COMP2201 – Discrete Mathematics
Complexity analysis

1. Determine whether the following statement is true or false. If the statement is false give a counterexample. Assume that the functions f, g and h take on only positive values.

$$f(n) = \Theta(h(n)) \text{ and } g(n) = \Theta(h(n)), \text{then } f(n) + g(n) = \Theta(h(n))$$

Solution

Given $f(n) = \Theta(h(n))$ and $g(n) = \Theta(h(n))$
Show that $f(n) + g(n) = \Theta(h(n))$

$f(n) = \Theta(h(n))$

$\Rightarrow \quad C_1 \mid h(n) \mid \quad \le \quad \mid f(n) \mid \quad \le \quad C_2 \mid h(n) \mid$
where C_1 and C_2 are constants

$g(n) = \Theta(h(n))$

$\Rightarrow \quad C_3 \mid h(n) \mid \quad \le \quad \mid g(n) \mid \quad \le \quad C_4 \mid h(n) \mid$
where C_3 and C_4 are constants

By Addition

$(C_1 + C_3) \mid h(n) \mid \quad \le \quad \mid f(n) \mid + \mid g(n) \mid \quad \le \quad (C_2 + C_4) \mid h(n) \mid$

$\Rightarrow \quad C_5 \mid h(n) \mid \quad \le \quad \mid f(n) \mid + \mid g(n) \mid \quad \le \quad C_6 \mid h(n) \mid$
where C_5 and C_6 are constants

As f, g and h take on only positive values

$\mid f(n) \mid + \mid g(n) \mid \quad = \quad \mid f(n) + g(n) \mid$

Therefore

$C_5 \mid h(n) \mid \quad \le \quad \mid f(n) + g(n) \mid \quad \le \quad C_6 \mid h(n) \mid$

$\therefore \quad f(n) + g(n) = \Theta(h(n))$

2. Determine the order of growth of $\sum_{k=1}^{n+1} k^{m-1}$, if m is a positive integer?

Solution

If m is a positive integer,

$$
\begin{aligned}
\text{Then, } 1^{m-1} + 2^{m-1} + 3^{m-1} + \ldots + n^{m-1} + (n+1)^{m-1} \\
&\le (n+1)^{m-1} + (n+1)^{m-1} + \ldots + (n+1)^{m-1} \\
&\le (n+1) * (n+1)^{m-1} \\
&\le (n+1)^m
\end{aligned}
$$

For all $n \ge 1$.

Therefore $1^{m-1} + 2^{m-1} + 3^{m-1} + \ldots + (n+1)^{m-1} = \mathrm{O}((n+1)^m)$.

We may obtain a lower bound by examining the full series in relation to a portion of the series :

$$
\begin{aligned}
1^{m-1} + 2^{m-1} + 3^{m-1} + \ldots + n^{m-1} + (n+1)^{m-1} \\
&\ge \left\lceil \frac{n+1}{2} \right\rceil^{m-1} + \ldots + (n\text{-}1)^{m-1} + n^{m-1} + (n+1)^{m-1} \\
&\ge \left\lceil \frac{n+1}{2} \right\rceil^{m-1} + \ldots + \left\lceil \frac{n+1}{2} \right\rceil^{m-1} + \left\lceil \frac{n+1}{2} \right\rceil^{m-1} \\
&\ge \left\lceil \frac{n+1}{2} \right\rceil * \left\lceil \frac{n+1}{2} \right\rceil^{m-1} \\
&\ge \left(\frac{n+1}{2} \right)\left(\frac{n+1}{2} \right)^{m-1} \\
&\ge \left(\frac{1}{2} \right)(n+1)\left(\frac{1}{2^{m-1}} \right)(n+1)^{m-1} \\
&\ge \left(\frac{1}{2^{1+m-1}} \right)(n+1)^m \\
&\ge \left(\frac{1}{2^m} \right)(n+1)^m
\end{aligned}
$$

Hence,

$$1^{m-1} + 2^{m-1} + 3^{m-1} + \ldots + n^{m-1} + (n+1)^{m-1} = \Omega((n+1)^m)$$

Therefore,

$$1^{m-1} + 2^{m-1} + 3^{m-1} + \ldots + n^{m-1} + (n+1)^{m-1} = \Theta((n+1)^m)$$

3. Let $f(n)$ and $g(n)$ be functions defined on the set of positive integers
Prove or disprove the following:

if $f(n) = \Theta(k(n))$ and $g(n) = \Theta(h(n))$
then $f(n)g(n) = \Theta(h(n)k(n))$.

