DG Methods for Advection Problems in Spherical Coordinates in (r, θ).

Consider the following transport equation written in conservation form:

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} + \operatorname{div}(a(x)u(x,t)) = 0, \qquad & x \in \Omega, t > 0. \\ u(x,t) = g(x,t) \qquad & x \in \partial\Omega_{\text{in}} := \{x \in \partial\Omega : a(x)\cdot \boldsymbol{n} > 0\}, t > 0 \\ u(x,0) = u_0(x), \qquad & x \in \Omega, \end{aligned}$$

where a is the transport vector and $\boldsymbol{n}$ is the unit outward normal of $\partial\Omega$.

Here we are representing the solution u and the domain Ω in spherical coordinates. Let $x = (r, \theta, \varphi)$ be the spherical coordinate representation of x and let

$$\Omega := \{(r, \theta, \varphi) : 0 \le r_0 < r < R, 0 \le \theta_0 \le \theta \le \Theta < \pi, 0 \le \varphi < 2\pi\}.$$

with $u(r, \theta, \varphi) = u(x)$.

Let $\hat{r} = \hat{r}(r, \theta, \varphi), \hat{\theta} = \hat{\theta}(r, \theta, \varphi), \hat{\varphi} = \hat{\varphi}(r, \theta, \varphi)$ be the orthornormal unit vectors in the r, θ, and φ directions respectively. We note that $\hat{r}$, $\hat{\theta}$ and $\hat{\varphi}$ are not constant. Let $a = [a_r, a_\theta, a_\varphi]$ be the coordinates of a w.r.t the basis vectors $\hat{r}, \hat{\theta}, \hat{\varphi}$, that is, $a = a_r\hat{a_r} + a_\theta\hat{\theta} + a_\varphi\hat{\varphi}$. For this problem, we assume $a_r \ge 0$, $a_\theta \ge 0$, and $a_\varphi = 0$ are constant. We note that a is not divergence free, so $\operatorname{div}(au) \ne a \cdot \nabla u$. Additionally, since $a_r, a_\theta \ge 0$, we have a representation for $\partial\Omega_{\text{in}}$:

$$\begin{aligned} \partial\Omega_{\text{in}} &= \{(r_0, \theta, \varphi) : \theta_0 \le \theta \le \Theta, 0 \le \varphi < 2\pi\} \cup \{(r, \theta_0, \varphi) : r_0 < r < R : 0 \le \varphi < 2\pi\} \\ &=: \Gamma_r \cup \Gamma_\theta \end{aligned}$$

Finally, we note that since $a_\varphi = 0$, then u is constant in φ and thusly $u = u(r, \theta, t)$.

We can now formulate the DG scheme. Let T be a control volume in Ω that is a logical rectangle in our spherical coordinates, that is,

$$T = \{(r, \theta, \varphi) : \alpha < r < \beta, \gamma < \theta < \delta, 0 \le \varphi < 2\pi\}.$$

We test the PDE by a smooth function $v(r, \theta)$ defined on T to see

$$\int_T \frac{\partial u(r,\theta,t)}{\partial t} v(r,\theta)\,\mathrm{d}T + \int_T \operatorname{div}(a(r,\theta,\varphi)u(r,\theta,t))v(r,\theta)\,\mathrm{d}T = 0. \tag{1}$$

We parameterize the first integral in (1) and pull out the φ integral to see

$$\begin{aligned} \int_T \frac{\partial u(r,\theta,t)}{\partial t} v(r,\theta)\,\mathrm{d}T &= \int_{r=\alpha}^{\beta}\int_{\theta=\gamma}^{\delta}\int_{\varphi=0}^{2\pi} \frac{\partial u(r,\theta,t)}{\partial t} v(r,\theta) r^2 \sin(\theta)\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}r \\ &= \int_{\varphi=0}^{2\pi} 1\,\mathrm{d}\varphi \int_{r=\alpha}^{\beta}\int_{\theta=\gamma}^{\delta} \frac{\partial u(r,\theta,t)}{\partial t} v(r,\theta) r^2 \sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}r \\ &= 2\pi \int_{r=\alpha}^{\beta}\int_{\theta=\gamma}^{\delta} \frac{\partial u(r,\theta,t)}{\partial t} v(r,\theta) r^2 \sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}r \end{aligned}$$

