The Earthquake Dissipative Engine: Energy Budget and Partition

PART A: Intro and Energy release rate

S. Nielsen

Short course on earthquake physics

Geoscienze, Università di Padova Oct 2025

27 Oct. - 16:30-18:30 classroom 2G
28 Oct. - 14:30-16:30 classroom 2H
29 Oct. - 10:30-12:30 classroom Lab Paleo
31 Oct. - 14:30-16:30 classroom 2L

Foreword

The earthquake cycle can be represented as an engine with fuel (elastic strain energy), dissipative processes (friction, rupture, anelastic strain) and –when the process is at non-equilibrium– a net production of kinetic energy (waves). The energy budget is challenging because:

- energy transfer is non-local, with focussed areas of energy sinks
- strain energy is quadratic, therefore the amount of energy release (or strain energy work) depends on the absolute strain (which is challenging to measure in the Earth) and not only on the strain change (easier to measure)
- dissipative processes, or friction(s) –the plural indicating friction-like processes in the wider sense– are not completely understood under the extreme conditions of earthquake rupture

We can explore the earthquake engine in the following framework:

- triggering of dynamic rupture as an energy barrier problem –you can get started with very little fuel.
- dynamic rupture propagation as a budget: energy flow and energy sinks at the rupture tip, moderated by rupture propagation velocity (not to confuse with slip velocity)
- energy scaling with rupture linear dimension (characteristic radius or width of rupture area). Energy is force $\times$ length, there appears indeed the length of rupture in the earthquake energy formula! Stopping bigger ruptures requires proportionally larger energy sinks... How to generate them?

Here we will focus on the dissipative aspects (energy sinks) of rupture and consider how this estimate of dissipation can inform the earthquake energy balance. We will be looking at how frictions can be quantified through

- observation of friction in every day's life –how can this apply to earthquake faults?
- observations of fault geology –the magnifying lens
- seismological data –looking from too far
- laboratory experimentation –looking from too close
- modelling –implicit complexity and the limits of homogenisation

1 The engine

Slab pull, slab push

The ultimate source of energy that fuels tectonic deformation in the Earth is heat from radioactive decay within the Earth mantle (releasing a power of approximately $42 \pm 2\ 10^{12}$ Watts (J. H. Davies and D. R. Davies 2010). The resulting temperature increase causes negative and positive gravity buoyancy that drives flow, convection and magma ascent inside the Earth, and descent of colder lithospheric plates at subduction zones.

The two main drivers of plate motion are ridge push and slab pull. It is believed that slab pull is the dominant one. The stress and deformation arising from plate motion in the Earth crust and at subduction zones allows elastic energy build-up that is, at least in part, released during earthquakes **(Fig. 1)**.

Slab pull-push

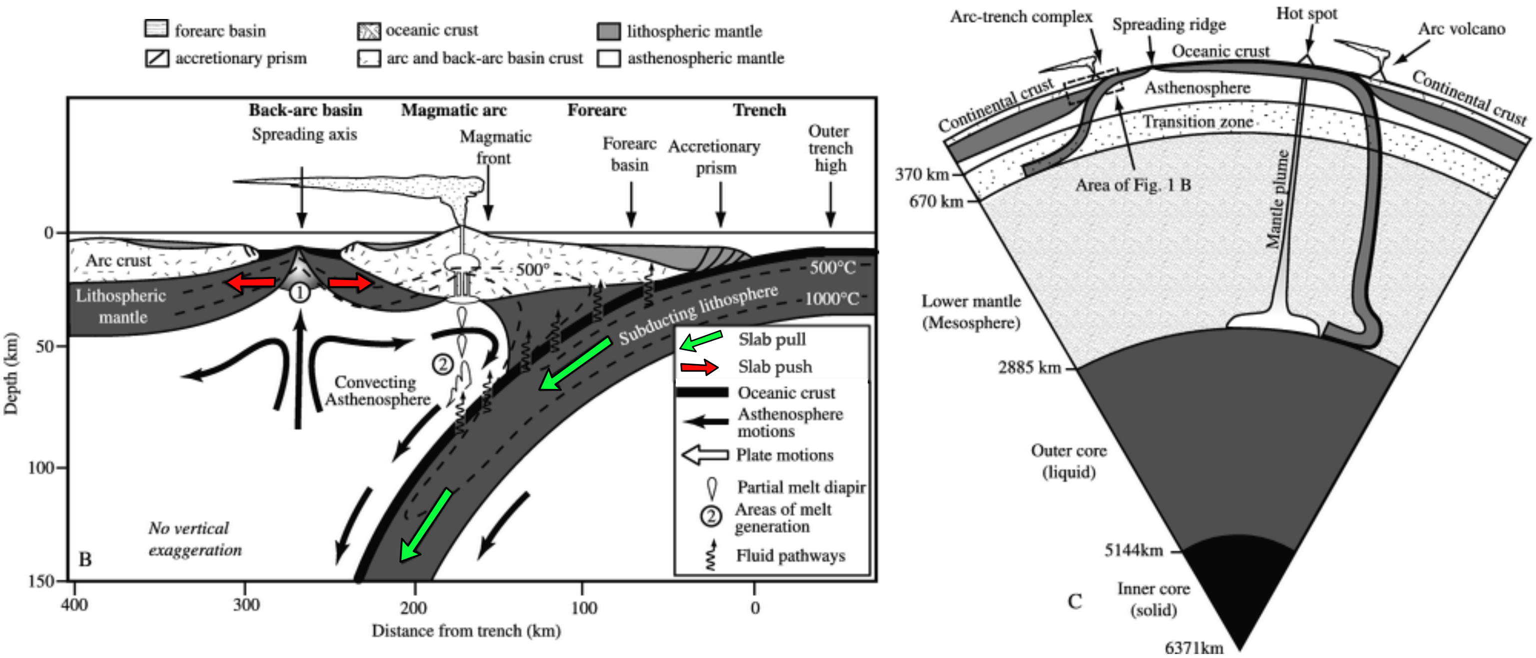


Figure 1: Schematic illustration of the oceanic plate structure, kinematics and dynamics. The main forces enabling plate motion and tectonic stress are the slab push (red arrow, expansion at the mid-oceanic ridgese, a consequence of positive buoyancy) and slab pull (negative buoyancy in the cold plunging plate). Modified from (Stern 2002).

These forces act either directly on the oceanic plates, or indirectly on the continental plate that they are in contact with. To accommodate the ensuing deformation and in response to the stress, major earthquakes take place at plate boundaries (interplate earthquakes). However, the push-pull game allows to build some amount of stress also within the plates, causing faulting and earthquakes there too (intraplate earthquakes), though in lesser magnitude and number.

For an indicative comparison with the radioactive heat power, a gross estimate[1] of average power released by earthquakes is $\approx 7\ 10^9$ Watts (i.e., a fraction of 0.017% of the energy produced by radioactive decay is released by earthquakes).

Exercise:

Estimate the slab pull per unit plate boundary width, due to a density contrast of 200 kg/m3 in a slab of 70 km thickness and 500 km length. Convert the force in stress in the slab plunge direction, close to the surface.

Elastic and anelastic strain

We may define the finite shear strain (excluding rigid rotation and displacement) using the Eulerian-Almansi tensor:

$$\gamma_{zx} = \gamma_{xz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} - \frac{\partial u_x}{\partial z}\frac{\partial u_x}{\partial x} - \frac{\partial u_z}{\partial z}\frac{\partial u_z}{\partial x}\right) \tag{1}$$

[1]The estimate can be obtained from the estimate of seismic moment ($4\ 10^{23}$ N m) released over a period of 90 years (Pacheco and Sykes 1992). The moment then needs to be converted to energy by multiplying by $0.5 10^4$ (an indicative moment to energy factor) and divided by the number of seconds in 90 years.

that uses the gradient of particle displacement (or deformation gradient) in reference to the initial (undisturbed) position. Under the assumption of infinitesimal strain, the quadratic terms can be neglected, to obtain:

$$\varepsilon_{zx} = \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) \tag{2}$$

(equivalent expressions can be obtained for all 9 strain tensor elements in 3D by replacing x and z with any of x, y, z components. Because these tensors are symmetric ($\gamma_{zx} = \gamma_{xz}$) only 6 tensor elements are sufficient to characterise strain).

In most rocks, strain beyond about 2% will be accommodated by nonlinear, anelastic permanent deformation. As a consequence, to solve elastic problems (wave propagation, elastic rebound, or elastic strain energy in seismology) the assumption of infinitesimal deformation is usually made and the form (2 is used). The finite deformation arising from earthquake faulting is treated separately, often introduced in the problem as a boundary condition (or displacement discontinuity) on a surface of zero thickness (Aki and Richards 2002).

