The Earthquake Dissipative Engine: Energy Budget and Partition

PART B: Radiation damping

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Short course on earthquake physics

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27 Oct. - 16:30-18:30 classroom 2G
28 Oct. - 14:30-16:30 classroom 2H
29 Oct. - 10:30-12:30 classroom Lab Paleo
31 Oct. - 14:30-16:30 classroom 2L

Setup for anti-plane (Mode III), fault is y=0, identical half-spaces

$$\delta(x,t) = u_3(x,0^+,t) - u_3(x,0^-,t), \qquad v(x,t) = \dot{\delta}(x,t),$$

and the shear traction on the plane is $\tau(x,t) = \sigma_{23}(x,0^\pm,t) = \mu\partial_y u_3(x,0^\pm,t)$.

Representation theorem (Somigliana identity, dislocation source).

With no body forces, the displacement produced by a displacement jump Δu_m on a (time-independent) surface Γ is

$$u_i(\mathbf{x},t) = \int_\Gamma \int_{-\infty}^{t} \Delta u_m(\xi,\tau)\, T_{im}(\mathbf{x},\xi,t-\tau)\, \mathrm{d}\tau\, \mathrm{d}S(\xi), \tag{1}$$

where $T_{im}(\mathbf{x},\xi,t) = \Sigma_{ij}(\mathbf{x},t;\xi)\, n_j(\xi)$ is the traction Green tensor (traction at $\mathbf{x}$ due to a unit impulsive point force at ξ in the m-direction), and n is the unit normal on Γ. Reciprocity implies $T_{im}(\mathbf{x},\xi,t) = T_{mi}(\xi,\mathbf{x},t)$.

For Mode III on $\Gamma = \{y=0\}$, only $i=m=3$ remains and $\mathrm{d}S \to \mathrm{d}\xi$:

$$u_3(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{t} \delta(\xi,\tau)\, T_{33}\big((x,y),(\xi,0),t-\tau\big)\, \mathrm{d}\tau\, \mathrm{d}\xi. \tag{2}$$

Taking $\mu\partial_y$ and the limit $y \to 0^+$ (traction is continuous):

$$\tau(x,t) = \mu\partial_y u_3(x,0^+,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{t} \delta(\xi,\tau)\, \underbrace{\mu\partial_y T_{33}\big((x,0^+),(\xi,0),t-\tau\big)}_{=:K_0(x-\xi,t-\tau)} \mathrm{d}\tau\, \mathrm{d}\xi. \tag{3}$$

For anti-plane SH in 2D, the free-space scalar Green function is

$$G_{33}(r,t) = \frac{\mathrm{H}\left(t - \frac{r}{c_s}\right)}{2\pi\mu} \frac{1}{\sqrt{t^2 - \frac{r^2}{c_s^2}}}, \qquad r = \sqrt{(x-\xi)^2 + y^2},$$

and the traction Green function on the plane (via reciprocity) is

$$T_{33}\big((x,0),(\xi,0),t\big) = -\frac{1}{\pi}\frac{\partial}{\partial x}\left[\frac{\mathrm{H}\left(t - \frac{|x-\xi|}{c_s}\right)}{\sqrt{c_s^2 t^2 - (x-\xi)^2}}\right]. \tag{4}$$

Substitute (4) into (3) and integrate by parts in ξ (boundary terms vanish for localized slip):

$$\tau(x,t) = -\frac{\mu}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{t} \partial_\xi \delta(\xi,\tau) \frac{\mathrm{H}\left(t - \tau - \frac{|x-\xi|}{c_s}\right)}{\sqrt{c_s^2(t-\tau)^2 - (x-\xi)^2}}\, \mathrm{d}\tau\, \mathrm{d}\xi. \tag{5}$$

Moving the time derivative from the kernel onto δ (or differentiating (5) w.r.t. t and using $\partial_t \delta = \dot{\delta}$) yields the clean causal form

$$\tau(x,t) = -\frac{\mu}{c_s}\dot{\delta}(x,t) + \tau_{\mathrm{hist}}(x,t), \tag{6}$$

$$\tau_{\mathrm{hist}}(x,t) = \frac{\mu}{\pi}\int_{-\infty}^{\infty}\int_{0}^{t-\frac{|x-\xi|}{c_s}} \frac{\partial_\xi \dot{\delta}(\xi,\tau)}{\sqrt{c_s^2(t-\tau)^2 - (x-\xi)^2}}\, \mathrm{d}\tau\, \mathrm{d}\xi, \tag{7}$$

where the upper limit enforces strict causality ($|x-\xi| \leq c_s(t-\tau)$). The short-time/short-distance singular part at the retarded front contributes the local term $-\mu/c_s\, v(x,t)$; the remainder τ_{hist} is a finite, strictly history-dependent convolution.

Equivalently,

$$\tau_{\text{hist}}(x,t) = \frac{\mu}{\pi}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{0}^{t} \frac{\partial_\xi \delta(\xi,\tau)\,\mathrm{H}\big(t-\tau-\frac{|x-\xi|}{c_s}\big)}{\sqrt{c_s^2(t-\tau)^2-(x-\xi)^2}}\,\mathrm{d}\tau\,\mathrm{d}\xi, \tag{8}$$

with the understanding that the instantaneous $-\mu/c_s\, v$ has been extracted from the wavefront singularity.