For the second integral in (1) we perform integration by parts to achieve

$$\begin{aligned}\int_T \operatorname{div}(a(r,\theta,\varphi)u(r,\theta,t))v(r,\theta)\,\mathrm{d}T = &-\int_T a(r,\theta,\varphi)u\cdot\nabla v(r,\theta)\,\mathrm{d}T \\ &+\int_{\partial T} \widehat{a(r,\theta,\varphi)u}\cdot\boldsymbol{n}(r,\theta,\varphi)v(r,\theta)\,\mathrm{d}S.\end{aligned} \tag{2}$$

where $\widehat{au}$ is our numerical flux to be determined. We parameterize the first integral on the right hand size of (2) as

$$\begin{aligned}&\int_T a(r,\theta,\varphi)u\cdot\nabla v(r,\theta)\,\mathrm{d}T \\ &= \int_{r=\alpha}^{\beta}\int_{\theta=\gamma}^{\delta}\int_{\varphi=0}^{2\pi}(a_r\hat{r}+a_\theta\hat{\theta}+0\hat{\varphi})u\cdot\left(\frac{\partial v}{\partial r}(r,\theta)\hat{r}+\frac{1}{r}\frac{\partial v}{\partial\theta}(r,\theta)\hat{\theta}+\frac{1}{r\sin(\theta)}\frac{\partial v}{\partial\varphi}(v,\varphi)\hat{\varphi}\right)r^2\sin(\theta)\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}r.\end{aligned} \tag{3}$$

Note $\hat{r},\hat{\theta},\hat{\varphi}$ are orthonormal, so (3) becomes

$$\begin{aligned}\int_T a(r,\theta,\varphi)u\cdot\nabla v(r,\theta)\,\mathrm{d}T &= \int_{r=\alpha}^{\beta}\int_{\theta=\gamma}^{\delta}\int_{\varphi=0}^{2\pi} a_r u(r,\theta,t)\frac{\partial v}{\partial r}(r,\theta)r^2\sin(\theta)\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}r \\ &\quad+\int_{r=\alpha}^{\beta}\int_{\theta=\gamma}^{\delta}\int_{\varphi=0}^{2\pi} a_\theta u(r,\theta,t)\frac{1}{r}\frac{\partial v}{\partial\theta}(r,\theta)r^2\sin(\theta)\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}r \\ &= 2\pi\int_{r=\alpha}^{\beta}\int_{\theta=\gamma}^{\delta} a_r u(r,\theta,t)\frac{\partial v}{\partial r}(r,\theta)r^2\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}r \\ &\quad+2\pi\int_{r=\alpha}^{\beta}\int_{\theta=\gamma}^{\delta} a_\theta u(r,\theta,t)\frac{\partial v}{\partial\theta}(r,\theta)r\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}r\end{aligned} \tag{4}$$

We note that the dependence on φ in a is lost due to the orthonormality of $\hat{r},\hat{\theta},\hat{\varphi}$. Specifically, $\hat{r}(r,\theta,\varphi)\cdot\hat{r}(r,\theta,\varphi)=1$ which is constant in φ. Identical results hold for $\hat{\theta}$ and $\hat{\varphi}$.

For the surface integrals in (2), we first parameterize the surface into four components with explicit unit outward normals given:

$$\begin{aligned}\partial T =&\{(\alpha,\theta,\varphi):\gamma<\theta<\delta,0\le\varphi<2\pi\} && \boldsymbol{n}=-\hat{r}\\ &\cup\{(\beta,\theta,\varphi):\gamma<\theta<\delta,0\le\varphi<2\pi\} && \boldsymbol{n}=\hat{r}\\ &\cup\{(r,\gamma,\varphi):\alpha<r<\beta,0\le\varphi<2\pi\} && \boldsymbol{n}=-\hat{\theta}\\ &\cup\{(r,\delta,\varphi):\alpha<r<\beta,0\le\varphi<2\pi\} && \boldsymbol{n}=\hat{\theta}.\end{aligned}$$

We then define the numerical flux as the upwind flux defined by

$$\widehat{au}=a\{u\}+\frac{|a(r,\theta,\varphi)\cdot\boldsymbol{n}(r,\theta,\varphi)|}{2}[u]$$

where $\{u\}=\frac{u^++u^-}{2}$ and $[u]=u^+\boldsymbol{n}^++u^-\boldsymbol{n}^-$. Note that since $\boldsymbol{n}^\pm$ are only signed versions of $\hat{r}$ and $\hat{\theta}$, then the orthonormality of $\hat{r},\hat{\theta},\hat{\varphi}$ removes any dependence on φ in the integral evaluation, thus we explicitly evaluate the surface integral in (2) as

$$
\begin{aligned}
\int_{\partial T} & \widehat{a(r,\theta,\varphi)u} \cdot \boldsymbol{n}(r,\theta,\varphi) v(r,\theta)\,\mathrm{d}S = \\
& -2\pi \int_{\theta=\gamma}^{\delta} (a_r \hat{r}\{u(\alpha,\theta,t)\} + \tfrac{|a_r|}{2}[u(\alpha,\theta,t)]) \cdot \hat{r} v(\alpha,\theta)\alpha^2 \sin(\theta)\,\mathrm{d}\theta \\
& +2\pi \int_{\theta=\gamma}^{\delta} (a_r \hat{r}\{u(\beta,\theta,t)\} + \tfrac{|a_r|}{2}[u(\beta,\theta,t)]) \cdot \hat{r} v(\beta,\theta)\beta^2 \sin(\theta)\,\mathrm{d}\theta \\
& -2\pi \int_{r=\alpha}^{\beta} (a_\theta \hat{\theta}\{u(r,\gamma)\} + \tfrac{|a_\theta|}{2}[u(r,\gamma)]) \cdot \hat{\theta} v(r,\gamma) r \sin(\gamma)\,\mathrm{d}r \\
& +2\pi \int_{r=\alpha}^{\beta} (a_\theta \hat{\theta}\{u(r,\delta)\} + \tfrac{|a_\theta|}{2}[u(r,\delta)]) \cdot \hat{\theta} v(r,\delta) r \sin(\delta)\,\mathrm{d}r
\end{aligned} \tag{5}
$$

