Course Notes for Math 320: Fundamentals of Mathematics Chapter 2: Sets.

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1 Introdction to Sets

Definition 1.1. *A set A is a well defined collection of objects. If a is an element of A then we write $a \in A$.*

Remark 1.2. *Well defined means there is no ambiguity as to whether or not an object is in a given set.*

Example 1.3. *Which of the following are sets? How big are they? What are thier elements?*

1. *The collection of great novels.*
2. *The truth set for $x^2 + y^2 = 34$ for $x, y \in R$.*
3. *The collection of members of the math department.*
4. *The students in math 320.*
5. $\{x \in \mathbb{R} | e^x < x\}$.
6. *The collection of all twin primes.*
7. *The set of all natural numbers from 5 to 10.*
8. $\{5, 6, 7, 8, 9, 10\}$.

Remark 1.4. *Three ways to describe a set are below: Try each with example 8 above.*

1. *In words.*
2. *Listing elements.*
3. *Set builder notation.*

Example 1.5. *Translate the sets below into or out of the various forms.*

1. *The truth set for* $x^2 + y^2 = 34$ *for* $x, y \in R$.

2. $\{x \in \mathbb{R} | e^x < x\}$.

3. *Let* $\mathbb{P}$ *be the collection of all prime numbers. The collection of all twin primes.*

4. *The set of all natural numbers from 5 to 10.*

5. $\{5, 6, 7, 8, 9, 10\}$.

Example 1.6. *Further examples of sets:*

1. *Natural Numbers:* $\mathbb{N} = \{1, 2, 3, 4, ...\}$
2. *Integers:* $\mathbb{Z} = \{... - 2, -1, 0, 1, 2, 3, ...\}$.
3. *Positive Integers:* $\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, ...\}$.
4. *Strictly Positive Integers:* $\mathbb{Z}_{>0} =$

5. *Even Integers:* $2\mathbb{Z} =$

6. *Integers that are perfect squares*

7. *Rational Numbers:* $\mathbb{Q} = \{\frac{a}{b} | a, b \in \mathbb{Z}, b \neq 0\}$

8. *Real numbers: Defn???*
9. $\mathbb{R}^n = \{(x_1, x_2, ..., x_n) | x_i \in \mathbb{R}\}$.
10. *Matrices:* $M_{(r,s)}(\mathbb{R})$ *is the set of all* $r \times s$ *matrices with entries in* $\mathbb{R}$.

$$M_{(3,3)}(\mathbb{R}) = \left\{ \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} | x_{11}, x_{12}, ..., x_{32}, x_{33} \in \mathbb{R} \right\}.$$

11. *Upper triangular Matrices:* $U_{(3,3)}(\mathbb{R})$

12. *Polynomials over* $\mathbb{R}$*:* $\mathbb{R}[x] = \{a_0 + a_1x + a_2x^2 + ... + a_nx^n | a_i \in \mathbb{R}, n \in \mathbb{Z}_{\geq 0}\}$.

Definition 1.7. *Assume that* A *and* B *are sets.*

1. B *is a* **subset** *of* A*, if* $\forall\, b \in B$ *we have that* $b \in A$*. Notation:* $B \subset A$.
2. B *is* **equal** *to* A *if* $B \subset A$ *and* $A \subset B$*. Notation:* $B = A$.
3. *If* B *is a subset of* A *and* $B \neq A$ *then* B *is called a* **proper subset** *of* A*. Notation:*

Example 1.8. *Give some examples of subsets from the set examples above.*

Remark 1.9. *Is a "set" with no elements a set? How many such are there?*

AXIOM: We consider a collection of objects such that there are no objects in it to be a set and we call it the empty set. Notation: $\emptyset$.

Definition 1.10. *The number of elements in a finite set* A *is called its* **cardinality***. Notation:* $|A|$.

Remark 1.11. *If* A *is an infinite set, we can also define its cardinality, but this is trickier: Rank* $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ *in terms of containment.*

Proposition 1.12. *Let* A *be a set. Then* $\emptyset \subset A$ *and* $A \subset A$.

Proposition 1.13. *Let A, B, and C be sets. If $A \subset B$ and $B \subset C$ then $A \subset C$.*

Proof see book.

