The Earthquake Dissipative Engine: Energy Budget and Partition

PART D: Moment, magnitude and energy

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Short course on earthquake physics

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27 Oct. - 16:30-18:30 classroom 2G
28 Oct. - 14:30-16:30 classroom 2H
29 Oct. - 10:30-12:30 classroom Lab Paleo
31 Oct. - 14:30-16:30 classroom 2L

1 Moment and strain energy

One of the most widely used measure of earthquake size is the seismic moment:

$$M_o = \mu \overline{D}\, S \tag{1}$$

where μ is the shear stiffness, $\overline{D}$ is the mean co-seismic slip and S is the rupture surface. You will notice that if μ is in Pascal units, D and S in metres and metres square, resp., then M_o has dimensions of energy in Joules, or Newton meters, therefore (1) is a moment, i.e. force $\times$ distance. (Incidentally, in the USA units are often changed to cm and dyne, resulting in ergs as opposed to Joules, and 1 erg = 10^{-7} J.) Despite its dimensionality, the moment is not a direct indication of the earthquake energy (though it is proportional to earthquake energy). The moment represents the integral of stress change over the source region, and it is much larger (up to 0.5×10^4 times larger) than the total earthquake energy E_{tot}, which can computed as the elastic strain energy released during an earthquake, namely:

$$E_{tot} = \frac{\overline{\sigma_0 + \sigma_1}}{2}\, \overline{D}\, S \tag{2}$$

where the prestress σ_0, final stress σ_1 and $\overline{\text{average}}$ slip $\overline{D}$ are represented in figure (1), and S is the area of the rupture. Here σ is the shear stress component on the fault plane, parallel to the slip direction. Interestingly, although the elastic strain is altered in a large volume surrounding the fault, the total energy change E_{tot} can be represented as in eq. (2) using only quantities defined directly on the fault surface. This result can be obtained by using the divergence theorem (also known as Gauss's theorem or Ostrogradsky's theorem) which relates the divergence of a given vector quantity **F** within a volume V to the flow of the same vector **F** across the surface S surrounding V How to derive eq. (2) from first principles is explained in appendix (available on demand - not part of the learning outcomes).

E_{tot} (2) represents the elastic energy which is stored in the rock volume surrounding the fault and which is released during the earthquake rupture. It can be seen as potential energy which is suddenly released, and transformed into kinetic

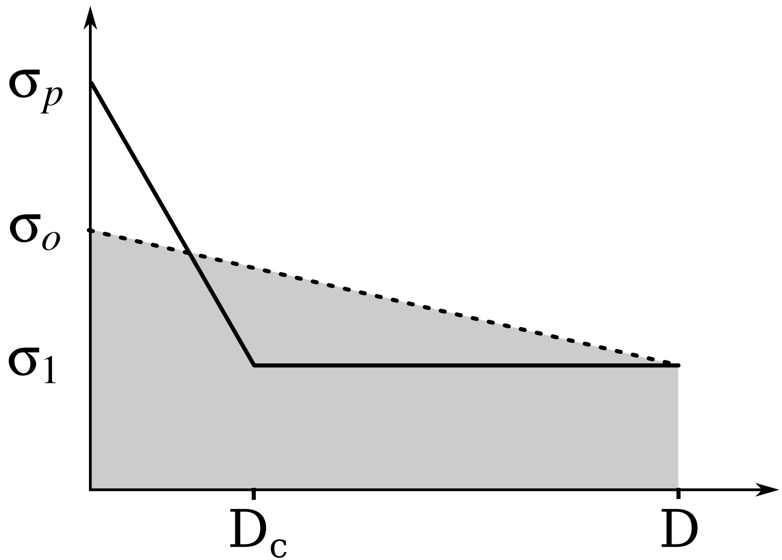


Figure 1: Schematic representation of shear stress change with fault slip D during an earthquake (average values on the rupture surface). σ_0 is stress on the fault before the earthquake; σ_p is peak stress reached before sliding starts; σ_1 is minimum sliding stress, also assumed here to be the same as final stress. The strain work per unit surface of fault $\frac{1}{S}E_{tot}$ (eq. 2) is represented by the grey area.

