COMP2201 – Discrete Mathematics
Binomial Theorem and Master Theorem

1. Expand $(2c - 3d)^5$ using the Binomial Theorem.

Solution

$$(a+b)^n = \sum_{k=0}^{n} C(n,k)a^{n-k}b^k$$

Required $(2c - 3d)^5$

Let $a = 2c$, $b = -3d$, $n = 5$

$$(2c-3d)^5 = \sum_{k=0}^{5} C(5,k)(2c)^{5-k}(-3d)^k$$

$$= C(5,0)a^5b^0 + C(5,1)a^{5-1}b^1 + C(5,2)a^{5-2}b^2 + C(5,3)a^{5-3}b^3 + C(5,4)a^{5-4}b^4 + C(5,5)a^{5-5}b^5$$

$$= 1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5$$

By Substitution

$$= (2c)^5(-3d)^0 + 5(2c)^4(-3d)^1 + 10(2c)^3(-3d)^2 + 10(2c)^2(-3d)^3 + 5(2c)^1(-3d)^4 + (2c)^0(-3d)^5$$

$$= (32c^5) + 5(-48c^4d) + 10(72c^3d^2) + 10(-108c^2d^3) + 5(162cd^4) + (-243d^5)$$

$$= 32c^5 - 240c^4d + 720c^3d^2 - 1080c^2d^3 + 810cd^4 - 243d^5$$

2. Use the Binomial Theorem to show that

$$\sum_{k=0}^{n} 2^k C(n,k) = 3^n$$

Solution

We know that $(a+b)^n = \sum_{k=0}^{n} C(n,k)a^{n-k}b^k$

We attempt to eliminate a^{n-k} and to substitute values for the proof

Let $a = 1$, $b = 2$

$$\sum_{k=0}^{n} C(n,k)a^{n-k}b^k = (a+b)^n$$

$$\sum_{k=0}^{n} C(n,k)1^{n-k}2^k = (1+2)^n$$

As $1^x = 1$ for $x \in \mathbb{Z}$

$$\sum_{k=0}^{n} 2^k C(n,k) = 3^n$$

3. What is the row of Pascal's triangle containing the binomial coefficients $\binom{9}{k}$, $0 \le k \le 9$

Solution

$$\binom{9}{0}, \binom{9}{1}, \binom{9}{2}, \binom{9}{3}, \binom{9}{4}, \binom{9}{5}, \binom{9}{6}, \binom{9}{7}, \binom{9}{8}, \binom{9}{9},$$

i.e.	1	9	36	84	126	126	84	36	9	1

OR

Using Pascal's Theorem

Pascal's Identity states $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

or $C(n+1,k) = C(n,k-1) + C(n,k)$ *for* $1 \le k \le n$

Therefore

$$C(8+1,1) = C(8,1-1) + C(8,1) = 1+8 = 9$$
$$C(8+1,2) = C(8,2-1) + C(8,2) = 8+28 = 36$$
$$C(8+1,3) = C(8,3-1) + C(8,3) = 28+56 = 84$$
$$C(8+1,4) = C(8,4-1) + C(8,4) = 56+70 = 126$$
$$C(8+1,5) = C(8,5-1) + C(8,5) = 70+56 = 126$$
$$C(8+1,6) = C(8,6-1) + C(8,6) = 56+28 = 84$$
$$C(8+1,7) = C(8,7-1) + C(8,7) = 28+8 = 36$$
$$C(8+1,8) = C(8,8-1) + C(8,8) = 8+1 = 9$$

i.e.	1	9	36	84	126	126	84	36	9	1

4. Let a,b,c be integers such that $a \ge 1$, $b > 1$ and $c > 0$. Let $f\colon N \rightarrow R$ be functions where N is the set of Natural numbers and R is the set of Real numbers such that

$$f(n) = cf(n/b) + a^b c$$

By using the principles of Recurrence Relation, find a general formula for $f(n)$

<u>Solution</u>

$$f(n) = c\,f(n/b) + a^b c \qquad .....1$$

Using equ. 1, As $f(n/b) = c\,f(n/b\,/\,b) + a^b c$
$= c\,f(n/b^2) + a^b c$

Substituting for $f(n/b)$

$$= c\,[\,c\,f(n/b^2) + a^b c\,] + a^b c$$

$$= c^2 f(n/b^2) + a^b c(c+1) \qquad .....2$$

Using equ. 1, As $f(n/b^2) = c\,f(n/b^2\,/\,b) + a^b c$
$= c\,f(n/b^3) + a^b c$

Substituting for $f(n/b^2)$

$$= c^2\,[\,c\,f(n/b^3) + a^b c\,] + a^b c(c+1)$$

$$= c^3 f(n/b^3) + a^b c\,(c^2 + c + 1) \qquad .....3$$

Using equ. 1, As $f(n/b^3) = c\,f(n/b^3\,/\,b) + a^b c$
$= c\,f(n/b^4) + a^b c$

Substituting for $f(n/b^3)$

$= \quad c^3 [\, c\, f(n/b^4) + a^b c\,] + a^b c\, (c^2 + c + 1)$

$= \quad c^4 f(n/b^4) + a^b c\, (c^3 + c^2 + c + 1)$ 4

$= \quad c^4 f(n/b^4) + a^b (c^4 + c^3 + c^2 + c)$ 4

$= \quad ......$

Given that k is a positive integer greater than 1

$$f(n) = c^k f(n/b^k) + a^b \sum_{i=1}^{k} c^i$$

OR

$$f(n) = c^k f(n/b^k) + a^b c \sum_{i=0}^{k-1} c^i$$

5. For each of the following recurrences,
   - Give an expression for the runtime $T(n)$ if the recurrence can be solved with the Master Theorem.
   - Otherwise, indicate that the Master Theorem does not apply and state why the problem cannot be solved by the Master Theorem.

