COMP2201 – Discrete Mathematics
Limits, Sequence and Series

Question 1

For the sequence x defined by

$x_1 = 2, \quad x_n = 3 + x_{n-1}, \; n \geq 2$

Find $\sum_{i=1}^{10} x_i$ and $\prod_{i=3}^{6} x_i$

Solution 1

Find $\sum_{i=1}^{10} x_i = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10}$

$= 2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29$

$= 155$

Find $\prod_{i=3}^{6} x_i = x_3 * x_4 * x_5 * x_6$

$= 8 * 11 * 14 * 17$

$= 20{,}944$

Question 2

Find a formula for the sequence c defined by

$$c_n = \sum_{i=1}^{n} b_i$$

i. For the sequence b defined by $b_1 = 2, \; b_n = 3 + b_{n-1}, \; n \geq 2$

ii. For the sequence b defined by $b_n = n(-1)^n, \; n \geq 1$

Solution 2

Find a formula for the the sequence c defined by

$$c_n = \sum_{i=1}^{n} b_i$$

i. For the sequence b defined by $b_1 = 2, \; b_n = 3 + b_{n-1}, \; n \geq 2$

$b_1 = 2,$ $c_1 = 2$
$b_2 = 5,$ $c_2 = 7$
$b_3 = 8,$ $c_1 = 15$

a = 2
d or r?, d = 3
AP
$l = 2+(n-1)3,$

$S_n = n(a + l)/2$

$$
\begin{aligned}
c_n &= 2 + 5 + 8 + ... + [2 + (n-1)3] \\
&= n(2 + 2 + 3(n-1))/2 \\
&= n(4 + 3(n-1))/2 \\
&= 4n/2 + 3n(n-1)/2 \\
&= 2n + 3n(n-1)/2
\end{aligned}
$$

ii. For the sequence b defined by $b_n = n(-1)^n,\ n \geq 1$

SIMPLE SOLUTION

$b_1 = -1,\qquad c_1 = -1$
$b_2 = 2,\qquad c_2 = 1$
$b_3 = -3,\qquad c_3 = -2$
$b_4 = 4,\qquad c_4 = 2$
$b_5 = -5,\qquad c_5 = -3$
$b_6 = 6,\qquad c_6 = 3$

$$c_n = -1 + 2 + -3 + 4 + -5 + 6 + ...$$

Based on the trend

$$
c_n = \begin{cases} n/2 & \text{if n is even,} \\ (-n - 1)/2 & \text{if n is odd} \end{cases}
$$

COMPLEX SOLUTION

$b_1 = -1,$
$b_2 = 2,$
$b_3 = -3,$
$b_4 = 4,$
$b_5 = -5,$
$b_6 = 6,$

$$
\begin{aligned}
c_n &= -1 + 2 + -3 + 4 + -5 + 6 + ... \\
&= -1 + -3 + -5 + ... \\
&\quad + 2 + 4 + 6 + ...
\end{aligned}
$$

We have 2 AP series

AP_1

$a = -1$
$d = -2$

AP_2

$a = 2$
$d = 2$

$S_n = (n/2)[2a + (n-1)d]$

As n is the number of terms in the series

Let N denote the number of terms in the sub-series i.e. AP_1 and AP_2

$S_n = (N_{AP1}/2)[2a + (N_{AP1}-1)d] + (N_{AP2}/2)[2a + (N_{AP2}-1)d]$

If n is even, $N = n/2$ (for AP_1), $N = n/2$ (for AP_2)
If n is odd, $N = (n+1)/2$ (for AP_1), $N = (n-1)/2$ (for AP_2)

If n is even

$$S_n = (N_{AP1}/2)[2a + (N_{AP1}-1)d] + (N_{AP2}/2)[2a + (N_{AP2}-1)d]$$

$$\begin{aligned} S_n &= (N_{AP1}/2)[2a_{AP1} + (N_{AP1}-1)d_{AP1}] &&+ (N_{AP2}/2)[2a_{AP2} + (N_{AP2}-1)d_{AP2}] \\ &= ((n/2)/2)[2(-1) + ((n/2)-1)(-2)] &&+ ((n/2)/2)[2(2) + ((n/2)-1)(2)] \\ &= (n/4)[-2 - n + 2] &&+ (n/4)[4 + n - 2] \\ &= -n^2/4 &&+ n^2/4 + 2n/4 \\ &= n/2 \end{aligned}$$

If n is odd

$$S_n = (N_{AP1}/2)[2a + (N_{AP1}-1)d] + (N_{AP2}/2)[2a + (N_{AP2}-1)d]$$

$$\begin{aligned} S_n &= (N_{AP1}/2)[2a_{AP1} + (N_{AP1}-1)d_{AP1}] &&+ (N_{AP2}/2)[2a_{AP2} + (N_{AP2}-1)d_{AP2}] \\ &= (((n+1)/2)/2)[2(-1) + (((n+1)/2)-1)(-2)] &&+ (((n-1)/2)/2)[2(2) + (((n-1)/2)-1)(2)] \\ &= ((n+1)/4)[-2 - (n+1) + 2] &&+ ((n-1)/4)[4 + (n-1) - 2] \\ &= ((n+1)/4)[-(n+1)] &&+ ((n-1)/4)[(n-1) + 2] \\ &= ((n+1)/4)[-n-1] &&+ ((n-1)/4)[n + 1] \\ &= ((n+1)/4)[\quad -n \quad -1 &&+ \quad n \quad - \quad 1] \\ &= ((n+1)/4)[\quad -2] \\ &= (-n - 1)/2 \end{aligned}$$

