to the graph b over (X, n)

$$\lambda_0(L) \geq 1 - \sqrt{1 - \alpha_n^2},$$

where the left hand side can be seen to be larger than $\alpha_n^2/2$ by the Taylor expansion of the square root.

9.3. **Cheeger estimates involving intrinsic metrics.** The considerations in the previous sections suggest that intrinsic metrics allow to prove the analogous results for general graph Laplacians. So, the question is where the metric actually appears in the definition of the isoperimetric constant.

Revisiting the definition of the area of the boundary of the set W above we find that

$$b(\partial W) = \sum_{(x,y) \in \partial W} b(x,y) = \sum_{(x,y) \in \partial W} b(x,y)d(x,y)$$

with the combinatorial graph distance d on the right hand side. Remember that d is an intrinsic metric for the graph b over (X, n).

Hence, replacing d by an intrinsic metric ρ for a graph b over (X, m) we define

$$\text{Area}(\partial W) = \sum_{(x,y) \in \partial W} b(x,y)\rho(x,y).$$

That is we take the length of an edge into consideration to measure the area of the boundary. We define

$$\alpha = \inf_{W \subseteq X \text{ finite}} \frac{\text{Area}(\partial W)}{m(W)},$$

and obtain the following theorem which is found in [6].

Theorem 9.2 (Theorem 1 in [6]). *Let b be a graph over (X, m) and let ρ be an intrinsic metric. Then,*

$$\lambda_0(L) \geq \frac{\alpha^2}{2}.$$

The proof of the theorem is based on an area and a co-area formula. For $f \geq 0$ let

$$\Omega_t = \{x \in X \mid f(x) > t\}.$$

Then one can prove using Fubini's theorem for $f \in C_c(X)$

$$m(\Omega_t) = \sum_{x \in X} f(x) m(x)$$

$$\text{Area}(\partial \Omega_t) = \sum_{x,y \in X} b(x, y) \rho(x, y) |f(x) - f(y)|.$$

The proof is then basically Cauchy-Schwarz inequality and various algebraic manipulations.

One may also include potentials $c \geq 0$ in the estimate by introducing edges from vertices x with $c(x) > 0$ to virtual sibling vertices $\dot{x}$ with edge weight $b(x, \dot{x}) = c(x)$. The union of vertices $x \in X$ and $\dot{x}$ is denoted by $\dot{X}$. Furthermore, we extend an intrinsic metric ρ on X to the new edges via

$$\rho(x, \dot{x}) = \frac{(m(x) - \sum_{y \in X} b(x, y) \rho(x, y)^2)^{\frac{1}{2}}}{c(x)}.$$

The extension of ρ becomes an intrinsic metric when choosing $m(\dot{x}) = m(x)$. Now, we define α by taking the infimum of the quotient with the extension of b and ρ but as above only over subsets of X.

10. Exponential volume growth and upper spectral bounds

In this section, we discuss upper bounds for the bottom of the essential spectrum

$$\lambda_0^{\text{ess}}(L) = \inf \sigma_{\text{ess}}(L).$$

The essential spectrum of an operator is the part of the spectrum which does not include discrete eigenvalues of finite multiplicity. Clearly, $\lambda_0(L) \leq \lambda_0^{\text{ess}}(L)$.

We discuss the classical result on Riemannian manifolds going back to Brooks first. Then we present a corresponding result for the normalized Laplacian and show how the result fails in the case of Laplacian with respect to the counting measure. Finally, we employ intrinsic metrics to recover Brooks' result for general graph Laplacians.

