Course Notes for Math 320:
Fundamentals of Mathematics
Chapter 1: Generalities on proofs.

August 12, 2013

1 Propositions, Truth

Definition 1.1 *A **proposition** is a sentence that is either true or false.*

Example 1.2 *Which one is a proposition?*

1. *This summer I spent a month in Cortina, Italy.*

2. *Are Nilla wafers your favorite cookie?.*

3. *Nilla wafers are the best cookies.*

4. $5^2 + 3^2 = 34$.

5. $5^2 + 3^2 = 35$.

6. $5^2 + 3^2 < 35$.

7. $x + 3$.

8. $x + 3 > 5$.

9. *If Alice is registered for math 320 then she must have received a C or better in math 150B (or its equivalent).*

Making new propositions out of old ones: Negation, conjunction (and), disjunction (or), implication (if, then).

Definition 1.3 *Given a proposition P, the further proposition that P is false is called the **negation** of P (or NotP).*

Example 1.4 *Negate the following:*

1. *This summer I spent a month in Cortina, Italy.*

2. $5^2 + 3^2 = 34$.

3. $5^2 + 3^2 = 35$.

4. $5^2 + 3^2 < 35$.

5. If Alice is registered for math 320 then she must have received a C or better in math 150B (or its equivalent). [Hmm... maybe leave this for later...]

Notice that if the statement P is true, then the statement Not$P$ is false and if the statement P is false then the statement Not$P$ is true. We summarize this in a "Truth Table:"

P	Not P
T	F
F	T

Definition 1.5 *A proposition of the form "P and Q" for two propositions P and Q is called the* **conjunction** *of P and Q. Notation: $P \cap Q$.*

Definition 1.6 *A proposition of the form "P or Q" for two propositions P and Q is called the* **disjunction** *of P and Q. Notation: $P \cup Q$.*

Example 1.7 *Using common sense, which of the following would you say are true?*

1. *This summer I spent a month in Cortina, Italy and $5^2 + 3^2 = 34$.*

2. *This summer I spent a month in Cortina, Italy and $5^2 + 3^2 = 35$.*

3. *$5^2 + 3^2 = 35$ and $5^2 + 3^2 = 34$.*

4. *$5^2 + 3^2 = 35$ and $5^2 + 3^2 > 35$.*

Example 1.8 *Using common sense, which of the following would you say are true?*

1. *This summer I spent a month in Cortina, Italy or $5^2 + 3^2 = 34$.*

2. *This summer I spent a month in Cortina, Italy or $5^2 + 3^2 = 35$.*

3. *$5^2 + 3^2 = 35$ or $5^2 + 3^2 = 34$.*

4. *$5^2 + 3^2 = 35$ or $5^2 + 3^2 > 35$.*

So "P and Q" is true only when *both* P and Q are true, and "P or Q" is true when either P or Q is true. Again this can be summarized a truth table.

P	Q	P and Q	P or Q
T	T	T	
T	F		T
F	T		T
F	F	F	

Definition 1.9 *Given two propositions P and Q, the proposition "P **implies** Q" is called a* **conditional** *proposition. Notation $P \Rightarrow Q$.*

Example 1.10 *Conditional propositions:*

1. *If Giuseppe is nice to Matteo then he will have an icecream.*

2. *Giuseppe is in his carseat implies he is dressed.*
   Or equivalently, if Giuseppe is in his carseat then he is dressed.

3. *If Giuseppe is dressed then he is in his carseat.*

4. *If Sam is allowed to register for math 320 then he received a C or better in math 150B (or its equivalent).*

The validity of an expression "P implies Q" is defined by the following truth table.

P	Q	P implies Q
T	T	T
T	F	F
F	T	T
F	F	T

In this case P is called the **hypothesis** and Q is called the **conclusion**. Equivalent expressions (see pg 5 of book) "If P, then Q" or "Q is true whenever P is true."

Example 1.11 *Which of the following are true?*

1. *If I spent a month in Cortina, Italy this summer, then $5^2 + 3^2 = 34$.*

2. *If I spent a month in Cortina, Italy this summer, then $5^2 + 3^2 = 35$.*

3. *If $5^2 + 3^2 = 35$, then $5^2 + 3^2 = 34$.*

4. If $5 + 3 = 7$, then $5^2 + 3^2 > 35$.

Example 1.12 *Mount Holyoke College (MHC) is a women's college. Assume Mary is a woman and a student at MHC. Assume also that Sam is not a student at MHC and that Sam is a man.*

1. *If Mary is a student at MHC then Mary is a woman.*

2. *If Sam is a student at MHC then Sam is a man.*

3. *If a student is a woman, then the student attends MHC.*

Definition 1.13 *Given two propositions P and Q, the* **converse** *to the proposition "P implies Q" is "Q implies P."*

Example 1.14 *Write the converse to the following. Which do your intuition tell you are true?*

1. *If Giuseppe is in his carseat then he is dressed.*

2. *To register for math 320, it is necessary that a student received a C or better in math 150B (or its equivalent).*

3. *If you are over 21 years old you may legally drink alcohol.*

Thus the truth or falsehood of the converse to the expression "P implies Q" is defined by the following truth table.