If a portion of the surface lies on $\partial\Omega_{\text{out}}$, that is, $\alpha = r_0$ or $\gamma = \theta_0$, then replace $\widehat{au}$ with ag. Those surface contributions become

$$
\begin{aligned}
& -2\pi \int_{\theta=\gamma}^{\delta} a_r g(r_0,\theta) v(r_0,\theta) r_0^2 \sin(\theta)\,\mathrm{d}\theta, \text{ and} \\
& -2\pi \int_{r=\alpha}^{\beta} a_\theta g(r,\theta_0) v(r,\theta_0) r \sin(\theta_0)\,\mathrm{d}r
\end{aligned}
$$

respectively.

Note that each integral contains a 2π that can be dropped.

Since u and v in the DG scheme are meant to be a sum of separable functions, then if we let $u = \psi_r(r)\psi_\theta(\theta)$ and $v = \xi_r(r)\xi_\theta(\theta)$, the volume integrals in (4) become

$$
\begin{aligned}
\int_T & a(r,\theta,\varphi) u \cdot \nabla v(r,\theta)\,\mathrm{d}T \\
& = 2\pi \int_{r=\alpha}^{\beta} a_r \psi_r(r) \xi_r'(r) r^2\,\mathrm{d}r \int_{\theta=\gamma}^{\delta} \psi_\theta(\theta)\xi_\theta(\theta)\sin(\theta)\,\mathrm{d}\theta \\
& + 2\pi \int_{r=\alpha}^{\beta} \psi_r(r)\xi_r(r) r\,\mathrm{d}r \int_{\theta=\gamma}^{\delta} a_\theta \psi_\theta(\theta)\xi_\theta'(\theta)\sin(\theta)\,\mathrm{d}\theta
\end{aligned} \tag{6}
$$

and the surface integrals in (5) become

$$
\begin{aligned}
\int_{\partial T} & \widehat{a(r,\theta,\varphi)u} \cdot \boldsymbol{n}(r,\theta,\varphi) v(r,\theta)\,\mathrm{d}S = \\
& -2\pi(a_r\{\psi_r(\alpha)\} + \tfrac{|a_r|}{2}[\psi_r(\alpha)])\xi_r(\alpha)\alpha^2 \int_{\theta=\gamma}^{\delta} \varphi_\theta(\theta)\xi_\theta(\theta)\sin(\theta)\,\mathrm{d}\theta \\
& +2\pi(a_r\{\psi_r(\beta)\} + \tfrac{|a_r|}{2}[\psi_r(\beta)])\xi_r(\beta)\beta^2 \int_{\theta=\gamma}^{\delta} \varphi_\theta(\theta)\xi_\theta(\theta)\sin(\theta)\,\mathrm{d}\theta \\
& -2\pi \int_{r=\alpha}^{\beta} \psi_r(r)\xi_r(r) r\,\mathrm{d}r (a_\theta\{\psi_\theta(\gamma)\} + \tfrac{|a_\theta|}{2}[\psi_\theta(\gamma)])\xi_\theta(\gamma)\sin(\gamma) \\
& +2\pi \int_{r=\alpha}^{\beta} \psi_r(r)\xi_r(r) r\,\mathrm{d}r (a_\theta\{\psi_\theta(\delta)\} + \tfrac{|a_\theta|}{2}[\psi_\theta(\delta)])\xi_\theta(\delta)\sin(\delta).
\end{aligned} \tag{7}
$$

Therefore, we need to calculate the following 1D integrals:

$$\int_{r=\alpha}^{\beta} a_r \psi_r(r) \xi_r'(r) r^2 \, \mathrm{d}r$$
$$\int_{r=\alpha}^{\beta} \psi_r(r) \xi_r(r) r \, \mathrm{d}r$$
$$\int_{\theta=\gamma}^{\delta} a_\theta \psi_\theta(\theta) \xi_\theta'(\theta) \sin(\theta) \, \mathrm{d}\theta$$
$$\int_{\theta=\gamma}^{\delta} \psi_\theta(\theta) \xi_\theta(\theta) \sin(\theta) \, \mathrm{d}\theta$$