2

Lagrange's Equations

2.1 Introduction

The dynamical equations of J.L. Lagrange were published in the eighteenth century some one hundred years after Newton's *Principia*. They represent a powerful alternative to the Newton–Euler equations and are particularly useful for systems having many degrees of freedom and are even more advantageous when most of the forces are derivable from potential functions.

The equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathfrak{L}}{\partial \dot{q}_i}\right) - \left(\frac{\partial \mathfrak{L}}{\partial q_i}\right) = Q_i \quad 1 \le i \le n \tag{2.1}$$

where

$\mathfrak{L}$ is the Lagrangian defined to be $T-V$,
T is the kinetic energy (relative to inertial axes),
V is the potential energy,
n is the number of degrees of freedom,
q_1 to q_n are the generalized co-ordinates,
Q_1 to Q_n are the generalized forces

and $\mathrm{d}/\mathrm{d}t$ means differentiation of the scalar terms with respect to time. *Generalized co-ordinates* and *generalized forces* are described below.

Partial differentiation with respect to $\dot{q}_i$ is carried out assuming that all the other $\dot{q}$, all the q and time are held fixed. Similarly for differentiation with respect to q_i all the other q, all $\dot{q}$ and time are held fixed.

We shall proceed to prove the above equations, starting from Newton's laws and D'Alembert's principle, during which the exact meaning of the definitions and statements will be illuminated. But prior to this a simple application will show the ease of use.

EXAMPLE

A mass is suspended from a point by a spring of natural length *a* and stiffness *k*, as shown in Fig. 2.1. The mass is constrained to move in a vertical plane in which the gravitational field strength is *g*. Determine the equations of motion in terms of the distance *r* from the support to the mass and the angle θ which is the angle the spring makes with the vertical through the support point.

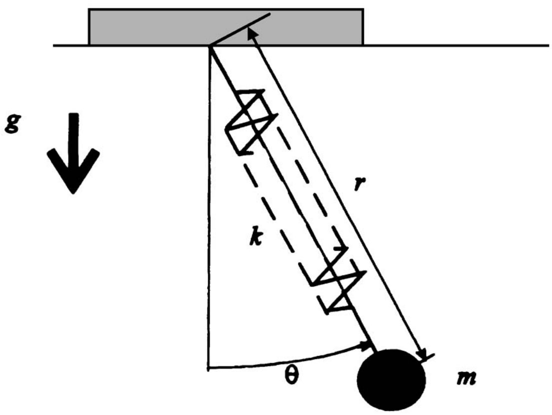

Fig. 2.1

The system has two degrees of freedom and r and θ, which are independent, can serve as generalized co-ordinates. The expression for kinetic energy is

$$T = \frac{m}{2}\left[\dot{r}^2 + (r\dot{\theta})^2\right]$$

and for potential energy, taking the horizontal through the support as the datum for gravitational potential energy,

$$V = -mgr\cos\theta + \frac{k}{2}(r-a)^2$$

so

$$\mathfrak{L} = T - V = \frac{m}{2}\left[\dot{r}^2 + (r\dot{\theta})^2\right] + mgr\cos\theta - \frac{k}{2}(r-a)^2$$

Applying Lagrange's equation with $q_1 = r$ we have

$$\frac{\partial\mathfrak{L}}{\partial\dot{r}} = m\dot{r}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathfrak{L}}{\partial\dot{r}}\right) = m\ddot{r}$$

and

$$\frac{\partial\mathfrak{L}}{\partial r} = mr\dot{\theta}^2 + mg\cos\theta - k(r-a)$$

From equation (2.1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathfrak{L}}{\partial\dot{r}}\right) - \left(\frac{\partial\mathfrak{L}}{\partial r}\right) = Q_r$$

$$m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta + k(r-a) = 0 \tag{i}$$

The generalized force $Q_r = 0$ because there is no externally applied radial force that is not included in V.

Taking θ as the next generalized co-ordinate

$$\frac{\partial \mathfrak{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathfrak{L}}{\partial \dot{\theta}}\right) = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta}$$

and

$$\frac{\partial \mathfrak{L}}{\partial \theta} = mgr \sin \theta$$

Thus the equation of motion in θ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathfrak{L}}{\partial \dot{\theta}}\right) - \left(\frac{\partial \mathfrak{L}}{\partial \theta}\right) = Q_\theta = 0$$

$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} - mgr \sin \theta = 0 \tag{ii}$$

The generalized force in this case would be a torque because the corresponding generalized co-ordinate is an angle. Generalized forces will be discussed later in more detail.

Dividing equation (ii) by r gives

$$2m\dot{r}\dot{\theta} + mr\ddot{\theta} - mg \sin \theta = 0 \tag{iia}$$

and rearranging equations (i) and (ii) leads to

$$mg \cos \theta - k(r - a) = m(\ddot{r} - r\dot{\theta}^2) \tag{ia}$$

and

$$-mg \sin \theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta}) \tag{iib}$$

which are the equations obtained directly from Newton's laws plus a knowledge of the components of acceleration in polar co-ordinates.

In this example there is not much saving of labour except that there is no requirement to know the components of acceleration, only the components of velocity.

2.2 Generalized co-ordinates

A set of generalized co-ordinates is one in which each co-ordinate is independent and the number of co-ordinates is just sufficient to specify completely the configuration of the system. A system of N particles, each free to move in a three-dimensional space, will require $3N$ co-ordinates to specify the configuration. If Cartesian co-ordinates are used then the set could be

$$\{x_1\ y_1\ z_1\ x_2\ y_2\ z_2 \ldots x_N\ y_N\ z_N\}$$

or

$$\{x_1\ x_2\ x_3\ x_4\ x_5\ x_6 \ldots x_{n-2}\ x_{n-1}\ x_n\}$$

where $n = 3N$.

This is an example of a set of generalized co-ordinates but other sets may be devised involving different displacements or angles. It is conventional to designate these co-ordinates as

$$\{q_1 \; q_2 \; q_3 \; q_4 \; q_5 \; q_6 \; \cdots \; q_{n-2} \; q_{n-1} \; q_n\}$$

If there are constraints between the co-ordinates then the number of independent co-ordinates will be reduced. In general if there are r equations of constraint then the number of degrees of freedom n will be $3N - r$. For a particle constrained to move in the xy plane the equation of constraint is $z = 0$. If two particles are rigidly connected then the equation of constraint will be

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = L^2$$

That is, if one point is known then the other point must lie on the surface of a sphere of radius L. If $x_1 = y_1 = z_1 = 0$ then the constraint equation simplifies to

$$x_2^2 + y_2^2 + z_2^2 = L^2$$

Differentiating we obtain

$$2x_2\,\mathrm{d}x_2 + 2y_2\,\mathrm{d}y_2 + 2z_2\,\mathrm{d}z_2 = 0$$

This is a perfect differential equation and can obviously be integrated to form the constraint equation. In some circumstances there exist constraints which appear in differential form and cannot be integrated; one such example of a rolling wheel will be considered later. A system for which all the constraint equations can be written in the form $f(q_1 \ldots q_n)$ = constant or a known function of time is referred to as *holonomic* and for those which cannot it is called *non-holonomic*.

If the constraints are moving or the reference axes are moving then time will appear explicitly in the equations for the Lagrangian. Such systems are called *rheonomous* and those where time does not appear explicitly are called *scleronomous*.

Initially we will consider a holonomic system (rheonomous or scleronomous) so that the Cartesian co-ordinates can be expressed in the form

$$x_i = x_i\,(q_1\, q_2 \ldots q_n t) \tag{2.2}$$

By the rules for partial differentiation the differential of equation (2.2) with respect to time is

$$v_i = \frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_j \frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t} \tag{2.3}$$

so

$$v_i = v_i\,(q_1 \; q_2 \cdots q_n \; \dot{q}_1 \; \dot{q}_2 \cdots \dot{q}_n t) \tag{2.4}$$

thus

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = \sum_j \frac{\partial v_i}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial v_i}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial^2 x_i}{\partial t^2} \tag{2.5}$$

Differentiating equation (2.3) directly gives

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = \sum_j \frac{\partial \dot{x}_i}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial x_i}{\partial q_j}\ddot{q}_j + \frac{\partial^2 x_i}{\partial t^2} \tag{2.6}$$

and comparing equation (2.5) with equation (2.6), noting that $v_i = \dot{x}_i$, we see that

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j} \tag{2.7}$$

a process sometimes referred to as the cancellation of the dots.

