Course Notes for Math 320: Fundamentals of Mathematics Chapter 4: Relations, Equivalence Relations and Congruence.

October 1, 2013

1 Relations

Definition 1.1. *(Kazimierz Kuratowski) Let S be a set and let a and b be members of S. The* **ordered pair** *(a, b) is the set $\{\{a\}, \{a, b\}\}$. The element a is called the* **first term** *of (a, b) and the element b is called the* **second term** *of (a, b).*

Lemma 1.2. *Let (a, b) and (c, d) be ordered pairs. Then $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.*

Definition 1.3.

1. *A* **relation** *is a collection of ordered pairs.*
2. *If R is a relation then the collection of all first terms is called the* **domain** *and the collection of all second terms is called the* **range**.
3. *If x is in the domain of R let $R[x] = \{y \in \text{range of } R \,|(x, y) \in R\}$.*
4. *If A and B are sets then $A \times B = \{(a, b)|a \in A \text{ and } b \in B\}$. This is a relation, called the* **Cartesian product**.

Example 1.4. *Find the domain A and range B and if possible express as $A \times B$.*

1. $$\begin{aligned} R_1 = \{&(Pietro, Giuseppe), (Pietro, Matteo), (Giuseppe, Pietro), (Giuseppe, Matteo),\\ &(Giuseppe, Marco), (Marco, Giuseppe), (Marco, Matteo), (Pietro, Enrico),\\ &(Marco, Enrico), (Giuseppe, Enrico), (Giuseppe, Giuseppe)\}.\end{aligned}$$

   What is $R_1[Giuseppe]$? What is $R_1[Pietro]$?

2. $R_2 = \{(1,\pi),(3,1),(5,1),(2,2),(\pi,1)\}$. *What is* $R_2[1]$*?*

3. $\mathbb{R}^2 = \{(x,y)|x,y \in \mathbb{R}\}$. *What is* $\mathbb{R}^2[1]$*?*

4. $R_3 = \{(x,y)|x,y \in R \text{ and } y = 2x\}$. *What is* $R_3[1]$*?*

5. $R_4 = \{(x,y)|x,y \in R \text{ and } y = x^2\}$. *What is* $R_3[1]$*?*

6. *Let* $A = \{a,b,c\}$ *and* $B = \{1,2\}$ *Find* $A \times B$.

7. *Let* $A = \{x|1 < x \leq 2\}$, $B = \{1,2,3\}$, *and* $C = \{x|2 < x < 3\}$. *Compute* $A \times B$, $B \times A$, $A \times C$, *and* $A \times \mathbb{R}$.

8. *Let* $A = \{1\}$*,* $B = \{3\}$*,* $C = \{4\}$ *and* $D = \{2\}$*. Compute* $(A \times B) \cup (C \times D)$ *and* $(A \cup C) \times (B \cup D)$*.*

Proposition 1.5. *If* A*,* B*,* C*, and* D *are sets then the following hold:*

1. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
2. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
3. $(A - B) \times B = (A \times B) - (B \times B)$ *(F&P Prop 4.2)*
4. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
5. $(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$.

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2 Graphs... ways of depicting relations

Graphs are a natural way of describing certain relations and motivating proofs.

1. $$\begin{aligned} R_1 = \{&(Pietro, Giuseppe), (Pietro, Matteo), (Giuseppe, Pietro), (Giuseppe, Matteo),\\ &(Giuseppe, Marco), (Marco, Giuseppe), (Marco, Matteo), (Pietro, Enrico),\\ &(Marco, Enrico), (Giuseppe, Enrico), (Giuseppe, Giuseppe)\}. \end{aligned}$$

2. $R_2 = \{(1,2),(3,1),(5,1),(2,2),(2,1)\}$.

3. $R_2 \cup R_2^{-1}$.

4. $\mathbb{R}^2 = \{(x,y)|x, y \in \mathbb{R}\}$.

5. $R_3 = \{(x,y)|x, y \in \mathbb{R}$ and $y = 2x\}$.

6. $R_4 = \{(x,y)|x, y \in \mathbb{R}$ and $y = x^2\}$.

CAUTION: Graphs $\neq$ Proofs. Having said that they can be informative.

