NPTEL MOOC,JAN-FEB 2015
Week 6, Module 3

DESIGN AND ANALYSIS OF ALGORITHMS

Greedy algorithms: Interval scheduling

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Greedy Algorithms

- Need to make a sequence of choices to achieve a global optimum
- At each stage, make the next choice based on some local criterion
  - Drastically reduces space to search for solutions
- Never go back and revise an earlier decision
- How to prove that local choices achieve global optimum?

Examples so far

Dijkstra's algorithm

- Local rule:
  Freeze the distance of nearest unburnt vertex
- Global optimum:
  Distance assigned to each vertex is shortest distance from source

Examples so far

Prim's algorithm

- Local rule:
  Add to the spanning tree the nearest vertex not yet in the tree
- Global optimum:
  Final spanning tree constructed is a minimum cost spanning tree

Examples so far

Kruskal's algorithm

- Local rule:
  Add to the current set of edges the next smallest edge that does not form a cycle
- Global optimum:
  Edges collected form a minimum cost spanning tree

Interval scheduling

- CMI has a special video classroom for delivering online lectures
- Different teachers want to book the classroom — the slot for each instructor i starts at s(i) and finishes at f(i)
- Slots may overlap, so not all bookings can be honoured
- Choose a subset of bookings to maximize the number of teachers who get to use the room

Interval scheduling …

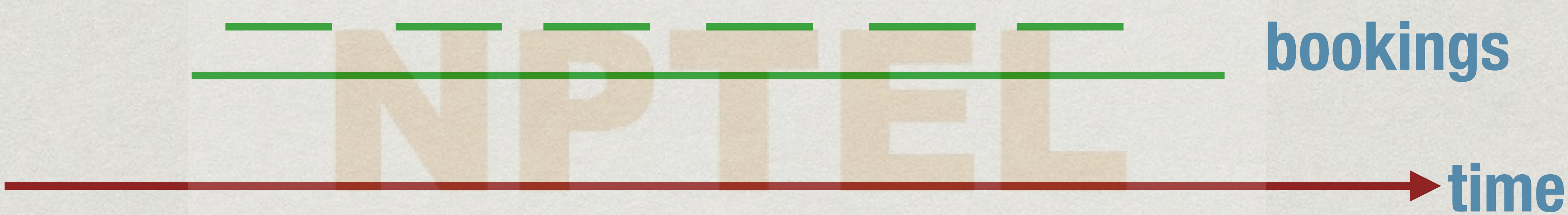
Greedy approach

- Pick the next booking to allot based on a local strategy
- Remove all bookings that overlap with this slot
- Argue that this sequence of bookings will maximize the number of teachers who get to use the room

Interval scheduling …

Greedy strategy 1

- Choose the booking whose start time is earliest
- Counterexample



Interval scheduling …

Greedy strategy 2

- Choose the booking whose interval is shortest
- Counterexample

Interval scheduling …

Greedy strategy 3

- Choose the booking that overlaps with minimum number of other bookings
- Counterexample

