THE UNIVERSITY OF THE WEST INDIES
Mona Campus

Semester I ■ Semester II ☐ Supplemental/Summer School ☐

Mid-Semester Examinations of: October ■ /February/March ☐ /June ☐ 2016/2017

Course Code and Title: **COMP2201 Discrete Mathematics for Computer Scientists**

Date:	**Friday, October 28, 2016**	Time:	**2:00 p.m.**
Duration:	**1 Hour.**	Paper No:	**1 (of 1)**

Materials required:

Answer booklet:	**Normal**	■	**Special**	☐	**Not required**	☐		
Calculator: *(where applicable)*	**Programmable**	☐	**Non Programmable**	■	**Not required**	☐		
Multiple Choice answer sheets:	**numerical**	☐	**alphabetical**	☐	**1-20**	☐	**1-100**	☐

Auxiliary/Other material(s) – Please specify: None

Candidates are permitted to bring the following items to their desks: Pencil or pen, Ruler, ID card, Exam card

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Instructions to Candidates: This paper has 2 pages & 6 questions.

Candidates are reminded that the examiners shall take into account the proper use of the English Language in determining the mark for each response.

All questions are COMPULSORY.

Calculators are allowed.

1. (a) Find the coefficient of x^6y^3 in the expansion $(x+y)^9$ **[2]**

   (b) Use Pascal's triangle to compute the values of

$$\binom{6}{3} \quad \textit{and} \quad \binom{7}{5}$$ **[2]**

2. Consider the recurrence function

$$T(n) = 9T(n/3) + 4n^3$$

Give an expression for the runtime $T(n)$ if the recurrence can be solved with the Master Theorem. Assume that $T(n) = 1$ for $n \le 1$. **[6]**

3. (a) In a given university only 5 percent of the students arrive at an examination one hour before it begins. Find the probability that among 120 students in that university, at least three will arrive at an examination one hour before it begins. **[3]**

   (b) If a student does not study at all for this COMP2201 Mid-term examination, the probability of passing the examination is 2%. If one studies at an average level, the probability of passing the examination is 48% whereas if study is done intensely, the probability of passing the COMP2201 Mid-term examination is 92%. The course lecturer is sure that 5% of students do not study at all, 75% of them study at an average level and 20% of them study intensely.
   Draw the Probability Tree that represents the given scenario. **[3]**

4. Seven (7) marbles of different colours and varied weights are placed in a bag. The Red marble is two times as likely to be pulled as the Orange and Yellow marbles. The Orange marble is three times as likely to be pulled as the Green and Blue marbles. The Blue marble is three times as likely to be pulled as the Indigo and Violet marbles. Assign probabilities to the seven outcomes in the sample space. **[5]**

5. Let $f_1(x)$, $f_2(x)$, $g_1(x)$ and $g_2(x)$ be functions defined $f_i : \mathbb{Z}^+ \rightarrow \mathbb{R}$, $g_i : \mathbb{Z}^+ \rightarrow \mathbb{R}$
   where $\mathbb{Z}^+$ is the set of Positive integers and $\mathbb{R}$ is the set of Real numbers

   Prove the following statement

$$\textit{If } f_1(x) = \Theta(g_1(x)) \textit{ and } f_2(x) = \Theta(g_2(x)), \text{ then } (f_1 f_2)(x) = \Theta((g_1 g_2)(x))$$ **[4]**

6. (a) Using the sequences y and z defined by $x_n = 3^n + 1$, $y_n = n(n - 1)$

   Find $$\left(\sum_{i=1}^{3} x_i\right)\left(\sum_{i=2}^{4} y_i\right)$$ **[1]**

   (b) Consider the arithmetic series:

$$5/2 + 11/2 + 17/2 + 23/2 + \ldots$$

   What is the smallest value of n such that $S_n > 300$,
   where S_n is the sum of the first n terms of the series? **[4]**

END OF QUESTION PAPER

1. (a) Find the coefficient of x^6y^3 in the expansion $(x+y)^9$ **[2]**

(b) Use Pascal's triangle to compute the values of

$$\binom{6}{3} \quad and \quad \binom{7}{5}$$

[2]

Solution 1

[(a) Proof using Binomial Theorem - 2 marks]
[Correct Binomial Theorem Formula - 1 mark]
[Logical steps of the Proof - 1 mark]
[(b) The row of Pascal's Triangle - 2 marks]
[Correct Rows 1-3 - ½ mark]
[Correct Rows 4-7 - ½ mark]
[Correct values for C(6, 3) and C(7,5) - 1 mark]

