Linear Programming

What is a linear program?

$$\min c^\top x \quad \text{s.t.}$$
$$Ax \leq b$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -3 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 7 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad c = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

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- set of variables x
- set of linear constraints $Ax \leq b$
- linear objective function $\min c^\top x$

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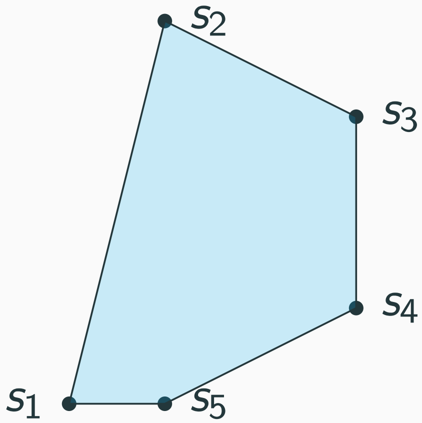

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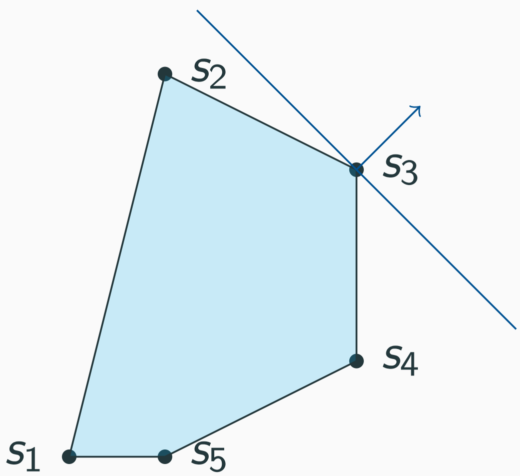

What is a mixed integer linear program?

$$\begin{aligned} \min c^\top x \quad & \text{s.t.} \\ Ax \leq b \\ x \in \mathbb{Z}^n \end{aligned}$$

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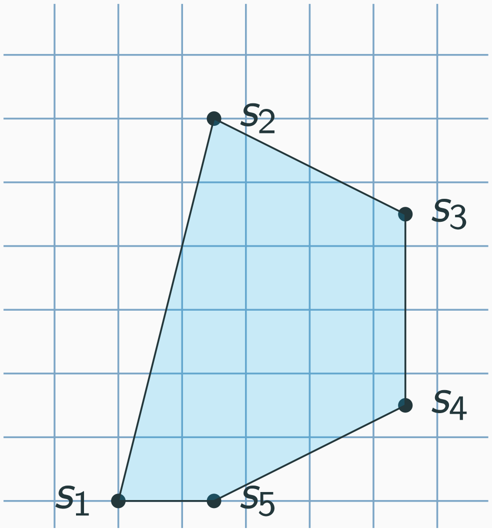

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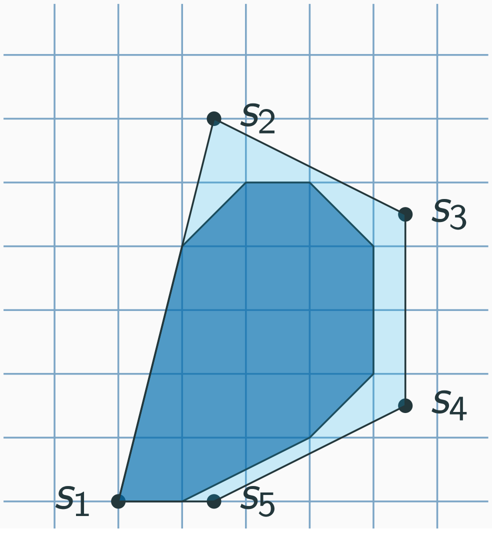

