0.1 Steady-state Approximation of Effective Viscosity

We begin with a calculation of a strain rate estimate of the effective viscosity for a network described by our model in the limit of highly rigid filaments. We carry this out by assuming we have applied a constant stress along a transect of the network. With moderate stresses, we assume the network reaches a steady state affine creep. In this situation, we would find that the stress in the network exactly balances the sum of the drag-like forces from cross-link slip. So for any transect of length D, we have a force balance equation.

$$\sigma = \frac{1}{D} \sum_{filaments} \sum_{crosslinks} \xi \cdot (\mathbf{v_i}(\mathbf{x}) - \mathbf{v_j}(\mathbf{x})) \tag{1}$$

where $\mathbf{v_i}(\mathbf{x}) - \mathbf{v_j}(\mathbf{x})$ is the difference between the velocity of a filament, i, and the velocity of the filament, j, to which it is attached at the cross-link location, $\mathbf{x}$. We can convert the sum over cross-links to an integral over the length using the average density of cross-links, $1/l_c$ and invoking the assumption of (linear order) affine strain rate, $\mathbf{v_i}(\mathbf{x}) - \mathbf{v_j}(\mathbf{x}) = \dot{\gamma} x$. This results in

$$\begin{aligned}\sigma = \frac{1}{D} \sum_{filaments} \int_0^L \xi \cdot \ (\mathbf{v_i}(\mathbf{s}) - \mathbf{v_j}(\mathbf{s}))\, \frac{ds \cos\theta}{l_c} \\ = \sum_{filaments} \frac{\xi \dot{\gamma} L}{l_c} \cos\theta \cdot (x_l + \frac{L}{2}\cos\theta)\end{aligned} \tag{2}$$

Here we have introduced the variables x_l, and θ to describe the leftmost endpoint and the angular orientation of a given filament respectively. Next, to perform the sum over all filaments we convert this to an integral over all orientations and endpoints that intersect our line of stress. We assume for simplicity that filament stretch and filament alignment are negligible in this low strain approximation. Therefore, the max distance for the leftmost endpoint is the length of a filament, L, and the maximum angle as a function of endpoint is $\arccos(x_l/L)$. The linear density of endpoints is the constant $D/l_c L$ so our integrals can be rewritten as this density over x_l and θ between our maximum and minimum allowed bounds.

$$\sigma = \frac{1}{D} \int_0^L dx_l \int_{-\arccos(\frac{x_l}{L})}^{\arccos(\frac{x_l}{L})} \pi d\theta \frac{\xi \dot{\gamma} L}{l_c} \cdot \frac{D}{L l_c} \cdot (x_l \cos\theta + \frac{L}{2} cos^2\theta) \tag{3}$$

Carrying out the integrals and correcting for dangling filament ends leaves us with a relation between stress and strain rate.

$$\sigma = 4\pi \left(\frac{L}{l_c} - 1 \right)^2 \xi \dot{\gamma} \tag{4}$$

We recognize the constant of proportionality between stress and strain rate as a viscosity (in 2 dimensions). Therefore, our approximation for the effective viscosity, η_c, at steady state creep in this low strain limit is

$$\sigma = 4\pi \left(\frac{L}{l_c} - 1 \right)^2 \xi \tag{5}$$

0.2 Steady-state Approximation of Effective Viscosity

We seek to determine a critical filament lifetime, τ_{crit} , below which the density of filaments approaches a stable steady state under constant extensional strain. To this end, let ρ be the filament density (i.e. number of filaments per unit area). We consider a simple coarse grained model for how ρ changes as a function of filament assembly k_{ass}, filament disassembly k_{diss}, ρ and strain thinning $\dot{\gamma}\rho$. Using $\rho_0 = \frac{k_{ass}}{k_{diss}}$, $\tau_r = \frac{1}{k_{diss}}$, and $\dot{\gamma} = \frac{\sigma}{\eta_c}$.

$$\frac{d\rho}{dt} = \frac{1}{\rho}\left(\rho_0 - \rho - \frac{\sigma\tau_r}{\eta_c(\rho)}\right) \tag{6}$$

where $\eta_c = \eta_c(\rho)$ on the right hand side reflects the dependence of effective viscosity on network density. The strength of this dependence determines whether a stable steady state, representing continuous flow, can exit. Using $\eta_c(\rho) \sim \xi\left(\frac{L}{l_c(\rho)} - 1\right)^2$ from above (ignoring the numerical prefactor) and $\rho \sim \frac{2}{Ll_c(\rho)}$, we obtain:

$$\frac{d\rho}{dt} = \frac{1}{\rho}\left(\rho_0 - \rho - \frac{\sigma\tau_r}{\xi(\rho L^2/2 - 1)^2}\right) \tag{7}$$

plotting at ρ_0 vs $\rho + \frac{\sigma\tau_r}{\eta_c(\rho)}\rho$, we can see that for sufficiently large τ_r, there is no stable state, i.e. strain thinning will occur. However, as τ_r deceases below a critical value τ_{crit}, a stable steady state appears. Note that when $\tau_r \tau_{crit}$, $\rho + \frac{}{den}\rho$ passes through a minimum value ρ_0 at $\rho = \rho^*$. Accordingly, we solve:

$$0 = \frac{d}{d\rho}\left(\rho + \frac{\sigma\tau_r}{\eta_c(\rho)}\rho\right) = 1 - \frac{\sigma\tau_r}{\xi(\rho L^2/2 - 1)^3} \tag{8}$$