From equation (2.2) we may write

$$\mathrm{d}x_i = \sum_j \frac{\partial x_i}{\partial q_j}\,\mathrm{d}q_j + \frac{\partial x_i}{\partial t}\,\mathrm{d}t \tag{2.8}$$

Since, by definition, virtual displacements are made with time constant

$$\delta x_i = \sum_j \frac{\partial x_i}{\partial q_j}\,\delta q_j \tag{2.9}$$

These relationships will be used in the proof of Lagrange's equations.

2.3 Proof of Lagrange's equations

The proof starts with D'Alembert's principle which, it will be remembered, is an extension of the principle of virtual work to dynamic systems. D'Alembert's equation for a system of N particles is

$$\sum_i (\boldsymbol{F}_i - m_i\ddot{\boldsymbol{r}}_i)\cdot\delta\boldsymbol{r}_i = 0 \quad 1 \le i \le N \tag{2.10}$$

where $\delta\boldsymbol{r}_i$ is any virtual displacement, consistent with the constraints, made with time fixed.

Writing $\boldsymbol{r}_1 = x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}$ etc. equation (2.10) may be written in the form

$$\sum_i (F_i - m_i\ddot{x}_i)\delta x_i = 0 \quad 1 \le i \le n = 3N \tag{2.11}$$

Using equation (2.9) and changing the order of summation, the first summation in equation (2.11) becomes

$$\sum_i F_i\delta x_i = \sum_i F_i\left(\sum_j \frac{\partial x_i}{\partial q_j}\delta q_j\right) = \sum_j \left(\sum_i F_i\frac{\partial x_i}{\partial q_j}\right)\delta q_j = \delta W \tag{2.12}$$

the virtual work done by the forces. Now $W = W(q_j)$ so

$$\delta W = \sum_j \frac{\partial W}{\partial q_j}\,\delta q_j \tag{2.13}$$

and by comparison of the coefficients of δq in equations (2.12) and (2.13) we see that

$$\frac{\partial W}{\partial q_j} = \sum_i F_i\,\frac{\partial x_i}{\partial q_j} \tag{2.14}$$

This term is designated Q_j and is known as a generalized force. The dimensions of this quantity need not be those of force but the product of the generalized force and the associated generalized co-ordinate must be that of work. In most cases this reduces to force and displacement or torque and angle. Thus we may write

$$\delta W = \sum_j Q_j\,\delta q_j \tag{2.15}$$

In a large number of problems the force can be derived from a position-dependent potential V, in which case

$$Q_j = -\frac{\partial V}{\partial q_j} \tag{2.16}$$

Equation (2.13) may now be written

$$\sum_i F_i \delta x_i = \sum_j \left(-\frac{\partial V}{\partial q_j} + Q_j \right) \delta q_j \tag{2.17}$$

where Q_j now only applies to forces not derived from a potential.

Now the second summation term in equation (2.11) is

$$\sum_i m_i \ddot{x}_i \, \delta x_i = \sum_i m_i \ddot{x}_i \left(\sum_j \frac{\partial x_i}{\partial q_j} \delta q_j \right)$$

or, changing the order of summation,

$$\sum_i m_i \ddot{x}_i \, \delta x_i = \sum_j \left(\sum_i m_i \ddot{x}_i \frac{\partial x_i}{\partial q_j} \right) \delta q_j \tag{2.18}$$

We now seek a form for the right hand side of equation (2.18) involving the kinetic energy of the system in terms of the generalized co-ordinates.

The kinetic energy of the system of N particles is

$$T = \sum_{i=1}^{i=N} \frac{m_i}{2} \dot{r}_i \cdot \dot{r}_i = \sum_{i=1}^{i=3N} \frac{m_i}{2} \dot{x}_i \dot{x}_i$$

Thus

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \sum_i m_i \dot{x}_i \frac{\partial x_i}{\partial q_j}$$

because the dots may be cancelled, *see* equation (2.7). Differentiating with respect to time gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j} \right) = \sum_i m_i \ddot{x}_i \frac{\partial x_i}{\partial q_j} + \sum_i m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_j}$$

but

$$\frac{\partial T}{\partial q_j} = \sum_i m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_j}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = \sum_i m_i \ddot{x}_i \frac{\partial x_i}{\partial q_j} \tag{2.19}$$

Substitution of equation (2.19) into equation (2.18) gives

$$\sum_i m_i \ddot{x}_i \, \delta x_i = \sum_j \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j \tag{2.20}$$

Substituting from equations (2.17) and (2.20) into equation 2.11 leads to

$$\sum_j \left[-\frac{\partial V}{\partial q_j} + Q_j - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_j} \right] \delta q_j = 0$$

Because the q are independent we can choose δq_j to be non-zero whilst all the other δq are zero. So

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = Q_j$$

Alternatively since V is taken not to be a function of the generalized velocities we can write the above equation in terms of the Lagrangian $\mathfrak{L} = T - V$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathfrak{L}}{\partial \dot{q}_j}\right) - \left(\frac{\partial \mathfrak{L}}{\partial q_j}\right) = Q_j \quad 1 \le j \le n \tag{2.21}$$

In the above analysis we have taken n to be $3N$ but if we have r holonomic equations of constraint then $n = 3N - r$. In practice it is usual to write expressions for T and V directly in terms of the reduced number of generalized co-ordinates. Further, the forces associated with workless constraints need not be included in the analysis.

For example, if a rigid body is constrained to move in a vertical plane with the y axis vertically upwards then

$$T = \frac{m}{2}(\dot{x}_G^2 + \dot{y}_G^2) + \frac{I_G}{2}\dot{\theta}^2 \text{ and } V = mgy_G$$

The constraint equations are fully covered by the use of total mass and moment of inertia and the suppression of the z_G co-ordinate.

2.4 The dissipation function

If there are forces of a viscous nature that depend linearly on velocity then the force is given by

$$F_i = -\sum_j c_{ij}\dot{x}_j$$

where c_{ij} are constants.

The power dissipated is

$$P = \sum_i F_i \dot{x}_i$$

In terms of generalized forces

$$P = \sum_i Q_i \dot{q}_i$$

and

$$Q_i = -\sum_j C_{ij}\dot{q}_j$$

where C_{ij} are related to c_{ij} (the exact relationship does not concern us at this point).

The power dissipated is

$$P = -\sum_j Q_j \dot{q}_j$$

By differentiation

$$\frac{\partial P}{\partial \dot{q}_i} = -Q_i - \sum_j \frac{\partial Q_j}{\partial \dot{q}_i}\dot{q}_j = -Q_i + \sum_j C_{ij}\dot{q}_j = -2Q_i$$

If we now define $\mathfrak{F} = P/2$ then

$$\frac{\partial \mathfrak{F}}{\partial \dot{q}_i} = -Q_i \tag{2.22}$$

The term $\mathfrak{F}$ is known as *Rayleigh's dissipative function* and is half the rate at which power is being dissipated.

Lagrange's equations are now

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathfrak{L}}{\partial \dot{q}_j}\right) - \left(\frac{\partial \mathfrak{L}}{\partial q_j}\right) + \frac{\partial \mathfrak{F}}{\partial \dot{q}_j} = Q_j \tag{2.23}$$

where Q_j is the generalized force not obtained from a position-dependent potential or a dissipative function.

