The Earthquake Dissipative Engine: Energy Budget and Partition

PART X: XXXX

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Short course on earthquake physics

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27 Oct. - 16:30-18:30 classroom 2G
28 Oct. - 14:30-16:30 classroom 2H
29 Oct. - 10:30-12:30 classroom Lab Paleo
31 Oct. - 14:30-16:30 classroom 2L

Appendix I. Divergence theorem and energy

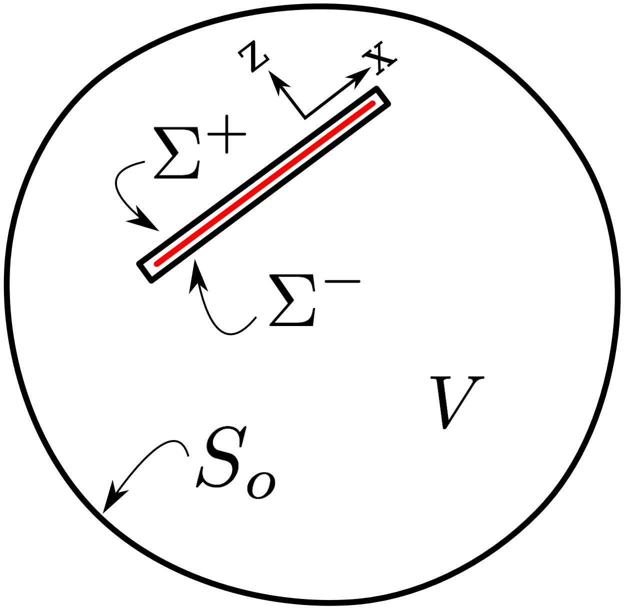

Figure 1: Schematic section of the Earth intersecting a fault surface (in red). The total Earth volume is V, S_o is the traction-free surface of the Earth, and the surface $\Sigma = \Sigma^+ + \Sigma^-$ snugly surrounds the fault. By making Σ^+ and Σ^- infinitesimally close, the volume between S and Σ can include all of V.

A full treatment of the energy change in earthquake can be found in Kostrov, Dhalen (...), which is here presented in a simplified fashion. First we state the divergence theorem. For a vector field $\mathbf{F}$ in a volume V, enclosed within a surface S:

$$\iiint_V (\nabla \cdot \mathbf{F})\, dV = \oiint_S (\mathbf{F} \cdot \mathbf{n})\, dS. \tag{1}$$

and its equivalent formulation applied to a tensor σ or σ_{ij} using Einstein notation (implicit summation on repeated indexes) for a tensor:

$$\iiint_V \partial_j \, (\sigma_{ij}) \, dV = \oiint_S n_j \, \sigma_{ij} \, dS. \tag{2}$$

where n_i is the i_{th} componenet of the local normal vector $\mathbf{n}$ pointing outwards from the surface S. Second, we recall the definintion of strain energy density (p.u. volume) as the product of stress and strain, namely

$$\rho_{\text{strain}} = {}^1\!/\!_2 \;\; \sigma_{ij} \, \varepsilon_{ij} \tag{3}$$

(with implicit summation over all indexes i, j). The earthquake "driving energy" E originates in the release of stored elastic strain, which is converted to in forms, including frictional dissipation, creation of new fractures and radiation of kinetic energy (waves). The difference between the stored strain energy *before* and *after* the rupture, in terms of local density, can be written as:

$$\rho_E \;\; = {}^1\!/\!_2 \; \left(\sigma^1_{ij}\varepsilon^1_{ij} \; - \sigma^0_{ij}\varepsilon^0_{ij}\right) \tag{4}$$

where the superscripts 1 and 0 refer to the final and initial conditions, respectively. then the total strain energy change E_{tot} can be obtained by integrating in the volume V surrounding the fault:

$$E_{tot} \;\; = {}^1\!/\!_2 \iiint_V \left(\sigma^1_{ij}\varepsilon^1_{ij} \; - \sigma^0_{ij}\varepsilon^0_{ij} \;\right) \; dV \tag{5}$$