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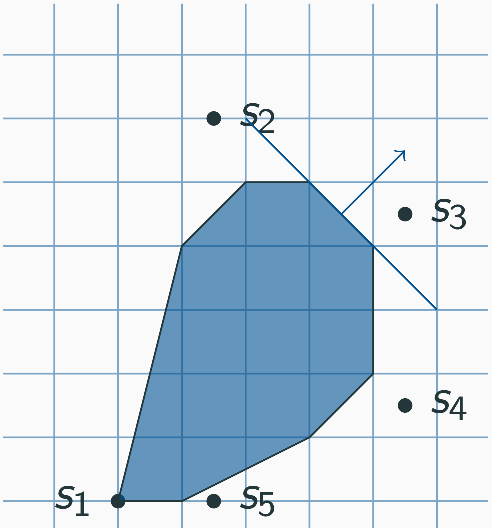

What is a mixed integer linear program?

$$
\begin{aligned}
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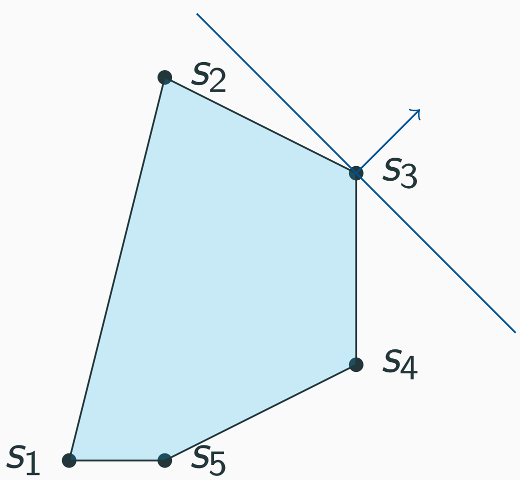
x \in \mathbb{Z}^n &
\end{aligned}
$$

Focus on Linear Programs

How to solve a linear program?

The Simplex Algorithm

- Find a feasible solution
- Travel along improving edges
- Terminate at optimal solution



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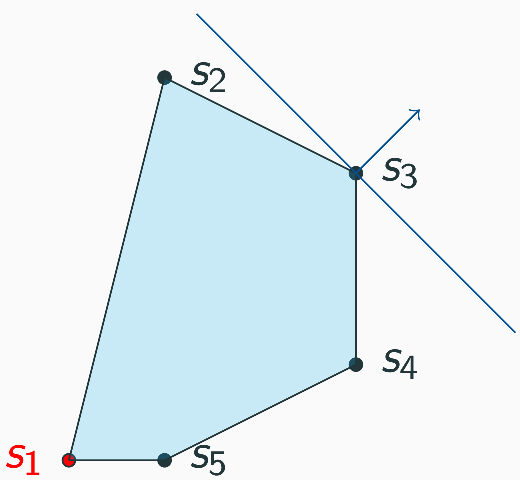

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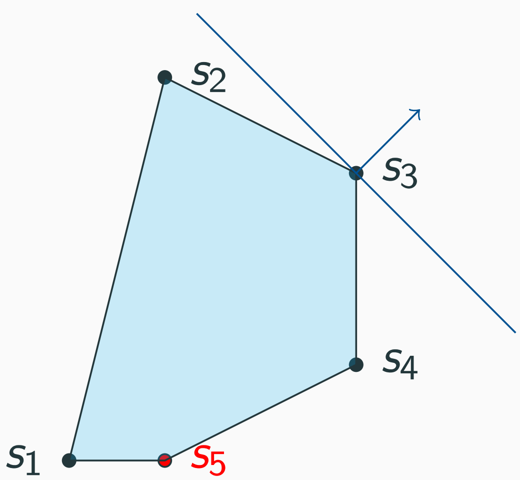

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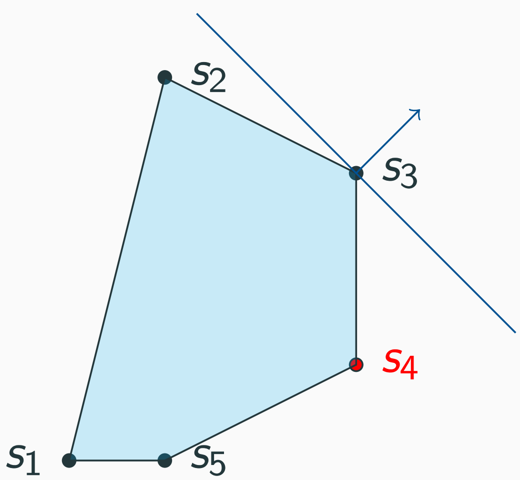