Definition 2.1. *Let R be a relation. Consider the set:*

$$R^{-1} = \{(x, y) | (y, x) \in \mathbb{R}\}.$$

*This set is call the R-**inverse**. We call a relation* **symmetric** *if $R = R^{-1}$.*

Example 2.2. *Find R^{-1}. Which are symmetric?*

1. $$\begin{aligned} R_1 = \{&(Pietro, Giuseppe), (Pietro, Matteo), (Giuseppe, Pietro), (Giuseppe, Matteo), \\ &(Giuseppe, Marco), (Marco, Giuseppe), (Marco, Matteo), (Pietro, Enrico), \\ &(Marco, Enrico), (Giuseppe, Enrico), (Giuseppe, Giuseppe)\}. \end{aligned}$$

2. $R_2 = \{(1, 2), (3, 1), (5, 1), (2, 2), (2, 1)\}$.

3. $R_2 \cup R_2^{-1}$.

4. $\mathbb{R}^2 = \{(x, y) | x, y \in \mathbb{R}\}$.

5. $R_3 = \{(x, y) | x, y \in \mathbb{R}$ and $y = 2x\}$.

6. $R_4 = \{(x,y)|x,y \in \mathbb{R} \text{ and } y = x^2\}$.

7. *Let* $A = \{a,b,c\}$ *and* $B = \{1,2\}$ *Find* $A \times B$.

8. *Let* $A = \{x|1 < x \leq 2\}$, $B = \{1,2,3\}$, *and* $C = \{x|2 < x < 3\}$. *Compute* $A \times B$.

Definition 2.3. *Let R be a relation with domain A and range B.*

1. *The cartesian graph of R is the plot of points $\{(x,y) \in A \times B|(x,y) \in R\}$ on a set of axes with horizonal axis representing A and verticle axis representing B.*

2. *Suppose that $A, B \in S$. The directed graph of R is a collection points, randomly placed, each representing exacly one element of S and each element of S being represented. Then given points representing $x, y \in S$ we connect them with an arrow from x to y if $(x,y) \in R$.*

Remark 2.4. *If a relation R is symmetric what can you say about its cartesian or directed graphs?*

Proposition 2.5. *Let R and S be relations. Then each of the following statements is true:*

1. $R = (R^{-1})^{-1}$
2. $R \subset S$ *if and only if* $R^{-1} \subset S^{-1}$
3. $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$
4. $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$
5. $R \cup R^{-1}$ *is symmetric*
6. $R \cap R^{-1}$ *is symmetric.*

3 Equivalence Relations

Example 3.1. *Some more examples:*

1. $$\begin{aligned} R_1 = \{&(Pietro, Giuseppe), (Pietro, Matteo), (Giuseppe, Pietro), (Giuseppe, Matteo), \\ &(Giuseppe, Marco), (Marco, Giuseppe), (Marco, Matteo), (Pietro, Enrico), \\ &(Marco, Enrico), (Giuseppe, Enrico), (Giuseppe, Giuseppe)\}. \end{aligned}$$

2. *Consider the following relation R on $\mathbb{Z}$.*

$$R_2 = \{(x, y) | x, y \in \mathbb{Z} \text{ and } x = -y\}.$$

3. *Consider the following relation R on $\mathbb{R}$.*

$$R_3 = \{(x, y) | x, y \in \mathbb{R} \text{ and } x - y = n \text{ for some } n \in \mathbb{Z}_{\geq 0}\}.$$

4. *Consider the following relation R on $\mathbb{Z}$.*

$$R_4 = \{(x, y) | x, y \in \mathbb{Z} \text{ and } x - y \text{ is even } \}.$$

Definition 3.2. *Let R be a relation on a set S.*

1. *We call R* **reflexive** *on S if for every $x \in S$ we have $(x, x) \in R$.*
2. *We call R* **symmetric** *if whenever $(x, y) \in R$ then $(y, x) \in R$.*
3. *We call R* **transitive** *if whenever $(x, y), (y, z) \in R$ then $(x, z) \in R$.*