10.1. **Brooks' theorem for manifolds.** Let M be a complete connected non-compact Riemannian manifold with infinite volume. Let $\lambda_0^{\text{ess}}(\Delta_M)$ be the bottom of the essential spectrum of the Laplace Beltrami operator Δ_M. Let $\overline{\mu}_M$ be the upper exponential growth rate of the distance balls

$$\overline{\mu}_M = \limsup_{r \to \infty} \frac{1}{r} \log \text{vol}(B_r(x)),$$

for an arbitrary $x \in M$. Brooks showed in 1981, [8],

$$\lambda_0^{\text{ess}}(\Delta_M) \leq \frac{\overline{\mu}_M^2}{4}.$$

Later in 1996 Sturm, [78], showed using the lower exponential growth rate of the distance balls with variable center

$$\underline{\mu}_M = \liminf_{r \to \infty} \inf_{x \in M} \frac{1}{r} \log \text{vol}(B_r(x))$$

the following bound

$$\lambda_0(\Delta_M) \leq \frac{\underline{\mu}_M^2}{4}.$$

Indeed, this result in [78] is proven for strongly local regular Dirichlet forms.

An immediate corollary is that 0 is in the spectrum of the Laplace Beltrami operator for M with subexponential growth, i.e., $\underline{\mu}_M = 0$.

10.2. **Brooks' theorem for graphs with standard weights.** For graphs with standard weights and the normalizing measure Dodziuk/ Karp [15] proved in 1987 the first analogue of Brooks' theorem for graphs. This result was later improved by Ohno/Urakawa [66] and Fujiwara [23] resulting in the estimate

$$\lambda_0^{\text{ess}}(\Delta_n) \leq 1 - \frac{2e^{\mu_n/2}}{e^{\mu_n} + 1}$$

with

$$\mu_n = \limsup_{r \to \infty} \frac{1}{r} \log n(B_r(x)),$$

for arbitrary $x \in X$ and $n = \deg$. It can be seen that the bound above is smaller than $\mu_n^2/8$.

Next, we discuss how for graphs with standard weights and the counting measure such a bound fails when volume growth is determined by the combinatorial graph distance. Again, we consider anti-trees which were introduced in Section 6.3. In [55] it was shown that

$$a = \Big(\sum_{r=0}^{\infty} \frac{v_r}{s_r s_{r+1}} \Big)^{-1}$$

is a lower bound on the spectrum of Δ on the anti-tree with s_r vertices in the r-th sphere and $v_r = s_0 + \ldots + s_r$, $r \geq 0$, (where $a = 0$ if

the sum diverges). Moreover, in the case where the sum converges the spectrum of Δ is purely discrete, i.e., there is no essential spectrum. In particular, this implies that anti-trees with

$$s_r \sim r^{2+\varepsilon}, \qquad \varepsilon > 0,$$

have positive bottom of the spectrum and no essential spectrum. However, for $s_r \sim r^{2+\varepsilon}$ we have $v_r \sim r^{3+\varepsilon}$, that is these are graphs of little more than cubic growth with positive bottom of the spectrum and no essential spectrum. Hence, there is no analogue to Brooks' or Sturm's theorem for Δ with respect to the combinatorial graph distance.

10.3. **Brooks' theorem involving intrinsic metrics.** Let b be a graph over (X, m) and ρ be an intrinsic metric. Let $B_r(x)$ be the distance r ball about a vertex x with respect to the metric ρ. We define

$$\mu = \liminf_{r \to \infty} \frac{1}{r} \log m(B_r(x)),$$

for fixed $x \in X$ and

$$\underline{\mu} = \liminf_{r \to \infty} \inf_{x \in X} \frac{1}{r} \log m(B_r(x)).$$

In [38] analogues of Brooks' and Sturm's theorem for regular Dirichlet forms are proven. As a special case the following theorem is obtained. Under somewhat stronger assumptions the estimate for the essential spectrum was also obtained independently by Folz [21].

Theorem 10.1 (Corollary 4.2 in [38]). *Let b be a connected graph over (X, m) and ρ be an intrinsic metric such that the balls are finite (B). Then,*

$$\lambda_0(L) \leq \frac{\mu^2}{8}.$$

If furthermore $m(X) = \infty$, then

$$\lambda_0^{\mathrm{ess}}(L) \leq \frac{\mu^2}{8}.$$

We indicate the idea of the proof.