(a) We know that

$$(a+b)^n = \sum_{k=0}^{n} C(n,k)a^{n-k}b^k$$

Considering

The term involving x^6y^3 arises in the Binomial Theorem by taking

n = 9 and k = 3:

$$C(n,k)x^{n-k}y^k = C(9,3)x^6y^3$$
$$= 84x^6y^3$$

Thus the coefficient of x^6y^3 is 84

(b)

							1							
						1		**1**						
					1		**2**		**1**					
				1		**3**		**3**		**3**				
			1		**4**		**6**		**4**		**1**			
		1		**5**		**10**		**10**		**5**		**1**		
	1		**6**		**15**		**20**		**15**		**6**		**1**	
1		**7**		**21**		**35**		**35**		**21**		**7**		**1**

Therefore

$$\binom{6}{3} = 20 \quad and \quad \binom{7}{5} = 21$$

OR

Using Pascal's Theorem

Pascal's Identity states $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

or $C(n+1,k) = C(n,k-1) + C(n,k)$ *for* $1 \le k \le n$

Therefore

$C(6,3) = C(5+1,3) = C(5,3-1) + C(5,3) = 10 + 10 = 20$

$C(7,5) = C(6+1,5) = C(6,5-1) + C(6,5) = 15 + 6 = 21$

2. Consider the recurrence function

$$T(n) = 9T(n/3) + 4n^3$$

Give an expression for the runtime $T(n)$ if the recurrence can be solved with the Master Theorem. Assume that $T(n) = 1$ for $n \le 1$. **[5]**

Solution 2

[Finding the critical exponent - ½ mark]
[Determining Order of Growth by comparison of *f(n)* and n^E - 1 mark]
[Proving Order of Growth by comparison of *f(n)* and $n^{E-\epsilon}$ - ½ mark]
[Proving Regularity Condition comparison of *f(n)* and $n^{E-\epsilon}$ - ½ mark]
[Stating clearly the Master Theorem Case selection - 1 mark]
[Correct solution stated using - 1 mark]

Given

$T(n) = 9T(n/3) + 4n^3$

Consider the recurrence:

$T(n) = aT(n/b) + f(n)$

where a, b are constants and g(n) is an arbitrary function in n,

Let the critical exponent, $E = \log_b a$

The critical exponent, E

$E = \log_3 9 = 2$

By examining the overhead function $f(n)$ with n^E

$f(n) = 4n^3$ and $n^E = n^2$

Therefore

$f(n) = 4n^3 = \Omega(n^2) = \Omega(n^E)$

We have

$f(n) = \Omega(n^E)$

We are on track for Master Theorem Case 3, if

(1) We can find $\epsilon > 0$ that allows $f(n) = \Omega(n^{E+\epsilon})$

AND

(2) $af(n/b) \le cf(n)$ for some constant c<1

(1) Furthermore, we find ϵ that allows $f(n) = O(n^{E+\epsilon})$

For definiteness, let $\epsilon = 0.5$

$E + \epsilon = 2 + 0.5 = 2.5$

It is clear that

$f(n) = 4n^3 = \Omega(n^{E+\epsilon}) = \Omega(n^{2.5})$

$f(n) = \Omega(n^{E+\epsilon}),$

(2) $af(n/b) \leq cf(n)$ for some constant $c<1$

As $a=9$, $b=3$, $f(n) = 4n^3$, it is clear that

$9(4(n/4)^3) \leq c\, 4n^3$ for some $c=0.9$

As for some $\epsilon > 0$, and $f(n) = \Omega(n^{E+\epsilon})$, and

f(n) satisfies the Regularity condition

$af(n/b) \leq cf(n)$ for some constant $c<1$ and all sufficiently large n

Master Theorem Case 3 holds

We conclude that

the solution for the given equation

$T(n) = 9T(n/3) + 4n^3$

is

$T(n) = \Theta(f(n))$

$T(n) = \Theta(n^3)$

3. (a) In a given university only 5 percent of the students arrive at an examination one hour before it begins. Find the probability that among 120 students in that university, at least three will arrive at an examination one hour before it begins. **[3]**

(b) If a student does not study at all for this COMP2201 Mid-term examination, the probability of passing the examination is 2%. If one studies at an average level, the probability of passing the examination is 48% whereas if study is done intensely, the probability of passing the COMP2201 Mid-term examination is 92%. The course lecturer is sure that 5% of students do not study at all, 75% of them study at an average level and 20% of them study intensely.
Draw the Probability Tree that represents the given scenario. **[3]**