EXAMPLE

For the system shown in Fig. 2.2 the scalar functions are

$$T = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2$$

$$V = \frac{k_1}{2}x_1^2 + \frac{k_2}{2}(x_2 - x_1)^2$$

$$\mathfrak{F} = \frac{c_i}{2}\dot{x}_1^2 + \frac{c_2}{2}(\dot{x}_2 - \dot{x}_1)^2$$

The virtual work done by the external forces is

$$\delta W = F_1\,\delta x_1 + F_2\,\delta x_2$$

For the generalized co-ordinate x_1 application of Lagrange's equation leads to

$$m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) + c_1\dot{x}_1 - c_2(\dot{x}_2 - \dot{x}_1) = F_1$$

and for x_2

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) = F_2$$

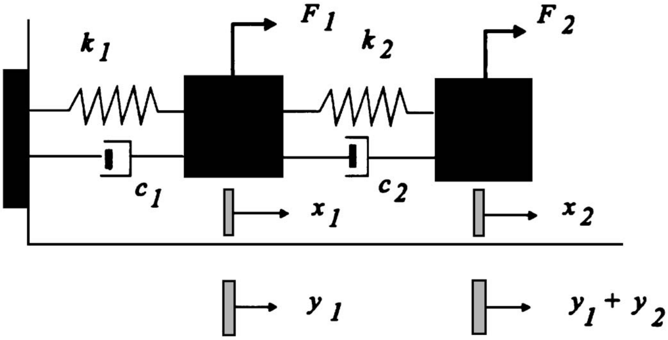

Fig. 2.2

Alternatively we could have used co-ordinates y_1 and y_2 in which case the appropriate functions are

$$T = \frac{m_1}{2}\dot{y}_1^2 + \frac{m_2}{2}(\dot{y}_1 + \dot{y}_2)^2$$

$$V = \frac{k_1}{2}y_1^2 + \frac{k_2}{2}y_2^2$$

$$\mathfrak{F} = \frac{c_1}{2}\dot{y}_1^2 + \frac{c_2}{2}\dot{y}_2^2$$

and the virtual work is

$$\delta W = F_1\delta y_1 + F_2\delta(y_1 + y_2) = (F_1 + F_2)\delta y_1 + F_2\delta y_2$$

Application of Lagrange's equation leads this time to

$$m_1\ddot{y}_1 + m_2(\ddot{y}_1 + \ddot{y}_2) + k_1 y_1 + c_1\dot{y}_1 = F_1 + F_2$$

$$m_2(\ddot{y}_1 + \ddot{y}_2) + k_2 y_2 + c_2\dot{y}_2 = F_2$$

Note that in the first case the kinetic energy has no term which involves products like $\dot{q}_i\dot{q}_j$ whereas in the second case it does. The reverse is true for the potential energy regarding terms like q_iq_j. Therefore the coupling of co-ordinates depends on the choice of co-ordinates and de-coupling in the kinetic energy does not imply that de-coupling occurs in the potential energy. It can be proved, however, that there exists a set of co-ordinates which leads to uncoupled co-ordinates in both the kinetic energy and the potential energy; these are known as principal co-ordinates.

2.5 Kinetic energy

The kinetic energy of a system is

$$T = \frac{1}{2}\Sigma\, m_i\dot{x}_i^2 = \frac{1}{2}\,(\dot{x})^{\mathrm{T}}[m](\dot{x})$$

where

$$(x) = (x_1\, x_2 \ldots x_{3N})^{\mathrm{T}}$$

and

$$[m] = \begin{bmatrix} m_1 & & \\ & \ddots & \\ & & m_{3N} \end{bmatrix}_{\text{diagonal}}$$

Now

$$x_i = x_i(q_1\, q_2 \ldots q_n t)$$

so

$$\dot{x}_i = \sum_j \frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t}$$

We shall, in this section, use the notation ( ) to mean a column matrix and [ ] to indicate a square matrix. Thus with

$$(\dot{x}) = (\dot{x}_1 \dot{x}_2 \ldots \dot{x}_n)^{\mathrm{T}}$$

then we may write

$$(\dot{x}) = [A](\dot{q}) + (b)$$

where

$$[A] = \begin{bmatrix} \dfrac{\partial x_1}{\partial q_1} \cdots \dfrac{\partial x_1}{\partial q_n} \\ \dfrac{\partial x_n}{\partial q_1} \cdots \dfrac{\partial x_n}{\partial q_n} \end{bmatrix} = f(q_1\, q_2 \ldots q_n)$$

$$(\mathrm{b}) = \left[\frac{\partial x_1}{\partial t} \cdots \frac{\partial x_n}{\partial t}\right]^{\mathrm{T}}$$

and

$$(\dot{q}) = (\dot{q}_1\, \dot{q}_2 \ldots \dot{q}_n)^{\mathrm{T}}$$

Hence we may write

$$T = \frac{1}{2}\Big((b)^{\mathrm{T}} + (\dot{q})^{\mathrm{T}}\,[A]^{\mathrm{T}}\Big)\,[m]\,\Big([A](\dot{q}) + (b)\Big)$$

$$= \frac{1}{2}(\dot{q})^{\mathrm{T}}\,[A]^{\mathrm{T}}\,[m]\,[A]\,(\dot{q}) + (b)^{\mathrm{T}}[m]\,[A]\,(\dot{q}) + (b)^{\mathrm{T}}\,[m]\,(b) \tag{2.24}$$

Note that use has been made of the fact that $[m]$ is symmetrical. This fact also means that $[A]^{\mathrm{T}}[m][A]$ is symmetrical.

Let us write the kinetic energy as

$$T = T_2 + T_1 + T_0$$

where T_2, the first term of equation (2.24), is a quadratic in $\dot{q}$ and does not contain time explicitly. T_1 is linear in $\dot{q}$ and the coefficients contain time explicitly. T_0 contains time but is independent of $\dot{q}$. If the system is scleronomic with no moving constraints or moving axes then $T_1 = 0$ and $T_0 = 0$.

T_2 has the form

$$T_2 = \frac{1}{2}\sum_{ij} a_{ij}\, \dot{q}_i \dot{q}_j, \quad a_{ij} = a_{ji} = f(q)$$

and in some cases terms like $\dot{q}_i\dot{q}_j$ are absent and T_2 reduces to

$$T_2 = \frac{1}{2}\sum_i a_i \dot{q}_i^2$$

Here the co-ordinates are said to be orthogonal with respect to the kinetic energy.

2.6 Conservation laws

We shall now consider systems for which the forces are only those derivable from a position-dependent potential so that Lagrange's equations are of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathfrak{L}}{\partial \dot{q}_i}\right) - \left(\frac{\partial \mathfrak{L}}{\partial q_i}\right) = 0$$

If a co-ordinate does not appear explicitly in the Lagrangian but only occurs as its time derivative then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathfrak{L}}{\partial \dot{q}_i}\right) = 0$$

Therefore

$$\frac{\partial \mathfrak{L}}{\partial \dot{q}_i} = \text{constant}$$

In this case q_i is said to be a *cyclic* or *ignorable co-ordinate.*

Consider now a group of particles such that the forces depend only on the relative positions and motion between the particles. If we choose Cartesian co-ordinates relative to an arbitrary set of axes which are drifting in the x direction relative to an inertial set of axes as seen in Fig. 2.3, the Lagrangian is

$$\mathfrak{L} = \sum_{i=1}^{i=N} \frac{1}{2} m_i \left[(\dot{X} + \dot{x}_i)^2 + \dot{y}_i^2 + \dot{z}_i^2\right] - V(x_i\, y_i\, z_i)$$

Because X does not appear explicitly and is therefore ignorable

$$\frac{\partial \mathfrak{L}}{\partial \dot{X}} = \sum_{i=1}^{i=N} m_i(\dot{X} + \dot{x}_i) = \text{constant}$$

If $\dot{X} \to 0$ then

$$\sum_{i=1}^{i=N} m_i \dot{x}_i = \text{constant} \tag{2.25}$$

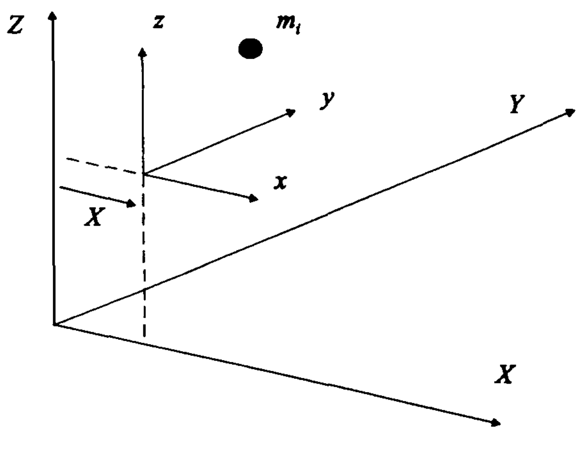

Fig. 2.3

This may be interpreted as consistent with the Lagrangian being independent of the position in space of the axes and this also leads to the linear momentum in the arbitrary x direction being constant or conserved.

Consider now the same system but this time referred to an arbitrary set of cylindrical co-ordinates. This time we shall superimpose a rotational drift of $\dot{\gamma}$ of the axes about the z axis, *see* Fig. 2.4. Now the Lagrangian is

$$\mathfrak{L} = \sum_i \frac{m_i}{2}\left[r_i^2(\dot{\theta}_i + \dot{\gamma})^2 + \dot{r}_i^2 + \dot{z}_i^2\right] - V(r_i, \theta_i, z_i)$$

Because γ is a cyclic co-ordinate

$$\frac{\partial \mathfrak{L}}{\partial \dot{\gamma}} = \sum_i m_i r_i^2(\dot{\theta}_i + \dot{\gamma}) = \text{constant}$$

If we now consider $\dot{\gamma}$ to tend to zero then

$$\frac{\partial \mathfrak{L}}{\partial \dot{\gamma}} = \sum_i m_i r_i^2 \dot{\theta}_i = \text{constant} \tag{2.26}$$

This implies that the conservation of the moment of momentum about the z axis is associated with the independence of $\partial\mathfrak{L}/\partial\dot{\gamma}$ to a change in angular position of the axes.