Idea of the proof. Let $\overline{\mu} = \limsup_{r \to \infty} \frac{1}{r} \log m(B_r(x))$. Then, the functions $f_a = e^{-a\rho(o, \cdot)}$ for $a > \overline{\mu}/2$ and fixed o are in $\ell^2(X, m)$. Moreover, by the mean value theorem and an estimate as in Section 3.5 we find that

$$\mathcal{Q}(f_a) \leq \frac{a^2}{2} \sum_{x \in X} |f_a(x)|^2 \sum_{y \in X} b(x, y) \rho(x, y)^2 \leq \frac{a^2}{2} \|f_a\|^2$$

To pass from $\overline{\mu}$ to μ or $\underline{\mu}$ we consider

$$g_{a,r} = (e^{2ar} f_a - 1) \vee 0.$$

Note that $g_{a,r}$ is supported on B_{2r} and, therefore, $g_{a,r}$ is in $C_c(X)$ whenever (B) applies. Finally, to see the statement for the essential spectrum we need to modify $g_{a,r}$ such that we obtain a sequence of functions that converge weakly to zero. We achieve this by cutting off $g_{a,r}$ at 1 on B_r, i.e.,

$$h_{a,r} = 1 \wedge g_{a,r}.$$

The weak convergence of $h_{a,r}$ to zero is ensured by the assumption $m(X) = \infty$. Now, the statement follows by a Persson-type theorem, [38, Proposition 2.1]. $\qquad\square$

Let us end this section with a few remarks.

In [38] it is also shown that the assumption (B) can be replaced by (A).

As a corollary we get under the assumption of the theorem $2\alpha \leq \mu$ for the Cheeger constant α of Section 9.3.

By comparing the degree path metric ρ_0 with the combinatorial graph distance d on anti-trees one finds that for $s_r \sim r^{2-\varepsilon}$ the balls with respect to ρ_0 grow polynomially, for $s_r \sim r^2$ they grow exponentially and for $s_r \sim r^{2+\varepsilon}$ the graph has finite diameter with respect to ρ_0. This shows that the examples in the section above are indeed sharp.

11. Uniform subexponential growth and p-independence of the spectrum

In the beginning of the 80's Simon [75] asked a famous question whether the spectra of certain Schrödinger operators on $\mathbb{R}^d$ are independent on which L^p space they are considered. Hempel/Voigt [39] gave an affirmative answer in 1986. Here, we consider a geometric analogue of this question. First, we discuss a theorem by Sturm on Riemannian manifolds, [77], and secondly, we present a result for weighted graphs involving intrinsic metrics.

11.1. p-independence for manifolds.
In 1993 Sturm [77] proved a theorem for uniformly elliptic operators on a complete Riemannian manifold M whose Ricci curvature is bounded below. We assume that M grows uniformly subexponentially if for any $\varepsilon > 0$ there is $C > 0$ such that for all $r > 0$ and all $x \in M$

$$\mathrm{vol}(B_r(x)) \leq Ce^{\varepsilon r}\mathrm{vol}(B_1(x)).$$

Then the spectrum of a uniformly elliptic operator on this manifold is independent of the space $L^p(M)$, $p \in [1, \infty]$ on which it is considered.

11.2. p-independence for weighted graphs with intrinsic metrics.
A graph (b, c) over (X, m) with an intrinsic metric ρ is said to

have *uniform subexponential growth* if for any $\varepsilon > 0$ there is $C > 0$ such that for all $r > 0$ and all $x \in M$

$$m(B_r(x)) \leq Ce^{\varepsilon r}m(x).$$

The proof of the following theorem follows closely the strategy of Sturm in [77].