Solution 3

[(a) Probability Distribution - 3 marks]

[Identification of correct Distribution - ½ mark]

[Correct Formula - ½ mark]

[Correct steps and solution - 2 marks]

[(b) Probability Tree - 3 marks]

[Correct Layout of Problem - ½ mark]

[Prior Probabilities - ½ mark]

[Conditional Probabilities - 1 mark]

[Joint Probabilities - 1 mark]

(a) Binomial

$$b(x; n, \theta) = C^n_x \, \theta^x (1-\theta)^{n-x} \text{ for } x = 0,1,2..,n$$

b(≥ 3; 120,0.05) = 1 – b(<3; 120, 0.05)

= 1 – [b(0; 120,0.05) + b(1; 120,0.05) + b(2; 120,0.05)]

= 1 – ($_{120}C_0$ x 0.05^0 x 0.95^{120} + $_{120}C_1$ x 0.05^1 x 0.95^{119} + $_{120}C_2$ x 0.05^2 x 0.95^{118})

= 1 – (0.002122 + 0.013405 + 0.041978)

= 1 – 0.057505

= 0.942495 or 0.9425 or 94.3%

Poisson

$$p(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$x = 0,1,2,...$

As λ = nθ = 120 x 0.05 = 6

p(≥3; 6) = 1 - p(<3; 6)

= 1 - [p(0; 6) + p(1; 6) + p(2; 6)]

= 1 - (0.002479 + 0.014873 + 0.044618)

= 1 – 0.061969

= 0.938031 or 0.9380 or 93.8%

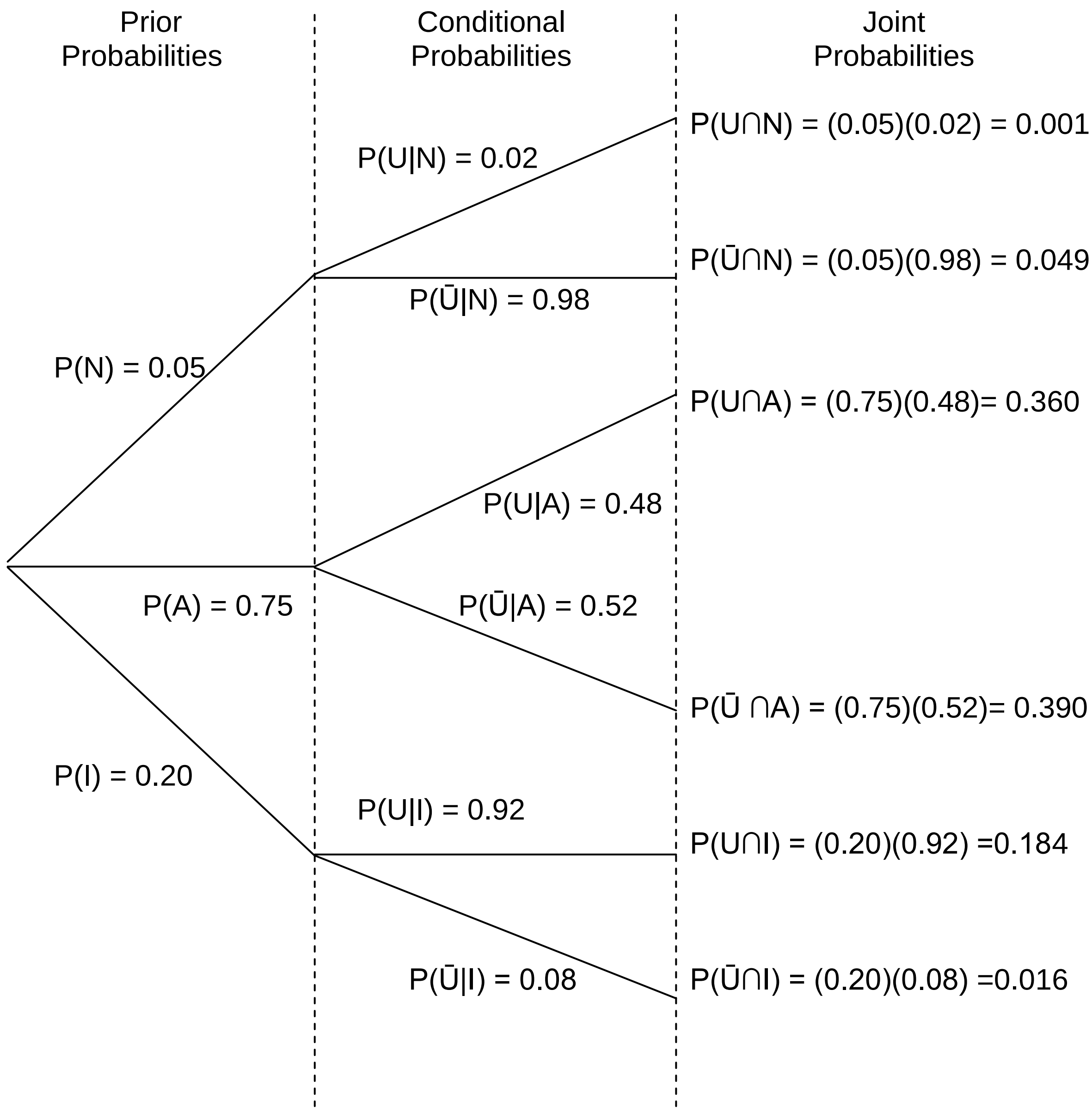
(b) Let N - Studies None at all
A - Studies at an Average Level
I - Studies Intensely