Interval scheduling …

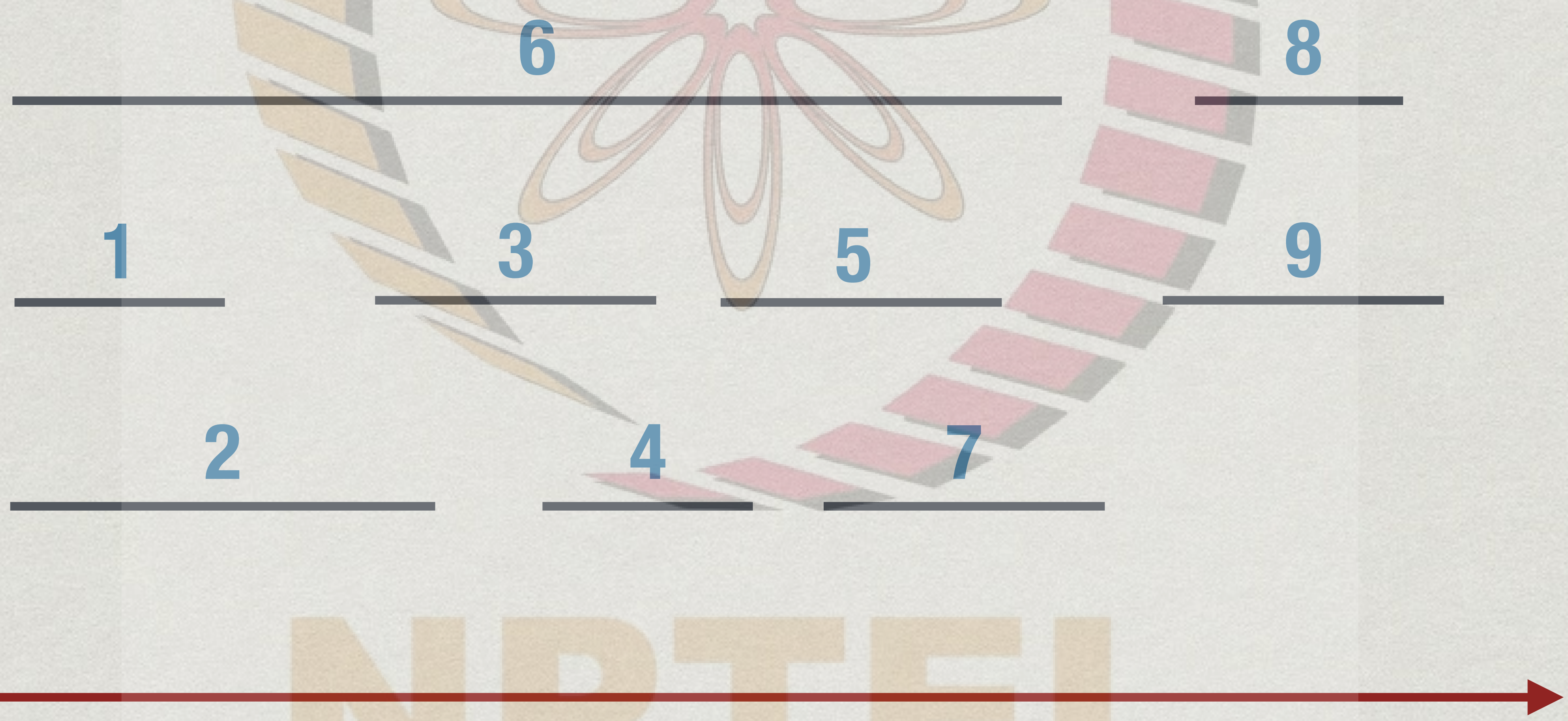
Greedy strategy 4

- Choose the booking that whose finish time is earliest
- Counterexample?
- Proof of correctness?

The algorithm

- B is the set of bookings
- A is the set of accepted bookings, initially empty
- While B is not empty
  - Pick b in B with smallest finishing time
  - Add b to A
  - Remove from B all bookings that overlap with b