Theorem 11.1 (Theorem 1 in [5])**.** *Let b be a connected graph over (X, m) and ρ be an intrinsic metric such that the balls are finite (B), which has finite jump size (J) and the graph grows uniformly subexponentially. Then,*

$$\sigma(L_p) = \sigma(L_2), \qquad p \in [1, \infty].$$

The statement of the theorem is in general wrong if one drops the growth assumption. In particular, if $\lambda_0(L_2) > 0$, then the graph grows exponentially, i.e., $\mu > 0$ by the section above. On the other hand, if the graph is stochastically complete, (i.e., $e^{-tL}1 = 1$, see Section 6.3), then 1 is an eigenfunction of L_∞ to the eigenvalue 0 and, therefore, $\lambda_0(L_1) = 0$ by duality. Hence, $\lambda_0(L_1) < \lambda_0(L_2)$.

On the other hand, it is an open question what happens for graphs that are subexponentially growing, i.e., $\mu = 0$, but not uniformly subexponentially growing.

REFERENCES

[1] L. Arlotti, J. Banasiak, Strictly substochastic semigroups with application to conservative and shattering solutions to fragmentation equations with mass loss, J. Math. Anal. Appl. **293** (2004), no. 2, 693–720.

[2] N. Alon, V. D. Milman, λ_1, isoperimetric inequalities for graphs, and super-concentrators, J. Combin. Theory Ser. B **38** (1985), no. 1, 73–88.

[3] R. Azencott, Behavior of diffusion semi-groups at infinity, Bull. Soc. Math. France **102** (1974), 193–240.

[4] M. T. Barlow, R. F. Bass. Stability of parabolic Harnack inequalities. Trans. Am. Math. Soc. **356** (2004), 1501–1533.

[5] F. Bauer, B. Hua, M. Keller, On the l^p spectrum of Laplacians on graphs, Advances in Mathematics **248** (2013), Issue 25, 717–735.

[6] F. Bauer, M. Keller, R. Wojciechowski, Cheeger inequalities for unbounded graph Laplacians, to appear in: Journal of the European Mathematical Society.

[7] A. Boutet de Monvel, D. Lenz, and P. Stollmann. Schnol's theorem for strongly local forms, Israel J. Math. **173** (2009), 189–211.

[8] R. Brooks, A relation between growth and the spectrum of the Laplacian, Math. Z. **178** (1981), no. 4, 501–508.

[9] R. Brooks, Spectral geometry and the Cheeger constant, Expanding Graphs, Proc. DIMACS Workshop, Amer. Math. Soc. (1993), 5–19.

[10] J. Cheeger, A lower bound for the smallest eigenvalue of the Laplacian, Problems in Analysis (Papers dedicated to Salomon Bochner, 1969), Princeton Univ. Press, Princeton, N. J., 1970, 195–199.

[11] P. R. Chernoff, Essential self-adjointness of powers of generators of hyperbolic equations, J. Funct. Anal. **12** (1973), 401–414.

[12] Y. Colin de Verdière, N. Torki-Hamza, F. Truc, Essential self-adjointness for combinatorial Schrödinger operators II–metrically non complete graphs, Math. Phys. Anal. Geom. **14** (2011), no. 1, 21–38.

[13] J. Dodziuk, Difference equations, isoperimetric inequality and transience of certain random walks, Trans. Amer. Math. Soc. **284** (1984), no. 2, 787–794.

[14] J. Dodziuk, Elliptic operators on infinite graphs, Analysis, geometry and topology of elliptic operators, World Sci. Publ., Hackensack, NJ, 2006, 353–368.

[15] J. Dodziuk, L. Karp, Spectral and function theory for combinatorial Laplacians, Geometry of random motion (Ithaca, N.Y., 1987), Contemp. Math. **73**, Amer. Math. Soc., Providence, RI, 1988, 25–40.

[16] J. Dodziuk, W. S. Kendall, Combinatorial Laplacians and isoperimetric inequality, From local times to global geometry, control and physics, Pitman Res. Notes Math. Ser. **150**, Longman Sci. Tech., Harlow, 1986, 68–74.

