The Earthquake Dissipative Engine: Energy Budget and Partition

PART F: Double couple

S. Nielsen

Short course on earthquake physics

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27 Oct. - 16:30-18:30 classroom 2G
28 Oct. - 14:30-16:30 classroom 2H
29 Oct. - 10:30-12:30 classroom Lab Paleo
31 Oct. - 14:30-16:30 classroom 2L

1 The double couple force equivalent

Fault motion can be considered as a discontinuity of displacement across a surface (dislocation) within an elastic body, which induces a perturbation and a re-arrangement of the surrounding medium. If the displacement is relatively impulsive, waves can be generated as a consequence of disequilibrium dynamics (change of momentum).

In the 50' there was some controversy about the precise mathematical significance of a dislocation and how to describe the source of perturbation that it would induce. Although many elastodynamic solutions had been obtained already in the 19th century and the beginning of the 20th century (Lamb, 1904), those were mostly concerned with the response to a single unidirectional force applied at a point. This type of simple source would correspond, for example, to a falling object (a meteor which hitting the surface of the Earth) and the solutions would allow to compute the triggered waves.

Clearly the dislocation associated to earthquake faulting constitutes a more complex source of perturbation. Because elastodynamic solutions were traditionally derived in the presence of force impulses, there were discussions about what combination of forces could best describe an earthquake. Because the earthquake corresponds to an release of internal stress, there should be no net change in momentum (contrary to the case of a falling meteor). One proposal was to use a force couple: two equal forces applied in the opposite directions in the vicinity of a point, but with an infinitesimal offset.

In fact, it was shown by Burridge and Knopoff (1964) that the simplest mathematical description of an earthquake source requires at least two force couples, i.e., the double couple solution. The elegant and rigorous demonstration was done by integrating the displacement discontinuity in the solution to the momentum equation (c.f. Aki and Richards book chapter 4, 1980), and showing that it was equivalent to a system of four forces resulting in no net change of momentum or moment.

We develop here a more intuitive derivation, starting from remarking that an *anelastic stress drop* takes place on a bound source surface in correspondence to an earthquake.

Let σ_{ij} be the elastic stress, and τ_{ij} the non-elastic perturbation (also known as the stress glut), resulting in a total stress $\sigma_{ij} + \tau_{ij}$. The stress glut is the perturbation of stress resulting form slip motion on a fault or crack. For example, as a failure occurs the fault will slip and release a given amount of shear stress. The perturbation due to fault slip can be introduced in a model of elastic behaviour by imposing a *boundary condition* on the fault surface. In this case there are two possible options: (1) impose a displacement condition on the fault or (2) impose a stress condition on the fault. The two options are equivalent and produce the same result, the choice of one or another depends on what is more convenient for the solution of the problem.

Using the change of momentum equation we may write

$$\rho\, \partial_t^2 u_i = \partial_j \sigma_{ij} + \partial_j \tau_{ij} + f_i \tag{1}$$

where ρ is mass density, u_i is the i_{th} component of particle displacement vector $\mathbf{u}$, σ_{ij} the stress due to elastic strain, τ_{ij} the stress due anelastic strain (including the fault slip), f_i the i_{th} component of the body forces vector $\mathbf{f}$.

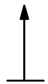
Assuming that the fault normal direction is z, and that the stress glut will affect the shear stress componenet τ_{xz} (and, due to symmetry, also τ_{zx}) alone, we can write the explicit cases:

$$\begin{aligned}
\rho\, \partial_t^2 u_x &= \partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz} + \partial_z \tau_{xz} + f_x \\
\rho\, \partial_t^2 u_y &= \partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz} + 0 + f_y \\
\rho\, \partial_t^2 u_z &= \partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz} + \partial_x \tau_{zx} + f_z
\end{aligned} \tag{2}$$

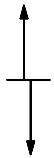
(the u_y component does not have any perturbative term because there is no contribution τ_{yz}).

Now let's chose a frame of reference such that the fault is centered at $(x,y,z) = (0,0,0)$. The rupture (or the stress glut) extends over a surface area L^2 of zero thickness. Although the source area is distributed, it will look like a point from a faraway distance; we can treat it as a point source. However, to account for net change of stress within the volume surrounding the extended source area, we will multiply the stress glut amplitude by L^3 when using the point source model. Finally, if we are looking at periods which are much longer than the duration of the source, the latter will look like an impulse in time, and we choose $t = 0$ as the origin time of the earthquake.

To describe a source that is located at a point, and which has zero thickness, we will use the Dirac delta function $\delta(x)$: a generalized function or *distribution* introduced by the physicist Paul Dirac. It is used to model an idealised function equal to zero everywhere, except at $x = 0$, and whose integral is equal to one. The Dirac function, and its derivative $\delta'(x)$ can be symbolically represented as



and



Finally, with the above assumptions our source function can be mathematically equated to :

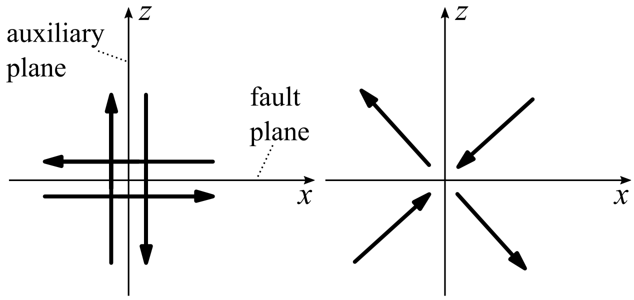
$$\begin{aligned} s(x,t) &= \Delta\tau\, L^3 \times \delta(x)\ \delta(y)\ \delta(z)\ \delta(t), \\ &= M_o \times \delta(x)\ \delta(y)\ \delta(z)\ \delta(t), \end{aligned} \tag{3}$$

where $\Delta\tau$ is the magnitude of stress drop; accessorily, $\Delta\tau\, L^3$ has dimensions of a moment (N m), we will hereafter call it seismic moment and re-name it M_o. Substitution of $s(x,t)$ into Equation 4 yeilds:

$$\begin{aligned} \rho\, \partial_t^2 u_x &= \partial_x\sigma_{xx} + \partial_y\sigma_{xy} + \partial_z\sigma_{xz} + M_o\, \delta(x)\, \delta(y)\, \delta'(z)\, \delta(t) + f_x \\ \rho\, \partial_t^2 u_z &= \partial_x\sigma_{zx} + \partial_y\sigma_{zy} + \partial_z\sigma_{zz} + M_o\, \delta'(x)\, \delta(y)\, \delta(z)\, \delta(t) + f_z \end{aligned} \tag{4}$$

By inspection, as the source terms $M_o\, \delta(x)\, \delta(y)\, \delta'(z)\, \delta(t)$ and $M_o\, \delta('x)\, \delta(y)\, \delta(z)\delta(t)$ are next to body forces f_x and f_y, they must have same dimensions and therefore correspond to forces.

In fact, because of the derivative $\delta'(z)$ in the first, and $\delta('x)$ in the second, they correspond to *two force couples* which exert an equal and opposite torque at point source location $(0,0,0)$. Indeed, the representation usually sketched for such system of forces is :



Double couple representation of the seismic source and an equivalent system of forces