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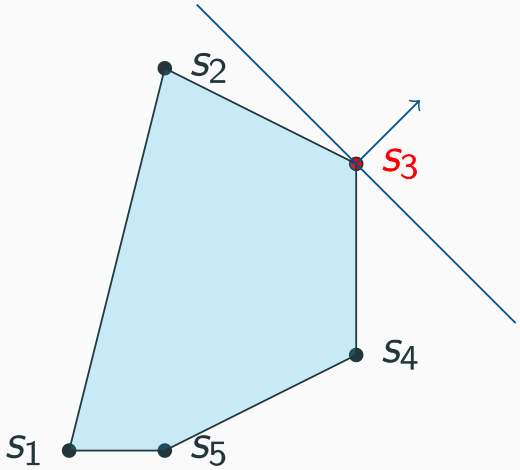

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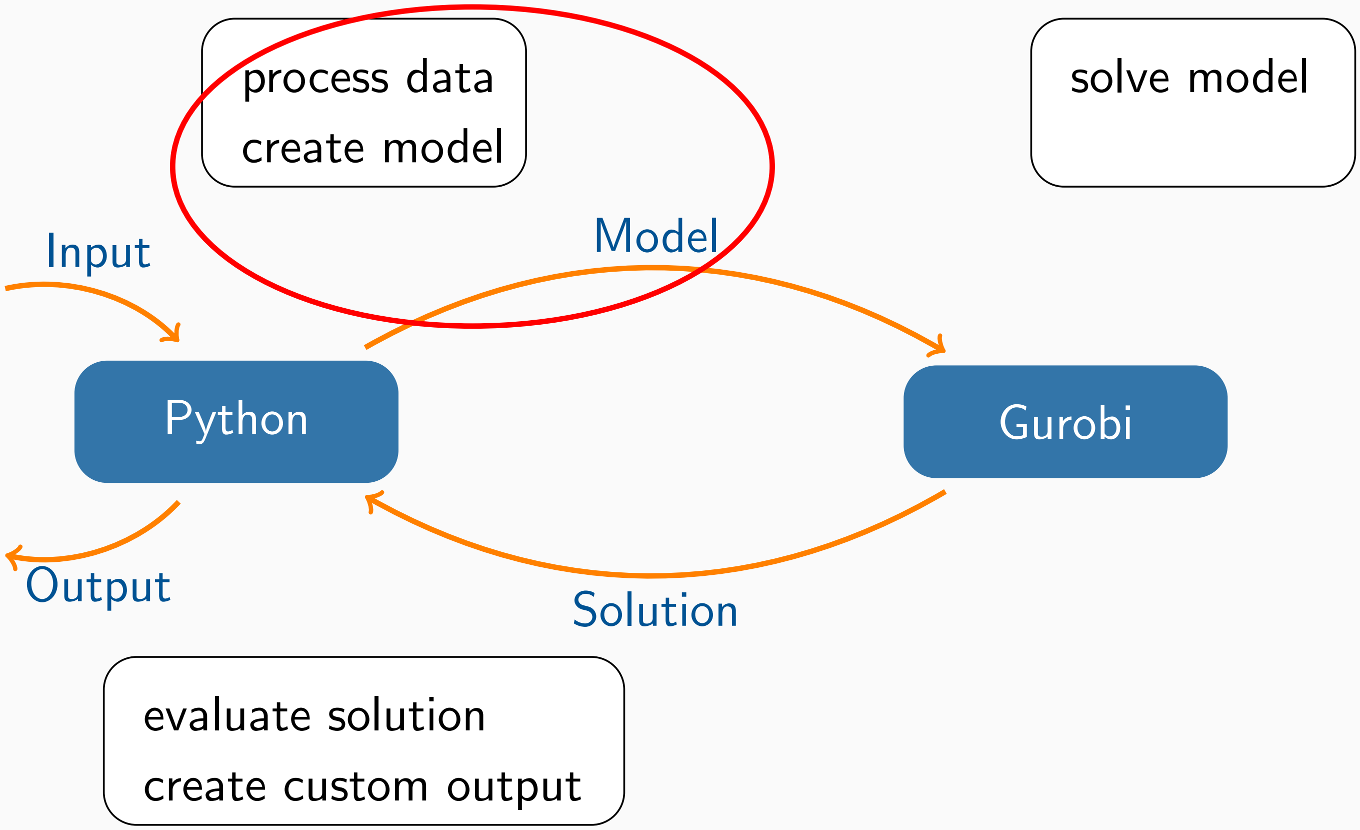
Good News:
Gurobi does that for us