[17] W. Feller, On boundaries and lateral conditions for the Kolmogorov differential equations. Ann. of Math. (2) **65** (1957), 527–570

[18] W. Feller, Notes to my paper "On boundaries and lateral conditions for the Kolmogorov differential equations." Ann. of Math. (2) **68** (1958), 735–736.

[19] M. Folz, Gaussian upper bounds for heat kernels of continuous time simple random walks, Electron. J. Probab. **16** (2011), no. 62, 1693–1722.

[20] M. Folz, Volume growth and stochastic completeness of graphs, Trans. Amer. Math. Soc. **366** (2014), no. 4, 2089–2119.

[21] M. Folz, Volume growth and spectrum for general graph Laplacians, Math. Z. **276** (2014), Issue 1-2, 115–131.

[22] L. Frank, D. Lenz, D. Wingert, Intrinsic metrics for non-local symmetric Dirichlet forms and applications to spectral theory, J. Funct. Anal **266** (2014), no. 8, 4765–4808.

[23] K. Fujiwara, Growth and the spectrum of the Laplacian of an infinite graph, Tohoku Math. J. (2) **48** (1996), no. 2, 293–302.

[24] K. Fujiwara, The Laplacian on rapidly branching trees, Duke Math. J. **83** (1996), no. 1, 191–202.

[25] M. Fukushima, Y. Ōshima, M. Takeda, Dirichlet forms and symmetric Markov processes, de Gruyter Studies in Mathematics, vol. 19, Walter de Gruyter Berlin, 1994.

[26] M. P. Gaffney, The harmonic operator for exterior differential forms, Proc. Nat. Acad. Sci. U. S. A. **37** (1951) 48–50.

[27] M. P. Gaffney, A special Stokes's theorem for complete Riemannian manifolds, Ann. of Math. (2) **60** (1954), 140–145.

[28] A. Georgakopoulos, S. Haeseler, M. Keller, D. Lenz, R. Wojciechowski, Graphs of finite measure, preprint 2013, arXiv:1309.3501.

[29] A. Grigor'yan, Stochastically complete manifolds, Dokl. Akad. Nauk SSSR **290** (1986),no. 3, 534–537.

[30] A. Grigor'yan, Analytic and Geometric Background of Recurrence and Non-Explosion of the Brownian Motion on Riemannian Manifolds, Bull. Amer. Math. Soc. **36**, (1999), no. 2, 135–249.

[31] A. Grigor'yan, Escape rate of Brownian motion on Riemannian manifolds, Appl. Anal. 71 (1999), no. 1–4, 63–89.

[32] A. Grigor'yan. Analysis on Graphs. Lecture Notes, University Bielefeld, 2009.

[33] A. Grigor'yan, E. P. Hsu, Volume growth and escape rate of Brownian motion on a Cartan- Hadamard manifold, Sobolev spaces in mathematics. II, Int. Math. Ser. (N. Y.), vol. 9, Springer, New York (2009), 209–225.

[34] A. Grigor'yan, X. Huang, J. Masamune, On stochastic completeness of jump processes, Math Z. **271** (2012), no. 3-4, 1211–1239.

[35] B. Güneysu, M. Keller, M. Schmidt, A Feynman-Kac-Itô formula for magnetic Schrödinger operators on graphs, preprint 2013, arXiv:1301.1304.

[36] S. Haeseler, M. Keller, Generalized solutions and spectrum for Dirichlet forms on graphs, in [57], 181–199.

[37] S. Haeseler, M. Keller, D. Lenz, R. K. Wojciechowski, Laplacians on infinite graphs: Dirichlet and Neumann boundary conditions, J. Spectr. Theory **2** (2012), no. 4, 397–432.

[38] S. Haeseler, M. Keller, R. K. Wojciechowski, Volume growth and bounds for the essential spectrum for Dirichlet forms, J. Lond. Math. Soc **88** (2013), no. 3, 883–898.