From this, with some algebra, we infer that

$$\rho^* = \frac{2}{L^2}\left(1 + \left(\frac{\sigma\tau_r}{\xi}\right)^{1/3}\right) \tag{9}$$

and

$$\frac{\sigma\tau_r}{\eta_c(\rho^*)} = \left(\frac{\sigma\tau_r}{\xi}\right)^{1/3} \tag{10}$$

We seek a value for $\tau_r = \tau_{crit}$ at which

$$\rho^* + \frac{\sigma\tau_{crit}}{\eta_c(\rho^*)}\rho^* = \rho_0 \tag{11}$$

Substituting from above, and using $\rho_0 = \frac{2}{Ll_c}$, we have:

$$\frac{2}{L^2}\left(1 + \left(\frac{\sigma\tau_r}{\xi}\right)^{1/3}\right)\left(1 + \left(\frac{\sigma\tau_r}{\xi}\right)^{1/3}\right) = \frac{2}{Ll_c} \tag{12}$$

Finally, rearranging terms, we obtain

$$\tau_{crit} = \frac{\xi}{\sigma}\left(\sqrt{\frac{L}{l_c}} - 1\right)^3 \tag{13}$$

Table 1. Simulation Parameter Values

Parameter	**Figure 3**	**Figure 4**	**Figure S2a,b**	**Figure S2c,d**	**Figure 7**	**Figure 9**
L	$1, 3, 5, 7, 10$	3	$3, 5$	$3, 5$	5	$3, 5, 8$
l_c	$0.2, 0.3, 0.5, 0.8$	$0.3, 0.5$	0.3	$0.15, 0.2, 0.3, 0.4$	$0.2, 0.3$	$0.15, 0.2, 0.3, 0.4$
μ_e/μ_c	100	100	$3-300$	100	100	100
μ_c	0.01	0.01	$0.01-0.3$	$0.001-0.03$	0.01	0.01
ξ	$0.1, 1$	$0.05, 0.1, 1$	$0.01, 0.1, 1$	$0.1, 1$	$0.1, 1, 3.3$	$0.1, 1$
υ			$0.1, 0.3, 1$	$0.1, 1$	$0.1, 1, 3$	0.1
ϕ			0.25	0.5	$0.25, 0.75$	0.25
τ_r		$0.1-10^4$			$0.01-10^3$	$0.01-10^3$
σ	$0.0002-0.01$	$0.00003-0.005$				