energy (the elastic waves that are generated by the sudden fault motion and produce ground shaking), on the one hand, and energy which is dissipated during the fracturing and fault sliding process, on the other side, and essentially lost to heat. We may write a simpified *earthquake energy budget* in the following form:

$$E_{tot} = W_f + E_k \tag{3}$$

where W_f is frictional work and E_k is kinetic energy (waves). The term W_f corresponds to the area that lies below the black solid curve in figure (1). As seen from the balance in (3), the amount of radiated elastic waves will be the difference between E_{tot} and the frictional dissipation W_f. In some simplified case as that of the schematic evolution of figure (1), where the final stress σ_1 is equal to the sliding stress during the earthquake slip (frictional stress), we may write that

$$W_f = \frac{(\sigma_p - \sigma_1)}{2} D_c\, S + \sigma_1\, D\, S \tag{4}$$

Using the expression thus obtained for W_f and (2), we can re-write equation (3) as follows

$$\begin{aligned} \frac{(\sigma_0 - \sigma_1)}{2} D\, S - \frac{(\sigma_p - \sigma_1)}{2} D_c\, S &= E_k \\ \frac{(\sigma_0 - \sigma_1)}{2} D\, S - G_c\, S &= E_k \end{aligned} \tag{5}$$

where the quantity defined as $G_c = \frac{(\sigma_p - \sigma_1)}{2} D_c$ is known as as *fracture energy*. Equation (5) tells how much energy is available to be radiated in the form of waves (kinetic energy). The case $E_k < 0$ is unphysical, as kinetic energy is produced and not *soaked up* by the earthquake process. Therefore the criterion $E_k \leq 0$ can be used as a criterion for the rupture propagation, and it can be indeed shown that it is equivalent to the Griffith energy criterion for rupture.

A few numbers

The slip during earthquakes scales with the length of the fault, and an indicative ratio of slip to fault length L is

$$\frac{D}{L} \approx 10^{-4}. \tag{6}$$

Taking this ratio as the strain released during earthquake rupture, and using a standard shear modulus $\mu = 50$ GPa for crustal rocks we can figure an indicative value of stress drop in earthquakes:

$$\Delta\sigma = \mu \times \varepsilon = (50\ 10^9) \times (10^{-4}) \approx 5\ 10^6 \text{ Pa} \tag{7}$$

where stress drop corresponds to $\Delta\sigma = \sigma_0 - \sigma_1$ in figure (1).

2 Scaling relations in earthquake rupture

Earthquake slip is proportional to stress drop and to the length of rupture, and for simple models it can be shown that

$$C\,\frac{D}{L} = \frac{\Delta\sigma}{\mu} \tag{8}$$

where $\overline{D}$ is the average slip inside the crack, μ is shear stiffness, L a typical rupture dimension (e.g. length or radius) and C a dimensionless shape factor. Here again, D/L is representative of strain change during the earthquake. It can be shown that for a circular crack if we take L as the radius then $C = {}^{7\pi}/_{16}$. Replacing $S = \pi L^2$ and the slip D from the scaling (8) in

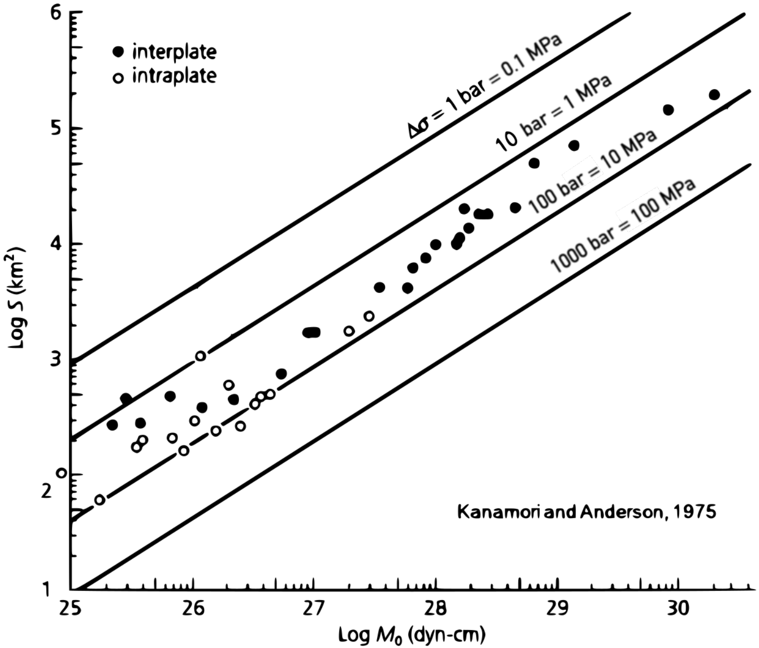


Figure 2: Scaling of seismic moment M_o and rupture size (surface S. The solid curves represent lines of constant stress drop. The stress drop does not show any evident trend with size; an indicative average value of $\Delta\sigma \approx 5$ MPa can be adopted but with a great scatter. More recent studies with an enhanced catalogue confirm that stress drop does not vary systematically with size, but that it is scattered over at least of 3 orders of magnitude.