[39] R. Hempel, J. Voigt, The spectrum of a Schrödinger operator in $L_p(\mathbb{R}^\nu)$ is p-independent, Comm. Math. Phys. **104** (1986), 243-250.

[40] I. Holopainen, P. M. Soardi, A strong Liouville theorem for p-harmonic functions on graphs, Ann. Acad. Sci. Fenn. Math. **22** (1997), 205–226.

[41] E. P. Hsu, G. Qin, Volume growth and escape rate of Brownian motion on a complete Riemannian manifold, Ann. Probab. **38** (2010), no. 4, 1570–1582.

[42] B. Hua, J. Jost, l^q harmonic functions on graphs, to appear in Israel J. Math., arXiv:1301.3403, 2013.

[43] B. Hua, M. Keller, Harmonic functions of general graph Laplacians, to appear in Calc. Var, arXiv:1303.7198.

[44] X. Huang, On stochastic completeness of weighted graphs, Ph.D. thesis, 2011.

[45] X. Huang, A note on the volume growth criterion for stochastic completeness of weighted graphs, Potential Anal. **40** (2014), Issue 2, 117–142.

[46] X. Huang, Escape rate of Markov chains on infinite graphs, to appear in J. Theoret. Probab., DOI: 10.1007/s10959-012-0456-x.

[47] X. Huang, private communication.

[48] X. Huang, Y. Shiozawa, Upper escape rate of Markov chains on weighted graphs. Stochastic Process. Appl. **124** (2014), no. 1, 317–347.

[49] X. Huang, M. Keller, J. Masamune, R. Wojciechowski, A note on self-adjoint extensions of the Laplacian on weighted graphs, J. Funct. Anal. **265** (2013), no. 8, 1556-1578.

[50] L. Karp. Subharmonic functions on real and complex manifolds. Math. Z. **179** (1982), 535-554.

[51] M. Keller, Lecture notes SS 2012 'Applications of operator theory - Discrete Operators', http://www.analysis-lenz.uni-jena.de/Team/Matthias+Keller.html.

[52] M. Keller, D. Lenz, Dirichlet forms and stochastic completeness of graphs and subgraphs, J. Reine Angew. Math. (Crelle's Journal) **666** (2012), 189–223.

[53] M. Keller, D. Lenz, Unbounded Laplacians on graphs: basic spectral properties and the heat equation, Math. Model. Nat. Phenom. **5** (2010), no. 4, 198–224.

[54] M. Keller, D. Lenz, M. Schmidt, M. Wirth, Diffusion determines the recurrent graph, preprint.

[55] M. Keller, D. Lenz, R. K. Wojciechowski, Volume growth, spectrum and stochastic completeness of infinite graphs, Math. Z. **274** (2013), no. 3-4, 905–932.

[56] P. Kuchment, Quantum graphs. I. Some basic structures, Waves Random Media, **14** (2004), 107–128.

[57] D. Lenz, F. Sobieczky, W. Woess (eds.), Random Walks, Boundaries and Spectra, Progress in Probability, vol. 64, Birkhäuser Verlag, Basel, 2011.

[58] R. Lyons with Y. Peres, *Probability on Trees and Networks,* Cambridge University Press, to appear; 2013 draft at http://mypage.iu.edu/~rdlyons/prbtree/book.pdf.

[59] J. Masamune, A Liouville property and its application to the Laplacian of an infinite graph, In Spectral analysis in geometry and number theory, volume **484** of Contemp. Math., 103–115, Providence, RI, 2009. Amer. Math. Soc.

[60] J. Masamune, T. Uemura, Conservation property of symmetric jump processes. Ann. Inst. Henri Poincaré Probab. Stat. **47** (2011), no. 3, 650–662.

[61] J. Masamune, T. Uemura, J. Wang, On the conservativeness and the recurrence of symmetric jump-diffusions. J. Funct. Anal. **263** (2012), no. 12, 3984–4008.