Both the above show that $\partial\mathfrak{L}/\partial\dot{q}$ is related to a momentum or moment of momentum. We now define $\partial\mathfrak{L}/\partial\dot{q}_i = p_i$ to be the *generalized momentum*, the dimensions of which will depend on the choice of generalized co-ordinate.

Consider the total time differential of the Lagrangian

$$\frac{\mathrm{d}\mathfrak{L}}{\mathrm{d}t} = \sum_j \frac{\partial \mathfrak{L}}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial \mathfrak{L}}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial \mathfrak{L}}{\partial t} \tag{2.27}$$

If all the generalized forces, Q_j, are zero then Lagrange's equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathfrak{L}}{\partial \dot{q}_j}\right) - \left(\frac{\partial \mathfrak{L}}{\partial q_j}\right) = 0$$

Substitution into equation (2.27) gives

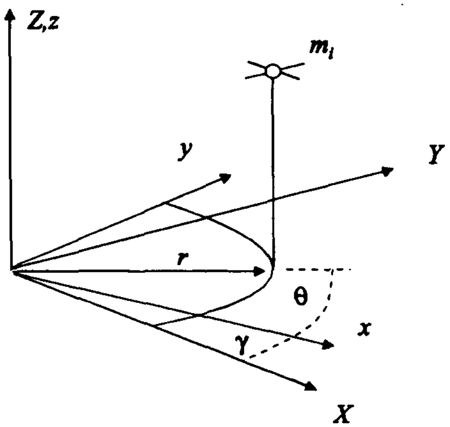

Fig. 2.4

$$\frac{\mathrm{d}\mathfrak{L}}{\mathrm{d}t} = \sum_j \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathfrak{L}}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_j \frac{\partial \mathfrak{L}}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial \mathfrak{L}}{\partial t}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_j \frac{\partial \mathfrak{L}}{\partial \dot{q}_j}\dot{q}_j - \mathfrak{L}\right) = -\frac{\partial \mathfrak{L}}{\partial t} \tag{2.28}$$

and if the Lagrangian does not depend explicitly on time then

$$\sum_j \frac{\partial \mathfrak{L}}{\partial \dot{q}_j}\dot{q}_j - \mathfrak{L} = \text{constant} \tag{2.29}$$

Under these conditions $\mathfrak{L} = T - V = T_2(q_j\dot{q}_j) - V(q_j)$. Now

$$T_2 = \frac{1}{2}\sum_j\sum_i a_{ij}\,\dot{q}_i\dot{q}_j, \qquad a_{ij} = a_{ji}$$

so

$$\frac{\partial T_2}{\partial \dot{q}_j} = \frac{1}{2}\sum_i a_{ij}\dot{q}_i + \frac{1}{2}\sum_j a_{ij}\dot{q}_j = \sum_i a_{ij}\dot{q}_i$$

because $a_{ij} = a_{ji}$.

We can now write

$$\sum_j \frac{\partial \mathfrak{L}}{\partial \dot{q}_j}\dot{q}_j = \sum_j \frac{\partial T_2}{\partial \dot{q}_j}\dot{q}_j = \sum_j\left(\dot{q}_j\sum_i a_{ij}\dot{q}_i\right) = 2T_2$$

so that

$$\begin{aligned}\sum_j \frac{\partial \mathfrak{L}}{\partial \dot{q}_j}\dot{q}_j - \mathfrak{L} &= 2T_2 - (T_2 - V)\\ &= T_2 + V = T + V\\ &= E\end{aligned}$$

the total energy.

From equation (2.29) we see that the quantity conserved when there are (a) no generalized forces and (b) the Lagrangian does not contain time explicitly is the total energy. Thus conservation of energy is a direct consequence of the Lagrangian being independent of time. This is often referred to as symmetry in time because time could in fact be reversed without affecting the equations. Similarly we have seen that symmetry with respect to displacement in space yields the conservation of momentum theorems.

2.7 Hamilton's equations

The quantity between the parentheses in equation (2.28) is known as the *Hamiltonian H*

$$H = \sum_j \frac{\partial \mathfrak{L}}{\partial \dot{q}_j}\dot{q}_j - \mathfrak{L}(q_j\dot{q}_jt) \tag{2.30}$$

or in terms of momenta

$$H = \sum_j p_j\dot{q}_j - \mathfrak{L}(q_j\dot{q}_jt) \tag{2.31}$$

Since $\dot{q}$ can be expressed in terms of p the Hamiltonian may be considered to be a function of generalized momenta, co-ordinates and time, that is $H = H(q_j p_j t)$. The differential of H is

$$\mathrm{d}H = \sum_j \frac{\partial H}{\partial q_j}\mathrm{d}q_j + \sum_j \frac{\partial H}{\partial p_j}\mathrm{d}p_j + \frac{\partial H}{\partial t}\mathrm{d}t \tag{2.32}$$

From equation (2.32)

$$\mathrm{d}H = \sum_j (p_j d\dot{q}_j + \dot{q}_j dp_j) - \sum_j \frac{\partial \mathfrak{L}}{\partial q_j}\mathrm{d}q_j - \sum_j \frac{\partial \mathfrak{L}}{\partial \dot{q}_j}\mathrm{d}\dot{q}_j - \frac{\partial \mathfrak{L}}{\partial t}\mathrm{d}t \tag{2.33}$$

By definition $\partial\mathfrak{L}/\partial\dot{q}_j = p_j$ and from Lagrange's equations we have

$$\frac{\partial \mathfrak{L}}{\partial q_j} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathfrak{L}}{\partial \dot{q}_j}\right) = \dot{p}_j$$

Therefore, substituting into equation (2.33) the first and fourth terms cancel leaving

$$\mathrm{d}H = \sum_j \dot{q}_j \mathrm{d}p_j - \sum_j \dot{p}_j \mathrm{d}q_j - \frac{\partial \mathfrak{L}}{\partial t}\mathrm{d}t \tag{2.34}$$

Comparing the coefficients of the differentials in equations (2.32) and (2.34) we have

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j \quad \frac{\partial H}{\partial p_j} = \dot{q}_j \tag{2.35}$$

and

$$\frac{\partial H}{\partial t} = -\frac{\partial \mathfrak{L}}{\partial t}$$

Equations (2.35) are called *Hamilton's canonical equations*. They constitute a set of $2n$ first-order equations in place of a set of n second-order equations defined by Lagrange's equations.

It is instructive to consider a system with a single degree of freedom with a moving foundation as shown in Fig. 2.5. First we shall use the absolute motion of the mass as the generalized co-ordinate.

$$\mathfrak{L} = \frac{m\dot{x}^2}{2} - \frac{k}{2}(x - x_0)^2$$

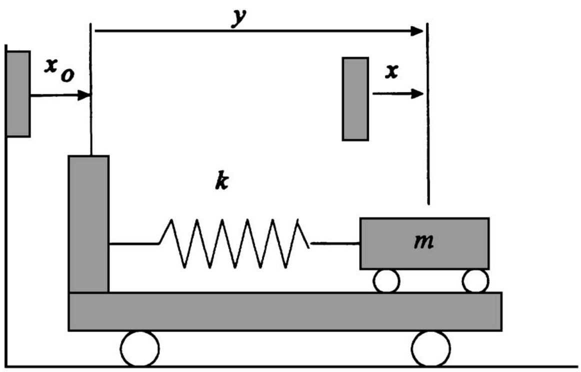

Fig. 2.5

$$\frac{\partial \mathfrak{L}}{\partial \dot{x}} = m\dot{x} = p$$

Therefore $\dot{x} = p/m$. From equation (2.32)

$$H = p(p/m) - \left[\frac{p^2}{2m} - \frac{k}{2}(x-x_0)^2\right]$$

$$= \frac{p^2}{2m} + \frac{k}{2}(x-x_0)^2 \tag{2.36}$$

In this case it is easy to see that H is the total energy but it is not conserved because x_0 is a function of time and hence so is H. Energy is being fed in and out of the system by whatever forces are driving the foundation.

