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(1.4-1.5) 63, 66, 70, 72, 75 Math 320
(2.2) 24, 29a, c, d, 30, 31 9/18/13

Foundations of Higher Mathematics
Homework 3

Chapter 1

Problem 63

Let A be an even integer and B be an odd integer. Prove that A + B is odd and AB is even.

Proof. If A is even and B is odd, then there exists integers k, n, such that $A = 2q$ and $B = 2k + 1$. It follows that,

$$\begin{aligned} A + B &= 2q + (2k + 1) \\ &= (2q + 2k) + 1 \end{aligned}$$

Thus, since $2q + 2k \in \mathbb{Z}$, A + B is odd. Similarly for AB,

$$\begin{aligned} AB &= 2q(2k + 1) \\ &= 4qk + 2q \\ &= 2(2qk + q) \end{aligned}$$

it follows that since $2qk + q$ is an integer, AB is even. ■

Problem 66

Let x be a real number. Prove $x = -1$ if and only if $x^3 + x^2 + x + 1 = 0$

Proof. $(\rightarrow)$
Assume $x = -1$. Then

$$\begin{aligned} x^3 + x^2 + x + 1 &= 0 \\ (-1)^3 + (-1)^2 + (-1) + 1 &= 0 \\ -1 + 1 - 1 + 1 &= 0 \\ 0 &= 0 \end{aligned}$$

Thus, $x = -1$ implies $x^3 + x^2 + x + 1 = 0$.

Proof. $(\leftarrow)$ Assume $x^3 + x^2 + x + 1 = 0$. Then

$$\begin{aligned} x^3 + x^2 + x + 1 &= 0 \\ (x^3 + x^2) + (x + 1) &= 0 \\ x^2(x + 1) + (x + 1) &= 0 \\ (x + 1)(x^2 + 1) &= 0 \end{aligned}$$

Since $x \in \mathbb{R}$ the only solution is $x = -1$. Therefore, $x^3 + x^2 + x + 1 = 0$ implies $x = -1$.

Proving both directions, we can conclude that $x = -1$ if and only if $x^3 + x^2 + x + 1 = 0$ ■

Problem 70

Let n be an integer such that n^2 is even. Prove that n^2 is divisible by 4.

Proof. Assume n^2 is even. Then by theorem 1.12 in the book, n is even. Since n is even, there exists a k such that, $n = 2k$. It follows that,

$$\begin{aligned} n^2 &= 2k \times 2k \\ &= 4k^2 \\ &= 4(k^2) \end{aligned}$$

Since k^2 is an integer, 4 divides n^2. ■

Problem 72

The student's proof is correct. To prove that a number divides another we must only show that some solution exists. So the approach they used to solve is logically correct.

Problem 75

Prove, by contradiction, that the sum of two even integers is even.

Proof. (Contradiction): Assume the sum of two even integers is odd. Then there exist integers k, q such that, 2k + 2q = 2(k + q). This is a contradiction, since we said the sum was odd, so therefore the sum of two even integers is even. ■

Chapter 2

Problem 24

Prove that $A \cap B = A$ if and only if $A \subseteq B$.

Proof. ($\rightarrow$) Assume $A \cap B = A$. Then $A \subseteq A \cap B$. If we take an arbitrary element x of A, then x is in A and B. It follows that every element in A is also in B. Thus, $A \subseteq B$. Therefore, $A \cap B = A$ implies $A \subseteq B$.

Proof. ($\leftarrow$) Assume $A \subseteq B$. Suppose x is an arbitrary element of A. Then x is in B. So if x is in A and x is in B, then for all x that are in A and B, x is in A. Thus, $A \cap B \subseteq A$. Similarly, $A \subseteq A \cap B$ which implies that $A \cap B = A$. Therefore, $A \subseteq B$ implies $A \cap B = A$.

Therefore, $A \cap B = A$ if and only if $A \subseteq B$ ■

Problem 29a

$\varnothing \cap A = \varnothing$ and $\varnothing \cup A = A$

Proof. ($\varnothing \cap A = \varnothing$) Assume an arbitrary x in $\varnothing$ and A. There are no elements that are in $\varnothing$ and A so it is (vacuously) true that x $\in \varnothing$. The statement $\varnothing \subseteq \varnothing \cap A$ follows directly from the fact the empty set is a subset of every set. Therefore $\varnothing \cap A = \varnothing$. ■

Proof. ($\varnothing \cup A = A$) Assume x is in $\varnothing$ or A. If x in A, $A \subseteq A$ from *proposition 2.2* in the book. If x in A, $\varnothing \subseteq A$ since $\varnothing$ is a subset of every set. Also, if x in A, then x in A or $\varnothing$. Thus, $\varnothing \cup A \subseteq A$ and $A \subseteq \varnothing \cup A$. Therefore, $\varnothing \cup A = A$. ■

Problem 29c

$A \subseteq A \cup B$

Proof. Assume x is an arbirary element of A. Then x is in A or x is in B. Therefore, $A \subseteq A \cup B$. ■

Problem 29d

$A \cup B = B \cup A$ and $A \cap B = B \cap A$

Proof. $(A \cup B = B \cup A)$
Suppose x is and element of $A \cup B$. Then x is in A or x is in B. Thus x is in B or x is in A. Hence, $A \cup B \subseteq B \cup A$ and similarly $B \cup A \subseteq A \cup B$. Therefore, $A \cup B = B \cup A$. ■

Proof. $(A \cap B = B \cap A)$
The proof is the same as above, replacing the words *or* for *and.* ■

Problem 30

Counterexample: $P = \varnothing, Q = \varnothing, R = \{1\}$

$$\begin{gathered}(P \cap Q) \cup R = P \cap (Q \cup R)\\ (\varnothing \cap \varnothing) \cup \{1\} = \varnothing \cap (\varnothing \cup \{1\})\\ \varnothing \cup \{1\} = \varnothing \cap \{1\}\\ \{1\} \neq \varnothing\end{gathered}$$

Problem 31

The student's proof was **wrong.** The mistake he or she made was in the step, $x \in A$ and $x \notin B$ or $x \notin C$. The statement should look like, $x \in A$ and $x \notin B$ **and** $x \notin C$.