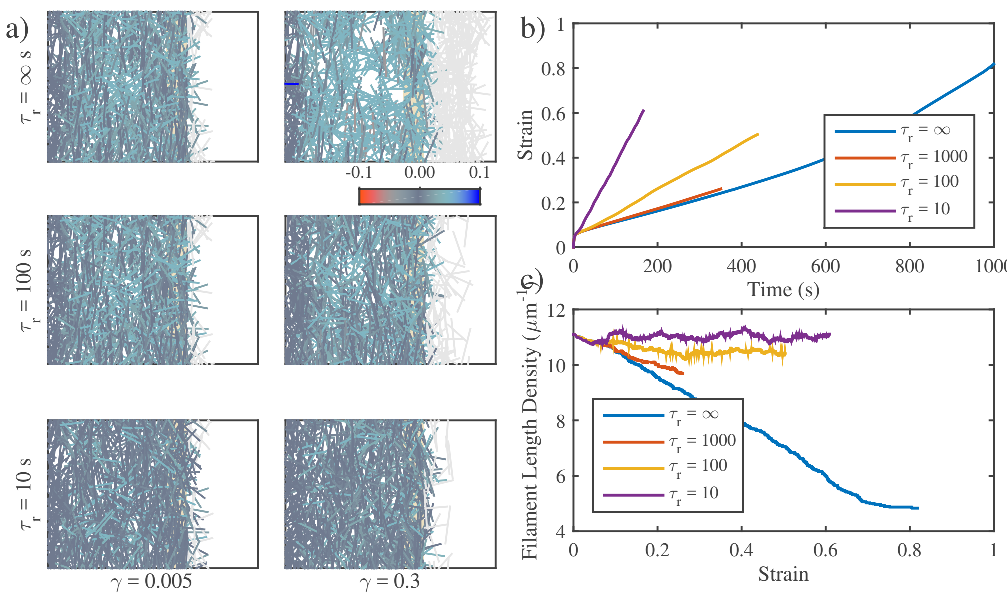

Figure 1. Filament recycling rescues strain thinning. **a)** Example networks with different levels of filament recycling. **b)** Strain curves for networks in (a). Note that the networks with faster recycling have higher strain rates and that their strain rates appear more constant than the $\tau_r = \infty$ case. **c)** Graph of filament density vs strain for the networks in (a). Note that networks with recycling appear to reach steady state densities, while the network without recycling continues to thin until it tore at 80% strain.

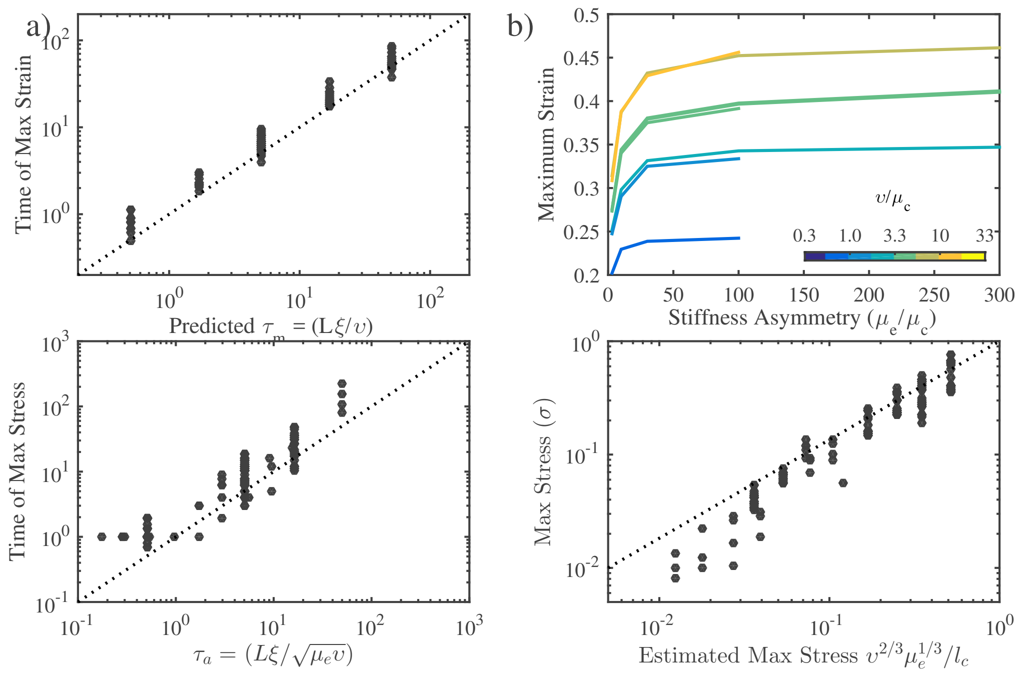

Figure 2. Mechanical properties of active networks. **a)** Timescale of maximum strain in networks free to contract. This relationship was found phenomenologically. **b** The network's ability to deform relies on the magnitude of asymmetric filament compliance. Total network strain also increases with the applied myosin force υ. Note that the extent of contraction approaches an asymptotic limit as the stiffness asymmetry approaches a ratio of ~ 100. **c** Time at which the network reached its maximum stress. τ_a was found phenomenologically. **d** Max stress of the network. Dependence was found phenomenologically.