However, looking more closely at the anelastic deformation processes involved in faulting, one notes that brittle, ductile or plastic deformation and flow mechanisms are triggered in a finite volume to accommodate finite deformations. If setting out to describe the latter processes explicitly rather than implicitly, a more suitable mathematical formulation of strain would arguably be (1).

Basic conventions for geometry and notation

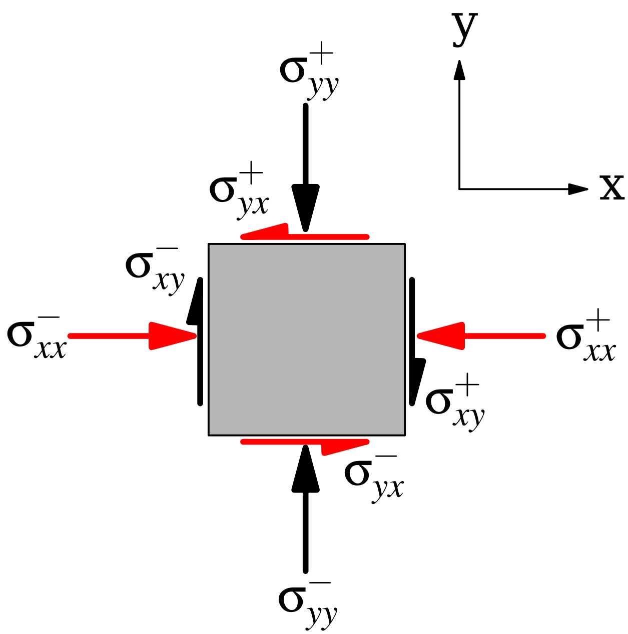


Figure 2: Stress components in 2D. Both σ and τ are used for stress here. τ is usually a shear component.

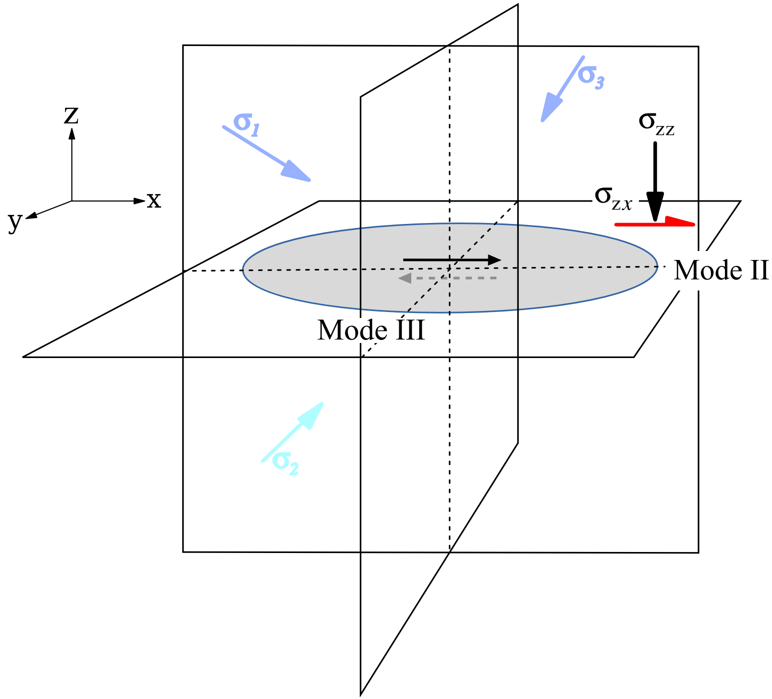


Figure 3: Fracture modes. We often use eiher 2D Mode II or 2D Mode III (anti-plane). For simplicty, we often adopt the notation $\tau = \sigma_{xy}$ (shear stress) and $\sigma = \sigma_n = \sigma_{yy}$ (normal stress).

Stress and elastic strain energy

Let's first For elastic isotropic media, two elastic parameters are sufficient to relate stress σ_{ij} and strain ε_{ij}. For example, using Lame's parameter λ and shear modulus G we can write Hooke's law of elasticity as

$$\begin{gathered}\sigma_{ij} = \lambda\, \delta_{ij}\, \varepsilon_{kk} + G\, (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\, \varepsilon_{kl} \\ \text{or} \\ \overline{\sigma} = \lambda\, \mathrm{trace}(\overline{\varepsilon})\, \mathbf{I} + 2\, G\, \overline{\varepsilon}\end{gathered} \tag{3}$$

where the Kroneker δ_{ij} is such that

$$\begin{array}{l}\delta_{ij} = 0 \quad \text{if} \quad i \neq j \\ \delta_{ij} = 1 \quad \text{if} \quad i = j\end{array} \tag{4}$$

and by expanding all possible individual index values we get:

$$\begin{aligned}\sigma_{xx} &= \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2G\, \varepsilon_{xx} \\ \sigma_{yy} &= \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2G\, \varepsilon_{yy} \\ \sigma_{zz} &= \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2G\, \varepsilon_{zz} \\ \sigma_{xy} &= 2G\, \varepsilon_{xy}; \sigma_{xz} = 2G\, \varepsilon_{xz}; \sigma_{yz} = 2G\, \varepsilon_{yz};\end{aligned} \tag{5}$$

Strain energy is the energy stored by a body undergoing deformation. It equates to the amount of work done by forces (or stresses) onto the body during the deformation of the body.

Take for example a case of uniaxial strain in direction x, (only $\varepsilon_{xx} \neq 0$. The stress and strain are related through Hooke's law $\sigma_{xx} = (\lambda + 2G)\, \varepsilon_{xx}$.

At each point inside a solid body (a rock for example) we can write an elementary strain energy *density* $d\rho_E$, produced by an elementary strain $d\varepsilon$:

$$d\rho_E = \sigma_{xx}(\varepsilon_{xx})\, d\varepsilon_{xx} \quad \text{work density for elementary strain } d\varepsilon_{xx} \tag{6}$$

We can obtain the energy density ρ_E resulting from a finite strain ε by integration:

$$\begin{aligned}
\rho_E &= \int_0^{\varepsilon_{xx}} \sigma_{xx}(\varepsilon_{xx})\, d\varepsilon \qquad \text{energy density} \\
\rho_E &= \int_0^{\varepsilon_{xx}} (\lambda + 2\mu)\, \varepsilon_{xx}\, d\varepsilon_{xx} \\
\rho_E &= \frac{1}{2} E\, \varepsilon_{xx}^2 = \frac{1}{2}\, \frac{1}{E} \sigma_{xx}^2
\end{aligned} \tag{7}$$

This is only for uniaxial strain ε_{xx} only; in general we would sum all possible stress components:

$$\rho_E = \sum_{i,j,k,l} \frac{1}{2}\, \frac{1}{c_{ijkl}} \sigma_{kl}^2$$

Note that the energy density is proportional to the square of the deformation, and the square of the stress; this is important because the same amount of deformation can preduce different energies depending on the initial condition of the solid (if it is already loaded with some stress, the work will be greater).

Finally, if the solid which is under study has a volume V, to obtain the net strain energy U we need to integrate over the volume,

$$U = \int_V \rho_E(\mathbf{x})\, d\mathbf{x} \quad \text{total energy in volume } V \tag{8}$$

which in case of *homogeneous* deformation will result in:

$$U = \rho_E \times V \quad \text{if } \rho \text{ is homogeneous} \tag{9}$$

Exercise: mind experiment

A short mind experiment to explore strain energy changes. (I) A linear elastic rubber band is stretched by an unknown amount from its rest position length. (II) The same band is now stretched again, adding 2 cm to the stretching of stage (I). (III) Finally, and additional 2 cm stretching are added to stage (II). Does stretching of stage III require the same amount of work than the stretching of stage II? The elastic band stiffness is 5 N/m. Is it possible to compute the work involved in stretching stage II or stage III? Based on your conclusions, do you think that it is possible to infer the change of energy in the Earth's lithosphere, due to a deformation episode (earthquake, post-glacial rebound, dam flooding)?

Definition and strain energy change, quadratic dependence, absolute value.
Divergence theorem and work?

A first peek into the energy budget

A very schematic representation of the elastic energy work and dissipation **(Fig. 4)** during the earthquake slip can be represented as an *average* value of work density per unit fault surface (work is in N m, the work density per unit fault surface is in Pa m^{-2} which equates to N m^{-1} or J m^{-2}). This type of graphic was introduced by Kanamori and Rivera 2006.

