COMP2201 – Discrete Mathematics
Generating Functions

Question 1
Find the generating function for the sequence {0, 1, 2, 3, 4, 5, …}

Solution 1

We have {1, 1, 1, 1, …} $\equiv$ $1 + x + x^2 + x^3 + x^4 + \ldots \equiv \dfrac{1}{1-x}$

By differentiating

$$\{1, 2, 3, 4, \ldots\} \equiv \frac{1}{(1-x)^2}$$

By right shifting, i.e. multiplying by x^1

$$\{0, 1, 2, 3, 4, \ldots\} \equiv \frac{x}{(1-x)^2}$$

Therefore the generating function for the sequence {0, 1, 2, 3, 4, 5, …} is $\dfrac{x}{(1-x)^2}$

Question 2
Find the generating function for the sequence {2, 6, 12, 20, …}

Solution 2

We have {1, 1, 1, 1, …} $\equiv$ $1 + x + x^2 + x^3 + x^4 + \ldots \equiv \dfrac{1}{1-x}$

By differentiating

$$\{1, 2, 3, 4, \ldots\} \equiv \frac{1}{(1-x)^2}$$

By differentiating, again

For LHS

{2, 6, 12, 20, …}

For RHS

$$\text{Let} \qquad y = \frac{1}{(1-x)^2} = (1-x)^{-2}$$

$$\text{Let} \qquad p = 1 - x \Rightarrow y = p^{-2}$$

$$\text{then} \qquad \frac{dy}{dx} = \frac{dy}{dp}\frac{dp}{dx} = \frac{d(p^{-2})}{dp}\frac{d(1-x)}{dx}$$

$$= -2p^{-3} \cdot (-1)$$

$$= -2(1-x)^{-3}(-1)$$

$$= 2(1-x)^{-3}$$

$$= \frac{2}{(1-x)^3}$$

Therefore

$$\{2, 6, 12, 20, \ldots\} \qquad \equiv \qquad \frac{2}{(1-x)^3}$$

Therefore the generating function for the sequence $\{2, 6, 12, 20, \ldots\}$ is $\dfrac{2}{(1-x)^3}$

Question 3
Find the generating function for the sequence

$$\{1, \quad 8, \quad 21, \quad 40, \quad 65, \quad \ldots\}$$

Solution 3
Recognize that

1	=	1 * 1	=	1 * 1	=	(1 + 0 * 3)	*	1
8	=	2 * 4	=	4 * 2	=	(1 + 1 * 3)	*	2
21	=	3 * 7	=	7 * 3	=	(1 + 2 * 3)	*	3
40	=	4 * 10	=	10 * 4	=	(1 + 3 * 3)	*	4
65	=	5 * 13	=	13 * 5	=	(1 + 4 * 3)	*	5
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This suggests

Initial sequence,I	{1, 1, 1, 1, 1, ...}
Differentiation	{1, 2, 3, 4, 5, ...}
Right Shifting,x	{0, 1, 2, 3, 4, ...}
Scaling by 3	{0, 3, 6, 9, 12, ...}
Addition to I	{1, 4, 7, 10, 13, ...}

Right Shifting,x $\{0, 1, 4, 7, 10, 13, ...\}$
Differentiation $\{1*1, 2*4, 3*7, 4*10, 5*13, ...\} = \{1,8,21,40,65,...\}$

We have $\{1, 1, 1, 1, \ldots\} \equiv 1 + x + x^2 + x^3 + x^4 + \ldots \equiv \dfrac{1}{1-x}$

By Differentiating

$$\{1, 2, 3, 4,5 \ldots\} \equiv \frac{1}{(1-x)^2}$$

By Right-shifting, 1 place

$$\{0, 1, 2, 3, 4, \ldots\} \equiv \frac{x}{(1-x)^2}$$

By Scaling by 3

$$\{0, 3, 6, 9, 12, \ldots\} \equiv \frac{3x}{(1-x)^2}$$

By Addition to {1, 1, 1, 1,...}i.e. $1/(1-x)$

$$\{1, 4, 7, 10, 13, \ldots\} \equiv \frac{3x}{(1-x)^2} + \frac{1}{1-x}$$

$$\equiv \frac{2x+1}{(1-x)^2}$$

By Right Shifting, x

$$\{0, 1, 4, 7, 10, 13, \ldots\} \equiv \frac{x(2x+1)}{(1-x)^2}$$

By Differentiating, again

For LHS

$\{1*1, 2*4, 3*7, 4*10,5*13, \ldots\}$

$\{1, 8, 21, 40, 65, \ldots\}$

For RHS

$$Let \qquad y = \frac{x(2x+1)}{(1-x)^2} = x(2x+1)(1-x)^{-2}$$

Let $u = x(2x+1), v = (1-x)^{-2}, y = u \cdot v$

then $\frac{dy}{dx} = V\frac{du}{dx} + U\frac{dv}{dx}$

Further

Let $p = 1-x, \Rightarrow v = p^{-2}$

and $\frac{dv}{dx} = \frac{dv}{dp}\frac{dp}{dx}$

then $\frac{dy}{dx} = V\frac{du}{dx} + U\frac{dv}{dp}\frac{dp}{dx}$

$$\frac{dy}{dx} = (1-x)^{-2}\frac{d[x(2x+1)]}{dx} + x(2x+1)\frac{d(p^{-2})}{dp}\frac{d(1-x)}{dx}$$

$$= (1-x)^{-2}[(2x+1)\frac{dx}{dx} + x\frac{d(2x+1)}{dx}] + x(2x+1)(-2p^{-3})\cdot(-1)$$

$$= (1-x)^{-2}[(2x+1)(1) + x(2)] + x(2x+1)(-2p^{-3})\cdot(-1)$$

$$= (1-x)^{-2}[(2x+1)(1) + x(2)] + x(2x+1)[(-2)(1-x)^{-3}]\cdot(-1)$$

$$= (1-x)^{-2}[4x+1] + [2x(2x+1)(1-x)^{-3}]$$

$$= (1-x)^{-2}[(4x+1)(1-x)^{-1}(1-x)^{1}] + [2x(2x+1)(1-x)^{-3}]$$

$$= (1-x)^{-3}[(4x+1)(1-x)] + [2x(2x+1)(1-x)^{-3}]$$

$$= (1-x)^{-3}[-4x^2 + 3x + 1] + [(4x^2 + 2x)(1-x)^{-3}]$$

$$= (1-x)^{-3}(-4x^2 + 3x + 1 + 4x^2 + 2x)$$

$$= (1-x)^{-3}(5x+1)$$

$$= \frac{5x+1}{(1-x)^3}$$

Therefore the generating function for sequence {1, 8, 21, 40, 65, …} is $\frac{5x+1}{(1-x)^3}$