LECTURE - COMP2201 – Discrete Mathematics

Generating Functions and Recurrence Relations

Question 0

A coding system encodes messages using strings of base 3 digits. A codeword is considered valid if and only if it contains an odd number of 2s.
Find a recurrence relation for the number of valid codewords of length n.

i. State initial conditions.
ii. Solve this recurrence relation using generating functions.

Solution 0

i.
Set of strings of {0, 1, 2}

Valid – String contains an odd number of 2s e.g. 210200122112 or 201202 or 2

Let n be the length of the codeword

S_n be the number of valid codewords of length n or

S_n is the number of codewords of length n with an odd number of 2s

i.e. S_{n-1} is the number of codewords of length n-1 with an odd number of 2s

S_{n-2} is the number of codewords of length n-2 with an odd number of 2s

etc.

By Calculation
$S_0 = 0$
$S_1 = 1$
$S_2 = 4$
.
.

Firstly,
The valid codewords begins with a '2' or Not

Let S_A be the number of valid codewords of length n that Do Not begin with 2
S_B be the number of valid codewords of length n that begins with 2
Then

$S_n = S_A + S_B$

Consider S_A [Begins with ‘2’]

The other n-1 base-3 digits must contain an even number of 2s

For these n-1 base-3 digits

The number of evens = Total number of codewords minus The number of odds

Therefore,

As we are now considering the n-1 base-3 digits

$S_A = 3^{n-1} - S_{n-1}$

Consider S_B [Does Not Begin with ‘2’, i.e. begins with 0 or 1]

Let us say a valid codeword that begins with 0

The other n-1 base-3 digits must contain an odd number of 2s i.e. S_{n-1}

As there are 2 such cases

$S_B = 2S_{n-1}$

Therefore, As

$S_n = S_A + S_B$

The Recurrence Relation

$S_n = 3^{n-1} - S_{n-1} + 2S_{n-1}$

Simplified

$S_n = S_{n-1} + 3^{n-1}$ for $n \geq 1$

The Initial Conditions

$S_0 = 0$

$S_1 = 1$

ii. $S_0 = 0$

$S_1 = 1$

$S_n = S_{n-1} + 3^{n-1}$ for $n \geq 1$

Let $S = s_0 + s_1x + s_2x^2 + s_3x^3 + \ldots + s_nx^n + \ldots$

$xS = s_0x + s_1x^2 + s_2x^3 + \ldots + s_{n-1}x^n + \ldots$

We know that

$S_n - S_{n-1} = 3^{n-1}$ for $n \geq 1$

By Subtraction

$S(1- x) = s_0 + (s_1 - s_0)x + (s_2\text{-}s_1)x^2 + (s_3\text{-}s_2)x^3 + \ldots + (s_n\text{-}s_{n-1})x^n + \ldots$

$S_2 = 4$ [By observation] or

$S_2 = 1 + 3^{2-1} = 4$ [By calculation]

$S_3 = 4 + 3^{3-1} = 13$

$S(1-x) = 3^{1-1}x + 3^{2-1}x^2 + 3^{3-1}x^3 + 3^{4-1}x^4 + \ldots + 3^{n-1}x^n + \ldots$

$S = [3^0x + 3^1x^2 + 3^2x^3 + 3^3x^4 + \ldots + 3^{n-1}x^n + \ldots] * (1 / (1 - x))$

As $1/(1-x) = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$

$S = (3^0x + 3^1x^2 + 3^2x^3 + 3^3x^4 + \ldots + 3^{n-1}x^n + \ldots) *$
$(1 + x + x^2 + x^3 + \ldots + x^n + \ldots)$

$$\begin{aligned}
S = \; & 3^0x\,(1 + x + x^2 + x^3 + \ldots + x^n + \ldots) + \\
& 3^1x^2\,(1 + x + x^2 + x^3 + \ldots + x^n + \ldots) + \\
& 3^2x^3\,(1 + x + x^2 + x^3 + \ldots + x^n + \ldots) + \\
& 3^3x^4\,(1 + x + x^2 + x^3 + \ldots + x^n + \ldots) + \\
& . \\
& . \\
& 3^{n-1}x^n\,(1 + x + x^2 + x^3 + \ldots + x^n + \ldots) + \\
& \ldots \\
= \; & 3^0x + (3^0+3^1)x^2 + (3^0+3^1+3^2)x^3 + (3^0+3^1+3^2+3^3)x^4 + \\
& (3^0+3^1+3^2+3^3+3^4)x^5 + \ldots + \\
& (3^0+3^1+3^2+3^3+3^4+\ldots+3^{n-1})x^n + \ldots
\end{aligned}$$

Therefore (before simplification) the closed form solution of the recurrence relation is
$[\; 3^0 + 3^1 + 3^2 + 3^3 + 3^4 + \ldots + 3^{n-1} \;]$
Simplified

$$\left[\sum_{k=0}^{n-1} 3^k\right]$$

The closed form solution of the recurrence relation is

$$\left[\sum_{k=0}^{n-1} 3^k\right]$$

Question 1

Solve the following recurrence relation:

$s_0 = 0$

$s_1 = 1$

$s_n = 2s_{n-1} - s_{n-2}$ for $n \geq 2$

Solution 1

Let $S = s_0 + s_1x + s_2x^2 + s_3x^3 + \ldots + s_nx^n + \ldots$

$2xS = 2s_0x + 2s_1x^2 + 2s_2x^3 + \ldots + 2s_{n-1}x^n + \ldots$

$$x^2S = \quad s_0x^2 + s_1x^3 + \ldots + s_{n-2}x^n + \ldots$$

By Subtraction and Addition

$$S(1-2x+x^2) = s_0 + (s_1 - 2s_0)x + (s_2-2s_1+s_0)x^2 + \ldots + (s_n-2s_{n-1} + s_{n-2})x^n + \ldots$$

As $s_n-2s_{n-1} + s_{n-2} = 0$

$$S(1-x)^2 = x$$

$$S = x / (1-x)^2$$

As $1/(1-x)^2 = 1 + 2x + 3x^2 + 4x^3 + \ldots + (n+1)x^n + \ldots$

$$S = x[1 + 2x + 3x^2 + 4x^3 + \ldots + (n+1)x^n + \ldots]$$

$$= x + 2x^2 + 3x^3 + 4x^4 + \ldots + nx^n + (n+1)x^{n+1} + \ldots \quad +$$

Therefore the closed form solution of the recurrence relation is

[n]

Question 2

Solve the following recurrence relation:

$s_0 = 0$

$s_1 = 4$

$s_n = 2s_{n-1} - s_{n-2}$ for $n \geq 2$

Solution 2

Let $S = s_0 + s_1x + s_2x^2 + s_3x^3 + \ldots + s_nx^n + \ldots$

$$2xS = \quad 2s_0x + 2s_1x^2 + 2s_2x^3 + \ldots + 2s_{n-1}x^n + \ldots$$

$$x^2S = \quad s_0x^2 + s_1x^3 + \ldots + s_{n-2}x^n + \ldots$$

By Subtraction and Addition

$$S(1-2x+x^2) = s_0 + (s_1 - 2s_0)x + (s_2-2s_1+s_0)x^2 + \ldots + (s_n-2s_{n-1} + s_{n-2})x^n + \ldots$$

As $s_n-2s_{n-1} + s_{n-2} = 0$

$$S(1-x)^2 = 4x$$

$$S = 4x / (1-x)^2$$

As $1/(1-x)^2 = 1 + 2x + 3x^2 + 4x^3 + \ldots + (n+1)x^n + \ldots$

$$S = 4x[1 + 2x + 3x^2 + 4x^3 + \ldots + (n+1)x^n + \ldots]$$

$$= 4x + 8x^2 + 12x^3 + 16x^4 + \ldots + 4nx^n + 4(n+1)x^{n+1} + \ldots \quad +$$

Therefore the closed form solution of the recurrence relation is

[4n]

Question 3

Solve the following recurrence relation:

$s_0 = 3$

$s_n = -s_{n-1} + 2$ for $n \geq 1$

Solution 3

Given

$s_0 = 3$

$s_n = -s_{n-1} + 2$ for $n \geq 1$

Consider the generating function

$$f(x) = s_0 + s_1 x + s_2 x^2 + \ldots + s_n x^n + \ldots$$
$$xf(x) = s_0 x + s_1 x^2 + \ldots + s_{n-1} x^n + \ldots$$

Adding…

$f(x) + xf(x) = s_0 + (s_1 + s_0)x + (s_2 + s_1)x^2 + \ldots + (s_n + s_{n-1})x^n + \ldots$

Now Substituting $s_0 = 3$, $s_1 = -s_0 + 2, \ldots, s_n = -s_{n-1} + 2$

$(1+x)f(x) = 3 + 2x + 2x^2 + \ldots + 2x^n + \ldots$

$f(x) = (3 + 2x + 2x^2 + \ldots + 2x^n + \ldots) * (1 / (1+x))$

$f(x) = (3 + 2x + 2x^2 + \ldots + 2x^n + \ldots) + (1 - x + x^2 - x^3 \ldots + (-1)^n x^n + \ldots)$

$$\begin{aligned} f(x) = \; & 3 - 3x + 3x^2 - 3x^3 \ldots + 3(-1)^n x^n + \ldots \\ & + 2x - 2x^2 + 2x^3 - 2x^4 \ldots + 2(-1)^{n-1} x^n + \ldots \\ & + 2x^2 - 2x^3 + 2x^4 - 2x^5 \ldots + 2(-1)^{n-2} x^n + \ldots \\ & \ldots \\ & + 2(-1)^0 x^n \end{aligned}$$

$f(x) = \ldots [(-1)^n + 2 * ((-1)^n + (-1)^{n-1} + \ldots + (-1)^2 + (-1)^1 + (-1)^0] x^n + \ldots$

Therefore closed form solution is

$[(-1)^n + 2 ((-1)^n + (-1)^{n-1} + \ldots + (-1)^2 + (-1)^1 + (-1)^0)]$

or

$$\left[(-1)^n + 2\sum_{k=0}^{n}(-1)^k\right]$$