Schematic energy budget

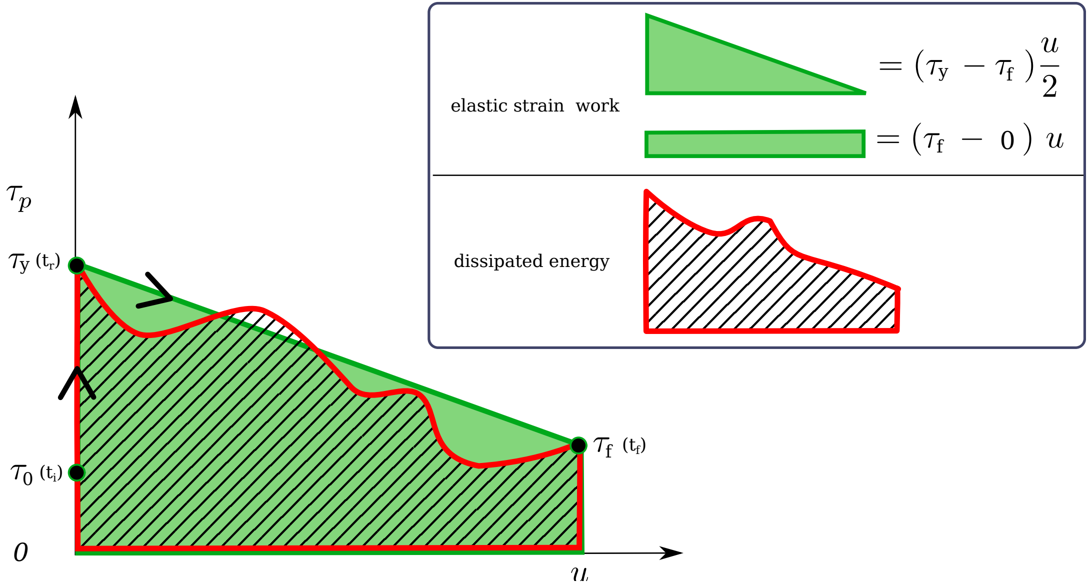
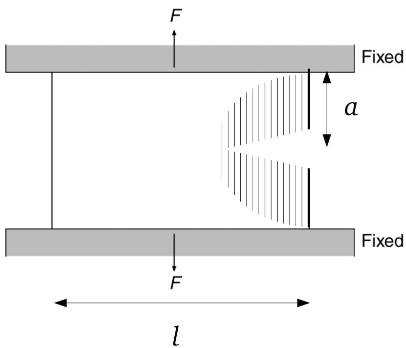


Figure 4: The work of the elastic strain energy equates to the whole green area. The dissipation is the hashed area enclosed in red curve. The dissipation area cannot exceed the elastic strain energy available (green area), or this would imply that earthquakes have a net energy creation! owever, if dissipation is *lower* than ESE, there is a net energy available for *waves* (kinetic energy)

Caveats of the point source picture. TOTAL enery balance from *Magnitude* section

2 The trigger

Irwin criterion based on energy: G is the energy released by unit crack advancement.

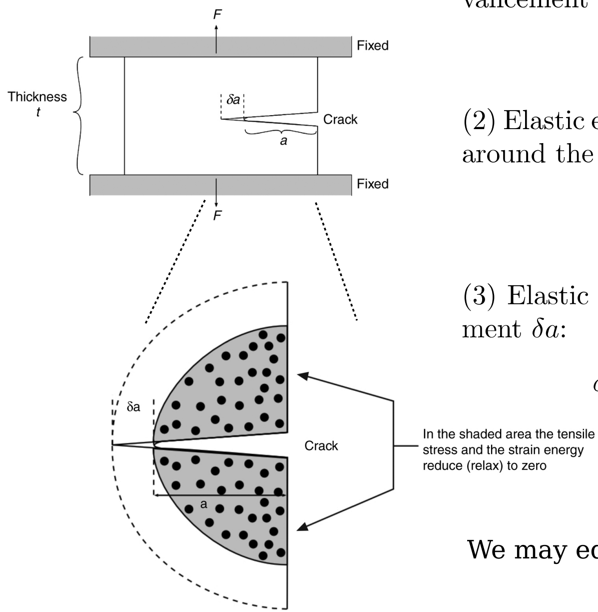


If a crack of length a appears, then elastic tension is released *roughly* in the hatched area, releasing the elastic energy$^{(*)}$:

$$U = (\frac{1}{2}\sigma\ \varepsilon) \times (\pi\ a^2) = \frac{1}{2}\frac{\sigma^2}{E}\pi a^2$$

(this is per unit length in the third direction).

(*) Notice I use twice the surface of the hatched area, πa^2. The crude argument is that the free edge on the right creates a virtual "mirror" image , equating to a full disk instead of half-disk. Not a rigorous argument, but it leads to the same solution as the more complete, rigorous derivation.



(1) Energy released by fracture advancement of δa :

$$G\ \delta a.$$

(2) Elastic energy release in the medium around the crack:

$$U = \frac{1}{2}\frac{\sigma^2}{E}\pi a^2$$

(3) Elastic energy release per advancement δa:

$$\begin{aligned} \delta U &= \delta a \times \partial_a U \\ &= \delta a\ \tfrac{\sigma^2}{E}\ \pi\ a \end{aligned}$$

We may equate (1) and (3) to get G

$$G = \pi a \frac{\sigma^2}{E} \quad \text{(mode I opening crack)}$$

$$G = \pi L \frac{\tau^2}{\mu'}(1-\nu) \quad \text{(mode II shear crack)} \tag{10}$$

The crack advancement dissipates energy. (For example, creation of new surface is a mechanical endothermal process, because surface energy is higher than volume energy. You may also assume that you are severing molecular bonds, therefore some work is input to separate the molecules until the bond is broken. Plastic deformation around the tip of the crack will also dissipate energy. In the case of a shear crack (although this probably was not on Irwin's mind when he designed the criterion), there is also friction, as we'll see below).

Irwin argued that the criterion for the crack advancement should be formulated as an energy budget problem. The energy dissipated should be matched by the release of elastic strain energy. Assuming that the energy dissipation per unit crack advancement G_c (critical fracture energy) in a specific material can be measured, then the criterion for crack advancement is

$$C\,L\frac{\Delta\tau^2}{\mu'} \geq G_c \tag{11}$$

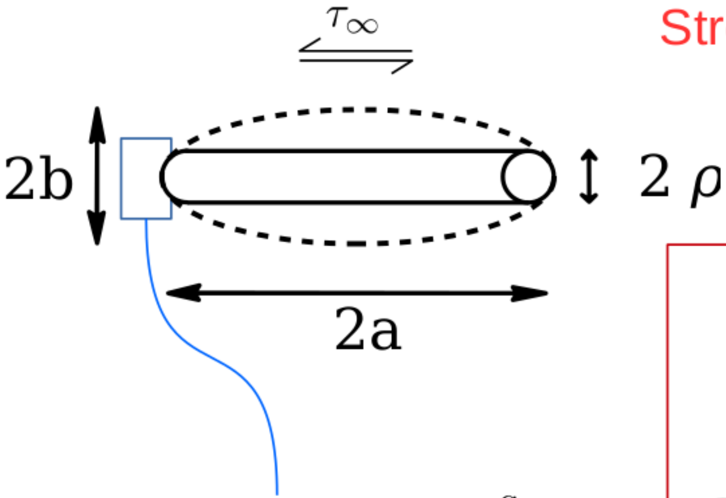
Importantly we see that the energy flow increases as the square of the stress (σ here, similar to τ in case of shear crack) and also in proportion to the crack length (a here, also called L in further slides).

Note that I sneakily substituted τ with $\Delta\tau$ in the equation. $\Delta\tau$ is the stress drop, used in case of non total loss of frictional strength across a fault.

The meaning of G_c. This can be interpreted as how *blunt* ot *sharp* the crack edge is, in an analogy to a blunt or sharp knife. This is in princliple defined as a material property; however, what we do observe in earthquake rupture, where it is possible to estimate a seismological equivalent of G_c (Abercrombie and Rice 2005), is that G_c scales with the rupture linear dimension.

An interpretation for the origin of G_c is the equivalence between crack radius of curvature and bluntness, and how they influence the stress concentration.