The algorithm in action



The algorithm in action

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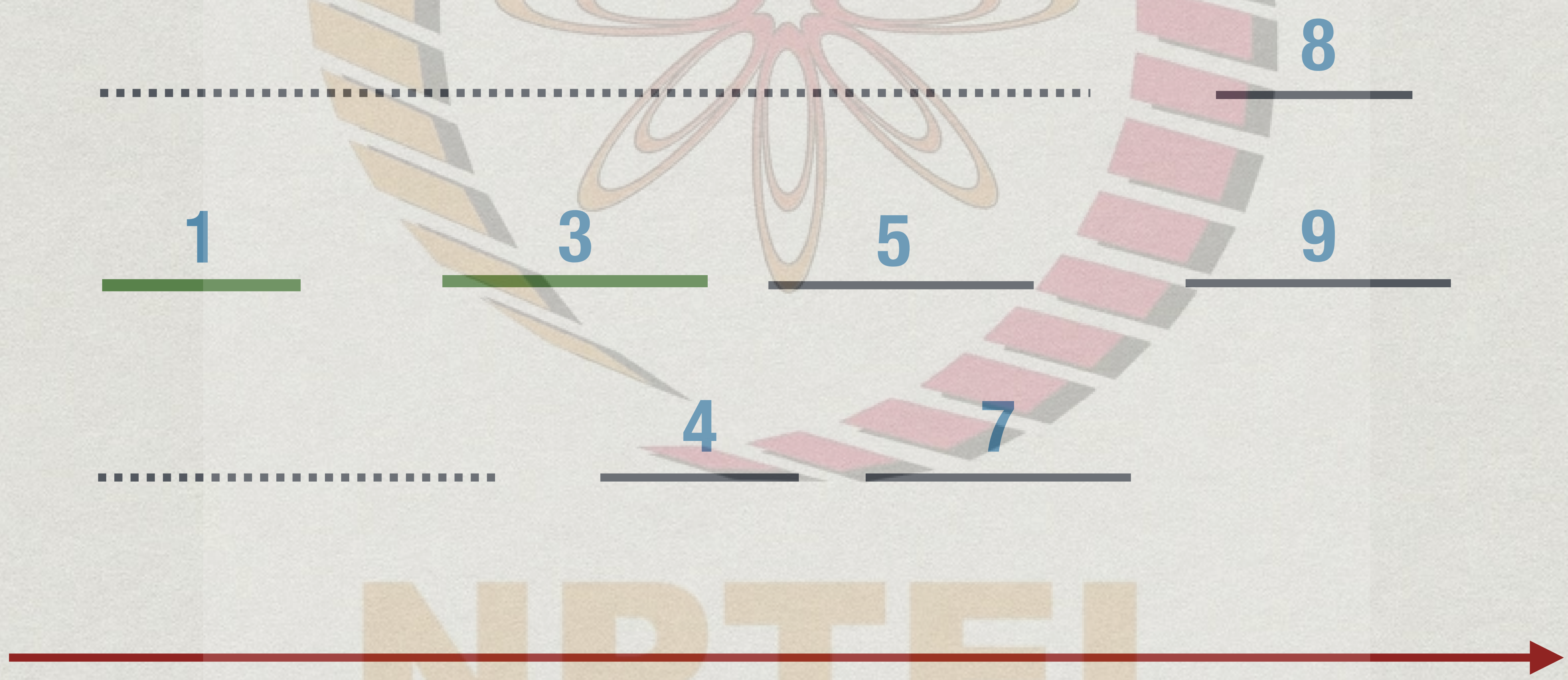
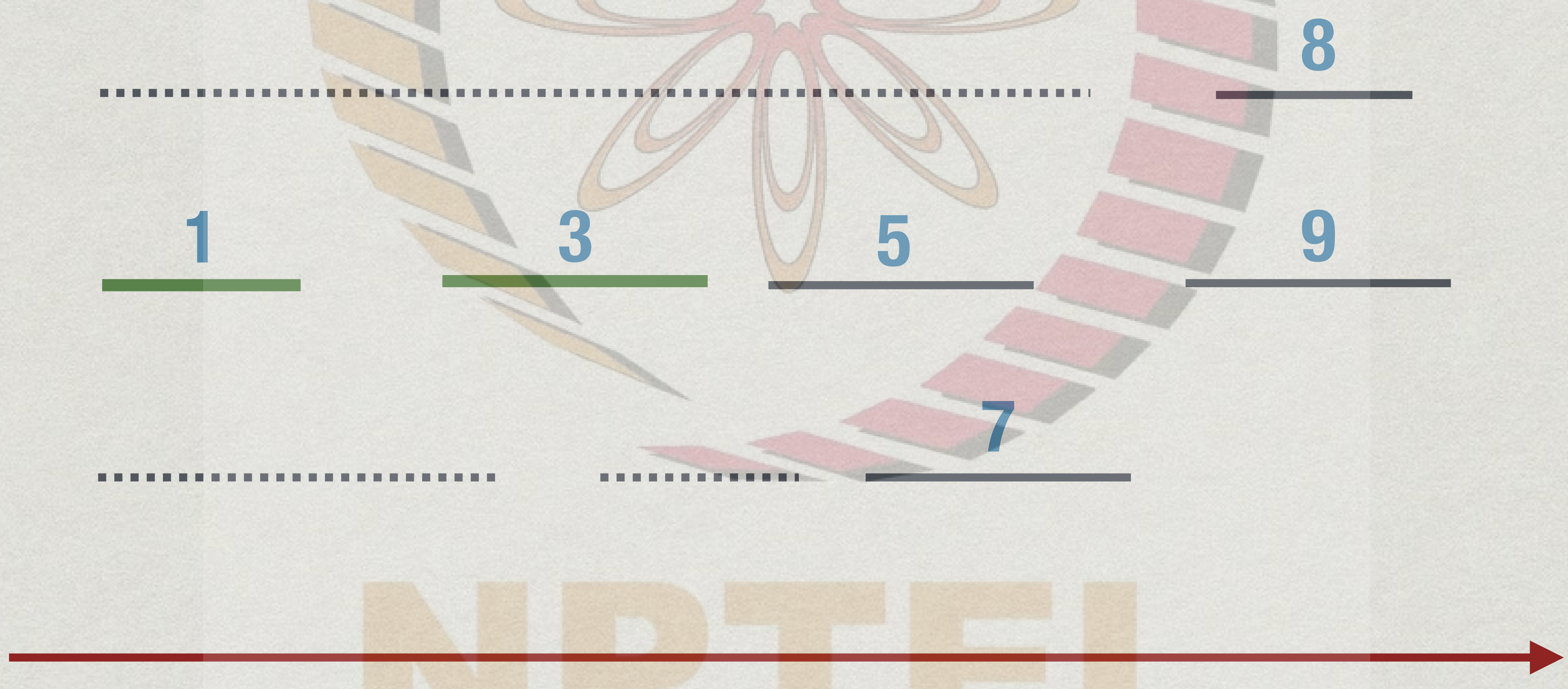
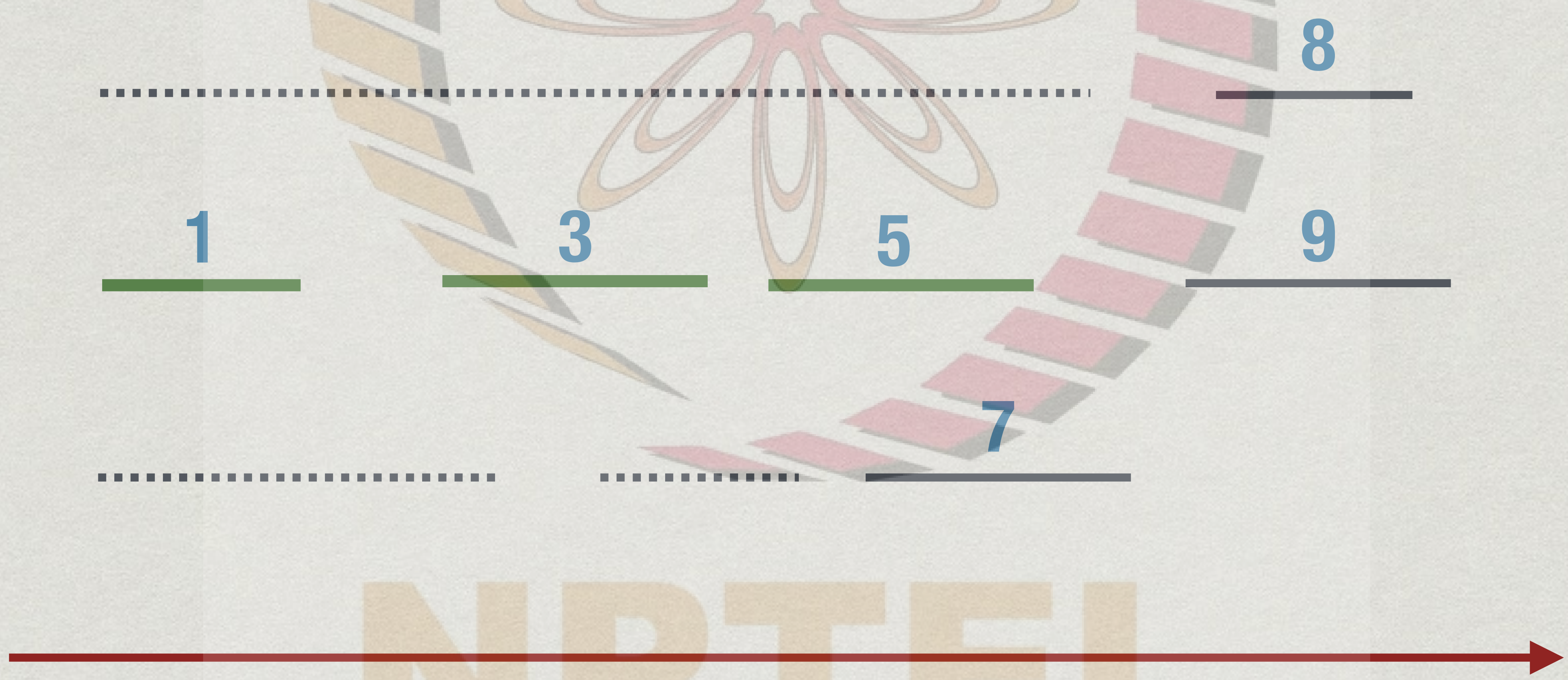