Using y as the generalized co-ordinate we obtain

$$\mathfrak{L} = \frac{m}{2}(\dot{y} + \dot{x}_0)^2 - \frac{k}{2}y^2$$

$$\frac{\partial \mathfrak{L}}{\partial \dot{y}} = m(\dot{y} + \dot{x}_2) = p$$

Therefore $\dot{y} = (p/m) - \dot{x}_0$ and

$$H = p\left[\frac{p}{m} - \dot{x}_0\right] - \left[\frac{p^2}{2m} - \frac{ky^2}{2}\right]$$

$$= \frac{p^2}{2m} - p\dot{x}_0 + \frac{ky^2}{2} \tag{2.37}$$

Taking specific values for x_0 and x (and hence y) it is readily shown that the numerical value of the Lagrangian is the same in both cases whereas the value of the Hamiltonian is different, in this example by the amount $p\dot{x}_0$.

If we choose $\dot{x}_0$ to be constant then time does not appear explicitly in the second case; therefore H is conserved but it is not the total energy. Rewriting equation (2.37) in terms of y and x_0 we get

$$H = \left(\frac{1}{2}m\dot{y}^2 + \frac{1}{2}ky^2\right) - \frac{1}{2}m\dot{x}_0^2 \tag{2.38}$$

where the term in parentheses is the total energy as seen from the moving foundation and the last term is a constant providing, of course, that $\dot{x}_0$ is a constant.

We have seen that choosing different co-ordinates changes the value of the Hamiltonian and also affects conservation properties, but the value of the Lagrangian remains unaltered. However, the equations of motion are identical whichever form of $\mathfrak{L}$ or H is used.

2.8 Rotating frame of reference and velocity-dependent potentials

In all the applications of Lagrange's equations given so far the kinetic energy has always been written strictly relative to an inertial set of axes. Before dealing with moving axes in general we shall consider the case of axes rotating at a constant speed relative to a fixed axis.

Assume that in Fig. 2.6 the *XYZ* axes are inertial and the *xyz* axes are rotating at a constant speed Ω about the *Z* axis. The position vector relative to the inertial axes is $\boldsymbol{r}$ and relative to the rotating axes is $\boldsymbol{\rho}$.

Now

$$\boldsymbol{r} = \boldsymbol{\rho}$$

and

$$\dot{\boldsymbol{r}} = \frac{\partial \boldsymbol{\rho}}{\partial t} + \boldsymbol{\Omega}\times\boldsymbol{\rho}$$

The kinetic energy for a particle is

$$T = \frac{1}{2} m\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}}$$

or

$$T = \frac{m}{2}\left(\frac{\partial \boldsymbol{\rho}}{\partial t}\cdot\frac{\partial \boldsymbol{\rho}}{\partial t} + (\boldsymbol{\Omega}\times\boldsymbol{\rho})\cdot(\boldsymbol{\Omega}\times\boldsymbol{\rho}) + 2\frac{\partial \boldsymbol{\rho}}{\partial t}\cdot(\boldsymbol{\Omega}\times\boldsymbol{\rho})\right) \tag{2.39}$$

Let $\boldsymbol{\Omega}\times\boldsymbol{\rho} = \boldsymbol{A}$, a vector function of position, so the kinetic energy may be written

$$T = \frac{m}{2}\left(\frac{\partial \boldsymbol{\rho}}{\partial t}\right)^2 + \frac{m}{2}A^2 + m\frac{\partial \boldsymbol{\rho}}{\partial t}\cdot\boldsymbol{A}$$

and the Lagrangian is

$$\mathfrak{L} = \frac{m}{2}\left(\frac{\partial \boldsymbol{\rho}}{\partial t}\right)^2 - \left(-\frac{m}{2}A^2 - m\frac{\partial \boldsymbol{\rho}}{\partial t}\cdot\boldsymbol{A}\right) - V \tag{2.39a}$$

The first term is the kinetic energy as seen from the rotating axes. The second term relates to a position-dependent potential function $\phi = -A^2/2$. The third term is the negative of a velocity-dependent potential energy U. V is the conventional potential energy assumed to depend only on the relative positions of the masses and therefore unaffected by the choice of reference axes

$$\mathfrak{L} = \frac{m}{2}\left(\frac{\partial \boldsymbol{\rho}}{\partial t}\right)^2 - \left(m\phi + U\right) - V \tag{2.39b}$$

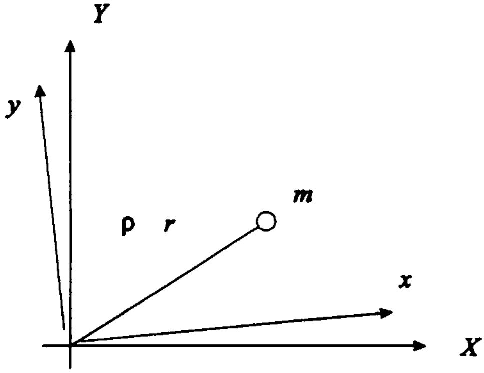

Fig. 2.6

It is interesting to note that for a charged particle, of mass m and charge $\bar{q}$, moving in a magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, where $\boldsymbol{A}$ is the magnetic vector potential, and an electric field $\boldsymbol{E} = -\nabla ø - \frac{\partial \boldsymbol{A}}{\partial t}$, where ø is a scalar potential, the Lagrangian can be shown to be

$$\mathcal{L} = -\frac{m}{2}\left(\frac{\partial \boldsymbol{\rho}}{\partial t}\right)^2 - \left(\bar{q}ø - \bar{q}\frac{\partial \boldsymbol{\rho}}{\partial t}\cdot\boldsymbol{A}\right) - V \tag{2.40}$$

This has a similar form to equation (2.39b).

From equation (2.40) the generalized momentum is

$$p_x = m\dot{x} + \bar{q}A_x$$

From equation (2.40b) the generalized momentum is

$$p_x = m\dot{x} + mA_x = m\dot{x} + m(\omega_y z - \omega_z y)$$

In neither of these expressions for generalized momentum is the momentum that as seen from the reference frame. In the electromagnetic situation the extra momentum is often attributed to the momentum of the field. In the purely mechanical problem the momentum is the same as that referenced to a coincident inertial frame. However, it must be noted that the xyz frame is rotating so the time rate of change of momentum will be different to that in the inertial frame.

EXAMPLE

An important example of a rotating co-ordinate frame is when the axes are attached to the Earth. Let us consider a special case for axes with origin at the centre of the Earth, as shown in Fig. 2.7 The z axis is inclined by an angle α to the rotational axis and the x axis initially intersects the equator. Also we will consider only small movements about the point where the z axis intersects the surface. The general form for the Lagrangian of a particle is

$$\begin{aligned}\mathcal{L} &= \frac{m}{2}\frac{\partial \boldsymbol{\rho}}{\partial t}\cdot\frac{\partial \boldsymbol{\rho}}{\partial t} + \frac{m}{2}(\boldsymbol{\Omega}\times\boldsymbol{\rho})\cdot(\boldsymbol{\Omega}\times\boldsymbol{\rho}) + m\frac{\partial \boldsymbol{\rho}}{\partial t}\cdot(\boldsymbol{\Omega}\times\boldsymbol{\rho}) - V \\ &= T' - U_1 - U_2 - V\end{aligned}$$

with

$$\boldsymbol{\Omega} = \omega_x \boldsymbol{i} + \omega_y \boldsymbol{j} + \omega_z \boldsymbol{k} \quad \text{and} \quad \boldsymbol{\rho} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

$$\boldsymbol{A} = \boldsymbol{\Omega}\times\boldsymbol{\rho} = \boldsymbol{i}(\omega_y z - \omega_z y) + \boldsymbol{j}(\omega_z x - \omega_x z) + \boldsymbol{k}(\omega_x y - \omega_y x)$$

$$\begin{aligned}\frac{m}{2}\boldsymbol{A}\cdot\boldsymbol{A} &= \frac{m}{2}[(\omega_y z - \omega_z y)^2 + (\omega_z x - \omega_x z)^2 + (\omega_x y - \omega_y x)^2] \\ &= -U_1\end{aligned}$$

and

$$\begin{aligned}m\frac{\partial \boldsymbol{\rho}}{\partial t}\cdot\boldsymbol{A} &= m\dot{x}(\omega_y z - \omega_z y) + m\dot{y}(\omega_z x - \omega_x z) + m\dot{z}(\omega_x y - \omega_y x) \\ &= -U_2\end{aligned}$$

where $\dot{x} = \frac{\partial x}{\partial t}$, etc. the velocities as seen from the moving axes.