Therefore

$$c_n = \begin{cases} n/2 & \text{if n is even,} \\ (-n-1)/2 & \text{if n is odd} \end{cases}$$

Question 3

Using the sequences y and z defined by

$y_n = 2^n - 1$ $\qquad z_n = n(n-1)$

Find $\left(\sum_{i=1}^{3} y_i\right)\left(\sum_{i=1}^{3} z_i\right)$

Solution 3

$$\begin{aligned} \left(\sum_{i=1}^{3} y_i\right)\left(\sum_{i=1}^{3} z_i\right) &= (y_1 + y_2 + y_3)\,(z_1 + z_2 + z_3) \\ &= (1 + 3 + 7)\,(0 + 2 + 6) \\ &= 88 \end{aligned}$$

Question 4

Determine the limit of $f(x)$ as $x \to \infty$ for the following:

i. $f(x)=\dfrac{3x}{x+5}$ ii. $f(x)=\dfrac{x}{-3+5x}$

Solution 4

i. $f(x)=\dfrac{3x}{x+5}$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{3x}{x+5}=3$$

ii. $f(x)=\dfrac{x}{-3+5x}$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x}{-3+5x}=\frac{1}{5}$$

Question 5

Let f and g be functions $N\rightarrow R$, where N is the set of Natural numbers and R is the Real numbers

In notation formats, what is meant by f is "Omega" of g denoted

$$f(x)=\Omega(g(x))$$

Solution 5

$f(x)=\Omega(g(x))$ means

i. if there exist a positive constant *C1* such that

$$|f(n)| \geq C1|g(n)|$$

for all but finitely many positive integers *n*.

ii.

If $\lim_{x\to\infty}\left|\dfrac{f(x)}{g(x)}\right|$ exists, then

$$\lim_{x\to\infty}\left|\frac{f(x)}{g(x)}\right|>0$$

Question 6

If the first term of a series is *a*, the nth term is *l* and the common difference is *d* for AP or common ratio is *r* for GP, determine for both AP and GP

i. the sum S_n of the first n terms
ii. the nth term, *l*

Solution 6

For AP $S_n = (n/2)(a + l)$

or $(n/2)(2a + (n-1)d)$

$l = a + (n-1)d$

For GP $S_n = a(r^n - 1)/(r - 1)$

or $a(1 - r^n)/(1 - r)$

$l = ar^{n-1}$

Question 7

Three consecutive terms of an arithmetic series are $4x + 11$, $2x + 11$ and $3x + 17$.

Find the value of x.

Solution 7

For AP, the difference between consecutive terms is the common difference, d

Therefore

$$d = (2x + 11) – (4x + 11)$$
$$= -2x$$

Also

$$d = (3x + 17) – (2x + 11)$$
$$= x + 6$$

Equating

$$d = -2x = x + 6$$
$$\Rightarrow -3x = 6$$
$$x = -2$$

Question 8

Consider the geometrical progression:

2 + 5/3 + 25/18 + 125/84 +…

What is

i. S_n

ii. the limit of S_n

Solution 8

i.

The series is a GP

$$a = 2$$

As $a = 2$

and $ar = 5/3$

$$(2)r = 5/3$$
$$r = (5/3)/2$$
$$= 5/6$$

For GP

$$S_n = a(r^n - 1)/(r - 1)$$
$$= 2 ((5/6)^n – 1) / ((5/6) – 1)$$
$$= 2 ((5/6)^n – 1) / (-1/6)$$
$$= -12 ((5/6)^n – 1)$$

ii

Where $|r| < 1$

$$\text{Limit of } S_n = a / (1 - r) = 2 / (1 - (5/6)) = 2 / (1/6) = 12$$

Question 9

In an arithmetic series, the sum of the second term and the fifth term is 18. The sixth term is greater than the third term by 9.

i. Find the first term and the common difference.

ii. What is the smallest value of n such that $S_n > 600$, where S_n is the sum of the first n terms of the series?

Solution 9

i.

$u_2 = a + d$ $\qquad u_n = a + (n\text{-}1)d$ **2**

$u_3 = a + 2d$

$\qquad S_n = (n/2)(2a + (n\text{-}1)d)$ **3**

The fifty-sixth term of an arithmetic sequence: $u_{56} = a + 55d=$

Summing formula:

$u_2 + u_5 = 18$

$\Rightarrow a + d + a + 4d = 18$

$\Rightarrow 2a + 5d = 18$ ....(**1**)

$u_6 = u_3 + 9$

$\Rightarrow a + 5d = a + 2d + 9$

$\Rightarrow 3d = 9$

$\Rightarrow d = 3$ ....(**2**)

Substituting the value for d into equation (**2**):

$\Rightarrow 2a + 5(3) = 18 \Rightarrow 2a = 3 \Rightarrow a = 3/2$

ii.

$S_n = (n/2)(2a + (n\text{-}1)d) > 600$

$(n/2)[2 (3/2) + (n\text{-}1) (3)] > 600$

$(n/2)[3 + 3n - 3] > 600$

$3n^2 > 1200$

$n > 20$

As n can only be of the set of Natural Numbers

$n = 21$

The question asks what is the smallest value of n for the sum to exceed 600. 21 terms are needed to exceed this value.