[62] O. Milatovic, Essential self-adjointness of discrete magnetic Schrödinger operators on locally finite graphs, Integr. Equ. Oper. Theory **71** (2011), 13–27.

[63] O. Milatovic, A Sears-type self-adjointness result for discrete magnetic Schrödinger operators, Journal of Mathematical Analysis and Applications **396**, (2012) 801–809.

[64] O. Milatovic, F. Truc, Self-adjoint extensions of discrete magnetic Schrödinger operators, Ann. Henri Poincaré **15** (2014), 917–936.

[65] M. Mokhtar-Kharroubi, J. Voigt, On honesty of perturbed substochastic C_0-semigroups in L^1-spaces, J. Operator Theory **64** (2010), no. 1, 131–147.

[66] Y. Ohno, H. Urakawa, On the first eigenvalue of the combinatorial Laplacian for a graph, Interdiscip. Inform. Sci. **1** (1994), no. 1, 33–46.

[67] S. Ouyang, Volume growth, Comparison theorem and Escape Rate of Diffusion Process, Preprint arXiv:1310.3996, 2013.

[68] M. A. Pinsky, Mean exit time of a diffusion process from a small sphere, Proc. Amer. Math. Soc. **93** (1985), no. 1, 157-158.

[69] G. E. H. Reuter, Denumerable Markov processes and the associated contraction semigroups on l. Acta Math. **97** (1957), 1–46.

[70] M. Rigoli, M. Salvatori, M. Vignati, Subharmonic functions on graphs. Israel J. Math. **99**, (1997), 1–27.

[71] W. Roelcke, Über den Laplace-Operator auf Riemannschen Mannigfaltigkeiten mit diskontinuierlichen Gruppen, Math. Nachr. **21** (1960), 131–149.

[72] M. Schmidt, Global properties of Dirichlet forms on discrete spaces, Diplomarbeit 2011, arXiv:1201.3474v2.

[73] I. E. Shnol', On the behaviour of the eigenfunctions of Schrödinger's eqution, Mat.Sb., **42** (1957), 273–286. erratum 46(88), 259, 1957.

[74] B. Simon, Spectrum and continuum eigenfunctions of Schrödinger operators, J. Funct. Anal., **42** (1981), 347–355.

[75] B. Simon, Schrödinger semigroups, Bull. Amer. Math. Soc., **7**(3), 447-526, 1982.

[76] R. S. Strichartz, Analysis of the Laplacian on the complete Riemannian manifold, J. Funct. Anal. **52** (1983), no. 1, 48–79.

[77] K. T. Sturm. On the lp-spectrum of uniformly elliptic operators on riemannian manifolds, J. Funct. Anal. **118** (1993), 442–453.

[78] K.-T. Sturm, Analysis on local Dirichlet spaces. I. Recurrence, conservativeness and L^p Liouville properties, J. Reine Angew. Math **456** (1994), 173–196.

[79] N. Torki-Hamza, Laplaciens de graphes infinis I–Graphes métriquement complets, Confluentes Math. **2** (2010), no. 3, 333–350.

[80] J. Weidmann, Linear operators in Hilbert spaces, Graduate Texts in Mathematics, vol. 68, Springer-Verlag, New York, 1980.

[81] W. Woess, Random walks on infinite graphs and groups, Cambridge Tracts in Mathematics, vol. 138, Cambridge University Press, Cambridge, 2000.

[82] R. K. Wojciechowski, Stochastic completeness of graphs, ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–City University of New York.

[83] R. K. Wojciechowski, Heat kernel and essential spectrum of infinite graphs, Indiana Univ. Math. J. **58** (2009), 1419–1441.

[84] R. K. Wojciechowski, Stochastically incomplete manifolds and graphs, in [57], 163–179.

[85] S. T. Yau, Some function-theoretic properties of complete Riemannian manifold and their application to geometry, Indiana Univ. Math. J. **25** (1976), no. 7, 659–670.

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