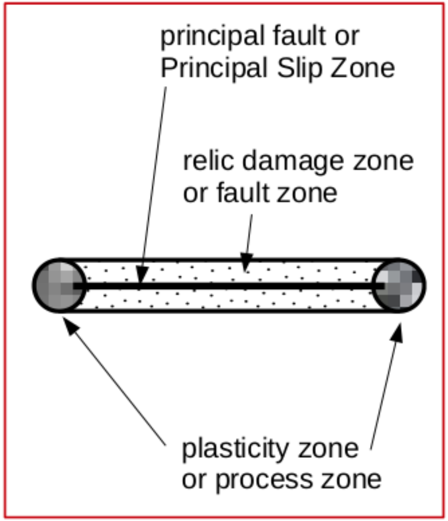
Stress concentration

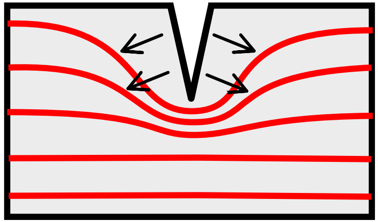


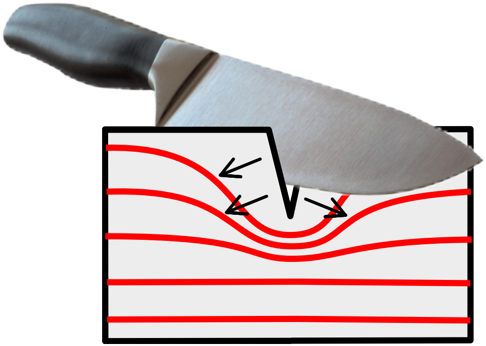
$$\tau_{\max} = \tau_\infty \frac{a}{b}$$

curvature radius $\rho = b^2/a$

$$\boxed{\tau_{\max} = \tau_\infty \sqrt{\frac{a}{\rho}}}$$







$$\tau_c = \left(\frac{\mu \ G_c}{\pi \ a} \right)^{\frac{1}{2}}$$

min. crack propagation stress

$$\text{curvature radius } \rho = b^2/a$$

$$\tau_{\max} = \tau_\infty \sqrt{\frac{a}{\rho}}$$

assuming we start to fail, then
τ_{max} is the yield strength
of the rock (under shear)

$$\text{equating } \tau_\infty \text{ and } \tau_c :$$

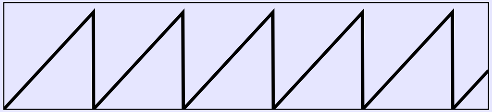
$$\sqrt{\frac{\mu G_c}{\pi \ a}} = \tau_{\max} \sqrt{\frac{\rho}{a}}$$

$$G_c = \frac{\pi \ \rho \ \tau_{\max}^2}{\mu}$$

material toughness increases
with the size of the process zone

Exercise:

Consider the saw tooth model of earthquake recurrence



where rupture is triggered by loading up to a critical stress, and the process is repeated n times.

In light of eq. 11 ($\pi L \frac{\Delta\tau^2}{\mu'} \geq G_c$) can we postulate different ways of triggering earthquake rupture?

There are indeed examples where the saw-tooth model does not seem to apply very well. This is the case for a number of intraplate earthquakes. Between 1811 and 1812 there were three large earthquakes (> 7.3) in the New Madrid Mississippi valley, in a very stable continental area with no record of tectonic strain. More recently, the 2017 $M_w 6.5$ Botswana quake, reactivated a 2 billion year old fracture zone, with no deformation measured across the region in the previous decades.

3 Energy transfers in the earthquake process

Further down the divergence theorem is used to compute the total energy change during an earthquake –for now let's look at the global energy partition and energy flows. There are three main energy classes to consider in the earthquake process:

- ✓ built up as elastic strain and volume of elastic strain release
- ✓ dissipation (work done on the boundary that is the fault surface).
- ✓ radiation (example of instant friction release –full conversion to kinetic energy)

One can consider the earthquake problem as a transfer of energy from one class to the other. In particular, there is a convergence of elastic energy towards the propagating rupture tip, where a net energy is dissipated in the process of rupture advancement. The importance of this energy loss and how it controls the rupture process is discussed in the section below.

The energy flow

- The model of LEFM and the singularity
- What happens at the rupture tip
- Stress and energy re-distribution

In linear elastic fracture mechanics (LEFM), the solution or the stress near the crack tip is computed assuming that there region o of anelastic deformation ids infinitesimally small, therefore resulting in a non-physical stress singularity at $x = 0$. For a crack of half-length L:

$$\tau = \frac{\Delta\tau\sqrt{\pi L}}{\sqrt{2\pi x}} \tag{12}$$

where τ is shear stress, $\Delta\tau$ is stress drop (or initial stress applied at infinity in case of total stress drop); x is the distance from the crack tip. A schematic graphic representation of the stress on the fault and around the rupture tip is shown in **(Fig. 5)**.

Stress concentration around rupture tip

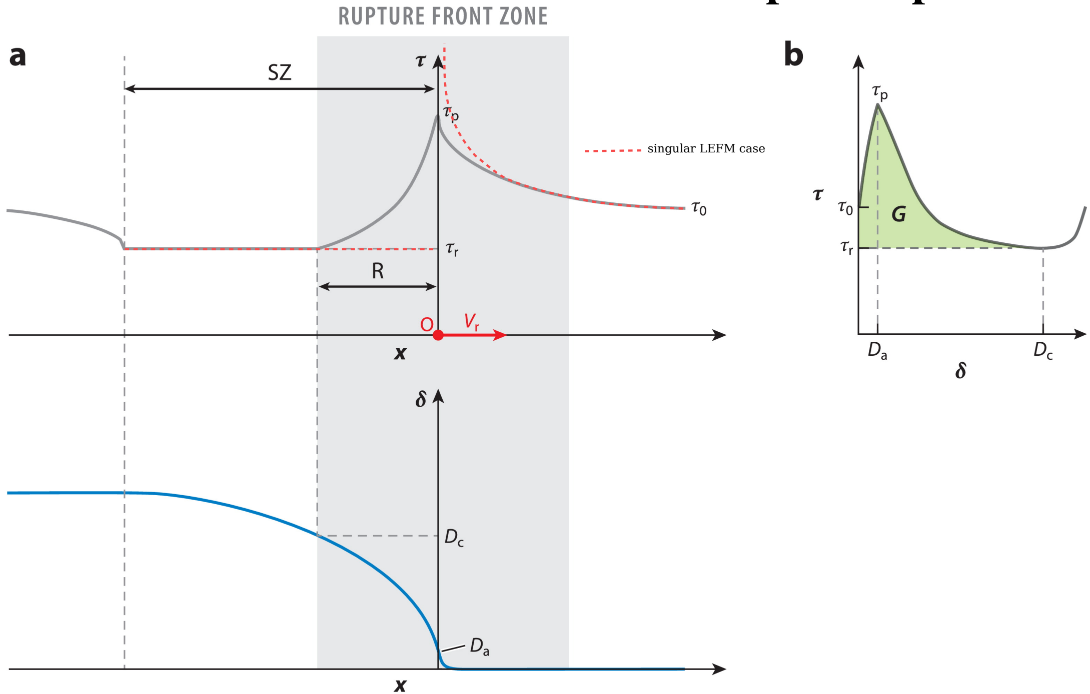


Figure 5: From Cocco, Aretusini, Cornelio, Nielsen et al. 2023

How to derive stress concentration in $r^{-1/2}$?

See slide set → A2_Derive_stress_con.pdf

To remove the singularity, multiply (12) by $\sqrt{2\pi r}$. This results in the definition of the stress intensity. For a in-plane crack of half-length L and stress drop $\Delta\tau$:

$$K = \sqrt{\pi L}\,\Delta\tau \tag{13}$$

There is no production or loss of net energy except at the tip. This can be shown by using the divergence theorem. During a deformation, the work done against a closed surface is equal to the change of elastic strain energy inside the volume enclosed by such surface, provided that the volume behaves as a linear elastic body and that all functions are harmonic inside it. This is the case everywhere except at the crack tip, due to the singularity. When the crack advances a finite energy sink appears at the crack tip.

The energy sink can be computed mathematically by integrating the singularity, which is a tedious complex analysis exercise, with use of the residue theorem. Jim Rice introduced the J integral, on a path including the plasticity region, and showed that that the result of such integration is path-independent (a ramification of the divergence theorem).

As a result of such integration, one obtains the static energy flow (slow growing or quasi-static crack) that can be written as:

$$G_0(L) = \frac{K^2}{2\mu'}(1-\nu) \tag{14}$$

(and we can now compare this result with equation 10). The dynamic energy flow (rupture propagating at v_r that is approaching the elastic wave velocity) requires a velocity correction:

$$G = G_0\, g(v_r) \tag{15}$$

However, we can use an integral solution for a case of inhomogeneous stress drop (Fossum and Freund 1975):

$$= K\,(L) = \frac{\sqrt{\pi}}{2}\int_0^L \frac{\Delta\tau\,(x)}{\sqrt{L\,(t)-x}}dx \tag{16}$$

In the case of two domains with different stress drops, $\Delta\tau$ for $0 < x < L_1$ and $\Delta\tau_1$ for $L_1 < x < L$, we can integrate by parts to obtain:

$$K(L) = \Delta\tau \sqrt{\pi L}\left(1 - \left(1 - \frac{\Delta\tau_1}{\Delta\tau}\right)\sqrt{1 - \frac{L_1}{L}}\right) \tag{17}$$

noting that, if $L < L_1$ the square root becomes imaginary, and we implicitly retrieve the solution corresponding to the homogeneous case (13) (isn't maths fun!?)