U - Passing the course COMP2201 Mid-term examination

Given

P(U|N) = 0.02
P(U|A) = 0.48
P(U|I) = 0.92

P(N) = 0.05
P(A) = 0.75
P(I) = 0.20



4. Seven (7) marbles of different colours and varied weights are placed in a bag. The Red marble is two times as likely to be pulled as the Orange and Yellow marbles. The Orange marble is three times as likely to be pulled as the Green and Blue marbles. The Blue marble is three times as likely to be pulled as the Indigo and Violet marbles. Assign probabilities to the seven outcomes in the sample space. **[5]**

Solution 4

[Probability Function - 5 marks]
[Correct initial layout of solution - 2 marks]
[Logical steps towards correct solution - 3 marks]

Let R represent Red marble

Let O represent Orange marble

Let Y represent Yellow marble

Let G represent Green marble

Let B represent Blue marble
Let I represent Indigo marble
Let V represent Violet marble

P(R) = 2 x P(O) = 2 x P(Y)
P(O) = 3 x P(G) = 3 x P(B)
P(B) = 3 x P(I) = 3 x P(V)

$$\sum P(x) = 1$$

P(R) + P(O) + P(Y) + P(G) + P(B) + P(I) + P(V) = 1

Interpreting in terms of P(V)
P(R) = 2 x P(O) = 2 x 3 x P(B) = 2 x 3 x 3 x P(V) = 18 P(V)
P(O) = P(Y) = 3 x P(B) = 3 x 3 x P(V) = 9 P(V)
P(Y) = 3 x P(B) = 3 x 3 x P(V) = 9 P(V)
P(G) = P(B) = 3P(V)
P(B) = 3P(V)
P(I) = P(V)

18P(V) + 9P(V) + 9P(V) + 3P(V) + 3P(V) + P(V) + P(V) = 1

(18 + 9 + 9 + 3 + 3 + 1 + 1)P(V) = 1
P(V) = 1/44

P(R) = 18/44 = 9/22
P(O) = 9/44
P(Y) = 9/44
P(G) = 3/44
P(B) = 3/44
P(I) = 1/44
P(V) = 1/44

5. Let $f_1(x)$, $f_2(x)$, $g_1(x)$ and $g_2(x)$ be functions defined $f_i : \mathbb{Z}^+ \rightarrow \mathbb{R}$, $g_i : \mathbb{Z}^+ \rightarrow \mathbb{R}$ where $\mathbb{Z}^+$ is the set of Positive integers and $\mathbb{R}$ is the set of Real numbers
Prove the following statement

$$\text{If } f_1(x) = \Theta(g_1(x)) \text{ and } f_2(x) = \Theta(g_2(x)), \text{ then } (f_1 f_2)(x) = \Theta((g_1 g_2)(x))$$ **[4]**

Solution 5

[Stating f_1 (n) and f_2 (n) in terms of the inequality - 1 mark]
[Applying Distributive Law of Function Composition - 1 mark]
[Logical steps of the Proof - 2 marks]

Given $f_1(x) = \Theta(g_1(x))$ and $f_2(x) = \Theta(g_2(x))$
Show that $f_1\, f_2(x) = \Theta(g_1g_2(x))$

$f_1(x) = \Theta(g_1(x))$

$\Rightarrow \quad C_1\,|\,g_1(x)\,| \quad \le \quad |\,f_1(x)\,| \quad \le \quad C_2\,|\,g_1(x)\,|$ …1
where C_1 and C_2 are constants