The algorithm in action



The algorithm in action



The algorithm in action

The algorithm in action

The algorithm in action

Correctness

- Our algorithm produces a solution A
- Let O be any optimal allocation of bookings
- A and O need not be identical
  - Can have multiple allocations of same size
- Instead, just show that $|A| = |O|$ — same size

Greedy allocation stays ahead

- Let $A = i_1, i_2, \ldots, i_k$
  - Assume sorted: $f(i_1) \leq s(i_2)$, $f(i_2) \leq s(i_3)$, …
- Let $O = j_1, j_2, \ldots, j_m$
  - Again, assume sorted: $f(j_1) \leq s(j_2)$, $f(j_2) \leq s(j_3)$, …
- To show that $k = m$

Greedy allocation stays ahead

Claim: For each $r \leq k$, $f(i_r) \leq f(j_r)$

- Our greedy solution "stays ahead" of O

Proof: By induction on r

- $r = 1$: our algorithm chooses booking i_1 with earliest overall finish time

Greedy allocation stays ahead

- $r > 1$: Assume, by induction that $f(i_{r-1}) \leq f(j_{r-1})$
- Then, it must be the case that $f(i_r) \leq f(j_r)$
- If not, algorithm would choose j_r rather than i_r

Greedy allocation is optimal

- Suppose $m > k$
- We know that $f(i_k) \leq f(j_k)$
- Consider booking j_{k+1} in O
  - Greedy algorithm terminates when B is empty
  - Since $f(i_k) \leq f(j_k) \leq s(j_{k+1})$, this booking is compatible with $A = i_1, i_2, \ldots, i_k$
  - After selecting i_k, B still contains j_{k+1}. Contradiction!

Implementation, complexity

- Initially, sort the n bookings by finish time, O(n log n)
  - Bookings are renumbered 1,2,…,n in this order
- Set up an array ST[1..n] so that ST[i] = s(i)
- Start with booking 1
- After choosing booking j, scan ST[j+1], ST[j+2], … and choose first k such that ST[k] > f(j)
- Second phase is O(n), so O(n log n) overall