From ρ_w of eq. (4) let's try to get an expression based only on on initial stress σ^0_{ij}, stress difference $\Delta\sigma_{ij} = \sigma^1_{ij} - \sigma^0_{ij}$ and strain difference $\Delta\varepsilon_{ij} = \varepsilon^1_{ij} - \varepsilon^0_{ij}$:

$$\begin{aligned} \rho_E &= {}^1\!/_2 \left(\sigma^0_{ij} + \Delta\sigma_{ij}\right)\left(\varepsilon^0_{ij} + \Delta\varepsilon_{ij}\right) - {}^1\!/_2\ \sigma^0_{ij}\varepsilon^0_{ij} \\ &= {}^1\!/_2 \left(\sigma^0_{ij}\Delta\varepsilon_{ij} + \Delta\sigma_{ij}\varepsilon^0_{ij} + \Delta\sigma_{ij}\Delta\varepsilon_{ij}\right) \end{aligned} \tag{6}$$

To further simplify the expression, we can assume purely elastic strain in the volume surrounding the fault, with a constitutive law (Hooke's law) relating stress term to strain via the linear elasticsity.

Take good note here that we apply the divergence theorem only to the elastic part of the deformation. Therefore, we will need to <u>exclude form this energy change using the divergence theorem, all volumes of rock where nonlinear or anelasic processes take place, and treat those separately</u>. In particular, volumes within which plastic deformation or frictional dissipative processes take place will be excluded. In this specific treatment, we will assume that they take place within a volume of infinitesimal thickness –a mathematical fault surface.

For convenience we may use the elastic modules tensor c_{ijkl} such that:

$$c_{ijkl} = \lambda\, \delta_{ij}\delta_{kl} + G\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{7}$$

and then write the stress-strain Hooke's law as:

$$\begin{aligned} \sigma_{ij} &= c_{ijkl}\ \varepsilon_{kl} \\ \Delta\sigma_{ij} &= c_{ijkl}\Delta\varepsilon_{kl} \\ \varepsilon^0_{ij} &= \frac{\sigma^0_{kl}}{c_{klij}} \end{aligned} \tag{8}$$

therefore

$$\begin{aligned}
\rho_E &= 1/2\left(\sigma^0_{ij}+\Delta\sigma_{ij}\right)\left(\varepsilon^0_{ij}+\Delta\varepsilon_{ij}\right)-1/2\;\;\sigma^0_{ij}\varepsilon^0_{ij}\\
&= 1/2\left(\sigma^0_{ij}\Delta\varepsilon_{ij}+\cancel{c_{ijkl}}\Delta\varepsilon_{kl}\,\frac{\sigma^0_{kl}}{\cancel{c_{klij}}}+\Delta\sigma_{ij}\Delta\varepsilon_{ij}\right)\\
&= 1/2\left(\sigma^0_{ij}\Delta\varepsilon_{ij}+\Delta\varepsilon_{kl}\;\sigma^0_{kl}+\Delta\sigma_{ij}\Delta\varepsilon_{ij}\right)
\end{aligned} \tag{9}$$

where the symmetry property $c_{ijkl}=c_{klij}$ has been used to eliminate the elastic moduli. Because of the implicit summation all repeated indexes are dummies so we know that $\Delta\varepsilon_{kl}\;\sigma^0_{kl}=\Delta\varepsilon_{ij}\;\sigma^0_{ij}$ and therefore after replacing the $\Delta\sigma_{ij}$ we obtain:

$$\rho_E \quad = 1/2\;\Delta\varepsilon_{ij}\left(\sigma^0_{ij}+\sigma^1_{ij}\right) \tag{10}$$

The total energy change is otained by integrating the density on all the volume

$$E_{tot} \quad = 1/2\iiint_V \Delta\varepsilon_{ij}\left(\sigma^0_{ij}+\sigma^1_{ij}\right)dV \tag{11}$$

Furthermore we have that

$$\begin{aligned}
\partial_i\left(\sigma_{ij}u_j\right) &= \cancel{\partial_i\left(\sigma_{ij}\right)}u_j+\sigma_{ij}\;\partial_i\left(u_j\right)\\
&= \sigma_{ij}\;\partial_i\left(u_j\right)
\end{aligned} \tag{12}$$

because at equilibrium $\partial_i\left(\sigma_{ij}\right)=0$. If u_i is the displacement of particles between before and after the rupture, by definition of the infinitesimal strain we have

$$\Delta\varepsilon_{ij}=1/2\left(\partial_i u_j+\partial_j u_i\right) \tag{13}$$

and because of the summation on all indexes we are allowed to write:

$$\sigma_{ij}\Delta\varepsilon_{ij}=\sigma_{ij}\partial_i u_j=\partial_i\left(\sigma_{ij}u_j\right) \tag{14}$$