the expression of the seismoc moment (1), we obtain a relation between moment and rupture dimension:

$$\begin{aligned} M_o &= \frac{16}{7}\Delta\sigma\, L^3 \\ &= \frac{16}{7}\Delta\sigma\, \left(\frac{S}{\pi}\right)^{3/2} \\ \log M_o &= {}^3/_2\, \log S + \log\Delta\sigma + \log\frac{16}{7\pi^{3/2}} \end{aligned} \tag{9}$$

therefore plotting the moment versus rupture surface should yield a slope of $3/2$ in log and an offset proportional to stress drop. Among the first to show the stress drop and the scaling of moment in such way was the study partly reproduced in figure **??**.

This type of scaling was repeated many times since with an ever increasing dataset. It appears that the stress drop does have a large varibility, but it is not easy to discern a systematic dependance with the size of the event. An indicative average of $\Delta\sigma \approx 5$ MPa can be used, but with a variability of more than one order of magnitude. More recent studies with enhanced catalogue indicate that stress drop can vary at least of 3 orders of magnitude (0.1 MPa to 100 MPa, or even reaching GPa values in some extreme cases for very small earthquakes.

The scaling relation 8, and the invariance (on average) of stress drop with magnitude, allow to define the earthquakes as a "self-similar" process. Small and large events behave in a similar way once re-scaled by the dimension of the rupture.

This invariance in stress drop appears in catalogs assembling data from different earthquake sequences and fault regions. Other studies argue that an increase in stress drop with magnitude may be apparent when focussing on specific regional sequences, as expected from laboratory experiments that show gradual frictional weakening of rocks under earthquake-like conditions However such increase will be masked by the variability due to geological and tectonic setting when using a mixed catalogue of earthquakes. Assuming self-similarity and adopting the observed average values for standard earthquakes, we can establish a correlation between rupture size, magnitude, slip, moment and the frequency range of the radiated waves, as shown in figure 2

In addition, combining equations 8, 2 and 1 we can write that:

$$E_{tot} = \frac{\sigma_0 + \sigma_1}{2\,\mu}\, M_o \tag{10}$$

Assuming an absolute lower bound on the final stress as $\sigma_1 = 0$, and using the indicative value 5 MPa for the stress drop, we obtain $\sigma_0 = \sigma_0 - \sigma_1 = \Delta\sigma \approx 5$ MPa. Using an indicative value of $\mu = 50\ 10^9$ Pa for the shear modulus, we would obtain for the seismic energy ${}^{(\sigma_0+\sigma_1)}/_{(2\,\mu)}\ M_o = {}^{(5\ 10^6)}/_{(2\ 50\ 10^9)}\ M_o = {}^1/_2\ 10^{-4}\ M_o$; therefore an upper bound for seismic energy in relation to the moment is

$$E_{tot} = \frac{1}{2}\, 10^{-4}\, M_o. \tag{11}$$

3 Magnitude scales

Historically, earthquake size has been defined based on the felt intensity of shaking, because that was the main available observation – and also because the shaking had direct consequences in the destruction toll. Depending on how the shaking is measured (instrument), at what distance from the epicenter, and in what type of environment, different types of scales have been developed. For example, a scale depending on the amount of damage incurred was developed around the end of the 19th century by the italian geophysicist Giuseppe Mercalli. Around 1935, Charles Richter working at the Caltech defined another scale based on the maximum ground displacement measured at 100 km from the epicenter, and subsequent variations of such scale were proposed (body wave magnitude, surface wave magnitude, and other magnitudes). Different scales are still in use today, depending on the type of available measurements, but since the 1970s one the most widely used is the *moment magnitude scale*.

