DERIVATION

Terms:

z, a dichotomous random variable that alternates over time between two states

$z \in \{0, 1\}$

T_0, random duration of state 0, with density function $f_{T_0}(t)$

T_1, random duration of state 1, with density function $f_{T_1}(t)$

$\tilde{F}_{T_i} = \int_t^\infty f_{T_i}(t')dt'$, the complementary cumulative distribution function for durations

$s_i(t)$, random elapsed time since entering state i

$h_{T_i}(s) = \frac{f_{T_i}(s)}{\tilde{F}_{T_i}(s)}$, the hazard function. $h_{T_i}(s)dt$ gives the probability of exiting state i in time interval $(t, t + dt)$, given that $s_i(t) = s$.

$\hat{f}(\omega)$, Fourier transform of a function $f(t)$

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For each of the two alternating states z = {0,1}, the rate of change in probability over time depends on the evolution of the s variable, the time since entering the current states. As ds/dt = 1, the advective flux of probability is simply f(x). From this quantity is subtracted the probability of exiting the current state.

$$\frac{\partial f_0}{\partial t}(s,t) = -\frac{\partial}{\partial s}(1 \cdot f_0(s,t)) - h_{T_0}(s) f_0(s,t) \qquad (1)$$

$$\frac{\partial f_1}{\partial t}(s,t) = -\frac{\partial}{\partial s}(1 \cdot f_1(s,t)) - h_{T_0}(s) f_1(s,t) \qquad (2)$$

The value of s is reset to 0 whenever an alternation between states occurs. The initial probability source at s = 0 is determined by the probability of switches out of the previous state, leading to the following boundary conditions:

$$f_0(0,t) = \int_0^\infty h_{T_1}(s) f_1(s,t) ds \qquad (3)$$

$$f_1(0,t) = \int_0^\infty h_{T_0}(s) f_0(s,t) ds \qquad (4)$$

Initial Conditions:

$$f_1(s,t=0)=\delta(s); f_0(s,t=0)=0 \tag{5}$$

We first solve for the case of starting in state 1.

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Using these conditions, a solution can be found for :

$$p_{1|1}(t)=\tilde{F}_{T_1}(t)+\frac{1}{2\pi}\int_{-\infty}^{\infty}dw\frac{1}{i\omega}\left[\hat{\tilde{F}}_{T_1}(\omega)\hat{f}_{T_0}(\omega)\hat{f}_{T_1}(\omega)\frac{i\omega}{1-\hat{f}_{T_0}(\omega)\hat{f}_{T_1}(\omega)}\right]e^{i\omega t} \tag{6}$$

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To find the probability $p_{1|0}$, simply time shift the above expression by the durations of the initial state, T^0_0, whose density function is $f_{T^0_0}(t)$-- the switch times are then described by $f_{T^0_0}(s)ds$. This amounts to a convolution of the density for initial durations with the previous solution:

$$p_{1|z(0)=0}(t)=\int_0^t f_{T^0_0}(s)p_{1|1}(t-s)ds \tag{7}$$

Thus the solution can be given in the Fourier domain as:

$$\hat{p}_{1|z(0)=0}(\omega)=\hat{f}_{T^0_0}(\omega)\hat{p}_{1|1}(\omega). \tag{8}$$

Using the simplifying assumption that $f_{T^0_0}(t)=f_{T_0}(t)$, eg the first duration is from the same probability density as all other T_0, we find that:

$$\hat{p}_{1|z(0)=0}(\omega)=\hat{\tilde{F}}_{T_1}(\omega)\left(\hat{f}_{T_0}+\frac{\hat{f}^2_{T_0}(\omega)\hat{f}_{T_1}(\omega)}{1-\hat{f}_{T_0}(\omega)\hat{f}_{T_1}(\omega)}\right) \tag{9}$$

(Finishing up: at omega = 0, f_hat(omega) = mu_T1/(mu_T1 + muT0); ifft, integrate from 0 to t.)