Solution

$f(n) = \Theta(k(n))$

$\Rightarrow \quad C_1 |k(n)| \le |f(n)| \le C_2 |k(n)|$

where C_1 and C_2 are constants

$g(n) = \Theta(h(n))$

$\Rightarrow \quad C_3 |h(n)| \le |g(n)| \le C_4 |h(n)|$

where C_3 and C_4 are constants

By Multiplication

$C_1 C_3 |k(n)| |h(n)| \le |f(n)| |g(n)| \le C_2 C_4 |k(n)| |h(n)|$

As $f(n)$ and $g(n)$ are functions defined on the set of positive integers

$|f(n)| |g(n)| = |f(n) g(n)|$

Where $k(n)$ and $h(n)$ are functions defined on the set of positive integers

$|k(n)| |h(n)| = |k(n) h(n)|$

$\therefore \quad C_5 |k(n) h(n)| \le |f(n) g(n)| \le C_6 |k(n) h(n)|$

where C_5 and C_6 are new constants

$\therefore \quad f(n) g(n) = \Theta(h(n) k(n))$

4. Show that $n + 2n + 3n + \ldots + (n-1)n + n^2$ is of order n^3

Solution

$$
\begin{aligned}
n + 2n + 3n + \ldots + (n-1)n + n^2 &\le n^2 + n^2 + n^2 + \ldots + n^2 \\
&\le n * n^2 \\
&\le n^3
\end{aligned}
$$

For all $n \ge 1$.

Therefore $n + 2n + 3n + \ldots + (n-1)n + n^2 = \mathrm{O}(n^3)$.

We may obtain a lower bound by examining the full series in relation to a portion of the series :

$$
\begin{aligned}
& n + 2n + 3n + \ldots + (n-1)n + n^2 \\
&\ge \left\lceil \frac{n+1}{2} \right\rceil n + \ldots + (n-1)n + n^2 \\
&\ge \left\lceil \frac{n+1}{2} \right\rceil n + \ldots + \left\lceil \frac{n+1}{2} \right\rceil n + \left\lceil \frac{n+1}{2} \right\rceil n \\
&\ge \left\lceil \frac{n+1}{2} \right\rceil n * \left\lceil \frac{n+1}{2} \right\rceil \\
&\ge \left(\frac{n}{2} \right) n \left(\frac{n}{2} \right) \\
&\ge \left(\frac{1}{4} \right) n^3
\end{aligned}
$$

Hence,

$$n + 2n + 3n + \ldots + (n-1)n + n^2 = \Omega(n^3)$$

Therefore,

$$n + 2n + 3n + \ldots + (n-1)n + n^2 = \Theta(n^3)$$

5. Determine whether the following statement is true or false. If the statement is false give a counterexample. Assume that the functions f, g and h take on only positive values.

$$f(n) = O(g(n)) \textit{ and } g(n) = O(f(n)), \text{then } f(n) = \Theta(g(n))$$

Solution

Given $f(n) = O(g(n))$ and $g(n) = O(f(n))$

Show that $f(n) = \Theta(g(n))$

$f(n) = O(g(n))$

$\Rightarrow \quad |f(n)| \leq C_1 |g(n)|$ where C_1 is a constant

$g(n) = O(f(n))$

$\Rightarrow \quad |g(n)| \leq C_2 |f(n)|$ where C_2 is a constant

$|g(n)| / C_2 \leq |f(n)|$

$\Rightarrow \quad (1/C_2)|g(n)| \leq |f(n)|$

$C_3 |g(n)| \leq |f(n)|$ where C_3 is a new constant

$\therefore \quad C_3 |g(n)| \leq |f(n)|$ or $|f(n)| \geq C_3 |g(n)|$ or $f(n) = \Omega(g(n))$

$\Rightarrow \quad C_3 |g(n)| \leq |f(n)| \leq C_1 |g(n)|$

$\therefore \quad f(n) = \Theta(g(n))$

OR

As $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

$f(n) = \Theta(g(n))$