Modelling

Modelling


process data
create model
solve model
Input
Model
Python
Gurobi
Output
Solution
evaluate solution
create custom output

Problem: Crude Oil Refinement 1

10l crude oil

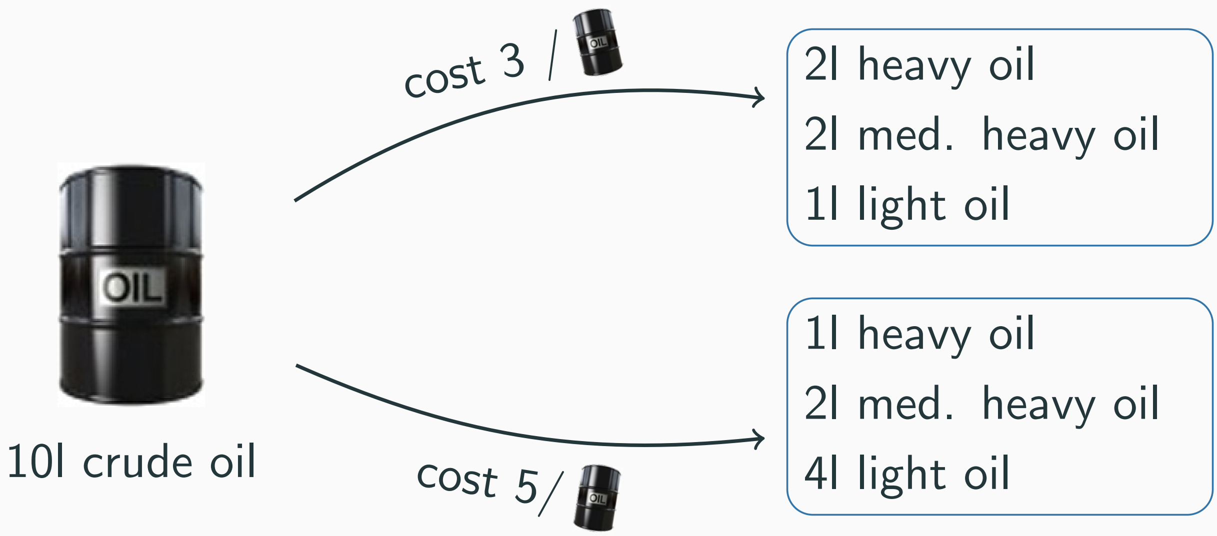
Problem: Crude Oil Refinement 1

cost 3 / 

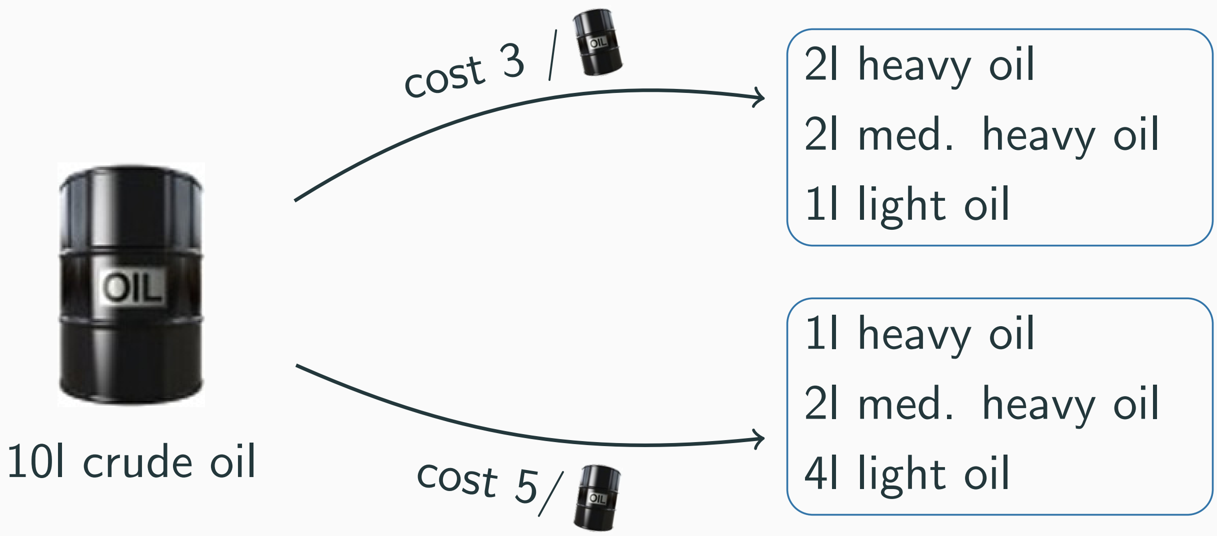
2l heavy oil
2l med. heavy oil
1l light oil

10l crude oil

Problem: Crude Oil Refinement 1



Problem: Crude Oil Refinement 1



demand: 3l heavy oil, 5l med. heavy oil, 4l light oil

Problem: Crude Oil Refinement 1

10l crude oil

cost 3 / 

- 2l heavy oil
- 2l med. heavy oil
- 1l light oil

cost 5/ 

- 1l heavy oil
- 2l med. heavy oil
- 4l light oil

demand: 3l heavy oil, 5l med. heavy oil, 4l light oil

objective: minimize cost

LP Model Problem 1

$$\min 3x_1 + 5x_2$$

s.t.

$$2x_1 + 1x_2 \geq 3$$
$$2x_1 + 2x_2 \geq 5$$
$$1x_1 + 4x_2 \geq 4$$
$$x_1, x_2 \geq 0$$

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Initialize gurobipy and create set of variables x

```
from gurobipy import *

# Create a new model
m = Model()

# Create variables
x = m.addVar(vtype=GRB.CONTINUOUS)
y = m.addVar(vtype=GRB.CONTINUOUS)
```

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Variable types

- GRB.CONTINUOUS
- GRB.BINARY
- GRB.INTEGER
- GRB.SEMICONT
- GRB.SEMIINT

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- GRB.CONTINUOUS $(-\infty, \infty)$
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- GRB.BINARY $\{0, 1\}$
