COMP2201 – Discrete Mathematics
Recurrence Relations

Question 1
Solve the following recurrence relation:

$s_0 = 0$

$s_1 = 1$

$s_n = 2s_{n-1} – s_{n-2}$ for $n \geq 2$

Solution 1

$$\begin{aligned} \text{Let}\quad S &= s_0 + s_1x + s_2x^2 + s_3x^3 + \ldots + s_nx^n + \ldots \\ 2xS &= 2s_0x + 2s_1x^2 + 2s_2x^3 + \ldots + 2s_{n-1}x^n + \ldots \\ x^2S &= s_0x^2 + s_1x^3 + \ldots + s_{n-2}x^n + \ldots \end{aligned}$$

By Subtraction and Addition

$$S(1- 2x+x^2) = s_0 + (s_1 - 2s_0)x + (s_2-2s_1+s_0)x^2 + \ldots + (s_n-2s_{n-1} + s_{n-2})x^n + \ldots$$

As $s_n-2s_{n-1} + s_{n-2} = 0$

$$S(1-x)^2 = x$$

$$S = x / (1-x)^2$$

As $1/(1-x)^2 = 1 + 2x + 3x^2 + 4x^3 + \ldots + (n+1)x^n + \ldots$

$$\begin{aligned} S &= x[1 + 2x + 3x^2 + 4x^3 + \ldots + (n+1)x^n + \ldots] \\ &= x + 2x^2 + 3x^3 + 4x^4 + \ldots + nx^n + (n+1)x^{n+1} + \ldots \quad + \end{aligned}$$

Therefore the closed form solution of the recurrence relation is

[n]

Question 2
Solve the following recurrence relation:

$s_0 = 1$

$s_n = 2s_{n-1} + 1$ for $n \geq 1$

Solution 2
Given

$s_0 = 1$

$s_n = 2s_{n-1} + 1$ for $n \geq 1$

Consider the generating function

$$S = s_0 + s_1x + s_2x^2 + \ldots + s_nx^n + \ldots$$
$$2xS = 2s_0x + 2s_1x^2 + \ldots + 2s_{n-1}x^n + \ldots$$

Subtracting…

$$S - 2xS = s_0 + (s_1 - 2s_0)x + (s_2 - 2s_1)x^2 + \ldots + (s_n - 2s_{n-1})x^n + \ldots$$

As $s_n = 2s_{n-1} + 1$

Now Substituting $s_0 = 1$, $s_1 = 2s_0 + 1, \ldots, s_n - 2s_{n-1} = 1$

$$S(1 - 2x) = 1 + x + x^2 + \ldots + x^n + \ldots$$
$$= 1 / (1 - x)$$
$$S = (1 / (1 - x)) * (1 / (1 - 2x))$$
$$S = (1 + x + x^2 + \ldots + x^n + \ldots) * (1 + 2x + 2^2x^2 + 2^3x^3 + \ldots + 2^nx^n + \ldots)$$

$$\begin{aligned} f(x) = {} & 1 * (1 + 2x + 2^2x^2 + 2^3x^3 + \ldots + 2^nx^n + \ldots) \\ & + x * (1 + 2x + 2^2x^2 + 2^3x^3 + \ldots + 2^nx^n + \ldots) \\ & + x^2 * (1 + 2x + 2^2x^2 + 2^3x^3 + \ldots + 2^nx^n + \ldots) \\ & \ldots \\ & + x^n * (1 + 2x + 2^2x^2 + 2^3x^3 + \ldots + 2^nx^n + \ldots) \end{aligned}$$

$$f(x) = 1 + (2+1)x + (2^2+2+1)x^2 + (2^3+2^2+2+1)x^3 + \ldots + (2^n+2^{n-1}+\ldots+2^2+2+1)x^n + \ldots$$

Therefore closed form solution is

$$[(2^n+2^{n-1}+\ldots+2^2+2+1)]$$

or

$$\left[\sum_{k=0}^{n} 2^k\right]$$

Recall that $\left[\sum_{k=0}^{n} 2^k\right]$ is a GP with first term, a = 1 and common ratio, r = 2

Considering the GP formula for summing to the term with nth index

The closed form solution is

$$[2^{n+1} - 1]$$

Question 3

Solve the following recurrence relation:

$$s_0 = 3$$
$$s_n = –s_{n-1} + 2 \quad \text{for } n \geq 1$$

Solution 3

Given

$$s_0 = 3$$
$$s_n = –s_{n-1} + 2 \quad \text{for } n \geq 1$$

Consider the generating function

$$f(x) = s_0+s_1x+ s_2x^2+\ldots+ s_nx^n+\ldots$$
$$xf(x) = s_0x+ s_1x^2+\ldots+ s_{n-1}x^n+\ldots$$

Adding…

$f(x) + xf(x) = s_0 + (s_1 + s_0)x + (s_2 + s_1)x^2+\ldots+ (s_n + s_{n-1})x^n +\ldots$

Now Substituting $s_0 = 3$, $s_1 = –s_0 + 2$,…, $s_n = –s_{n-1} + 2$

$(1+x)f(x) = 3 + 2x + 2x^2+\ldots+ 2x^n +\ldots$

$f(x) = (3 + 2x + 2x^2+\ldots+ 2x^n +\ldots) * (1 / (1+x))$

$f(x) = (3 + 2x + 2x^2+\ldots+ 2x^n +\ldots) + (1 - x + x^2 - x^3\ldots+ (-1)^nx^n +\ldots)$

$$\begin{aligned}
f(x) = \; & 3 - 3x + 3x^2 - 3x^3\ldots \quad + 3(-1)^nx^n +\ldots \\
& + 2x - 2x^2 + 2x^3 - 2x^4\ldots \quad + 2(-1)^{n-1}x^n +\ldots \\
& +2x^2 - 2x^3 + 2x^4 - 2x^5\ldots+ 2(-1)^{n-2}x^n +\ldots \\
& \ldots \\
& + 2(-1)^0x^n
\end{aligned}$$