Remark 3.3. *Caution:*

1. *To show that a relation R is reflexive, we take ONE arbitrary element $x \in S$ and show that $(x, x) \in R$. What should the cartesian graph or directed graph of R look like if it is reflexive?*
2. *To show that a relation R is symmetric, we take TWO arbitrary elements $x, y \in S$, such that $(x, y) \in R$ and we use this to show that $(y, x) \in R$. What should the cartesian graph or directed graph of R look like if it is symmetric?*
3. *To show that a relation R is transitive, we take THREE arbitrary elements $x, y, z \in S$ such that $(x, y), (y, z) \in R$ and use this to show that $(x, z) \in R$.What should the cartesian graph or directed graph of R look like if it is transitive?*

Example 3.4. *Which of the above is reflexive, symmetric, and/or transitive?*

Definition 3.5. *A relation that is reflexive, symmetric, AND transitive is called an equivalence relation.*

Example 3.6. *Which of the above is an equivalence relation?*

Remark 3.7. *Recall that given a relation R on a set S then $[x] = \{y \in S | (x, y) \in R\}$. Or in the 'tilde' notation: $[x] = \{y \in S | x \sim y\}$. Now we can rephrase the requirements to be an equivalence relation interms of this representative $[x]$.*

1. *For each $x \in S$...*
2. *If $x \in [y]$ then...*
3. *If $y \in [x]$ and ...*

Definition 3.8. *Given an equivalence relation R on a set S and an element $x \in S$, the set $[x]$ is called the* **equivalence class** *of x.*

Group Work problems for Relations, graphs, and equivalence relations:

1. Let $R = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a|b\ \}$. List five members of $R[7]$, and list five members of $R[14]$. For which $n \in \mathbb{N}$ is it true that $R[n] = \mathbb{N}$? Is R symmetric? Is it an equivalence relation?

2. Let $R = \{(x, y \in \mathbb{R} \times \mathbb{R} | x^2 - y^2 = 0\}$.

   (a) Prove that R is an equivalence relation on $\mathbb{R}$.

   (b) List all the elements of [3].

3. Let $S = \mathbb{N} \times \mathbb{N}$. For any two members (a, b) and (c, d) of S we will define a relation $(a, b) \sim (c, d)$ if $ad = bc$. Prove that $\sim$ is an equivalence relation.

4. Let $A = \{1, 2, 3, 4, 5\}$. Draw the directed graphs associated with each of the following relations on A. Which relations are equivalence relations?

   (a) $R = \{(a, b) \in A \times A \mid a|b\}$

   (b) $R = \{(a, b) \in A \times A \mid a \neq b\}$

   (c) $R = \{(a, b) \in A \times A \mid a + b \in 2\mathbb{Z}\}$

   (d) $R = \{(a, b) \in A \times A \mid a + b \text{ odd }\}$

5. Let R be a relation such that $R^{-1} \subset R$. Must R be symmetric? Prove your answer.

4 Congruence or clock arithmetic: Example of equivalence relation

Example 4.1. *Consider the following relation R on $\mathbb{Z}$.*

$$R_n = \{(x, y) | x, y \in \mathbb{Z} \text{ and } x - y = nk \text{ for some } k \in \mathbb{Z}\}.$$

Group Work

1. Let p be an odd prime and let $S = \{[1], [2], ..., [p-1]\}$ be the set of nonzero equivalence classes mod p.

   (a) Prove that for any integer a such that $[a] \in S$, we have that $[-a] \in S$ and that $[-a] \neq [a]$.

   (b) Prove that if $[a] \in S$ and $[b] \in S$, then $[a] \otimes [b] \in S$.

   (c) Prove that if $[a]^2 = [b]^2$ then $[a] = [b]$ or $[a] = -[b]$.

   (d) Prove that exactly $(p-1)/2$ members of S have square roots (i.e. $[a]$ has a square root provided that there is an integer x such that $[x]^2 = [a]$).
   Hint. First show that there are at most $(p-1)/2$ elements of S with square roots by using that $[x]^2 = [-x]^2$. Then consider the following subset of S: $\{[1]^2, [2]^2, ..., [(p-1)/2]^2\}$. Show that no two of these can be equal using above parts. This shows that there are at least $(p-1)/2$ elements of S with square roots.

2. In the previous problem, instead of taking p to be an odd prime number, take $p = 16$. Show by example that all four statements are then false.