Proposition 1.14. *Let A, B, and C be sets. If $A \subset B$ and $B \subset C$ and $C \subset A$. Then $A = C$.*

Group Work

Definition 1.15. *Power Set: Given a set A the* **power set** *of A is the set whose elements are the subsets of A.*

1. Find the power set of $A = \{1, 2, 3, 4\}$.
2. What is the cardinality of $P(A)$?
3. Is there a set with exactly 12 subsets?
4. (FP 18) Consider the following three "proofs" of the conjecture that: If A and B are sets and $A \subset B$ then $P(A) \subset P(B)$. Which, if any are correct. Justify your answers.
   (a) "Proof 1": Let $x \in P(A)$. Then $x \in A$. Since $A \subset B$, $x \in B$. Therefore $x \in P(B)$, so $P(A) \subset P(B)$.
   (b) "Proof 2": Let $A = \{1, 2\}$ and let $B = \{1, 2, 3\}$. Then $P(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ and $P(B) =) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Therefore $P(A) \subset P(B)$.
   (c) "Proof 3": Let $x \in A$. Since $A \subset B$, $x \in B$.Since $x \in A$ and $x \in B$, $\{x\} \in P(A)$ and $\{x\} \in P(B)$. Therefore $P(A) \subset P(B)$.
5. (FP 19) Modify an incorrect proof in the previous exercise to obtain a correct proof of the conjecture in that exercise.

2 Operations on Sets

Universal Set of discourse.
Venn Diagrams vs. Proofs

Definition 2.1. *Assume that U is the universal set of discourse and that A and B are two sets (i.e. subsets of U).*

1. *The* **union** *of A and B is the set of all elements of U that belong to A or B. Notation: $A \cup B = \{x \in U | x \in A \text{ or } x \in B\}$.*

2. *The* **intersection** *of A and B is the set of all elements of U that belong to A and B. Notation: $A \cap B = \{x \in U | x \in A \text{ and } x \in B\}$.*

3. *The* **complement** *of A is the set of all elements of U that do not belong to A. Notation: $A' = \{x \in U | x \notin A\}$.*

4. *The* **complement of** B **relative to** A *is the set of all elements in A that are not in B. Notation: $B - A = \{x \in U | x \in B \text{ and } x \notin A\}$.*

$$
\begin{tikzcd}
& U \arrow[d, dash] & \\
& A \cup B \arrow[dl, dash] \arrow[dr, dash] & \\
A \arrow[dr, dash] & & B \arrow[dl, dash] \\
& A \cap B \arrow[d, dash] & \\
& \emptyset &
\end{tikzcd}
$$

Example 2.2. *1. Intervals:* $[0, 1) \cup [1, 2) =$

2. *Intervals:* $[0, 1) \cup (1, 2) =$

3. *Intervals:* $[0, 1] - [1, 2) =$

4. *Recall that* $n\mathbb{Z} = \{na | a \in \mathbb{Z}\}$. *What is* $2\mathbb{Z} \cap 3\mathbb{Z}$*?*

5. *What is* $2\mathbb{Z} \cup 3\mathbb{Z}$*?*

6. *In* $\mathbb{R}^2$ *show that the line* $x + y = -2$ *does not intersect with the circle* $x^2 + y^2 = 1$.

Theorem 2.3. *Let A, B, and C be sets.*

1. $\emptyset \cap A = \emptyset$ *and* $\emptyset \cup A = A$
2. $A \cap B \subset A$
3. $A \subset A \cup B$
4. $A \cup B = B \cup A$ *and* $A \cap B = B \cap A$
5. $A \cup (B \cup C) = (A \cup B) \cup C$ *and* $A \cap (B \cap C) = (A \cap B) \cap C$
6. $A \cup A = A = A \cap A$
7. *If* $A \subset B$ *then* $A \cup C \subset B \cup C$ *and* $A \cap C \subset B \cap C$
8. $(A')' = A$
9. $(A \cup B)' = A' \cap B'$
10. $(A \cap B)' = A' \cup B'$
11. $A - B = A \cap B'$
12. $A \subset B$ *if and only if* $B' \subset A'$
13. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *and* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

More space