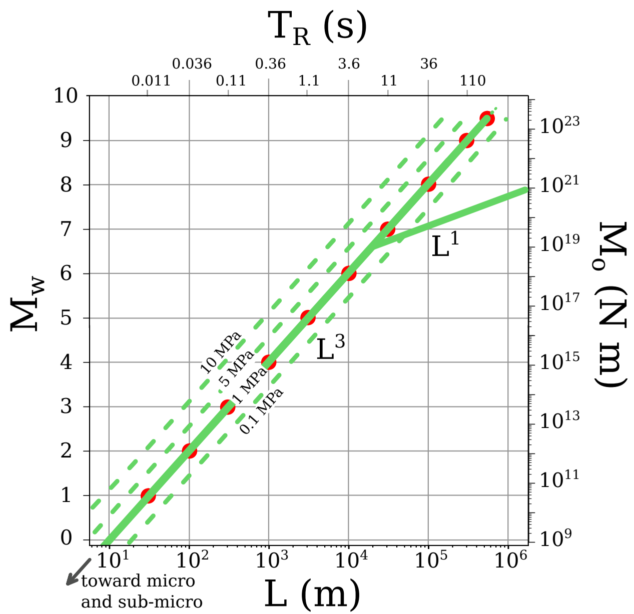


Figure 3: Scaling between magnitude M_w, moment M_o, fault dimension L and rupture duration T_R. These relations are INDICATIVE –departures from this graph do happen depending on fault style, seismic environnement, rupture geometry, stress drop, rupture velocity. It is assumed that the rupture area expands as L^2, at least until $M_w \approx 6.4$ (corresponding to a rupture which spans the entire width of the seismogenic crust, about 10 km). At $M_w > 6.4$ the rupture area can either continue to grow as L^2, or change to L^1 as indicated by the bifurcation in the solid line. This bifurcation would relflect the transition form a circular expanding rupture to a long, narrow rupture strip (e.g., a strike-slip fault). As shown here, a rule of thumb can be used that $L = 10^{1+\frac{M_w}{2}}$ m, and that $T_R = L/2800$ s. Finally, the well-known moment magnitude relation found in equation 12 allows to relate M_w and M_o.

4 Moment magnitude

This scale (as you may have guessed easily from its name) is based on moment, which represents somewhat a more robust quantification than other types measures (like the shaking) which may vary depending on setting and conditions. However moment (eq. 1) is proportional to the final slip D in an earthquake and is therefore a *static* measurement, not necessarily indicating how much shaking (kinetic energy) has been released by the rupture. An extreme example are slow earthquakes or slow slip events radiate hardly any waves at all – but their moment can be sometimes as large as in typical intermediate magnitude earthquakes.

Because the energy released by earthquakes spans many orders of magnitude, it is natural to adopt a logarithmic scale to define their size, and most generally you'll find base 10 log in the scale definition. It is the case for the *Moment Magnitude* usually noted M_w defined as:

$$M_w = \frac{2}{3}(\log M_o - 9.1) \tag{12}$$

where M_o is in N m. One limitation of the moment magnitude, is that it is based on the lowest frequency (or static slip) of the earthquake. Slow earthquakes and other special ruptures may have a large slip but a reduced wave radiation – therefore moment magnitude is not necessarily indicative of the shaking or the damage induced by the earthquake.

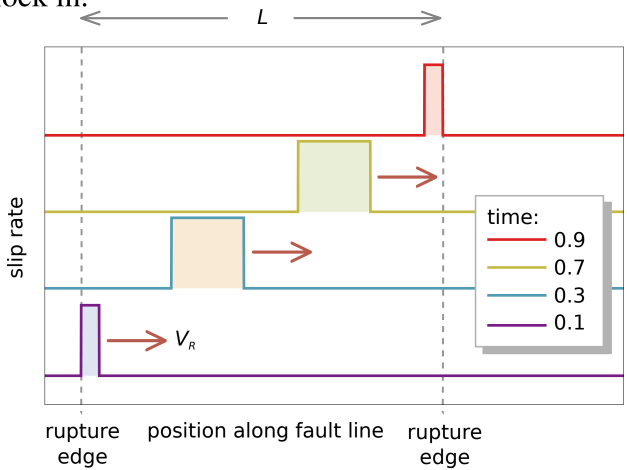
5 Estimating moment and rupture size from data