   In all cases, assume that $T(n) = 1$ for $n \leq 1$.

   (a) $T(n) = 4T(n/2) + n$

<u>Solution</u>

Consider the recurrence:

$T(n) = aT(n/b) + f(n)$

where a, b are constants and f(n) is an arbitrary function in n,
let the critical exponent, $E = \log_b a$

Given

$T(n) = 4T(n/2) + n$

The critical exponent, E

$E = \log_2 4 = 2$

By examining the overhead function f(n) with n^E

$f(n) = n$ *and* $n^E = n^2$

Therefore

$f(n) = n = O(n^E) = O(n^2)$

We find ϵ *that allows* $f(n) = O(n^{E-\epsilon})$

For definiteness, let $\epsilon = 0.5$

$E - \epsilon = 2 - 0.5 = 1.5$

It is clear that

$f(n) = n = O(n^{E-\epsilon}) = O(n^{1.5})$

As for some $\epsilon > 0$, *and* $f(n) = O(n^{E-\epsilon})$

Master Theorem Case 1 holds

We conclude that

the solution for the equation

$T(n) = 4T(n/2) + n$

is

$$T(n) = \Theta(n^E) \quad or \quad T(n) = \Theta(n^{\log_b a})$$

$$T(n) = \Theta(n^{\log_2 4})$$

$$T(n) = \Theta(n^2)$$

(b) $T(n) = 2^n T(n/2) + n^n$

Solution

$T(n) = 2^n T(n/2) + n^n \rightarrow$ Does not apply
(a is not constant)

(c) $T(n) = 2T(n/2) + n/\log n$

Solution

$T(n) = 2T(n/2) + n/\log n \rightarrow$ Does not apply
(non-polynomial difference between f(n) and $n^{\log_b a}$)

(d) $f(n) = 3f(n/3) + n/2$

Solution

Consider the recurrence:

$$f(n) = af(n/b) + g(n)$$

where a, b *are constants and* $g(n)$ *is an arbitrary function in n,*
let the critical exponent, $E = \log_b a$

Given

$$f(n) = 3f(n/3) + n/2$$

The critical exponent, E

$$E = \log_3 3 = 1$$

By examining the overhead function $g(n)$ *with* n^E

$$g(n) = n/2 = ½n \quad and \quad n^E = n^1$$

Therefore

$$g(n) = ½\,n = O(n^E) = O(n)$$

and

$$g(n) = ½\,n = \Omega(n^E) = \Omega(n)$$

We have

$$g(n) = \Theta(n^E)$$

As $g(n) = \Theta(n^E)$

Master Theorem Case 2 holds

We conclude that

the solution for the equation

$$f(n) = 3f(n/3) + n/2$$

is

$$f(n) = \Theta(g(n) \log n) \quad or \quad f(n) = \Theta(n^E \log n) \quad or \quad f(n) = \Theta(n^{\log_b a} \log n)$$

$$f(n) = \Theta(n \log n)$$

(e) $f(n) = 64f(n/8) - n^2 \log n$

Solution

$f(n) = 64f(n/8) - n^2 \log n$ → Does not apply

(g(n) is not positive)

(f) $f(n) = 16f(n/4) + n!$

Solution

Consider the recurrence:

$f(n) = af(n/b) + g(n)$

where a, b are constants and f(n) is an arbitrary function in n,

let the critical exponent, $E = \log_b a$

Given

$f(n) = 16f(n/4) + n!$

The critical exponent, E

$E = \log_4 16 = 2$

By examining the overhead function g(n) with n^E

$g(n) = n!$ *and* $n^E = n^2$

Therefore

$g(n) = n! = \Omega(n^E) = \Omega(n^2)$

Furthermore, we find δ *that allows* $g(n) = O(n^{E+\epsilon})$

For definiteness, let $\epsilon = 0.5$

$E + \epsilon = 2 + 0.5 = 2.5$

It is clear that

$g(n) = n! = \Omega(n^{E+\epsilon}) = \Omega(n^{2.5})$

$g(n) = \Omega(n^{E+\epsilon})$,

As for some $\epsilon > 0$, *and* $g(n) = \Omega(n^{E+\delta})$, *but for some* $\epsilon > 0$, *and* $g(n) = O(n^{E+\epsilon})$

Master Theorem Case 3 holds

We conclude that

the solution for the equation

$f(n) = 16f(n/4) + n!$

is

$f(n) = \Theta(g(n))$

$f(n) = \Theta(n!)$