$f_2(x) = \Theta(g_2(x))$

$\Rightarrow \quad C_3\,|\,g_2(x)\,| \quad \le \quad |\,f_2(x)\,| \quad \le \quad C_4\,|\,g_2(x)\,|$ …2
where C_3 and C_4 are constants

As $f_i : \mathbb{Z}^+ \rightarrow \mathbb{R}$
where $\mathbb{Z}^+$ is the set of Positive integers

As we are trying to prove the Composition function $f_1\, f_2(x)$
We may examine only those cases of Real Numbers where $f_2(x)$ is in $\mathbb{Z}^+$

By Function Substitution, where x is substituted by $f_2(x)$
$C_1|g_1\, C_3|g_2(x)|\,| \quad \le \quad |\,f_1\,|f_2(x)|\,| \quad \le \quad C_2|g_1\, C_4|g_2(x)|\,|$

By Distributive Law
$C_1C_3|g_1\,|g_2(x)|\,| \quad \le \quad |\,f_1\,|f_2(x)|\,| \quad \le \quad C_2C_4|g_1\,|g_2(x)|\,|$

$\Rightarrow \quad C_5|g_1\,|g_2(x)|\,| \quad \le \quad |\,f_1\,|f_2(x)|\,| \quad \le \quad C_6|g_1\,|g_2(x)|\,|$
where C_5 and C_6 are constants

As $f_i(n)$ and $g_i(n)$ are functions defined on the set of positive integers
$|\,h_i\,|h_j(n)|\,| \;=\; |\,h_j\,h_j(n)\,|$
where h represent functions f and g
and i and j are indexes for the functions f and g

$\therefore \quad C_5|g_1\,|g_2(x)|\,| \quad \le \quad |\,f_1\,|f_2(x)|\,| \quad \le \quad C_6|g_1\,|g_2(x)|\,|$

$\Rightarrow \quad C_5|g_1g_2(x)| \quad \le \quad |\,f_1f_2(x)| \quad \le \quad C_6|g_1g_2(x)|$

$\therefore \quad f_1\, f_2(x) = \Theta(g_1g_2(x))$

6. (a) Using the sequences y and z defined by $x_n = 3^n + 1$, $y_n = n(n - 1)$

Find $\left(\sum_{i=1}^{3} x_i\right)\left(\sum_{i=2}^{4} y_i\right)$ **[1]**

(b) Consider the arithmetic series:

$5/2 \; + \; 11/2 + 17/2 \; + \; 23/2 \; + \; \ldots$

What is the smallest value of n such that $S_n > 300$,
where S_n is the sum of the first n terms of the series? **[4]**

Solution 6

[(a) Correct solution - 1 mark]
[½ mark is awarded if answer is incorrect]
[and any part of the solution is correct]
[(b) Arithmetic Series - 4 marks]
[First term and Common difference]

[common difference, d - ½ mark]
[Correct formula for S_n - ½ mark]
[Solving S_n to produce quadratic equation - 1 mark]
[Correct formula for quadratic equation - ½ mark]
[Solving quadratic equation - 1 mark]
[Final Statement with Smallest Value for n - ½ mark]

(a)

$$\left(\sum_{i=1}^{3} x_i\right)\left(\sum_{i=2}^{4} y_i\right) = (x_1 * x_2 * x_3)\,(y_2 * y_3 * y_4)$$

$$= (4 + 10 + 28)(2 + 6 + 12)$$

$$= 42 \times 20$$

$$= 840$$

(b) $a = 5/2$

$d = 11/2 - 5/2 = 6/2 = 3$

$S_n = (n/2)(2a + (n-1)d)$

$$\begin{aligned}
S_n = \; & (n/2)(2a + (n-1)d) > 300 \\
& (n/2)[2\,(5/2) + (n-1)\,(3)] > 300 \\
& (n/2)[5 + 3n - 3\,] > 300 \\
& 3n^2 + 2n > 600 \\
& 3n^2 + 2n - 600 > 0
\end{aligned}$$

In view of $ax^2 + bx + c = 0$

$n > (-b \pm \sqrt{(b^2 - 4ac)})/2a$

$n > (-(2) \pm \sqrt{((2)^2 - 4(3)(-600))} / 2(3)$

$n > (-2 \pm \sqrt{(4 + 7200)}) / 6$

$n > (-2 \pm 84.9) / 6$

$n > 13.82$

As n can only be of the set of Natural Numbers

$n = 14$

The smallest value of *n* for the sum to exceed 300 is 14.