$f(x) = \ldots [(-1)^n + 2 * ((-1)^n+(-1)^{n-1}+\ldots+(-1)^2+(-1)^1+(-1)^0] x^n +\ldots$

Therefore closed form solution is

$$[(-1)^n + 2 ((-1)^n+(-1)^{n-1}+\ldots+(-1)^2+(-1)^1+(-1)^0)]$$

or

$$\left[(-1)^n + 2\sum_{k=0}^{n}(-1)^k \right]$$

Question 4

Solve the following recurrence relation:

$s_0 = 1$

$s_1 = 1$

$s_n = -2s_{n-1} - s_{n-2}$ for $n \geq 2$

Solution 4

$$
\begin{aligned}
\text{Let}\quad G &= s_0 + s_1x + s_2x^2 + s_3x^3 + \ldots + s_nx^n + \ldots \\
2xG &= 2s_0x + 2s_1x^2 + 2s_2x^3 + \ldots + 2s_{n-1}x^n + \ldots \\
x^2G &= s_0x^2 + s_1x^3 + \ldots + s_{n-2}x^n + \ldots
\end{aligned}
$$

By Addition

$$S(1+2x+x^2) = s_0 + (s_1+2s_0)x + (s_2+2s_1+s_0)x^2 + \ldots + (s_n+2s_{n-1}+s_{n-2})x^n + \ldots$$

As $s_n+2s_{n-1}+s_{n-2} = 0$

$$S(1+x)^2 = 1 + (1+2)x$$

$$S = (1+3x) / (1+x)^2$$

As $1/(1+x)^2 = 1 - 2x + 3x^2 - 4x^3 + \ldots + (-1)^n(n+1)x^n + \ldots$

$$
\begin{aligned}
S &= 1 * (1 - 2x + 3x^2 - 4x^3 + \ldots + (-1)^n(n+1)x^n + \ldots) \\
&\quad + 3x * (1 - 2x + 3x^2 - 4x^3 + \ldots + (-1)^n(n+1)x^n + \ldots) \\
&= 1 - 2x + 3x^2 - 4x^3 + \ldots + (-1)^n(n+1)x^n + \ldots \\
&\quad + 3x - 6x^2 + 9x^3 - 12x^4 + \ldots + 3(-1)^{n-1}(n)x^n + \ldots \\
&= 1 + x - 3x^2 + 5x^3 - 7x^4 + \ldots + [(-1)^n(n+1) + 3(-1)^{n-1}(n)]x^n + \ldots
\end{aligned}
$$

Therefore the closed form solution of the recurrence relation is

$$[(-1)^n(n+1) + 3(-1)^{n-1}(n)]$$

or simplified to

$$[(-1)^n[(n+1) + 3(-1)^{-1}]]$$

or

$$[(-1)^n(n - 2)]$$

Question 5

Solve the following recurrence relation:

$s_0 = 3$

$s_n = -3s_{n-1} + 2$ for $n \geq 1$

Solution 5

Given

$s_0 = 3$

$s_n = -3s_{n-1} + 2$ for $n \geq 1$

Consider the generating function

$f(x) = s_0+s_1x+ s_2x^2+\ldots+ s_nx^n+\ldots$

$3xf(x) = 3s_0x+3s_1x^2+\ldots+3s_{n-1}x^n+\ldots$

Adding…

$f(x) + 3xf(x) = s_0 + (s_1 + 3s_0)x + (s_2 + 3s_1)x^2+\ldots+ (s_n + 3s_{n-1})x^n +\ldots$

Now Substituting $s_0 = 3$, $s_1 = -3s_0 + 2 ,\ldots,\ s_n = -3s_{n-1} + 2$

$(1+3x)f(x) = 3 + 2x + 2x^2+\ldots+ 2x^n +\ldots$

$= 1 + 2 (1 + x + x^2+\ldots+ x^n +\ldots)$

$f(x) = (1 + 2 (1 + x + x^2+\ldots+ x^n +\ldots)) * (1 / (1+3x))$

$f(x) = (1 + 2 (1 + x + x^2+\ldots+ x^n +\ldots)) * (1 - 3x + 3^2x^2 - 3^3x^3\ldots+ (-3)^nx^n +\ldots)$

$f(x) = 1 * (1 - 3x + 3^2x^2 - 3^3x^3\ldots+ (-3)^nx^n +\ldots)$

$+ 2 [\ 1 * (1 - 3x + 3^2x^2 - 3^3x^3\ldots+ (-3)^nx^n +\ldots)$

$+ x * (1 - 3x + 3^2x^2 - 3^3x^3\ldots+ (-3)^nx^n +\ldots)$

$+ x^2 * (1 - 3x + 3^2x^2 - 3^3x^3\ldots+ (-3)^nx^n +\ldots)$

$\ldots$

$+ x^n * (1 - 3x + 3^2x^2 - 3^3x^3\ldots+ (-3)^nx^n +\ldots)\ ]$

$f(x) = 1 * (1 - 3x + 3^2x^2 - 3^3x^3\ldots+ (-3)^nx^n +\ldots)$

$+ 2 [1 + (-3 + 1)x + (3^2-3+1)x^2 + \ldots+ ((-3)^n+(-3)^{n-1}+\ldots+3^2-3+1) x^n +\ldots]$

Therefore closed form solution is

$[(-3)^n + 2 ((-3)^n+(-3)^{n-1}+\ldots+3^2-3+1)]$

or

$$\left[(-3)^n + 2\sum_{k=0}^{n}(-3)^k\right]$$