A fixed point theorem for COFEs

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1 INTRODUCTION

A COFE (*complete ordered family of equivalences*) is a set equipped with an approximate equality ($\equiv_n$) (where $a \equiv_n b$ is read as "a and b are equal for n steps") and the ability to take certain limits. These were introduced by Di Gianantonio and Miculan [1] to model recursive definitions, and have become a standard tool in step-indexed logics such as Iris [3].

The main theorem that makes COFEs useful for modelling recursive definitions is the following fixed-point theorem: If f is a function from a COFE to itself which is *contractive* in that:

$$a \equiv_n b \quad \longrightarrow \quad f(a) \equiv_{n+1} f(b)$$

then f has a unique fixed point.

The purpose of this note is to introduce and prove a stronger fixed-point principle for COFEs: to find a unique fixed point of f, it is sufficient that f be *contractive on fixed points*:

$$a \equiv_n b \equiv_n f(a) \equiv_n f(b) \quad \longrightarrow \quad f(a) \equiv_{n+1} f(b)$$

That is, when showing that $f(a) \equiv_{n+1} f(b)$, the user of this fixed-point principle may assume not only that $a \equiv_n b$, but also that a and b are already *partial fixed points*:

$$a \equiv_n f(a) \qquad\qquad b \equiv_n f(b)$$

This stronger principle, based on a result by Fisher [2], allows unique fixed points to be found for complex nested recursive definitions that cannot be handled by contractivity alone.

2 BACKGROUND

An *ordered family of equivalences* (OFE) consists of a set A and a sequence of equivalence relations $\equiv_n$ ($n \in \mathbb{N}$) on A, where:

- $\equiv_0$ is total: $\forall a, b.\ a \equiv_0 b$
- The equivalence relations $\equiv_n$ are finer for larger n:

$$(\equiv_0) \supseteq (\equiv_1) \supseteq (\equiv_2) \supseteq \dots$$

- Their intersection is equality: $(\forall n.\ a \equiv_n b) \longrightarrow a = b$

A sequence of elements x_n is *coherent* if $x_n \equiv_n x_{n+1}$ for all n, and it has *limit* a if $x_n \equiv_n a$ for all n. Limits are unique when they exist, and a *complete* OFE (or COFE) is an OFE in which all coherent sequences have a (necessarily unique) limit.

OFEs as metric spaces. An OFE can be viewed as a certain kind of metric space, by defining the distance function d as follows:

$$d(a, b) = \inf\{2^{-n} \mid n \in \mathbb{N},\ a \equiv_n b\}$$

The resulting metric is a *1-bounded bisected ultrametric*:

- 1-bounded, because the maximum value of d is 1 since $a \equiv_0 b$ always
- bisected, because d takes only values 2^{-n} and 0
- ultrametric, because it obeys a stronger version of the triangle inequality, with + replaced with max

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The reader familiar with metric spaces might have expected a different definition of completeness, in terms of *Cauchy sequences*. In OFE notation, a sequence x_n is *Cauchy* if its elements are eventually approximately equal:

$$\forall n\, \exists k\, \forall i \geq k,\ j \geq k.\ x_i \equiv_n x_j$$

and it converges to a if its elements are eventually approximately equal to a:

$$\forall n\, \exists k\, \forall i \geq k.\ x_i \equiv_n a$$

Some authors define COFEs as OFEs in which all Cauchy sequences converge. This definition is equivalent to the simpler definition above, because all coherent sequences are Cauchy and all Cauchy sequences contain a coherent subsequence, and their limits agree.

Contractive functions. A function $f : A \to A$ on a COFE A is said to be *contractive* if:

$$a \equiv_n b \longrightarrow f(a) \equiv_{n+1} f(b)$$

The fixed point principle for COFEs introduced by Di Gianantonio and Miculan [1] states that such functions have unique fixed points, and moreover that this fixed point can be obtained by iteration from an arbitrary starting point. From the point of view of COFEs as metric spaces, this principle is a restatement of Banach's fixed point theorem (from which we get the term "contractive").

