MAT 4010: Homework 1

Instructions

- Please write clearly and show all of your work. Answers without justification may be worth zero.
- When justifying your conclusions using a theorem, give the theorem name/number.
- Submit your homework to the appropriate assignment on Gradescope before the deadline.
- If you complete your homework on paper, you are encouraged to use a scanner app to convert photos of your work to a PDF before submitting to Gradescope.
- For programming questions, submit your source code (.m files) and any plots generated (.png files) to the appropriate "Programming" assignment on Gradescope before the deadline. Your .m files must follow the naming convention Hxx_Qyy.m where xx is a two-digit homework number and yy is a two-digit question number (e.g., H01_Q04.m).

Problem Set

1. In this problem we will investigate approximating the square root of a nonnegative real number.

   Write a program (using MATLAB/Octave) that approximates $\sqrt{a}$ using the sequence $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$. Your program should prompt the user for a nonnegative value a, a positive initial guess x_1, a convergence threshold ϵ, and the maximum iterate number `nMax`.

   Your program should calculate values for x_n until either the absolute value of the error estimate $|\hat{e}_{n+1}| = |x_{n+1} - x_n|$ is less than ϵ or the maximum iterate x_{nMax} is calculated. Each iteration, your program should print the calculated x_n and $|\hat{e}_n|$.

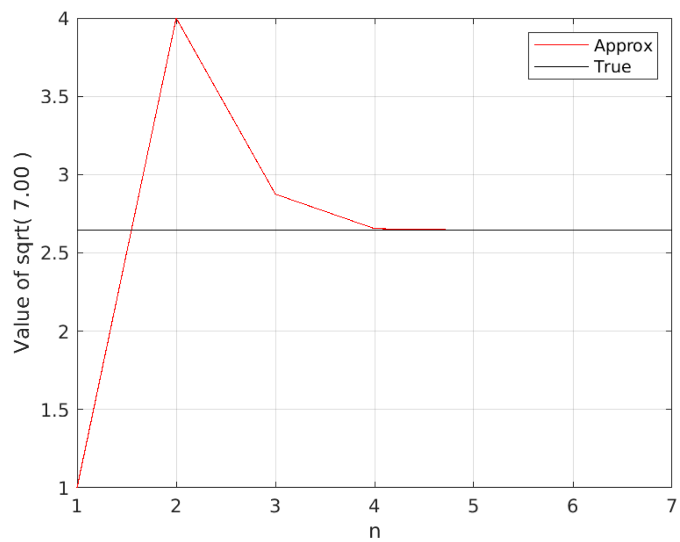
   Your program should also produce two plots. The first should plot x_n versus n on the same plot as $\sqrt{a}$ versus n, using different linestyle and/or markers and/or colors, and should include axes labels and a legend. The second should plot $|\hat{e}_n|$ versus n using `semilogy` and should include axes labels.

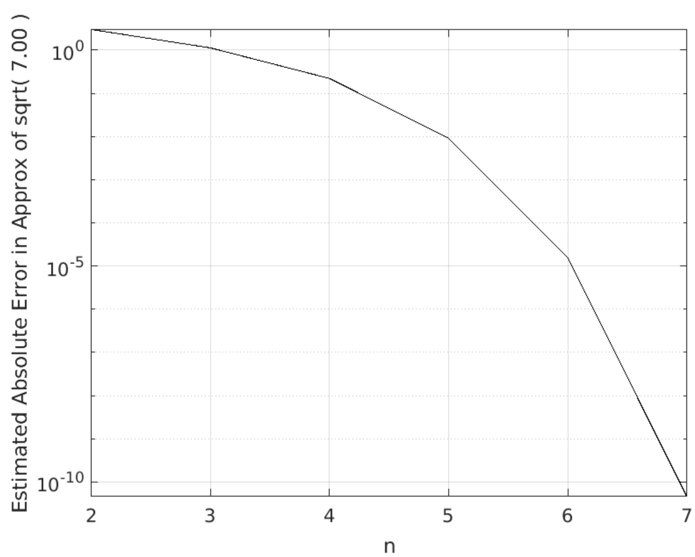
   Note: Comment out any plotting-related code before submitting to Gradescope.