When Lagrange's equations are applied to these functions U_1 gives rise to position-dependent fictitious forces and U_2 to velocity and position-dependent

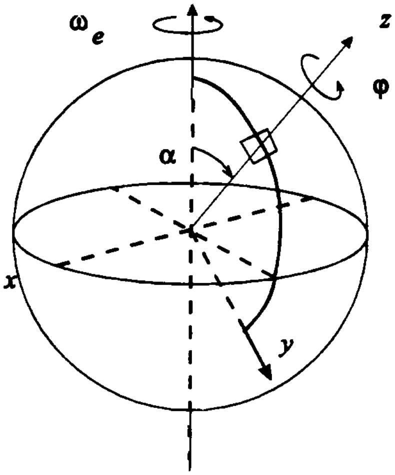

Fig. 2.7

fictitious forces. Writing $U = U_1 + U_2$ we can evaluate the x component of the fictitious force from

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \dot{x}}\right) - \left(\frac{\partial U}{\partial x}\right) = -Q_{\mathrm{f}x}$$

$$m(\omega_y \dot{z} - \omega_z \dot{y}) - m(\omega_z x - \omega_x z)\omega_z - m(\omega_x y - \omega_y x)(-\omega_y) - m(\dot{y}\omega_z - \dot{z}\omega_y) = -Q_{\mathrm{f}x}$$

or

$$-Q_{\mathrm{f}x} = m[(\omega_z^2 + \omega_y^2)x - \omega_x\omega_y y - \omega_x\omega_z z] + 2m(\omega_y \dot{z} - \omega_z \dot{y})$$

Similarly

$$-Q_{\mathrm{f}y} = m[(\omega_x^2 + \omega_z^2)y - \omega_y\omega_z z - \omega_y\omega_x x] + 2m(\omega_z \dot{x} - \omega_x \dot{z})$$

$$-Q_{\mathrm{f}z} = m[(\omega_y^2 + \omega_x^2)z - \omega_z\omega_x x - \omega_z\omega_y y] + 2m(\omega_x \dot{y} - \omega_y \dot{x})$$

For small motion in a tangent plane parallel to the xy plane we have $\dot{z} = 0$ and $z = R$, since $x \ll z$ and $y \ll z$, thus

$$-Q_{\mathrm{f}x} = m[-\omega_x\omega_z R] - 2m\omega_z \dot{y} \tag{i}$$

$$-Q_{\mathrm{f}y} = m[-\omega_y\omega_z R] + 2m\omega_z \dot{x} \tag{ii}$$

$$-Q_{\mathrm{f}z} = m(\omega_y^2 + \omega_x^2)R - 2m(\omega_x \dot{y} - \omega_y \dot{x}) \tag{iii}$$

We shall consider two cases:

Case 1, where the xyz axes remain fixed to the Earth:

$$\omega_x = 0 \quad \omega_y = -\omega_e \sin\alpha \quad \text{and} \quad \omega_z = \omega_e \cos\alpha$$

Equations (i) to (iii) are now

$$-Q_{\mathrm{f}x} = -2m\omega_e \cos\alpha\, \dot{y}$$

$$-Q_{\mathrm{f}y} = m(\omega_e^2 \sin\alpha\, \cos\alpha\, R) + 2m\omega_e \cos\alpha\, \dot{x}$$

$$-Q_{\mathrm{f}z} = m(\omega_e^2 \sin^2\alpha)R - 2m\omega_e \sin\alpha\, \dot{x}$$

from which we see that there are fictitious Coriolis forces related to $\dot{x}$ and $\dot{y}$ and also some position-dependent fictitious centrifugal forces. The latter are usually absorbed in the modified gravitational field strength. In practical terms the value of g is reduced by some 0.3% and a plumb line is displaced by about 0.1°.

Case 2, where the xyz axes rotate about the z axis by angle ϕ:

$$\omega_x = \omega_e \sin\alpha \, \sin\phi, \quad \omega_y = -\omega_e \sin\alpha \, \cos\phi \quad \text{and} \quad \omega_z = \omega_e \cos\alpha + \dot{\phi}$$

We see that if $\dot{\phi} = -\omega_e \cos\alpha$ then $\omega_z = 0$, so the Coriolis terms in equations (i) and (ii) disappear. Motion in the tangent plane is now the same as that in a plane fixed to a non-rotating Earth.

2.9 Moving co-ordinates

In this section we shall consider the situation in which the co-ordinate system moves with a group of particles. These axes will be translating and rotating relative to an inertial set of axes. The absolute position vector will be the sum of the position vector of a reference point to the origin plus the position vector relative to the moving axes. Thus, referring to Fig. 2.8, $\boldsymbol{r}_j = \boldsymbol{R} + \boldsymbol{\rho}_j$ so the kinetic energy will be

$$T = \sum_j \frac{m_j}{2} \dot{\boldsymbol{r}}_j \cdot \dot{\boldsymbol{r}}_j = \sum_j \frac{m_j}{2} (\dot{\boldsymbol{R}} \cdot \dot{\boldsymbol{R}} + \dot{\boldsymbol{\rho}}_j \cdot \dot{\boldsymbol{\rho}}_j + 2\dot{\boldsymbol{R}} \cdot \dot{\boldsymbol{\rho}}_j)$$

Denoting $\sum_j m_j = m$, the total mass,

$$T = \frac{m}{2} \dot{\boldsymbol{R}} \cdot \dot{\boldsymbol{R}} + \sum_j \frac{m_j}{2} \dot{\boldsymbol{\rho}}_j \cdot \dot{\boldsymbol{\rho}}_j = \dot{\boldsymbol{R}} \cdot \sum_j m_j \cdot \dot{\boldsymbol{\rho}}_j \tag{2.41}$$

Here the dot above the variables signifies differentiation with respect to time as seen from the inertial set of axes. In the following arguments the dot will refer to scalar differentiation.

If we choose the reference point to be the centre of mass then the third term will vanish. The first term on the right hand side of equation (2.41) will be termed T_0 and is the kinetic energy of a single particle of mass m located at the centre of mass. The second term will be

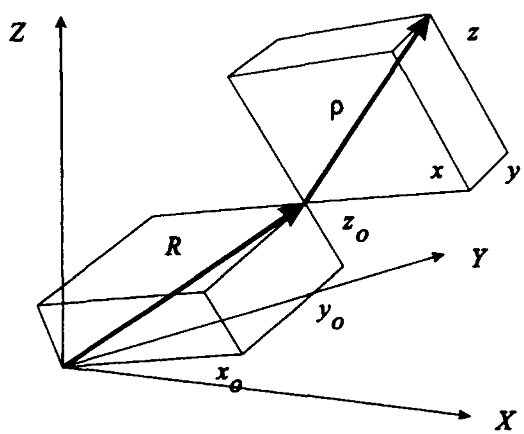

Fig. 2.8

denoted by T_G and is the kinetic energy due to motion relative to the centre of mass, but still as seen from the inertial axes.

The position vector $\boldsymbol{R}$ can be expressed in the moving co-ordinate system xyz, the specific components being x_0, y_0 and z_0,

$$\boldsymbol{R} = x_0\boldsymbol{i} + y_0\boldsymbol{j} + z_0\boldsymbol{k}$$

By the rules for differentiation with respect to rotating axes

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t_{xyz}} &= \frac{\partial \boldsymbol{R}}{\partial t_{xyz}} + \omega \times \boldsymbol{R} \\ &= \dot{x}_0\boldsymbol{i} + \dot{y}_0\boldsymbol{j} + \dot{x}_0\boldsymbol{k} + (\omega_y z_0 - \omega_z y_0)\boldsymbol{i} \\ &\qquad + (\omega_z x_0 - \omega_x z_0)\boldsymbol{j} + (\omega_x y_0 - \omega_y x_0)\boldsymbol{k}\end{aligned}$$

so

$$T_0 = \frac{m}{2}\Big[\dot{x}_0\boldsymbol{i} + \dot{y}_0\boldsymbol{j} + \dot{x}_0\boldsymbol{k} + (\omega_y z_0 - \omega_z y_0)\boldsymbol{i} + (\omega_z x_0 - \omega_x z_0)\boldsymbol{j} + (\omega_x y_0 - \omega_y x_0)\boldsymbol{k}\Big]^2 \tag{2.42}$$