Theorem 1 (Di Gianantonio and Miculan, Banach). *If f is contractive on an inhabited COFE, then f has a unique fixed point, which is the limit of the iterates $f^n(x)$ starting from an arbitrary point x.*

3 A FIXED POINT THEOREM

Say that a function $f : A \to A$ is *contractive on fixed points* if

$$a \equiv_n b \equiv_n f(a) \equiv_n f(b) \longrightarrow f(a) \equiv_{n+1} f(b)$$

All contractive functions are contractive on fixed points, but the converse is not true (for an example, see section 4). However, the fixed-point theorem still holds for functions merely contractive on fixed points:

Theorem 2. *[derived from Fisher] If f is contractive on fixed points on an inhabited COFE, then f has a unique fixed point, which is the limit of the iterates $f^n(x)$ starting from an arbitrary point x.*

Proof. First, show by induction on n that, for arbitrary x:

$$x \equiv_n f(x) \longrightarrow f(x) \equiv_{n+1} f(f(x))$$

We use the c.f.p. property of f and hence need that $x \equiv_n f(x) \equiv_n f(x) \equiv_n f(f(x))$. We already have $x \equiv_n f(x)$, so need only show $f(x) \equiv_n f(f(x))$. When $n = 0$, this holds by totality of $\equiv_0$. When $n = n' + 1$, we have $x \equiv_{n'} f(x)$ and hence $f(x) \equiv_{n'+1} f(f(x))$ by IH.

This property suffices to show that for arbitrary x, the sequence $f^n(x)$ is coherent, so by completeness has limit a. We show that it is a fixed point by proving $f(a) \equiv_n a$ by induction on n, where the $n = 0$ case is trivial. For $n = n' + 1$, by the IH and the convergence of $f^n(x)$ to a, we have:

$$a \equiv_{n'} f^{n'}(x) \equiv_{n'} f(a) \equiv_{n'} f^{n'+1}(x)$$

and so $f(a) \equiv_n f^n(x) \equiv_n a$.

Finally, for uniqueness of the fixed point, we suppose that f has two fixed points p, q. We show that $p \equiv_n q$ by induction on n, relying on the c.f.p. property of f to show that $p = f(p) \equiv_{n'+1} f(q) = q$ since $p \equiv_{n'} q \equiv_{n'} f(p) \equiv_{n'} f(q)$. □

4 EXAMPLE

Consider the following function, borrowed from Krstić and Matthews [4]:

$$f(x) \;=\; \text{if } x = 0 \text{ then } 0 \text{ else } f(f(x-1))$$

This function is total, always returns zero, and only ever recurses on strictly smaller arguments. Yet it is not syntactically structurally recursive, as it recurses on the argument $f(x-1)$.

We can view this definition as being the fixed point of an operator T:

$$T : (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})$$
$$Tf = \lambda x.\ \text{if } x = 0 \text{ then } 0 \text{ else } f(f(x-1))$$

We give the set $\mathbb{N} \to \mathbb{N}$ the structure of a COFE by saying that functions are approximately equal when they agree on a prefix:

$$f \equiv_n g \quad \text{iff} \quad \forall k < n.\ f(k) = g(k)$$

In this COFE, the operator T is not contractive, so Banach's fixed point theorem cannot be used to prove it has a unique fixed point. However, we can show that it is contractive on fixed points, by first proving the following lemma about partial fixed points of T:

Lemma 1. *If $T(f) \equiv_n f$, then $\forall k < n.\ f(k) = 0$*

Proof. Induction on n. □

The operator T is then shown to be contractive on fixed points, since any partial fixed points of T must satisfy the condition of the above lemma. By a further application of the above lemma, this unique fixed point is equal to $\lambda x.\ 0$.

5 NOTES

Theorem 2 is derived from a result of Fisher [2], which states that a function f on a compact metric space has a unique fixed point if:

$$d(f(x), f(y)) < \frac{1}{2}(d(x, f(y)) + d(f(x), y))$$

Theorem 2 is not directly implied by this result, but is a sort of ultrametric version of it, and can be proved by going through the steps of Fisher's proof. However, the actual proof in section 3 differs slightly from Fisher's, both in order to be constructive (Fisher's proof does not directly show that the fixed point can be found by iterating f), and because the bisected nature of COFEs removes the need for compactness.

The theorem and the example of section 4 have been formalised in Coq, and the development is available from:

`https://github.com/stedolan/cofe-fixpoints`

REFERENCES

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