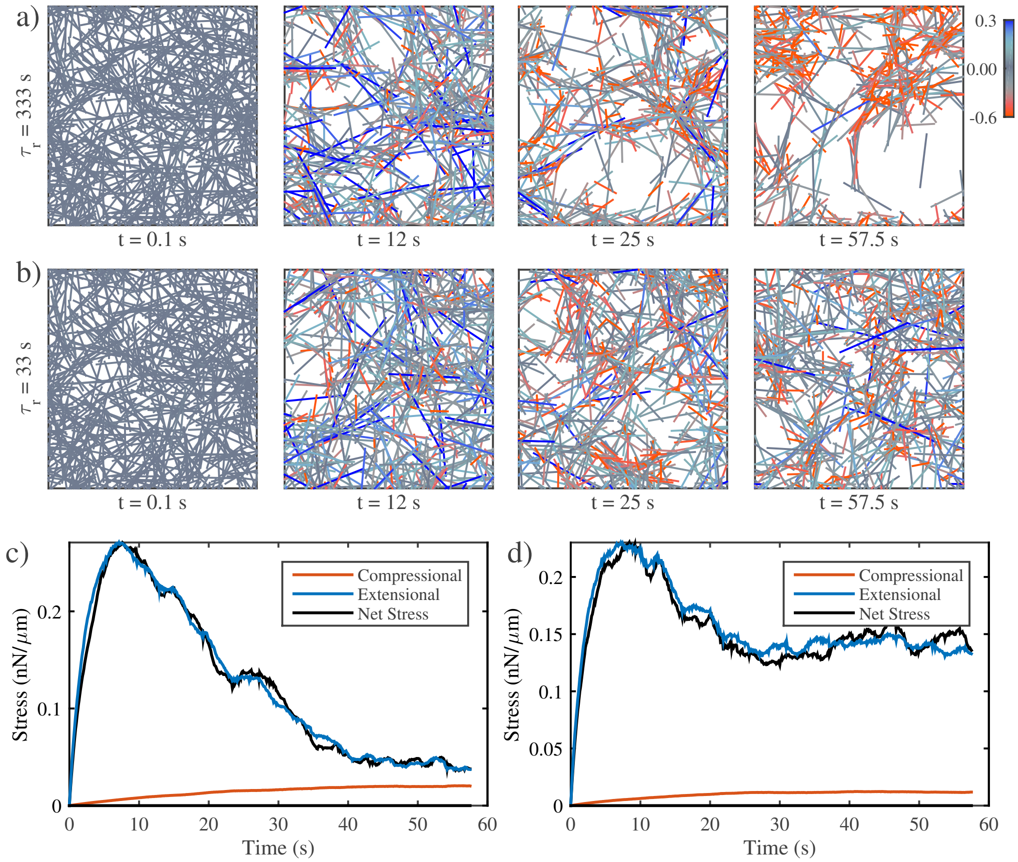

Figure 3. Tearing of active networks is prevented via recycling. **a)** An active network undergoing large scale deformations due to active filament rearrangements. **b)** The same network as in a) but with a shorter filament recycling time. **c)** Time trace of internal stresses for network in panel a. **d)** Time trace of internal stresses for network in panel b.

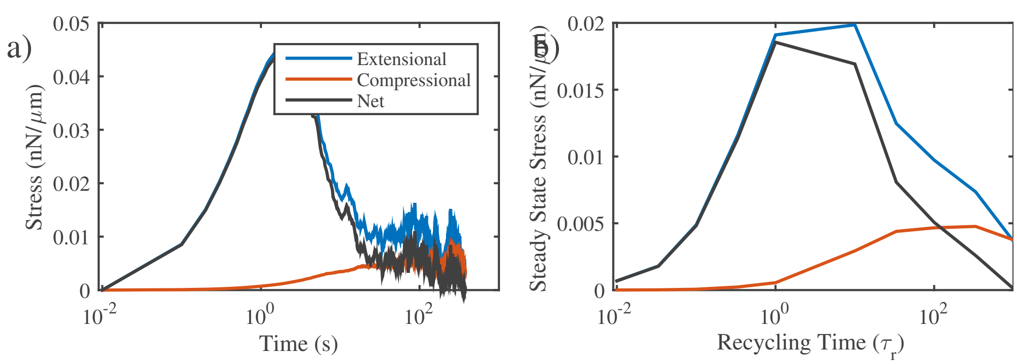

Figure 4. Bimodal recycling dependence mirrors bimodal stress buildup. **a)** Bimodal buildup of stress in a network with long filament lifetime ($\tau_r = 1000s$). **b)** Steady state stress for networks with same parameters as in a) with varying recycling times.

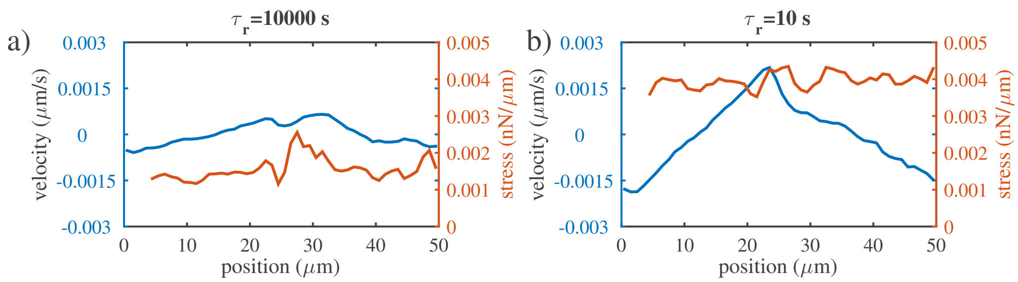

Figure 5. Stress and strain profiles of networks with contractile and passive domains. **a)** Blue line indicates strain velocity profile while orange represents net stress as measured in the main text. **b)** Same as panel a except for the condition where recycling time is 10 s. Note the increase in net stress and the corresponding increase in flow rate.