Now about the case where the crack is dynamically propagating. The analytical expression for $g(v_r)$ is quite complex, it will depend on the geometry (mode II or mode III in 2D, or mixed-mode depending on the position for an elliptical rupture). It is more complicated if one is considering the case of rupture velocity above the Rayleigh wave speed. Different cases are considered in Broberg 1999; Freund 1990 and the solution for an elliptical rupture is given by **Burridge1969**. However, using an analytical expression for $g(v_r)$ does not take into account waves diffracted by sudden changes in rupture velocity, so it will ever be an approximation. All forms of $g(v_r)$ decrease monotonously with slip velocity from $g(0) = 1$ gradually approaching zero for $g(c_r) = 0$, where c_r is a limiting velocity (e.g. S-wave velocity for mode III, Rayleigh-wave velocity for mode II sub-sonic propagation). For practical computational purposes, we can approximate $g(v_r)$ with the simplest form of monotonously decreasing function of v_r, a linear function such that:

$$g(v_r) \approx 1 - \frac{v_r}{c_r}. \tag{18}$$

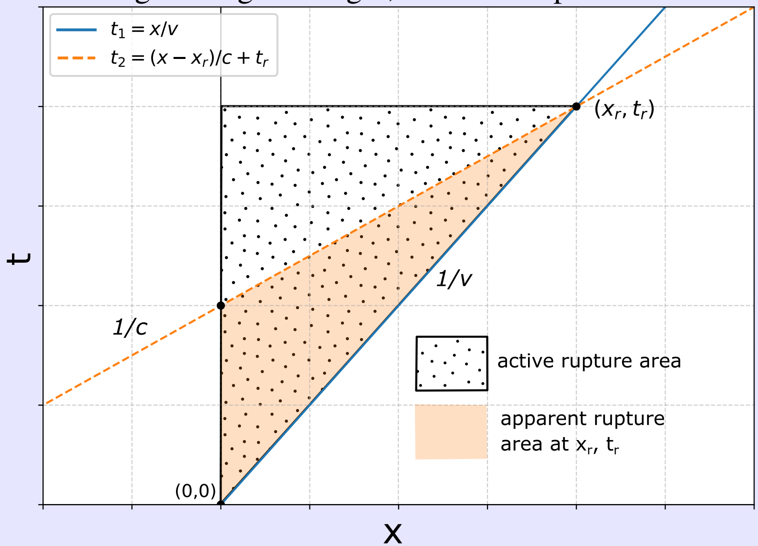
Then we can write an approximate balance between the available energy flow G and the critical fracture energy G_c as:

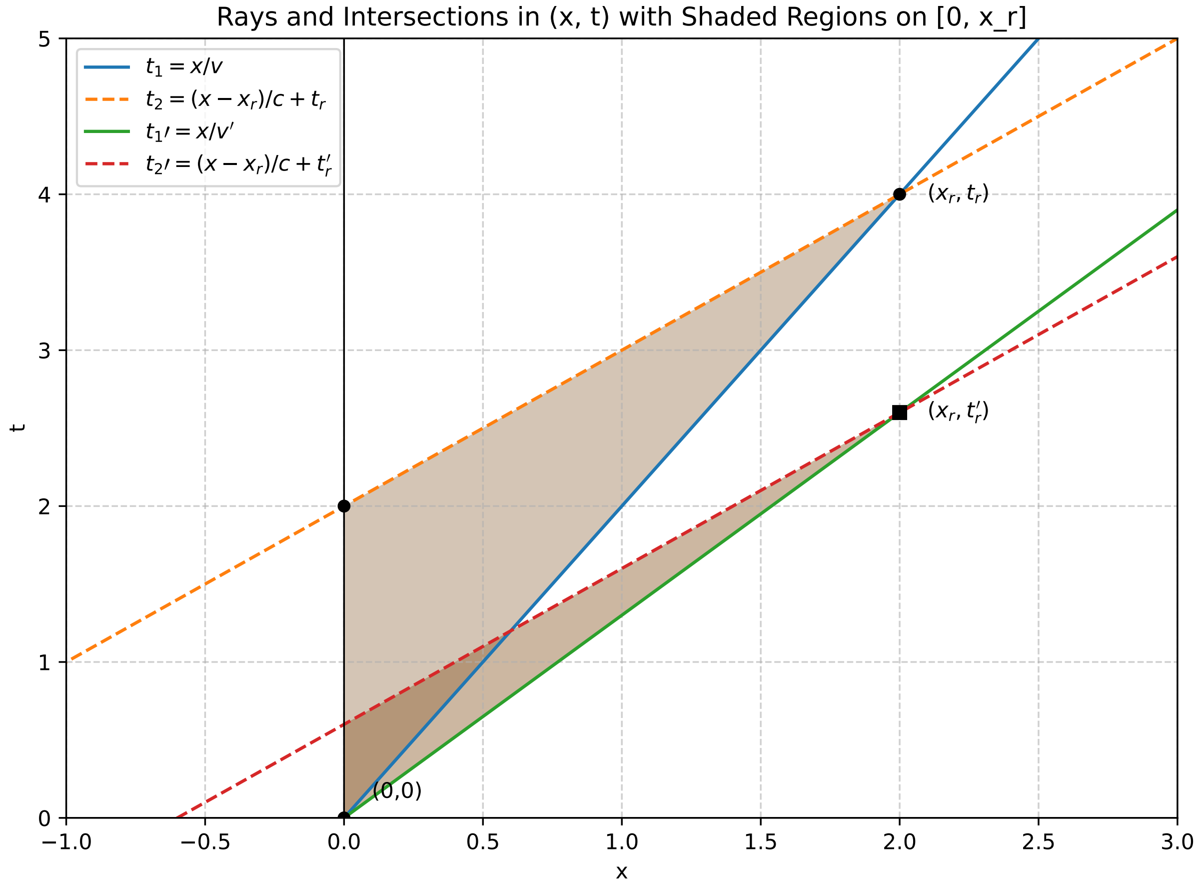
$$G_0\left(1 - \frac{v_r}{c_r}\right) \approx G_c$$

4 The kinematic contraction of the stress field

Where does this $\left(1-\frac{v_r}{c_r}\right)$ come from?

Using the figure below with rupture casuality cone, derive an expression for the 'apparent rupture area'. Try to factorise so that the $\left(1-\frac{v_r}{c_r}\right)$ factor appears. Note that this is an approximate solution designed to gain insigth, not the complete solution!




Rays and Intersections in (x, t) with Shaded Regions on [0, x_r]
$t_1 = x/v$
$t_2 = (x - x_r)/c + t_r$
$t_{1'} = x/v'$
$t_{2'} = (x - x_r)/c + t_r'$
(x_r, t_r)
(x_r, t_r')
(0,0)
t
x
5
4
3
2
1
0
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0

$$t_1(x) = \frac{x}{v}, \qquad \text{curve for rupture velocity @ v}$$

$$t_2(x) = \frac{x}{c} + t_r - \frac{x_r}{c} \qquad \text{causality cone for waves @ c}$$

$$t_2(x) - t_1(x) = \left(\frac{x}{c} + t_r - \frac{x_r}{c}\right) - \frac{x}{v} = \left(\frac{1}{c} - \frac{1}{v}\right)(x - x_r).$$

$$A_1 = \int_0^{x_r} \left[t_2(x) - t_1(x)\right] dx = \left(\frac{1}{c} - \frac{1}{v}\right) \int_0^{x_r} (x - x_r)\, dx$$

$$= \left(\frac{1}{c} - \frac{1}{v}\right) \left[\frac{x^2}{2} - x_r x\right]_0^{x_r} = \frac{x_r^2}{2}\left(\frac{1}{v} - \frac{1}{c}\right).$$

$$A = \frac{x_r^2}{2v}\left(1 - \gamma\right) \text{ where } \gamma = \frac{v}{c}.$$

This is the area of active rupture visible (i.e. within the causality cone) from point (x_r, t_r). The energy reaching (x_r, t_r) would be proportional to this apparent area, but to obtain a parameter that has the correct deimensions of energy flow (J m^{-2}) we need to scale A by a proper dimensional factor

$$\frac{2\Delta\tau^2 v}{\mu x_r}$$

to obtain an *approximate* energy flow at x_r

$$G \approx A_0 \left(1 - \gamma\right) \frac{x_r\, \Delta\tau^2}{\mu}$$

We could call this form a Doppler approximation of energy flow, due to the wavefield contraction being computed solely based on the Doppler shift $1 - v/c$.

But really –where does $\left(1 - \frac{v_r}{c_r}\right)$ come from?

The full solution involves, among other things, the Lorentz contraction instead... $\sqrt{1 - v^2/c^2}$

Exercise: derive Lorentz contraction!

