COMP2201 – Discrete Mathematics
Graph Theory including Planarity and Cycles

1. Prove that K_5 is not planar.

Answer 1.

Revision:

K_n

- the complete graph on n vertices
- the simple graph with n vertices in which there is an edge between every pair of distinct vertices

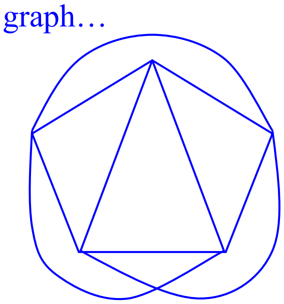
Planar graph

- a graph that can be drawn without its edges crossing

Solution:

i. Assuming that vertices are numbered from top to bottom proceeding firstly in a left to right direction, we have vertices numbered from 1 (topmost) to 5 (bottom-right most)

By attempting to re-draw graph…



We get edge 3-4 crossing 1-5 or 2-5.

$\therefore$ K_5 is not planar

ii. Where v = number of vertices
e = number of edges
f = number of faces, including the exterior face

$v = 5$
$e = 4 + 3 + 2 + 1 = 10$
$f = 8$

If G is *planar* graph, Then

$v - e + f = 2$ or $f = e - v + 2$

i.e. $5 - 10 + 8 = 3 \neq 2$

$\therefore$ K_5 is not planar

iii. If G is *planar* graph, Then

$e \leq 3v - 6$

$e = 10,\ v = 5$

$\Rightarrow 10 \leq 3 \times 5 - 6 \quad \text{or} \quad 10 \leq 9$

$\therefore$ K_5 is not planar

iv. Suppose that K_5 is planar

Since every cycle has at least 3 edges, each face is bounded by at least 3 edges.
Thus the number of edges that bound faces is at least 3f.

In a planar graph, each edge belongs to at most two bounding cycles.

Therefore $\quad 2e \geq 3f$

Using $\quad v - e + f = 2 \quad$ or $\quad f = e - v + 2$

We find that $\quad 2e \geq 3(e - v + 2)$

Now $\quad v = 5$

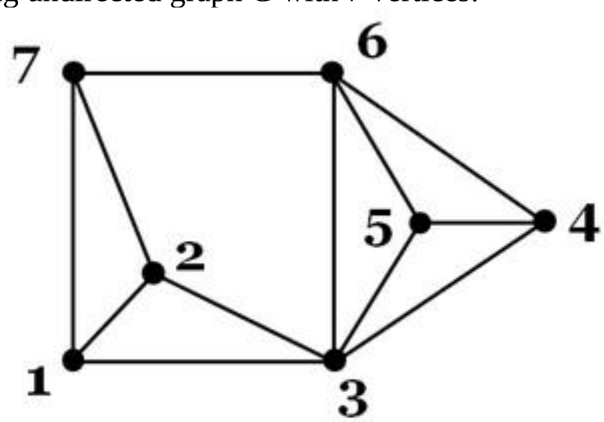
$e = 4 + 3 + 2 + 1 = 10$

$$2(10) \geq 3(10 - 5 + 2)$$

$$20 \geq 21$$

$\therefore$ K_5 is not planar

2. Given the following undirected graph G with 7 vertices:



i. List the clique(s) with 3 or more vertices that are within the graph

ii. Identify the maximal clique(s)

iii. Find the maximum clique(s) or state why there is no maximum cliques in the graph G

iv. Determine the clique number $\omega(G)$

Answer 2.

i. [A **clique**, C, in an undirected graph $G = (V, E)$ is a subset of the vertices, $C \subseteq V$, such that every two distinct vertices are adjacent.]

The cliques with 3 or more vertices are:

a. V = {1, 2, 3}
b. V = {1, 2, 7}
c. V = {3, 4, 5}
d. V = {3, 5, 7}
e. V = {4, 5, 6}
f. V = {3, 4, 5, 6}

ii. [A **maximal clique** is a clique that cannot be extended by including one more adjacent vertex, that is, a clique which does not exist exclusively within the vertex set of a larger clique.]

The maximal cliques are:

a. V = {1, 2, 3}
b. V = {1, 2, 7}
c. V = {3, 4, 5, 6}

Cliques {3, 4, 5}, {3, 5, 6} and {4, 5, 6} are not maximal as by including vertex 6, 4 and 3 respectively, a larger clique is identified.

iii. [A **maximum clique** of a graph, G, is a clique, such that there is no clique with more vertices.]

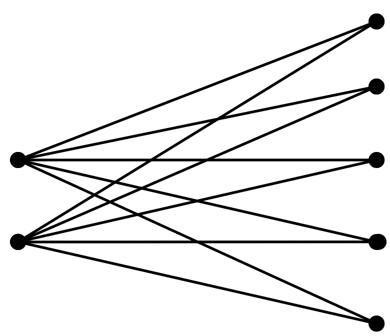
The given graph has one maximum clique. The maximum clique is {3, 4, 5, 6}

iv. [The **clique number** $\omega(G)$ of a graph G is the number of vertices in a maximum clique in G.]

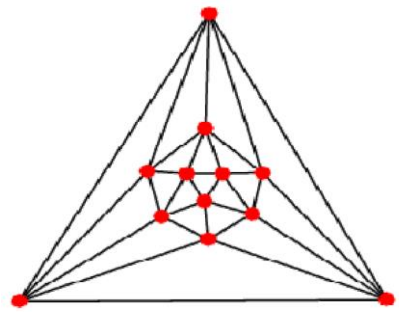
The maximum clique is {3, 4, 5, 6}.
The number of vertices in the maximum clique is 4.
Therefore the clique number $\omega(G)$ is 4.

3. Draw $K_{2,5}$, the complete bi-partite graph on 2 and 5 vertices.

Answer 3.



