# Foundations of Higher Mathematics
# Homework 10

1. (a) $f(x) = 3x - 5$
   **Injective**: Let $f(x) = f(y)$. Then $3x - 5 = 3y - 5$. It follows that:
   $$\begin{aligned} 3x &= 3y \\ x &= y \end{aligned}$$
   **Surjective**: Let $y = f(x)$. It follows that $y = 3x - 5$. Thus:
   $$\begin{aligned} y &= 3x - 5 \\ y - 5 &= 3x \\ \frac{y-5}{3} &= x \end{aligned}$$
   So, $f(\frac{y-5}{3}) = y$ is a solution.

   (b) $f(x) = x^2 + 5$
   **Not injective**: Let $a = 1$ and $b = -1$. We can see that $a \neq b$, but $f(a) = 1^2 + 5 = 6 = f(b)$.
   **Not surjective**: Take $f(x) = 0$.
   $$\begin{aligned} x^2 + 5 &= 0 \\ x^2 &= -5 \end{aligned}$$
   Since $x$ has no solutions in $\mathbb{R}$, $f$ is not a surjection.

   (c) $f(x) = x^2 - 5x - 6$
   **Not injective**: Let $a = 6$ and $b = -1$. Clearly $a \neq b$, but:
   $$\begin{aligned} f(x) &= x^2 - 5x - 6 \\ &= (x - 6)(x + 1) \end{aligned}$$
   Thus, $f(6) = 0 = f(-1)$.

   **Not Surjective**: Notice that $f'(x) = 2x - 5$. There is a critical point at $x = \frac{5}{2}$. (Minimum) It follows that $f(\frac{5}{2}) = -\frac{49}{4}$. Any $y$ value less than this value is impossible, and hence does not have a corresponding $x$ value. Thus, $f$ is not surjective.

   (d) $f(x) = x^3 - 5$
   **Injective**: Let $f(x) = f(y)$ it follows that:
   $$\begin{aligned} x^3 - 5 &= y^3 - 5 \\ x^3 &= y^3 \\ x &= y \end{aligned}$$
   **Surjective**: Let $y = f(x)$. It follows that $y = x^3 - 5$, so a solution is $x = \sqrt[3]{y + 5}$

   (e) $f(x) = x^3 - x$
   **Not injective**. Let $x = 1$ and $y = -1$. We can see that $x \neq y$, but $f(x) = 1 - 1 = 0$ and $f(y) = -1 + 1 = 0$. Thus, $f$ is not an injection.
   **Surjective** I graphed the function, and the function is continuous, and extends to all value of $y$.

2. (F&P #4)

*Proof.* Assume $f$ is strictly increasing. We want to show $f$ is one-to-one. Take an arbitary $x$ and $y$, $x \neq y$. Assume $x$ is the smaller value. Since $f$ is strictly increasing and $x < y$, it follows that $f(x) < f(y)$. If $y$ is the smaller value, swapping $x$ and $y$ in the previous argument we see that $f(y) < f(x)$. Thus, $f(x) \neq f(y)$. So $f$ is a one-to-one function.

Now we want to show that $f^{-1}$ is strictly increasing. Let $a$ and $b$ be arbitrary elements of $\mathbb{R}$ such that $f(a) < f(b)$ It follows that $a < b$ from the fact that $f$ is strictly increasing. Assume $f^{-1}(c) < f^{-1}(d)$. We can see that $f(f^{-1}(c)) = c$, and $f(f^{-1}(d)) = d$. It follows that $c < d$, since $f$ is strictly increasing. Therefore, $f^{-1}$ is strictly increasing. ■

3. (F&P #7)

(a) One-to-one but not onto: $f(n) = 5n$

(b) Onto but not one-to-one: $f(n) = \begin{cases} n & \text{if n is even} \\ n-1 & \text{if n is odd} \end{cases}$

(c) Neither onto nor one-to-one: $f(n) = n \mod 2$

(d) $X = \{1, 2, 3, 4\}$
There is *not* a function that is one-to-one that does not map $X$ onto $X$ because one-to-one implies a unique pairing of elements in the domain to the codomain. Since $X$ is mapping to itself, every element will be mapped onto. There is *not* a function that maps X onto X that is not one-to-one from similar logic. Every element in the range must be mapped onto the same set.

4. (F&P #73)

(a) Prove that if $y \in B$, then $f^{-1}(y) \subseteq f^{-1}(B)$.

*Proof.* Assume $y \in B$. It follows that $y \in Y$ because $B \subseteq Y$. Also assume that $x \in f^{-1}(y)$. Want to show $x \in f^{-1}(B)$ (or $f(x) \in B$). It follows that $f(x) \in \{y\}$. Thus, $f(x) = y$. It follows that $f(x) = y \in B$, or $x \in f^{-1}(B)$. ■

(b) Prove that if $f(x) \in f(A)$ and $f$ is a one-to-one map, then $x \in A$.

*Proof.* Assume $f(x) \in f(A)$ and $f$ is injective. By Theorem 5.11 (part d), $A = f^{-1}(f(A))$, since $f$ is one-to-one. It follows that $x \subseteq f^{-1}(f(x))$. So, since $f(x) \in f(A)$, and $A = f^{-1}(f(A))$, $x \in A$. ■

5. $Tr : M_2(\mathbb{R}) \to \mathbb{R}$

**Non-Injective**: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. $A \neq B$ but $Tr(A) = 2 = Tr(B)$ so $Tr$ is not injective.

**Surjective**: Let y be an arbitrary element in $\mathbb{R}$ such that $y = Tr(A)$. It follows that $y = a_{11} + a_{22}$. One solution is, $A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$. So $Tr$ is surjective.

6. **Non-Injective**: Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$. $A \neq B$ but $Det(A) = 0 = Det(B)$ so $Det$ is not injective.
**Surjective**: Assume $y \in \mathbb{R}$ such that, $y = Det(A)$. By definition, a solution to $y = Det(A)$ is

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Where $y = a_{11}a_{22} - a_{21}a_{12}$.

7. **Not injective**: Let $A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$ $A \neq B$, but $Char(A) = x^2 - 0x + 0 = Char(B)$, so $Char$ is not injective.

   **Surjective?:** Take $p \in P_2$ such that $p = ax^2 + bx + c$. We want to see if every solution in $P_2$ has a corresponding element in $M_2(\mathbb{R})$.

$$ax^2 + bx + c = x^2 - Tr(A)x + Det(A)$$
$$(a-1)x^2 + (b + Tr(A))x + c - Det(A) = 0$$

   Then, to solve, we see if there are any solutions from part of the quadratic formula:

$$\sqrt{b^2 - 4ac}$$
$$= \sqrt{(b + Tr(A))^2 - 4(a-1)(c - Det(A))}$$

   I assume that for some $A$ this works... I may have just overlooked a simple counter example.

8. *Proof.* Assume $g \circ f$ is injective. We want to show that $f$ is injective. Assume an arbitrary $a, b \in X$ such that $f(a) = f(b)$. Let's define $y = f(a) = f(b)$. Now we want to show that $a = b$. We know that $g \circ f$ is injective, so $g(f(a)) = g(f(b))$. It follows that $g(y) = g(y)$. Since $g \circ f$ in an injection, we can conclude $a = b$. Therefore, $f$ is injective. ∎