The out-of-plane (anti-plane) displacement $w(x, y, t)$ in a homogeneous elastic solid satisfies the shear-wave equation

$$\frac{\partial^2 w}{\partial t^2} = c_s^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right),$$

where c_s is the shear-wave velocity.
For a rupture front moving steadily at speed v in the $+x$ direction, introduce a **co-moving** coordinate

$$w(x, y, t) = W(\xi, y), \qquad \xi = x - vt.$$

Substitute (x, t) by $x - vt$ in the wave equation above, and try to single out the factor $\sqrt{1 - v^2/c^2}$

Substitution gives

$$(1-\gamma^2)\,W_{\xi\xi} + W_{yy} = 0, \qquad \gamma \equiv \frac{v}{c_s}.$$

Define the Lorentz-type factor

$$\alpha_s \equiv \sqrt{1-\gamma^2} = \sqrt{1-\frac{v^2}{c_s^2}}.$$

With the stretched coordinate $\tilde{\xi} = \xi/\alpha_s$, the governing equation becomes Laplace's equation,

$$W_{\tilde{\xi}\tilde{\xi}} + W_{yy} = 0.$$

Hence, the steady moving problem is equivalent to a static 2-D Laplace problem in a stretched coordinate system, where all distances parallel to the direction of propagation are *contracted* by the Lorentz-type factor α_s.
The near-tip fields therefore depend on the "elliptic radius"

$$\tilde{r} = \sqrt{\frac{(x-vt)^2}{\alpha_s^2} + y^2} = \sqrt{\frac{(x-vt)^2}{1-\gamma^2} + y^2},$$

and stresses take the familiar square-root form

$$\tau_{xy}(x,y,t) \sim \frac{K_{III}(v)}{\sqrt{2\pi\,\tilde{r}}}\, f_{III}(\tilde{\theta}),$$

where $K_{III}(v)$ is the dynamic stress intensity factor (SIF) and $\tilde{\theta}$ the polar angle in the stretched coordinates.

According to Freund (1990, Eq. 5.3.11), the dynamic SIF is related to the static one by

$$K_{III}(v) = \frac{K_{III}^{(0)}}{(1-\gamma^2)^{1/4}} = \frac{K_{III}^{(0)}}{\alpha_s^{1/2}},$$

showing the amplification of local stresses as the tip velocity approaches c_s. Evaluating the dynamic J-integral (energy release rate) around a moving contour yields (Freund 1990, Eq. 6.4.45)

$$G(v) = \frac{K_{III}^2(v)}{2\mu}\,\frac{1-\gamma^2}{1+\gamma},$$

where the factor $(1+\gamma)^{-1}$ originates from the *convective term* in the moving-contour J-integral: the control surface translating with the crack tip carries total energy $(W+K)$ downstream at velocity v, leading to an asymmetric energy flux between the leading and trailing faces of the contour.

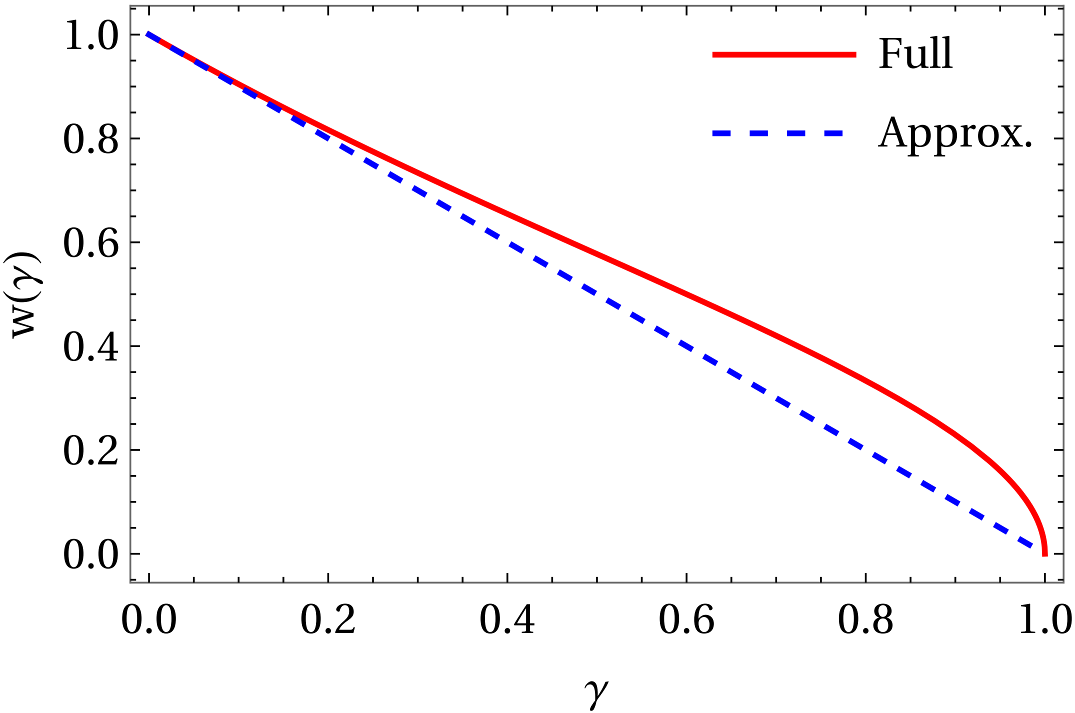
Substituting the above relation between $K_{III}(v)$ and $K_{III}^{(0)}$ gives the expression in terms of the static loading:

$$G(v) = G_0\,\frac{\sqrt{1-\gamma^2}}{1+\gamma}, \qquad G_0 = \frac{(K_{III}^{(0)})^2}{2\mu}.$$

Finally, the approximation is obtained by

$$\lim_{\gamma\to 0}\frac{\sqrt{1-\gamma^2}}{1+\gamma} = 1-\gamma, \quad G(v) \approx G_0(1-\gamma)$$

$$G(v) \approx A_0 \left(1 - \frac{v}{c}\right) \frac{x_r \, \Delta\tau^2}{\mu}$$



++

Thus the velocity dependence of $G(v)$ arises partly from the Lorentz-type contraction of the near-tip fields ($\sqrt{1-\gamma^2}$), and partly from the asymmetric convective energy flux ($(1+\gamma)^{-1}$) around the moving crack tip.

Mathematically, this transformation is *identical* to the Lorentz transformation of special relativity if the elastic shear-wave speed c_s is treated as the invariant velocity. The mapping

$$\tilde{\xi} = \frac{x - vt}{\sqrt{1 - v^2/c_s^2}}, \qquad \tilde{t} = \frac{t - (v/c_s^2)\,x}{\sqrt{1 - v^2/c_s^2}},$$

leaves the wave operator $\partial_t^2 - c_s^2 \nabla^2$ invariant, and the coordinate contraction by $\sqrt{1 - v^2/c_s^2}$ follows directly. This "elastic Lorentz invariance" is therefore exact in the mathematical sense, but it does *not* imply physical relativity: c_s is a material property, not a universal constant, and no time dilation or relativistic mass effects occur. Hence, the Lorentz contraction in dynamic rupture is best viewed as a *mathematical analogue* of special-relativistic contraction, arising within classical continuum mechanics.

A brief history of time (and space) contraction

Doppler (1842)

> "If a luminous body moves in the direction of the observer, or the observer moves toward the source, then the number of waves striking the eye per unit time increases by a ratio depending on their relative speed. If the body recedes, the number of waves received per second decreases in the same proportion."

Mathematically, in modern notation, Doppler expressed this as

$$f' = f\,\frac{c}{c \mp v},$$

where the upper (minus) sign applies when the source and observer approach each other, and the lower (plus) sign when they recede.

This is the *classical Doppler effect*, valid in the limit $v \ll c$.
He also expressed it in terms of the wavelength (since colour corresponds to wavelength):

$$\lambda' = \lambda \frac{c \mp v}{c} = \lambda \left(1 \mp \frac{v}{c}\right)$$

Lorentz (1895)

In his *"Versuch einer Theorie der elektrischen und optischen Erscheinungen in bewegten Körpern"* (1895), Lorentz wrote (in translation) something along the following lines:

> "If a system of molecules (or electrons) moves with velocity v through the ether in the x-direction, and if in the rest system the equilibrium of forces holds, then in the moving system the distances between molecules in the x-direction must be shortened by the factor
>
> $$\sqrt{1 - \frac{v^2}{c^2}},$$
>
> while the dimensions perpendicular to the motion remain unaltered."

Einstein (1905)

In his paper *"Zur Elektrodynamik bewegter Körper"* (1905), Einstein derives the same contraction from his two postulates (the principle of relativity and the constancy of the speed of light). In §4 he argues:

"We thus arrive at the result that the length of the rod measured in the stationary system is shorter than the length of the same rod at rest by the factor

$$\sqrt{1 - \frac{v^2}{c^2}}.$$

"

Recall that G_c is defined earlier as the amount of energy spent in propagating rupture of a unit length –usually considered an intrinsic property of the rock, but evidence that it is more than that and possibly a variable, with a dynamically determined value. G_c is a form of dissipation, but it does not include all the dissipation happening on a fault during rupture, only part of it, the one that affects rupture propagation and rupture velocity.

