# The Earthquake Dissipative Engine: Energy Budget and Partition

## PART E: Derivation of stress pattern at the rupture tip

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27 Oct. - 16:30-18:30 classroom 2G
28 Oct. - 14:30-16:30 classroom 2H
29 Oct. - 10:30-12:30 classroom Lab Paleo
31 Oct. - 14:30-16:30 classroom 2L

# 1 Static Mode III Stress Singularity

**Where does the $r^{-1/2}$ singularity in stress come from?**

Perhups the simplest derivation is proposed by Williams, M. L. (1957) – On the stress distribution at the base of a stationary crack Journal of Applied Mechanics, 24, 109–114. Mode III is the simplest configuration (only two shear stresses) but it allows some

We consider an anti-plane shear (Mode III) crack in an isotropic, linear elastic solid. The out-of-plane displacement $w(x,y)$ satisfies Laplace's equation

$$\nabla^2 w = 0, \tag{1}$$

with shear stresses

$$\tau_{xz} = \mu \frac{\partial w}{\partial x}, \qquad \tau_{yz} = \mu \frac{\partial w}{\partial y}. \tag{2}$$

Let the crack lie along the negative $x$-axis, and introduce polar coordinates $(r,\theta)$ centered at the crack tip. On the crack faces ($\theta = \pm\pi$) the shear traction must vanish:

$$\tau_{\theta z}(r,\theta) = \mu \frac{1}{r}\frac{\partial w}{\partial \theta} = 0 \quad \text{for} \quad \theta = \pm\pi. \tag{3}$$

## Separable (Williams) form

Seek separable solutions of the form

$$w(r,\theta) = r^{\lambda} f(\theta). \tag{4}$$

Laplace in ploar coordinates yields:

$$\nabla^2 w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = 0. \tag{5}$$

Substituting into Laplace's equation in polar coordinates gives

$$f''(\theta) + \lambda^2 f(\theta) = 0, \tag{6}$$

whose general solution is

$$f(\theta) = A\sin(\lambda\theta) + B\cos(\lambda\theta). \tag{7}$$

The traction-free condition requires

$$\frac{\partial w}{\partial \theta} = 0 \quad \text{at} \quad \theta = \pm\pi, \tag{8}$$

so that

$$\lambda\,[A\cos(\lambda\pi) - B\sin(\lambda\pi)] = 0. \tag{9}$$

## Mode III symmetry and eigenvalues

Mode III loading is antisymmetric about the crack plane, hence $B = 0$. A nontrivial solution then demands

$$\cos(\lambda\pi) = 0 \quad \Rightarrow \quad \lambda = \tfrac{1}{2}, \tfrac{3}{2}, \ldots \tag{10}$$

The dominant (most singular) term near the tip corresponds to $\lambda = 1/2$, giving

$$w(r,\theta) = A\, r^{1/2} \sin\left(\frac{\theta}{2}\right). \tag{11}$$

### Stress field and singularity

Differentiating,

$$\frac{\partial w}{\partial r} = \frac{A}{2} r^{-1/2} \sin\left(\frac{\theta}{2}\right), \tag{12}$$

$$\frac{\partial w}{\partial \theta} = \frac{A}{2} r^{1/2} \cos\left(\frac{\theta}{2}\right), \tag{13}$$

so that

$$\tau_{rz} = \mu \frac{\partial w}{\partial r} = \mu \frac{A}{2} r^{-1/2} \sin\left(\frac{\theta}{2}\right), \tag{14}$$

$$\tau_{\theta z} = \mu \frac{1}{r} \frac{\partial w}{\partial \theta} = \mu \frac{A}{2} r^{-1/2} \cos\left(\frac{\theta}{2}\right). \tag{15}$$

Both components vary as $r^{-1/2}$, showing the characteristic inverse-square-root singularity.

## Static Mode II Stress Singularity

For in–plane shear (Mode II) loading, the displacements $u_x, u_y$ satisfy the Navier equations of plane elasticity,

$$\mu \nabla^2 u_i + (\lambda + \mu) \frac{\partial}{\partial x_i} (\nabla \cdot \mathbf{u}) = 0, \tag{16}$$

which may equivalently be expressed in terms of the Airy stress function $\Phi(x,y)$, defined by

$$\sigma_{xx} = \frac{\partial^2\Phi}{\partial y^2}, \qquad \sigma_{yy} = \frac{\partial^2\Phi}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y}. \tag{17}$$

The function $\Phi$ satisfies the biharmonic equation

$$\nabla^4\Phi = 0. \tag{18}$$

## Williams eigenfunction form

Near a traction–free crack tip, we seek homogeneous solutions of the form

$$\Phi(r,\theta) = r^{\lambda+1} f(\theta), \tag{19}$$

where $(r,\theta)$ are polar coordinates centred at the crack tip. Substitution into the biharmonic equation yields an angular eigenvalue problem whose admissible exponents are

$$\lambda = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots$$

as first shown by Williams (1957). The leading term $\lambda = \frac{1}{2}$ gives stresses varying as $r^{-1/2}$.

## Stress field for Mode II

Enforcing traction–free boundary conditions on the crack faces ($\theta = \pm\pi$) and antisymmetry of shear about the $x$–axis leads to the standard near–tip field

$$\sigma_{xx} = -\frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\left(2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right), \tag{20}$$

$$\sigma_{yy} = \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}, \tag{21}$$

$$\tau_{xy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right), \tag{22}$$

where $K_{II}$ is the Mode II stress intensity factor. All components exhibit the same $r^{-1/2}$ singularity. Along the crack plane ahead of the tip ($\theta = 0$, $r = x$),

$$\tau_{xy}(x,0) = \frac{K_{II}}{\sqrt{2\pi x}}, \tag{23}$$

demonstrating the classical inverse–square–root stress concentration for static in–plane shear.

## Remarks

The functional form of the angular terms differs from Mode III, but the scaling of stress with distance from the crack tip is identical, following from the same eigenvalue $\lambda = 1/2$ in the Williams expansion. Equivalent derivations can be obtained through the complex–potential (Muskhelishvili) formulation, using analytic functions $\phi(z)$ and $\psi(z)$ for which the near–tip behaviour again yields the $1/\sqrt{r}$ singularity.

# References


[1] Williams, M. L. (1957). *On the stress distribution at the base of a stationary crack*. *Journal of Applied Mechanics*, **24**, 109–114.

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[5] Anderson, T. L. (2017). *Fracture Mechanics: Fundamentals and Applications*, 4th ed. CRC Press, Boca Raton.

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