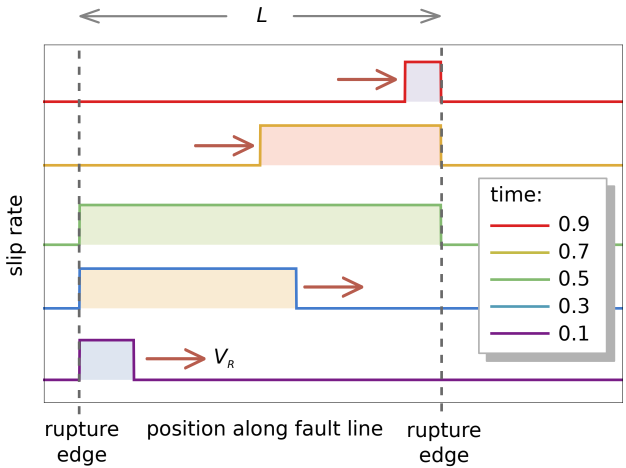
There are several ways to estimate the moment and rupture size, including mapping of surface ruptures or geodetic deformation maps when they are available. When seismological data alone is available –and this is rather frequent– a widely used method consists in fitting the displacement spectral amplitude with two parameters, the zero frequency amplitude Ω_0 and the corner frequency f_c.

Spectral amplitude is obtained by taking the Fourier transform of the signal. Let's use a model of simple seismic source that it is long and narrow, so we can assume a line source of length L. The rupture propagation across L at constant velocity V_r allows to define a characteristic *rupture time*:

$$T_R = {}^{L}\!/_{V_R} \tag{13}$$

If slip does not rise instantly at the onset of rupture, we can define a second characteristic time, the *rise time* T_D for slip D to lock in.





The far-field seismic *displacement* u for an earthquake source corresponds in first approximation to the *moment rate* at the source, scaled by a number of factors due to geometrical spreading, attenuation, radiation pattern, directivity wave velocity etc. For simplicity we can combine all corrective factors in the term $\wp$ such that:

$$u(t) = \wp\, \dot{M}_o \left(t - \frac{R}{c} \right) \tag{14}$$

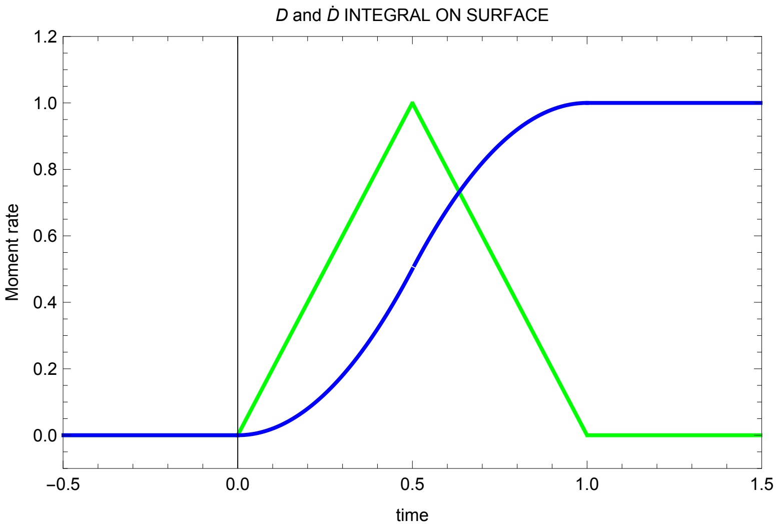


Figure 4: Moment rate and Moment as a function of time, in the two cases of slip corresponging to figure **??**.

where the dot denotes the time derivative and c is the wave (either P or S wave) velocity in the propagating medium between source and receiver, and R is the distance. The corrective factor can be defined as

$$\wp = \frac{e^{-\pi \omega R / c Q}}{4\,\pi\,\rho\,c^3} \frac{R_{ad}}{R} \tag{15}$$

where ρ is mass density, and the exponential represents the effect of attenuation, described here by a frequency-independent quality factor Q. The term R_{ad} represents the radiation pattern, i.e., the angular dependence in the amplitude. (For a double couple source R_{ad} depends on the particular type of wave which is being observed: P, S or one or another type of surface waves, but this is not developed in this section).

In the case of a source function illustrated in figure **??** with $T_D = T_R$, the *moment rate* function obtained by time derivative of 1 assuming $\overline{D} = \text{const.}$ yields:

$$\dot{M}_o = \mu\, D\, \dot{S}, \quad \text{with} \begin{cases} \dot{S} = a\, t & \text{for } \; 0 < t < T_R \\ \dot{S} = a\,(2\,T_R - t) & \text{for } \; T_R < t < 2\,T_R \end{cases} \tag{16}$$

which corresponds to a triangle $\blacktriangle(t)$ as illustrated in figure 5 so that $\dot{M}_o = \mu\, D\, \blacktriangle(t)$. Since $M_o = \int_0^{2\,T_R} \dot{M}_o\, dt$, in equation 16 the amplitude factor is $a = {}^{M_o}/{T_R^2}$.