Remark. For slip against a rigid half-space, multiply the right-hand sides of (6)–(8) by $1/2$, so the radiation impedance is $\mu/(2c_s)$.

Radiation damping and causal history term (Mode III)

Consider anti–plane (Mode III) slip $\delta(x,t)$ on a fault along $y=0$ in an infinite, homogeneous, isotropic elastic medium with shear modulus μ and shear-wave speed c_s. The shear traction on the fault can be written as the sum of a background loading, an instantaneous "radiation damping" term, and a causal nonlocal history integral:

$$\tau(x,t) = \tau_{\text{back}}(x,t) - \frac{\mu}{c_s}\dot{\delta}(x,t) + \tau_{\text{hist}}(x,t), \tag{9}$$

$$\tau(x,t) - \tau_{\text{back}}(x,t) = \Delta\tau$$

$$\boxed{\frac{\Delta\tau}{\mu} = -\frac{\dot{\delta}(x,t)}{c_s} + \frac{\tau_{\text{hist}}(x,t)}{\mu}.}$$

Exercise: estimate slip velocity

1) neglecting $\tau_{\rm hist}(x,t)$ in

$$\frac{\Delta\tau}{\mu} = -\frac{\dot{\delta}(x,t)}{c_s} \quad + \frac{\tau_{\rm hist}(x,t)}{\mu}.$$

and using indicative stress drop, shear modulus and shear wave speed, estimate the slip velocity during an earthquake!!

Compare to the apparent slip velocity in the Myanmar earthquake video

2) Using Hooke's law of elasticity and kinematic considerations, re-derive the above equation without the τ_{hist}. Start from $\varepsilon_{xy} = \delta/h$ where δ is slip at time t and h is the wave propagation distance away from the fault at time t.

3) In what situations can τ_{hist} be neglected ?

4) Finite-difference simulation : introduce an instantaneous crack failing along L of about 64 grid nodes.

Two fundamental scaling in earthquake sources:

$$\frac{\Delta\tau}{\mu} \propto \frac{v}{c_s} \qquad (\longrightarrow \dot{\delta} \approx 1 \text{ m/s})$$

$$\frac{\Delta\tau}{\mu} \propto \frac{u}{L} \qquad (\longrightarrow \delta \approx 10^{-4} L)$$

Exact causal history kernel

For two identical half–spaces the elastodynamic representation theorem gives, for Mode III, the history term

$$\boxed{\tau_{\text{hist}}(x,t) = \frac{\mu}{\pi}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} d\xi \int_0^t d\tau \, \frac{\partial_\xi \delta(\xi,\tau)\, H\left(t-\tau-\frac{|x-\xi|}{c_s}\right)}{\sqrt{c_s^2(t-\tau)^2-(x-\xi)^2}}}, \tag{10}$$

where H is the Heaviside step function enforcing strict causality: only points (ξ,τ) inside the shear–wave cone $|x-\xi| \leq c_s(t-\tau)$ contribute.

Alternatively, in terms of the slip rate $v = \dot{\delta}$,

$$\boxed{\tau_{\text{hist}}(x,t) = \frac{\mu}{\pi}\int_{-\infty}^{\infty} d\xi \int_0^{t-\frac{|x-\xi|}{c_s}} d\tau \, \frac{\partial_\xi v(\xi,\tau)}{\sqrt{c_s^2(t-\tau)^2-(x-\xi)^2}}}. \tag{11}$$

After integrating by parts in ξ, assuming δ vanishes at infinity, one may also write

$$\boxed{\tau_{\text{hist}}(x,t) = -\frac{\mu}{\pi}\int_{-\infty}^{\infty} d\xi \, \partial_\xi \left[\int_0^{t-\frac{|x-\xi|}{c_s}} d\tau \, \frac{v(\xi,\tau)}{\sqrt{c_s^2(t-\tau)^2-(x-\xi)^2}}\right]}. \tag{12}$$

Emergence of the local radiation damping term

Equations (10)–(12) contain an inverse–square–root singularity as the retarded front $|x-\xi| = c_s(t-\tau)$ is approached. Taking the short–distance, short–time limit of the integral gives

$$\int_{|x-\xi|<c_s\varepsilon} \frac{\partial_\xi v(\xi, t-\varepsilon)}{\sqrt{c_s^2\varepsilon^2 - (x-\xi)^2}}\, d\xi \;\longrightarrow\; -\frac{\pi}{c_s} v(x,t),$$

which yields precisely the instantaneous term

$$-\frac{\mu}{\pi} \times \frac{\pi}{c_s} v(x,t) = -\frac{\mu}{c_s} v(x,t),$$

identifying it as the *radiation damping* contribution in Eq. (9).

Physical meaning

The coefficient

$$\eta_{\mathrm{III}} = \frac{\mu}{c_s} \tag{13}$$

is the *radiation impedance* (per unit area) for Mode III slip on an internal fault in identical half–spaces. The instantaneous traction–velocity relation

$$\tau_{\mathrm{rad}} = -\eta_{\mathrm{III}}\, v = -\frac{\mu}{c_s} v$$

corresponds to an energy flux

$$\mathcal{P}_{\mathrm{rad}} = (-\tau_{\mathrm{rad}})v = \frac{\mu}{c_s} v^2 \geq 0,$$

equal to the shear–wave energy radiated away from the fault.

For slip against a rigid boundary (one–sided radiation) the impedance and damping coefficient are halved: $\eta_{\mathrm{III}} = \mu/(2c_s)$.