Mini project

Dynamics identification of a second-order object

Introduction

The project objective is to determine the parameters of a second-order dynamic system based on the measurements of a physical object. An example of such a system is a mass mounted on a spring whose free movement is obstructed by the friction forces. Using the provided measurement data (obtained experimentally), the system dynamics is to be described quantitatively and numerical studies are to be performed in a simulation environment.

Model of object dynamics

An example object, described via a second-order differential equation, is illustrated in Figure 1. The mass m is attached to the wall by a spring (the spring is characterized by constant elasticity $k > 0$). Once deflected from the equilibrium position, the mass enters into oscillatory motion inflicted by the spring force – F_S, damped by the friction force – F_T. If the energy losses are not compensated (e.g., by an external force), the oscillations will cease in time and the mass will eventually stop.

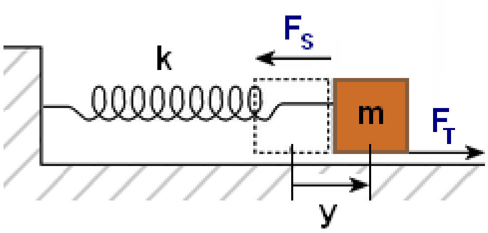

Figure 1. Mass-spring second-order system

Force F_S, directed opposite to the mass displacement, is determined by the formula

$$F_S(t) = -ky(t), \tag{1}$$

where y is the displacement and t is the independent variable (continuous) that tracks the evolution of time.

The friction force is given by

$$F_T(t) = -cv(t), \tag{2}$$

where $v(t)$ is the velocity of the mass center and $c > 0$ is a proportionality factor.

The resulting force acting on the mass $F_W = a(t)m$, where $a(t)$ is the acceleration, is given by

$$F_W = F_S + F_T. \tag{3}$$

Given the fundamental mechanical dependencies $v = \dot{y}$ and $a = \dot{v} = \ddot{y}$, after substituting (1) and (2) into (3), the following relation is obtained

$$m\ddot{y} = -ky - c\dot{y}, \tag{4}$$

which after dividing by m yields

$$\ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = 0. \tag{5}$$

Relation (5) constitutes the dynamic equation of mass motion.

It can be assumed that the mass of the object m has been measured with sufficiently high accuracy (m is treated as a predetermined constant in the identification process). Using the experimental data, retrieved for the mass motion initiated under different initial conditions, c and k parameters are to be calculated.

Harmonic oscillator

The dynamics of damped harmonic oscillator can be described in a general form as

$$\ddot{y} + 2\zeta\omega_0\dot{y} + \omega_0^2 y = 0, \tag{6}$$

where ω_0 is the angular frequency of undamped oscillator and ζ is the damping factor. For the considered mass-spring system, these quantities are specified by

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \text{and} \quad \zeta = \frac{c}{2\sqrt{km}}. \tag{7}$$

The solution of equation (6) under the initial displacement y_0 is given by

$$y(t) = y_0 e^{-\zeta\omega_0 t}\sin\left(\omega t + \varphi\right), \tag{8}$$

where φ – the phase shift – does not affect the model parameters. The frequency of damped oscillations is given by

$$\omega = \frac{2\pi}{T} = \omega_0\sqrt{1-\zeta^2}, \tag{9}$$

where T is the oscillation period. An example system response is illustrated in Figure 2.

