COMP2201 – Discrete Mathematics
Modular Arithmetic and Probability Theory – Part 1 (Special)
Tutorial solutions

1. Given that $73 \equiv 27 (mod\ 23)$

   List four other positive values and four negative values that are also congruent to 27.

   Solution

   The smallest positive value congruent to 27 (mod 23) is 4.

   The positive values are found by adding multiples of 23, therefore the values are

   4, 27, 50, 73, 96, 119, …

   The negative values are found by subtract multiples of 23, therefore the values are

   -19, -42, -65, -88, -111, -134, …

2. By using the Multiplicative Property of Congruence, find the smallest positive value which is congruent to

   $11 \times 15 \pmod 8$

   Solution

   By the Multiplicative Property of Congruence

   $[(a \bmod n) \times (b \bmod n)] \bmod n = (a \times b) \bmod n$

   We also have

   $(a \times b) \bmod n = [(a \bmod n) \times (b \bmod n)] \bmod n$

   $$\begin{aligned} 11 \times 15 \pmod 8 &= [(11 \bmod 8) \times (15 \bmod 8)] \bmod 8 \\ &= [\ 3 \times 7\ ] \bmod 8 \\ &= 21 \bmod 8 \\ &= 5 \end{aligned}$$

   Therefore

   $5 \equiv 11 \times 15 \pmod 8$

3. In Modular Arithmetic, define the set Z_n, list the residue classes (mod 5) and construct the Addition and Multiplication Tables in Z_5.

   Solution

   Define the set Z_n

The set Z_n as the set of nonnegative integers less than n. Therefore

$$Z_n = \{ 0, 1, 2, 3, \ldots, (n-1) \}$$

List the residue classes (mod 5)

The set of **residues**, or **residue classes** (mod n), is a class of [r] for each integer in Z_n where

$$[r] = \{a: a \text{ is an integer}, a \equiv r \pmod{n}\}.$$

The set of **residues**, or **residue classes** (mod 5), labeled as [0], [1], [2], [3], [4] are

$[0] = \{\ldots,-20, -15, -10, -5, 0, 5, 10, 15, 20, \ldots\}$
$[1] = \{\ldots,-19, -14, -9, -4, 1, 6, 11, 16, 21, \ldots\}$
$[2] = \{\ldots,-18, -13, -8, -3, 2, 7, 12, 17, 22, \ldots\}$
$[3] = \{\ldots,-17, -12, -7, -2, 3, 8, 13, 18, 23, \ldots\}$
$[4] = \{\ldots,-16, -11, -6, -1, 4, 9, 14, 19, 24, \ldots\}$

Construct the Addition and Multiplication Tables in Z_5.

+	**0**	**1**	**2**	**3**	**4**
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

×	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

4. Consider the random experiment of tossing eight fair coins simultaneously. What is the probability that the number of heads and the number of tails differ by at most 2?

Solution

The possible combinations for the exact number of heads and tails such that they differ by at **most 2** are

4 heads and 4 tails
3 heads and 5 tails
5 heads and 3 tails

These three cases are mutually exclusive and as such, the probability that the number of heads and the number of tails differ by at most 2 is given by;

$$\frac{{}^8C_4 + {}^8C_3 + {}^8C_5}{256} = \frac{70+56+56}{256} = \frac{91}{128}$$

5. Assuming that any seven-digits could be used to form a telephone number
   i. How many telephone numbers would not have any repeated digits?
   ii. How many seven-digit telephone numbers would have at least one repeated digit?
   iii. What is the probability that a randomly chosen seven-digit telephone number would have at least one repeated digit?

Solution

i. ${}^{10}P_7$ or 10 x 9 x 8 x 7 x 6 x 5 x 4 or 604800

ii. 10^7 - ${}^{10}P_7$ or 10000000 – 604800 or 9395200

iii. (10^7 - ${}^{10}P_7$) / 10^7 or 9395200 / 10000000 or 0.93952

6. A random experiment consists of rolling an unfair, six-sided die. The digit 6 is three times as likely to appear as each of the numbers 2 and 4. Each of the numbers 2 and 4 are twice as likely to appear as each of the numbers 1, 3 and 5
   i. Assign probabilities to the six outcomes in the sample space
   ii. Suppose that the random variable X, is assigned the value of the digit that appears when the die is rolled. What is the expected value of X?

Solution

i. P(6) = 3 x P(2)

P(2) = P(4) = 2 x P(1)

P(1) = P(3) = P(5)

P(6) = 6 x P(1)

$$\sum P(x) = 1$$

1 = P(6) + P(5) + P(4) + P(3) + P(2) + P(1)

1 = 6P(1) + P(1) + 2P(1) + P(1) + 2P(1) + P(1)

1 = 13P(1)

P(1) = P(3) = P(5) = 1/13
P(2) = P(4) = 2/13
P(6) = 6/13

ii. Expected Value = $\sum xP(x)$

where *x* is the possible outcome and *P(x)* is the probability associated with each outcome.

Expected Value
= 6P(6) + 5P(5) + 4P(4) + 3P(3) + 2P(2) + 1P(1)

= 6 x 6/13 + 5 x 1/13 + 4 x 2/13 + 3 x 1/13 + 2 x 2/13 + 1x1/13
= 4.384615 or 4.385

7. Suppose that if you study none at all, the probability of passing the course COMP2201 is 10%, if you studied lightly, the probability of passing the course is 50%, and if you studied intensely, the probability of passing the course is 85%. Suppose, we know that only 5% of the students do no study at all, 25% of them study lightly and 70% actually study intensely. What is the probability of passing the course?

Solution

Let N - Study None at all
L - Study Lightly
I - Studied Intensely

C - Passing the course COMP2201

Given

P(C|N) = 0.10
P(C|L) = 0.50
P(C|I) = 0.85

P(N) = 0.05
P(L) = 0.25
P(I) = 0.70

Required

P(C)

$$P(A) = P(A/B_1)P(B_1) + P(A/B_2)P(B_2) + P(A/B_3)P(B_3) + \ldots$$

Therefore

$$P(C) = P(C/N)P(N) + P(C/L)P(L) + P(C/I)P(I)$$

P(C) = (0.10)(0.05) + (0.50)(0.25) + (0.85)(0.70)

= 0.725