Similarly with $\boldsymbol{\rho}_j = x_j\boldsymbol{i} + y_j\boldsymbol{j} + z_j\boldsymbol{k}$ we have

$$T_G = \sum_j \frac{m_j}{2}\Big[\dot{x}_j\boldsymbol{i} + \dot{y}_j\boldsymbol{j} + \dot{x}_j\boldsymbol{k} + (\omega_y z_j - \omega_z y_j)\boldsymbol{i} + (\omega_z x_j - \omega_x z_j)\boldsymbol{j} + (\omega_x y_j - \omega_y x_j)\boldsymbol{k}\Big]^2 \tag{2.43}$$

The Lagrangian is

$$\mathfrak{L} = T_0(x_0\, y_0\, z_0\, \dot{x}_0\, \dot{y}_0\, \dot{z}_0) + T_G(x_j\, y_j\, z_j\, \dot{x}_j\, \dot{y}_j\, \dot{z}_j) - V \tag{2.44}$$

Let the linear momentum of the system be $\boldsymbol{p}$. Then the resultant force $\boldsymbol{F}$ acting on the system is

$$\boldsymbol{F} = \frac{\mathrm{d}}{\mathrm{d}t_{xyz}}\,\boldsymbol{p} = \frac{\mathrm{d}}{\mathrm{d}t_{xyz}}\,\boldsymbol{p} + \omega \times \boldsymbol{p}$$

and the component in the x direction is

$$F_x = \frac{\mathrm{d}}{\mathrm{d}t_{xyz}}\,p_x = \frac{\mathrm{d}}{\mathrm{d}t_{xyz}}\,p_x + (\omega_y p_z - \omega_z p_y)$$

In this case the momenta are generalized momenta so we may write

$$F_x = Q_x = \frac{\mathrm{d}}{\mathrm{d}t_{xyz}}\left(\frac{\partial \mathfrak{L}}{\partial \dot{x}_0}\right) - \left(\omega_z \frac{\partial \mathfrak{L}}{\partial \dot{y}_0} - \omega_y \frac{\partial \mathfrak{L}}{\partial \dot{z}_0}\right) \tag{2.45}$$

If Lagrange's equations are applied to the Lagrangian, equation (2.44), exactly the same equations are formed, so it follows that in this case the contents of the last term are equivalent to $\partial\mathfrak{L}/\partial x_0$.

If the system is a rigid body with the xyz axes aligned with the principal axes then the kinetic energy of the body for motion relative to the centre of mass T_G is

$$T_G = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$$, see section 4.5

The modified form of Lagrange's equation for angular motion

$$Q_{\omega x} = \frac{\mathrm{d}}{\mathrm{d}t_{xyz}}\left(\frac{\partial \mathfrak{L}}{\partial \dot{\omega}_x}\right) - \left(\omega_z \frac{\partial \mathfrak{L}}{\partial \dot{\omega}_y} - \omega_y \frac{\partial \mathfrak{L}}{\partial \dot{\omega}_z}\right) \tag{2.46}$$

yields

$$Q_{\omega x} = I_x \dot{\omega}_x - \left(\omega_z I_y \omega_y - \omega_y I_z \omega_z\right) \tag{2.47}$$

In this equation ω_x is treated as a generalized velocity but there is not an equivalent generalized co-ordinate. This, and the two similar ones in $Q_{\omega y}$ and $Q_{\omega z}$, form the well-known Euler's equations for the rotation of rigid bodies in space.

For flexible bodies T_G is treated in the usual way, noting that it is not a function of x_0, $\dot{x}_0$ etc., but still involves ω.

2.10 Non-holonomic systems

In the preceding part of this chapter we have always assumed that the constraints are holonomic. This usually means that it is possible to write down the Lagrangian such that the number of generalized co-ordinates is equal to the number of degrees of freedom. There are situations where a constraint can only be written in terms of velocities or differentials.

One often-quoted case is the problem of a wheel rolling without slip on an inclined plane (*see* Fig. 2.9).

Assuming that the wheel remains normal to the plane we can write the Lagrangian as

$$\mathfrak{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_1\dot{\varnothing}^2 + \frac{1}{2}I_2\dot{\psi}^2 - mg(\sin\alpha\, y + \cos\alpha\, r)$$

The equation of constraint may be written

$$\mathrm{d}s = r\,\mathrm{d}\varnothing$$

or as

$$\mathrm{d}x = \mathrm{d}s \sin\psi = r \sin\psi\, \mathrm{d}\varnothing$$
$$\mathrm{d}y = \mathrm{d}s \cos\psi = r \cos\psi\, \mathrm{d}\varnothing$$

We now introduce the concept of the *Lagrange undetermined multipliers* λ. Notice that each of the constraint equations may be written in the form $\Sigma a_{jk}\, dq_j = 0$; this is similar in form to the expression for virtual work. Multiplication by λ_k does not affect the equality but the dimensions of λ_k are such that each term has the dimensions of work. A modified virtual work expression can be formed by adding all such sums to the existing expression for virtual work. So $\delta W' = \delta W + \Sigma(\lambda_k \Sigma a_{jk} \mathrm{d}q_j)$; this means that extra generalized forces will be formed and thus included in the resulting Lagrange equations.

Applying this scheme to the above constraint equations gives

$$\lambda_1 \mathrm{d}x - \lambda_1(r \sin\psi)\mathrm{d}\varnothing = 0$$
$$\lambda_2 \mathrm{d}y - \lambda_2(r \cos\psi)\mathrm{d}\varnothing = 0$$

The only term in the virtual work expression is that due to the couple C applied to the shaft, so $\delta W = C\,\delta\varnothing$. Adding the constraint equation gives

$$\delta W' = C\,\delta\varnothing + \lambda_1 \mathrm{d}x + \lambda_2 \mathrm{d}y - [\lambda_1(r \sin\psi) + \lambda_2(r \cos\psi)]\mathrm{d}\varnothing$$

Applying Lagrange's equations to $\mathfrak{L}$ for $q = x, y, \varnothing$ and ψ in turn yields

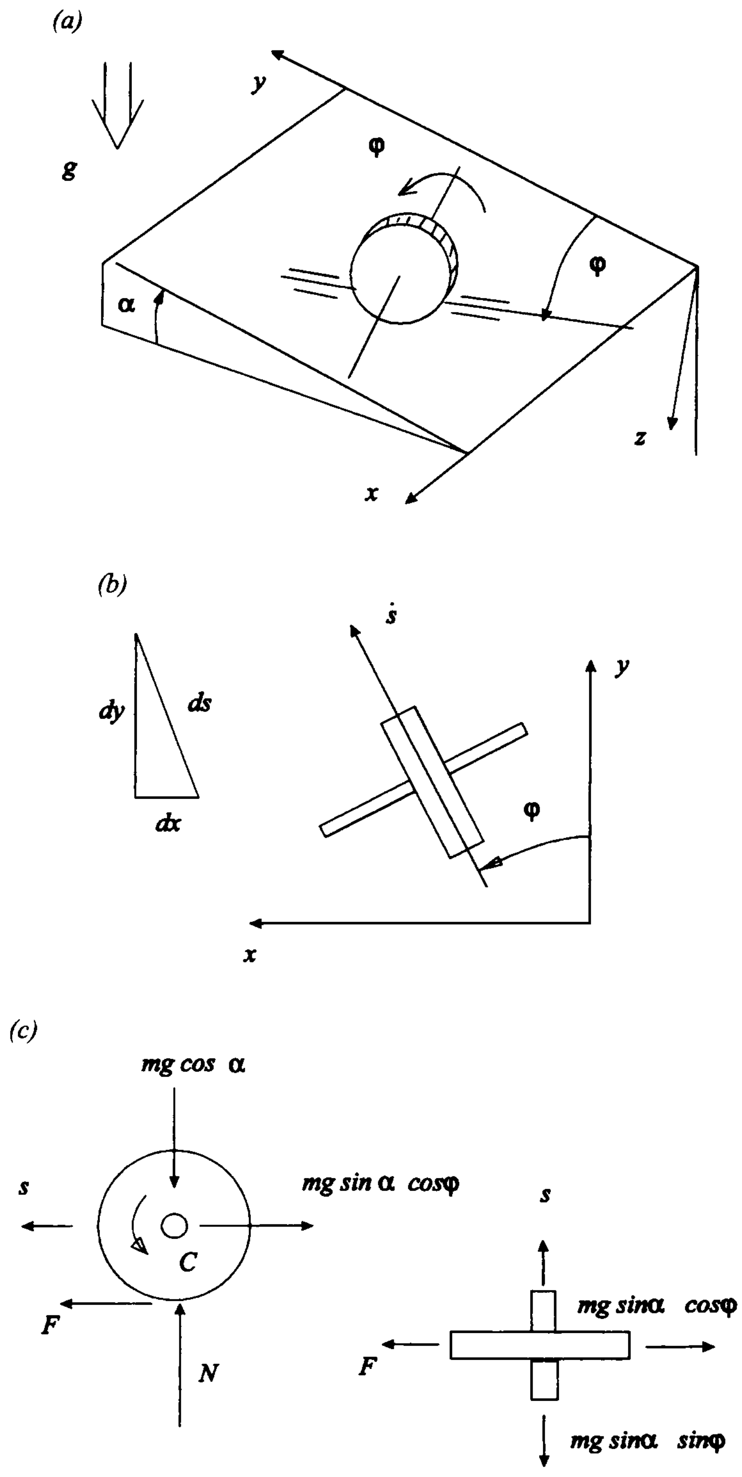

Fig. 2.9 (a), (b) and (c)