Sample output:

```
Enter the value for a: 7
Enter the value for x1: 1
Enter the value for epsilon: 1e-6
Enter the value for nMax: 10
For n =  2, x(  2 ) =   4.00 and |eHat(  2 )| = 3.00e+00
For n =  3, x(  3 ) =   2.88 and |eHat(  3 )| = 1.12e+00
For n =  4, x(  4 ) =   2.65 and |eHat(  4 )| = 2.20e-01
For n =  5, x(  5 ) =   2.65 and |eHat(  5 )| = 9.12e-03
For n =  6, x(  6 ) =   2.65 and |eHat(  6 )| = 1.57e-05
For n =  7, x(  7 ) =   2.65 and |eHat(  7 )| = 4.68e-11
```

Sample plots:





Submit: Your .m file and the .png files for inputs of `a = 14`, `x1 = 2`, `epsilon = 1e-6`, and `nMax = 15`.

2. Let $a > 0$. Consider the sequence defined recursively by

$$p_{n+1} = p_n(2 - ap_n)$$

where p_0 is any initial point satisfying $0 < p_0 < \frac{2}{a}$.

a) Assume (without justification) that the sequence converges to p. Determine the value of p. Hint: Note that for convergence we have $\lim_{n\to\infty} p_n = p$ and use this fact in the above recurrence to solve for p.

b) Assume that the sequence converges to $p = \frac{1}{a}$. Using the definition, determine the order of convergence and the asymptotic error constant.

3. a) Using the asymptotic relation $e_{n+1} \approx \lambda e_n^{\alpha}$, show that, given three successive errors e_n, e_{n-1}, and e_{n-2}, we can approximate the order of convergence by the formula

$$\alpha \approx \frac{\ln e_n - \ln e_{n-1}}{\ln e_{n-1} - \ln e_{n-2}}$$

and the asymptotic error constant by

$$\lambda \approx \frac{e_n}{e_{n-1}^{\alpha}}.$$

Hint: Write the asymptotic relation for both e_n and e_{n-1}, take ln of both equations, and subtract the resulting equations.

b) Consider the following table of errors arising when approximating some quantity using three different methods. Use the formulae from part a) to approximate the order of convergence and asymptotic error constant for each method.

n	e_n		
	Method 1	Method 2	Method 3
1	$1.0 \cdot 10^{-1}$	$1.0 \cdot 10^{-1}$	$1.0 \cdot 10^{-1}$
2	$9.5 \cdot 10^{-2}$	$3.3 \cdot 10^{-2}$	$8.0 \cdot 10^{-2}$
3	$8.8 \cdot 10^{-2}$	$1.1 \cdot 10^{-2}$	$5.1 \cdot 10^{-2}$

4. In this problem we will investigate approximating the number of iterations required to achieve an error below a threshold, given values for the convergence order α and the asymptotic error constant λ.

Write a program (using MATLAB/Octave) that prompts the user for values for errors e_1, e_2, and e_3. Your program should then prompt the user for values for the convergence order α, the asymptotic error constant λ, and a threshold ϵ. Your program should then use the asymptotic relation $e_{n+1} \approx \lambda e_n^{\alpha}$ to calculate and print e_{n+1} values to determine approximately how many iterations are expected to be required to achieve an error less than ϵ. Finally, your program should print the determined number of iterations.

Sample output:

```
Enter a value for e1: 1e-1
Enter a value for e2: 9.5e-2
Enter a value for e3: 8.8e-2
Enter a value for alpha: 1.5
Enter a value for lambda: 3
Enter a value for epsilon: 1e-6
n =  3 gives e(  4 ) = 7.83e-02
n =  4 gives e(  5 ) = 6.57e-02
n =  5 gives e(  6 ) = 5.06e-02
n =  6 gives e(  7 ) = 3.41e-02
n =  7 gives e(  8 ) = 1.89e-02
n =  8 gives e(  9 ) = 7.80e-03
n =  9 gives e( 10 ) = 2.07e-03
n = 10 gives e( 11 ) = 2.82e-04
n = 11 gives e( 12 ) = 1.42e-05
n = 12 gives e( 13 ) = 1.61e-07
We expect approximately 13 iterations to be required
```