Exercise:

What processes are likely to induce the energy loss represented by G_c?

In the case of constant stress drop $\Delta\tau$ we can obtain the rupture velocity (again taking approximation 18):

$$v_r \approx c_r \left(1 - \frac{G_c\, \mu'}{\Delta\tau^2\, L} \; \frac{2\,\pi}{1 - \nu} \right) \tag{19}$$

Again, we can compare in the above expression the term in red with expressions (11) and (10). This appears to be the *ratio of critical fracture energy to the energy flow*. Therefore the same dimensionless parameter controls (1) the triggering of rupture propagation, and (2) the velocity of propagation. Note that for $L < \frac{G_c\, \mu'}{\Delta\tau^2}$ the rupture velocity is negative, which means no propagation is possible at all. In fact (19) implicitly includes the Irwin criterion.

Models that include plastic deformation over an extended damage zone avoid the singularity. However, even these models with an extended plastic region are compatible with LEFM, provided that the damage zone is small with respect to the crack semi-length L.

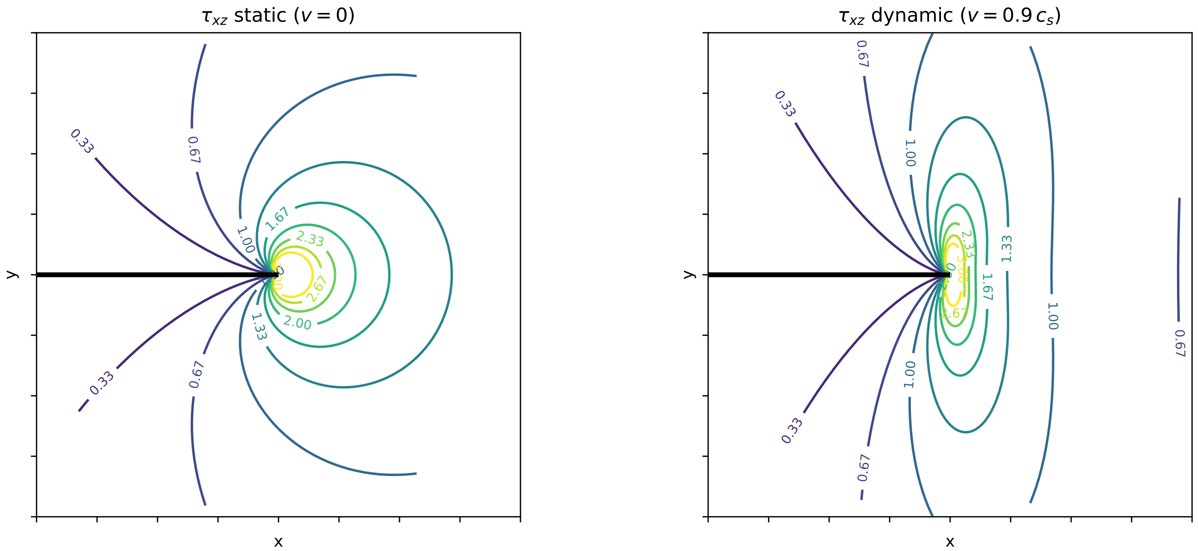


Figure 6: Effect of contraction on the single shear stress component

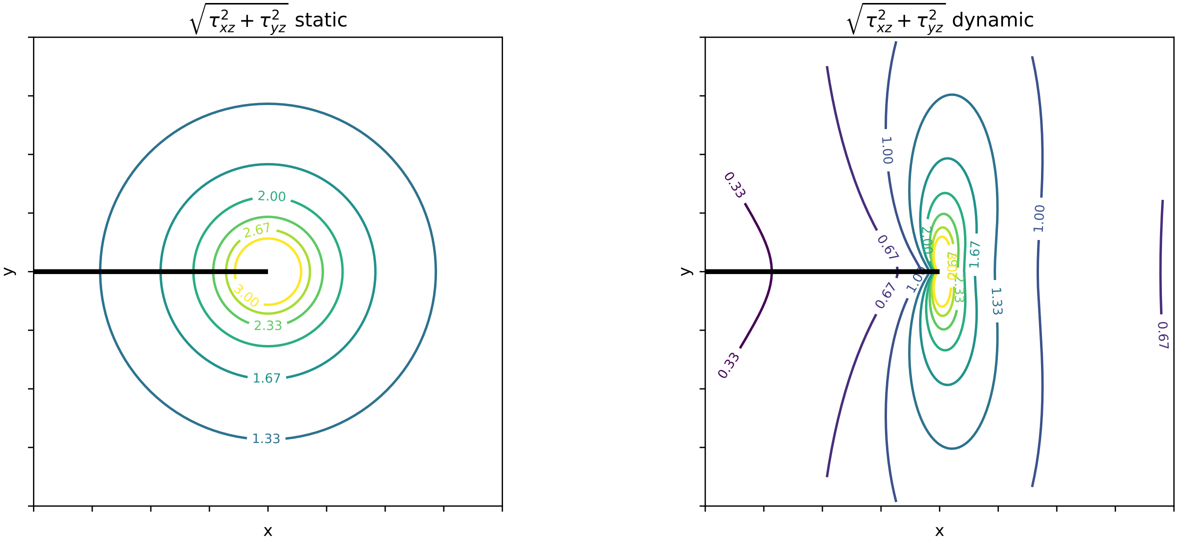


Figure 7: Effect of contraction on the stress magnitude second invariant of stress deviator

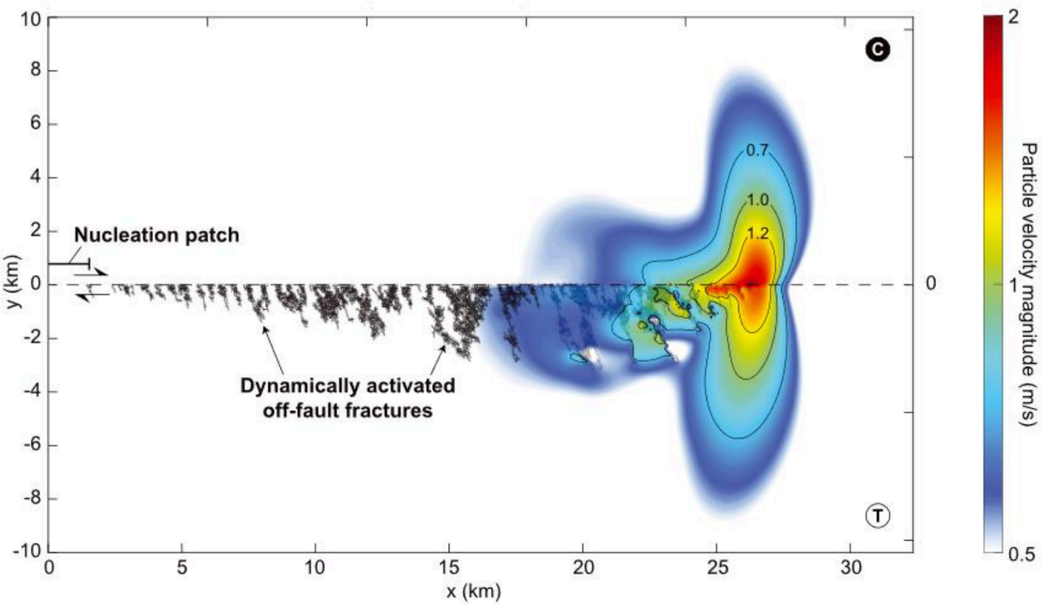


Figure 8: Full numerical computation of stress concentration AND off-fault damage (branching). From Okubo et al. 2021.

Why is the contraction important?

The contraction results in a reduced energy release rate when velocity increases
The contraction results in shrinkage of the process zone, so when rupture accelerates to $v \to c$, the stress field tends to a singularity
A singular tip induces a sharp slip velocity acceleration, this is extremely difficult to model numerically

Supershear rupture

An interesting note about the situation of super-shear rupture, or more specifically intersonic rupture. Stress propagates with waves during rupture, and there are two types of body waves in a solid (plus a variety of inhomogeneous waves, including Rayleigh wave). The fastest wave is the P-wave, so causality tells us that, in principle, a rupture (almost) as fast as P-waves may propagate. It could be faster than shear-wave velocity, implying that a supersonic shock wave is generated in the S-wave field. This is observed experimentally **(Fig. 9)**, and, within a reasonable doubt arising from data ambiguity, also in a number of natural earthquakes.

There is an important possible relation between *supershear* and *off-fault damage*. The shock-wave induces an extremely fast and high strain rate, this can allow the formation of pervasive microcracks that have no time to propagate to larger lengths, and do not have time to coalesce to form bigger fractures. This behaviour has been reproduced in the lab (Doan and Gary, 2009), and proposed as an interpretation for pulverised rocks observed in the side walls of seismic faults (e.g., Arima-Takatsuki fault, Japan).