$$
\begin{aligned}
m\ddot{x} &= \lambda_1 \\
m\ddot{y} + mg\sin\alpha &= \lambda_2 \\
I_1\ddot{\phi} &= C - [\lambda_1(r\sin\psi) + \lambda_2(r\cos\psi)] \\
I_2\ddot{\psi} &= 0
\end{aligned}
$$

In addition we still have the constraint equations

$$
\begin{aligned}
\dot{x} &= r\sin\psi\,\dot{\phi} \\
\dot{y} &= r\cos\psi\,\dot{\phi}
\end{aligned}
$$

Simple substitution will eliminate λ_1 and λ_2 from the equations.

From a free-body diagram approach it is easy to see that

$$
\begin{aligned}
\lambda_1 &= F\sin\psi \\
\lambda_2 &= F\cos\psi
\end{aligned}
$$

and

$$[\lambda_1(r\sin\psi) + \lambda_2(r\cos\psi)] = -Fr$$

The use of Lagrange multipliers is not restricted to non-holonomic constraints, they may be used with holonomic constraints; if the force of constraint is required. For example, in this case we could have included $\lambda_3 \mathrm{d}z = 0$ to the virtual work expression as a result of the motion being confined to the xy plane. (It is assumed that gravity is sufficient to maintain this condition.) The equation of motion in the z direction is

$$-mg\cos\alpha = \lambda_3$$

It is seen here that $-\lambda_3$ corresponds to the normal force between the wheel and the plane.

However, non-holonomic systems are in most cases best treated by free-body diagram methods and therefore we shall not pursue this topic any further. (*See* Appendix 2 for methods suitable for non-holonomic systems.)

2.11 Lagrange's equations for impulsive forces

The force is said to be impulsive when the duration of the force is so short that the change in the position co-ordinates is negligible during the application of the force. The variation in any body forces can be neglected but contact forces, whether elastic or not, are regarded as external. The Lagrangian will thus be represented by the kinetic energy only and by the definition of short duration $\partial T/\partial q$ will also be negligible. So we write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = Q_j$$

Integrating over the time of the impulse τ gives

$$\Delta\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \int_0^{t} Q_j\,\mathrm{d}t \tag{2.48}$$

or

$$\Delta\,[\text{generalized momentum}] = \text{generalized impulse}$$

$$\Delta p_j = J_j$$

EXAMPLE

The two uniform equal rods shown in Fig. 2.10 are pinned at B and are moving to the right at a speed V. End A strikes a rigid stop. Determine the motion of the two bodies immediately after the impact. Assume that there are no friction losses, no residual vibration and that the impact process is elastic.

The kinetic energy is given by

$$T = \frac{m}{2}\dot{x}_1^2 + \frac{m}{2}\dot{x}_2^2 + \frac{I}{2}\dot{\theta}_1^2 + \frac{I}{2}\dot{\theta}_2^2$$

The virtual work done by the impact force at A is

$$\delta W = F(-\mathrm{d}x_1 + a\,\mathrm{d}\theta_1)$$

and the constraint equation for the velocity of point B is

$$\dot{x}_1 + a\dot{\theta}_1 = \dot{x}_2 - a\dot{\theta}_2 \tag{ia}$$

or, in differential form,

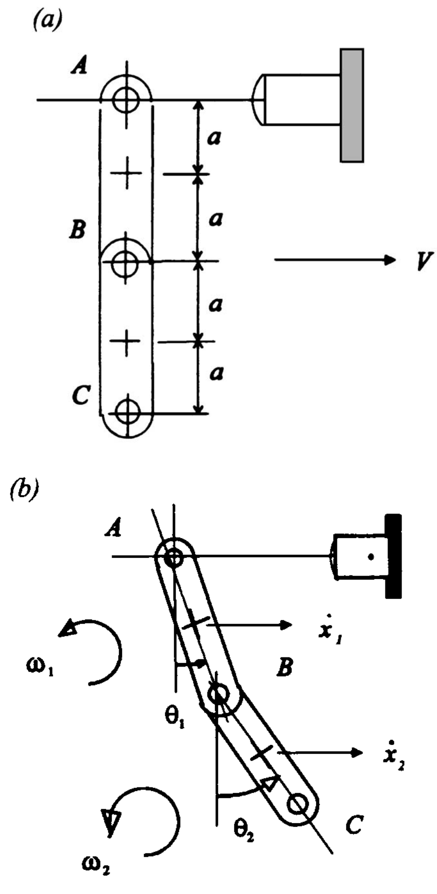

Fig. 2.10 (a) and (b)

$$dx_1 - dx_2 + a d\theta_1 + a d\theta_2 = 0 \tag{ib}$$

There are two ways of using the constraint equation: one is to use it to eliminate one of the variables in T and the other is to make use of Lagrange multipliers. Neither has any great advantage over the other; we shall choose the latter. Thus the extra terms to be added to the virtual work expression are

$$\lambda[dx_1 - dx_2 + a d\theta_1 + a d\theta_2]$$

Thus the effective virtual work expression is

$$\delta W' = F(-dx_1 + a d\theta_1) + \lambda[dx_1 - dx_2 + a d\theta_1 + a d\theta_2]$$

Applying the Lagrange equations for impulsive forces

$$m(\dot{x}_1 - V) = -\int F dt + \int \lambda dt \tag{ii}$$
$$m(\dot{x}_2 - V) = -\int \lambda\, dt \tag{iii}$$
$$I\dot{\theta}_1 = \int aF\, dt + \int a\lambda dt \tag{iv}$$
$$I\dot{\theta}_2 = \int a\lambda dt \tag{v}$$

There are six unknowns but only five equations (including the equation of constraint, equation (i)). We still need to include the fact that the impact is elastic. This means that at the impact point the displacement–time curve must be symmetrical about its centre, in this case about the time when point A is momentarily at rest. The implication of this is that, at the point of contact, the speed of approach is equal to the speed of recession. It is also consistent with the notion of reversibility or time symmetry.

Our final equation is then

$$V = a\dot{\theta}_1 - \dot{x}_1 \tag{vi}$$

Alternatively we may use conservation of energy. Equating the kinetic energies before and after the impact and multiplying through by 2 gives

$$mV^2 = m\dot{x}_1^2 + m\dot{x}_2^2 + I\dot{\theta}_1^2 + I\dot{\theta}_2^2 \tag{vi a}$$

It can be demonstrated that using this equation in place of equation (vi) gives the same result. From a free-body diagram approach it can be seen that λ is the impulsive force at B.

We can eliminate the impulses from equations (ii) to (v). One way is to add equation (iii) times 'a' to equation (v) to give

$$m(\dot{x}_2 - V)a + I\dot{\theta}_2 = 0 \tag{vii}$$

Also by adding 3 times equation (iii) to the sum of equations (ii), (iv) and (v) we obtain

$$m(\dot{x}_1 - V)a + 3m(\dot{x}_2 - V)a + I\dot{\theta}_1 + I\dot{\theta}_2 = 0 \tag{viii}$$

This equation may be obtained by using conservation of moment of momentum for the whole system about the impact point and equation (vii) by the conservation of momentum for the lower link about the hinge B.

Equations (ia), (vi), (vii) and (viii) form a set of four linear simultaneous equations in the unknown velocities $\dot{x}_1$, $\dot{x}_2$, $\dot{\theta}_1$ and $\dot{\theta}_2$. These may be solved by any of the standard methods.