The displacement spectrum (amplitude at different frequencies) is obtained by taking the Fourier Transform (FT) of the displacement $u(t)$ such that $\tilde{u}(\omega) = \mathcal{F}(u(t))$. and plotting the modulus $|\tilde{u}|$. The Fourier transform can be defined in several slightly different ways; we adopt here the form:

$$\tilde{u}(\omega) = \int_{-\infty}^{\infty} u(t)\, \exp^{-i\,\omega\,t}\, dt \tag{17}$$

where ω is angular frequency ($\omega = 2\pi f$ if f is frequency). The inverse Fourier transform in this case is:

$$u(t) = \frac{1}{2\pi} \int_{\infty}^{\infty} \tilde{u}(\omega)\ \exp^{i\ \omega\ t}\ dt \tag{18}$$

In the case of a triangle function as defined in equation 16 the FT yields:

$$|\tilde{u}(\omega)| = \wp\, M_o\ \left(\frac{\sin \frac{T_R \omega}{2}}{T_R\ \omega/2} \right)^2 \tag{19}$$

The envelope of the above spectrum at relatively high frequency can be approximated by

$$\omega \gg 2/T_R \longrightarrow \tilde{u}(\omega) \approx \wp\, M_o\ \frac{4}{T_R^2\ \omega^2} \tag{20}$$

and at relatively low frequency, by

$$\omega \ll 2/T_R \longrightarrow \tilde{u}(\omega)\ \approx \wp\, M_o \tag{21}$$

because $\dfrac{\sin x}{x} = 1$ in the limit $x \to 0$. If we plot equation 19 with log on both axes, we see quite clearly the transition from low frequency to high frequency, and we can define the corner frequency $\omega_c = 2/T_R$ graphically ($f_c = (\pi T_R)^{-1}$), by adding the two log-linear fit lines and looking for their intersection (figure **??**).

If $T_D <\ T_R$ (see simple source model of figure **??**) then the moment rate function is trapezoidal and the spectrum becomes:

$$|\tilde{u}(\omega)| = \wp M_o \left| \frac{\sin\left(\frac{\omega T_R}{2}\right)}{T_R\ \omega/2} \right| \left| \frac{\sin\left(\frac{\omega T_D}{2}\right)}{T_D\ \omega/2} \right| \tag{22}$$

where, in principle, two different corner frequencies could be distinguished (in practice with real earthquake data this is rather challenging and the single corner frequency is mostly used).

Rather than a triangle, a smoother, exponential time function can be used as a simple source model yielding the far-field spectrum

$$\tilde{u}(\omega) = \frac{\wp M_o}{1 + \frac{\omega^2}{\omega_c^2}} \tag{23}$$

as originally proposed by Brune

Importantly, all three theoretical spectra (19, 22, 23) show a decay in ω^{-2} at high frequency; a zero frequency limit proportional to the seismic moment; and a corner frequency inversely proportional to the duration of the source.

Log-log representation of the spectrum from simple seismic source models. (a) Spectrum from a triangular source function (equation 19); (b) "Brune" spectrum (equation 23). In both cases the characteristic source duration is $T_R = 1/2$ s, hence the corner frequency is $f_c = \dfrac{1}{\pi T_R} = \dfrac{2}{\pi}$ Hz (or, in angular frequency $\omega_c = 2/T_R = 4$ rad s^{-1}). The dashed lines represent the low frequency and high-frequency asymptotes; the vertical line indicates the corner frequency.

6 Relative frequency of large and small earthquakes: The GR distribution

This empirical law (i.e. based on observation) was first defined in the fifties by Beno Gutenberg and Charles Richter (same as the Richter magnitude scale). The law states that there are many more small earthquakes than large ones, and that the relative number decreases logarithmically with their magnitude. In a given time interval, letting N be the number of earthquakes having *at least* magnitude M_w, then

$$\log N = a - b\, M_w \tag{24}$$

where both a and b are positive real. Common values of b are close to 1. The value of b indicates the relative dominance of small earthquakes relative to large earthquakes, with larger values indicating that very few large earthquakes occur. Naturally the distribution is bounded in magnitude because of the limited size of the seismogenic faults– the largest magnitude ever recorded was M_w 9.5 (Valdivia, Chile, 1960). The a value is not quite relevant: it depends on the total number of earthquakes, and therefore, a sould increase with the extent of the area observed and with the extent of the time interval, while b would not.