4. Find a Hamilton cycle in the graph shown below



Revision:

Simple path (from v to w)

- a path from v to w with no repeated vertices

Cycle (or circuit)

- a path of non-zero length from v to v with no repeated edges

Simple cycle (from v to v)

- a cycle with no repeated vertices

Hamiltonian cycle

- A cycle in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice.

Hamiltonian path

- A simple path in a graph G that contains each vertex in G exactly once. (A Hamiltonian path begins and ends at different vertices)

Answer 4.

Solution:

Assuming that vertices are numbered from top to bottom proceeding firstly in a left to right direction, we have vertices numbered from 1 (topmost) to 12 (bottom-right most)

(1, 2, 3, 4, 5, 6, 9, 7, 8, 10, 12, 11, 1) or many others

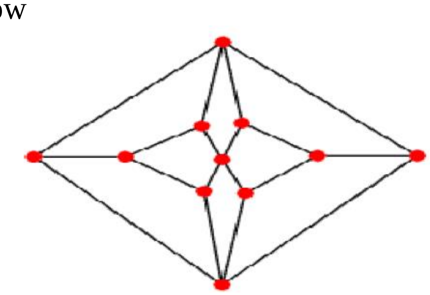
Revision:

Euler Cycle

- a cycle in a graph G that includes all of the edges and all of the vertices of G

Sum of the degrees of all the vertices in a graph is even

5. Consider the graph below



i. Prove that a Hamiltonian cycle does not exist in this graph

Hamiltonian cycle

- A cycle in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice.

Answer 5i.

A Hamiltonian cycle with n vertices should also have n edges. Likewise, each vertex in a Hamiltonian cycle has degree of 2.

Assuming that vertices are numbered from top to bottom proceeding firstly in a left to right direction, we have vertices numbered, we have vertices numbered from 1 (topmost) to 11 (bottommost).

There are 11 vertices and 18 edges.
Suppose that we could eliminate edges from the graph, leaving just a Hamiltonian cycle, as each vertex has degree of 2, we would have to eliminate edges leaving only 2 at each vertex. Consider…

Eliminate two edges from 1, 6, 11 and one edge from 2, 3, 4, 5, 7, 8, 9, 10.

Eliminate

Two edges at vertices 1, 6 and 11:

vertex 1 (edges 1-2, 1-3),
vertex 6 (edges 6-9, 6-10) and
vertex 11 (edges 11-4, 11-8)

resulting in a single edge being eliminated from (possibly)
vertices 2, 3, 9, 10, 4 an 8

One edge at vertices 2, 3, 4, 5, 7, 8, 9, 10

vertex 5 (edge 5-9 OR edge 5-4)
vertex 7 (edge 7-10 OR edge 7-8)

No. of edges eliminated would be 3 x 2 + 2 x 1 = 8

No. of edges remaining = 18 – 8 = 10

A Hamiltonian cycle with 11 vertices should also have 11 edges.

Therefore the graph does not contain a Hamiltonian cycle.

ii. If a Hamiltonian path exists, describe one; otherwise, prove that a Hamiltonian path does not exist.

Hamiltonian path

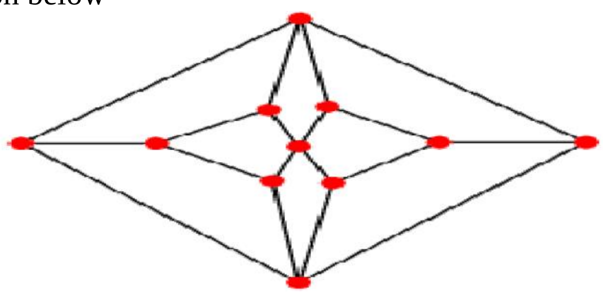
- A simple path in a graph G that contains each vertex in G exactly once. (A Hamiltonian path begins and ends at different vertices)

A Hamiltonian path with n vertices should have n-1 edges. Likewise, each vertex in a Hamiltonian cycle has degree of 2 (except for start and end).

Answer 5ii.

(4, 1, 8, 7, 3, 6, 2, 5, 9, 11, 10)

6. Consider the graph below



i. If an Euler cycle exists, describe it; otherwise state why one does not exist in this graph

ii. If an Euler path exists, describe one; otherwise, prove that an Euler path does not exist.

Answer 6.

i. [Revision:

Euler Cycle

- a cycle in a graph G that includes all of the edges and all of the vertices of G
- Each edge must exist in the cycle only once
- The degree of each vertex in the Euler cycle is even
- The sum of the degrees is even

]

There are 11 vertices and 18 edges.

If an Euler cycle exists, each vertex has an even degree…

Assuming that vertices are numbered from top to bottom proceeding in a left to right direction in each row, we have vertices numbered from 1 (topmost) to 11 (bottommost) similar to:

1
2, 3
4, 5, 6, 7, 8

9, 10
11

As there exists a vertex which does not have an even degree i.e. any of vertices 2, 3, 4, 5, 7, 8, 9 and 10 all with degree 3, and a graph with an Euler cycle must have every vertex having an even degree

Therefore the graph does not contain an Euler cycle.

ii. [Revision:

Euler Path

- a path in a graph G that includes all of the edges and all of the vertices of G
- Each edge must exist in the path only once
- The degree of each vertex in the Euler cycle is even except for the starting and ending vertex
- The sum of the degrees is even

]

The sum of the degrees is

$$4+3+3+3+3+4+3+3+3+3+4 = 36$$

However, as there exists more than two vertices with an odd degree i.e. any three of vertices 2, 3, 4, 5, 7, 8, 9 and 10 all with degree 3, and a graph with an Euler path must have exactly two vertices with odd degree

Therefore an Euler path does not exist.