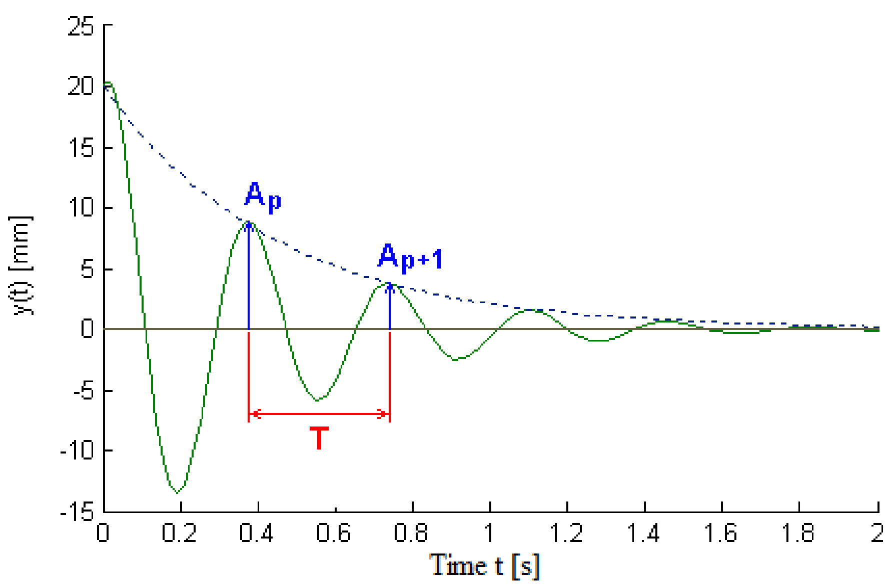

Figure 2. Impulse response of damped harmonic oscillator.

By analyzing the history of $y(t)$, one can determine the value of parameters c and k, thus identify model (5). In particular, using the information about the period of damped oscillations in relation to the amplitude suppression, c may be obtained from

$$c = -\frac{2m}{T}\ln\left(\frac{A_{p+1}}{A_p}\right), \tag{10}$$

where $\ln(x)$ is the natural logarithm of x, A_p – the response amplitude at time t_p (with $\sin(\cdot) = 1$), and A_{p+1} – the response amplitude T seconds afterwards (at time $t_p + T$). Then, according to (7) and (9), we can formulate an expression to calculate k:

$$k = \frac{(cT)^2 + (4\pi m)^2}{4mT^2}. \tag{11}$$

Tasks to perform

1. Based on the measurement data from the files: experiment_1.csv, experiment_2.csv, and experiment_3.csv, the nominal (mean) values of parameters c and k should be established. One needs also present the extreme values. In the calculations, assume the mass $m = 0.791$ kg.

   The values in the first column in the *.csv files reflect the time in seconds, and the values in the second column – the displacement in mm.

   Write a program (script) to automate the analysis of the measurement data and provide the information about:

   - the values of damped oscillation period T and exponential decay factor $\zeta\omega_0$ corresponding to equation (8),
   - the nominal values of T and $\zeta\omega_0$ and their range,
   - the nominal and extreme values of parameters c and k.

   The variable size of the data set (the number of records in the *.csv files) should be explicitly considered while reading the data from the files.

   Gather the results in tables.

2. Express the model in a state-space form:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t)\\ y(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned} \tag{12}$$

   where $\mathbf{x} = [x_1\ x_2]^T$ is the state vector with variables x_1 and x_2, and $[\cdot]^T$ – representing the transpose. The first state variable should reflect the displacement and the second variable – the mass center velocity. As the system output, choose the displacement.

   Determine the model eigenvalues and comment on the system dynamical properties (stability, oscillations).

   Perform numerical tests and plot the waveforms showing the oscillation, velocity, and acceleration for the initial condition $\mathbf{x}(0) = [70\ 120]$.

Recommended reading

1. John B. Little, *Modeling and Data Analysis: An Introduction with Environmental Applications*, American Mathematical Society, Providence, RI, 2019
2. Gene Franklin, J. David Powell, Abbas Emami-Naeini, *Feedback Control of Dynamic Systems*, Prentice Hall, Upper Saddle River, NJ, 2010
3. Gene Franklin, J. David Powell, Michael L. Workman, *Digital Control of Dynamic Systems*, Addison-Wesley Longman, Inc., Menlo Park, CA, 1998
4. Katsuhiko Ogata, *Modern Control Engineering*, Prentice Hall, Upper Saddle River, NJ, 2010
5. Katsuhiko Ogata, *Discrete-Time Control Systems*, Prentice Hall, Upper Saddle River, NJ, 1995