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```

Add Variables

```
addVar(lb=0,ub=GRB.INFINITY,obj=0.0,
                        vtype=GRB.CONTINUOUS,name="")
```

- *lb*, *ub*: variable lower and upper bound
- *obj*: coefficient of the linear objective function
- *vtype*: variable type
- *name*: name for further referencing

Add Variables

Different options:

- `model.addVar(...)`
- `model.addVars(...)`
- `model.addMVars(...)`

See also in the gurobipy manual

Create set of linear constraints $Ax \geq b$

```
# Add constraints
c1 = m.addConstr(2*x+y>=3)
c2 = m.addConstr(2*x+2*y>=5)
c3 = m.addConstr(x+4*y>=4)
c4 = m.addConstr(x>=0)
c5 = m.addConstr(y>=0)
```

Create set of linear constraints $Ax \geq b$

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```

Add Constraints

Basic form:

```
m.addConstr(LinExpr>=a)
```

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m.addConstr(LinExpr>=a)
```

Linear expressions can be created by:

- $le = 2 * x + 3 * y$
- $le = x.prod([2, 3])$
- $le = x.sum()$
- $le = quicksum([2 * x, 3 * y])$

Add Constraints

Different options:

- `model.addConstr(...)`
- `model.addLConstr(...)`
- `model.addConstrs(...)`
- `model.addMConstrs(...)`

See also in the gurobipy manual

Set linear objective function min $c^\top x$ and optimize the model

```
# Set objective function
m.setObjective(3*x+5*y,GRB.MINIMIZE)

# Optimize model
m.optimize()
```