Finally we can re-write the volume integral of W such that:

$$E_{tot} = 1/2 \iiint_V \partial_i \left[\left(\sigma^0_{ij} + \sigma^1_{ij} \right) \; u_j \right] dV \tag{15}$$

We may now apply the divergence theorem to write:

$$E_{tot} = 1/2 \iiint_V \partial_i \left[\left(\sigma^0_{ij} + \sigma^1_{ij} \right) \; u_j \right] dV = 1/2 \iint_S n_i \left(\sigma^0_{ij} + \sigma^1_{ij} \right) \; u_j \; dS \tag{16}$$

We may split S into two parts $S = S_o + \Sigma$ (Fig. 1). Σ is a surface snugly fitted around the fault, and S_o is the free surface of the Earth. Now because the free traction condition on the surface S_o is $\mathbf{T}(\mathbf{n}) = 0$ where $\mathbf{T}$ is traction; traction and stress are related by $T_j = \sigma_{ji} \, n_i$. Therefore the integral over S_o will vanish at any time, and we're left with

$$E_{tot} = 1/2 \iint_\Sigma n_i \left(\sigma^0_{ij} + \sigma^1_{ij} \right) \; u_j \; d\Sigma \tag{17}$$

So we have been able to show that the total energy change can be represented by a surface which can be made arbitrarily close to the fault surface, until we are for all practical purposes describing values of σ_{ij} and u_i which are effectively *on* the fault surface. To further simplify the expression, we can use a geometry where the fault normal is z, the fault is located along (x,y) plane at $z = 0$ and the slip occurs in direction x (Fig. 1). In this case the fault normal vector is $\mathbf{n} = (n_x, n_y, n_z) = (0,0,1)$, slip motion on the fault is $\mathbf{u} = (u_x, 0, 0)$ and

$$E_{tot} = 1/2 \iint_\Sigma \left(\sigma^0_{zx} + \sigma^1_{zx} \right) \; u_x \; d\Sigma \tag{18}$$

Letting Σ^- be adjacent to the "bottom" part of the fault ($z \to 0^-$) and Σ^+ be adjacent to the "top" part of the fault ($z \to 0^+$), defining slip $D = u_x^+ - u_x^-$, and using traction continuity such that $\sigma_{xz}^+ = \sigma_{xz}^-$ and symmetry ($\sigma_{zx} = \sigma_{xz}$) we can write

$$\begin{aligned} E_{tot} &= 1/2 \iint_{\Sigma^+} \left(\sigma_{xz}^0 + \sigma_{xz}^1\right) \; u_x \, d\Sigma - 1/2 \iint_{\Sigma^-} \left(\sigma_{xz}^0 + \sigma_{xz}^1\right) \; u_x \, d\Sigma \\ &= 1/2 \iint_{\Sigma} \left(\sigma_{xz}^0 + \sigma_{xz}^1\right) D \, d\Sigma \end{aligned} \tag{19}$$

and taking average values for stress and slip on the fault, and defining $\sigma = \sigma_{xz}$ where the subscripts are implicit we can write:

$$\begin{aligned} E_{tot} &= 1/2 \iint_{\Sigma} \left(\sigma_{xz}^0 + \sigma_{xz}^1\right) D \, d\Sigma \\ &= \frac{\overline{\sigma^0 + \sigma^1}}{2} \, \overline{D} \iint_{\Sigma} d\Sigma \\ E_{tot} &= \frac{\overline{\sigma^0 + \sigma^1}}{2} \overline{D} \, \Sigma \end{aligned} \tag{20}$$

Expression (20) shows that the total energy change in the volume can be calculated from the initial and final stress defined on the fault surface Σ alone. The only net work applied to the Earth-system during the earthquake is the result of the stress change on the fault surface (provided that all nonlinear processes are confined onto the surface).