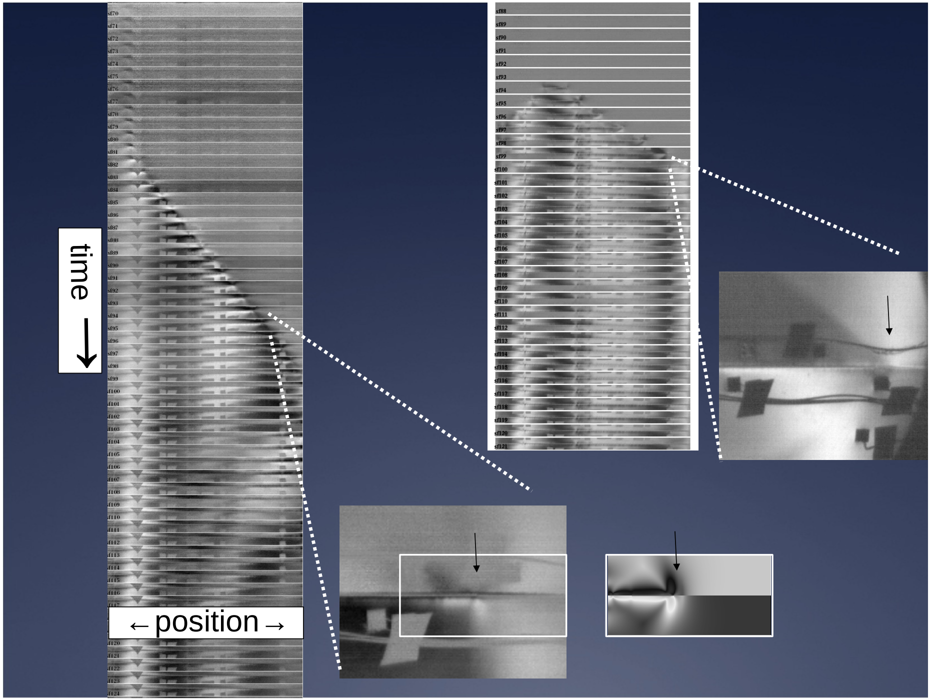


Figure 9: Laboratory experiment where both sub-sonic (left) and intersonic (right) ruptures have been produced from a spontaneously nucleating rupture on a precut polymer specimen. The shock wave or cone can be seen (black arrow) in photogram of the sample on the rightmost side of the figure. The time/position plots report time sequences of photograms to indicate the rupture propagation and to allow measuring its velocity.

Exercise:

What happens to the rupture velocity if the crack continues to propagate under constant G_c and $\Delta\tau$? How do earthquakes stop?

The direction of energy flow can be calculated in rupture propagation models; an example for 2D shear fracture (mode II) is shown in **(Fig. 10)**.

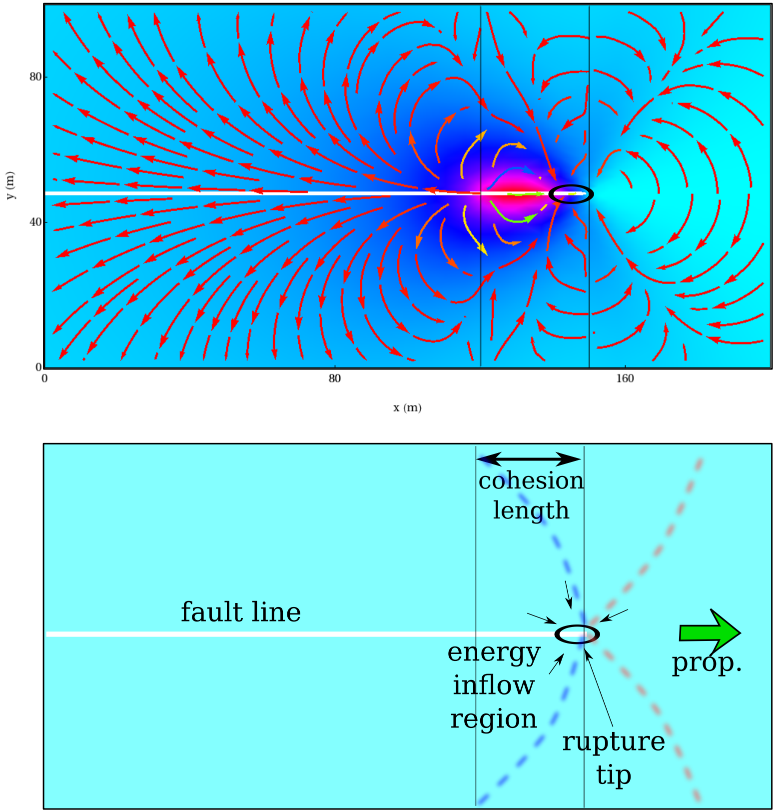


Figure 10: Energy flow in Mode II dynamic crack with a cohesion (slip weakening) zone

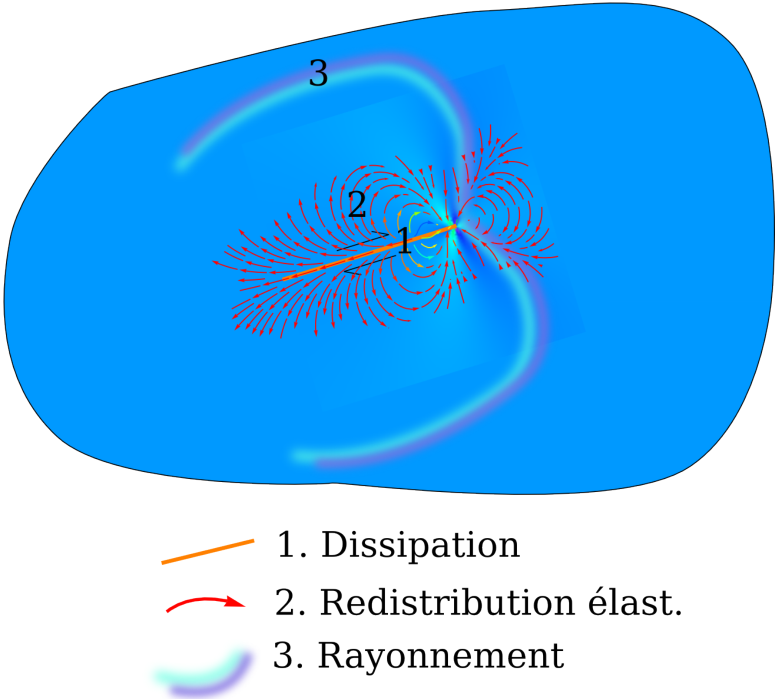


Figure 11: Aschematic visual representation of the energy redistribution during the earthquake

References

Abercrombie, Rachel E. and James R. Rice (Aug. 2005). “Can observations of earthquake scaling constrain slip weakening?” In: *Geophysical Journal International* 162.2, pp. 406–424. DOI: 10.1111/j.1365-246x.2005.02579.x.

Aki, K and P. Richards (2002). *Quantitative Seismology, second edition*. Science University Books.

Broberg, K. B. (1999). *Cracks and Fracture*. Academic Press. ISBN: 9780121341305.

Cocco, Massimo et al. (2023). “Fracture Energy and Breakdown Work During Earthquakes”. In: *Annual Review of Earth and Planetary Sciences* 51.1, pp. 217–252. DOI: 10.1146/annurev-earth-071822-100304.

Davies, J. H. and D. R. Davies (Feb. 2010). “Earth’s surface heat flux”. English. In: *Solid Earth* 1.1. Publisher: Copernicus GmbH, pp. 5–24. ISSN: 1869-9510. DOI: 10.5194/se-1-5-2010. (Visited on 09/27/2023).

Fossum, A. F. and L. B. Freund (Aug. 1975). “Nonuniformly moving shear crack model of a shallow focus earthquake mechanism”. In: *Journal of Geophysical Research* 80.23, pp. 3343–3347. DOI: 10.1029/jb080i023p03343.

Kanamori, Hiroo and Luis Rivera (2006). “Energy partitioning during an earthquake”. In: *Earthquakes: Radiated Energy and the Physics of Faulting*. American Geophysical Union, pp. 3–13. DOI: 10.1029/170gm03.

Pacheco, Javier F. and Lynn R. Sykes (June 1992). “Seismic moment catalog of large shallow earthquakes, 1900 to 1989”. In: *Bulletin of the Seismological Society of America* 82.3, pp. 1306–1349. ISSN: 0037-1106. DOI: 10.1785/BSSA0820031306.

Freund, L. B. (1990). *Dynamic fracture mechanics*. Cambridge University Press.

Stern, Robert J. (2002). “Subduction Zones”. en. In: *Reviews of Geophysics* 40.4, pp. 3–1–3–38. ISSN: 1944-9208. DOI: 10.1029/2001RG000108.