P	Q	P implies Q	Q implies P
T	T	T	
T	F	F	
F	T	T	
F	F	T	

Definition 1.15 *Given two propositions P and Q, the proposition "P* **if and only if** *Q" is true when both "P implies Q" and "Q implies P" are true. Notation $P \Leftrightarrow Q$.*

Example 1.16 *Which do your intuition tell you are true?*

1. *Giuseppe is in class $\Leftrightarrow$ he is dressed.*

2. *A student takes math 320 if and only if the student received a C or better in math 150B (or its equivalent).*

3. *You may legally drink alcohol if and only if you are over 21 years old.*

Example 1.17 *Which of the following are true?*

1. *Dr. Stevenson is italian $\Leftrightarrow 5^2 + 3^2 = 34$.*

2. *Dr. Stevenson is italian if and only if $5^2 + 3^2 = 35$.*

3. *$5^2 + 3^2 = 35$ if and only if $5^2 + 3^2 = 34$.*

4. *$5 + 3 = 7$ if and only if $5^2 + 3^2 > 35$.*

Definition 1.18 *Given two propositions P and Q, the* **contrapositive** *to "P implies Q" is the proposition "NOTQ implies NOTP."*

Example 1.19 *Find the contrapositive for each of the following. Which are true?*

1. *If Giuseppe is in class then he is dressed.*

2. *If Sam is allowed to register for math 320 then he received a C or better in math 150B (or its equivalent).*

3. *Dr. Stevenson is italian then $5^2 + 3^2 = 34$.*

4. *Dr. Stevenson is italian then $5^2 + 3^2 = 35$.*

5. *If $5^2 + 3^2 = 35$, then $5^2 + 3^2 = 34$.*

6. *If $5 + 3 = 7$, then $5^2 + 3^2 > 35$.*

P	Q	P implies Q	Not Q	Not P	$(Not\ Q \Rightarrow Not\ P)$	$(P \Rightarrow Q) \Leftrightarrow (Not\ Q \Rightarrow Not\ P)$
T	T	T	F			
T	F	F	T			
F	T	T	F			
F	F	T	T			

Definition 1.20 *A propositional expression is called a* **tautology** *if it yeilds a true proposition regardless of what propositions replace its variable expressions.*
A propositional expression is called a **contradiction** *if it yeilds a false proposition regardless of what propositions replace its variable expressions.*

Example 1.21 $(P \Rightarrow Q) \Leftrightarrow (Not\ Q \Rightarrow Not\ P)$. *We say that these expressions are logically equivalent.*

This can be useful when proving an if and only if statement, like

Example 1.22 *"You may legally drink if and only if you are over 21 years old."*

1. $(Q \Rightarrow P)$

2. $(Not\ Q \Rightarrow Not\ P)$;

Sometimes rephrasing things makes the truth or falsity of a statement jumps out. A few more useful logical equivalences (or tautologies) are De Morgan's Laws.

Proposition 1.23 *Each of the following propositional expressions is a tautology.*

1. $Not(P \Rightarrow Q) \Leftrightarrow P \cap Not\ Q$.

2. $Not(P \cup Q) \Leftrightarrow Not\ P \cap Not\ Q$.

3. $Not(P \cap Q) \Leftrightarrow Not\ P \cup Not\ Q$.

4. $Not(Not\ P) \Leftrightarrow P$.

5. $(P \cup Q) \Leftrightarrow (NOT\ P \Rightarrow Q)$.

These are mostly of use in negating complicated expressions.

Example 1.24 *Negate the following:*

1. *If Giuseppe is in class then he is dressed.*

2. *If $5 + 3 = 8$ then $f(5) + f(3) = f(8)$.*

3. *If $ab = 0$ then $a = 0$ or $b = 0$.*

2   Constants, Variables, and Quantifiers

Two types of objects: **constants** and **variables**.
Both live in some **universe**... that will have to be defined in each situation.

Example 2.1 $x + 3 > 5$
This is not a propostion.
Here 3 and 5 are a constants, x is a variable.
Where does x come from? Which "universe" of numbers? Real numbers ($\mathbf{R}$), the rational
numbers ($\mathbf{Q}$) or just integers ($\mathbf{Z}$) .

Example 2.2 $x + 3 > 5$ *for some real number x. Is this a proposition? Is it true?*

Definition 2.3 *Given a universe and a variable x, a **Propositional function** $P(x)$ in the*
variable x is a sentence such that for each value of x taken in the universe we obtain a
*proposition. The **truth set** of $P(x)$ is the set of all values x in the given universe such that*
$P(x)$ is a true proposition.

Example 2.4 *Find the truth set of each of the following propositional functions. Assume*
that all variables and constants are real number unless otherwise stated.

  1. $x + 3 > 5$

  2. $x^2 + y^2 = 4$

  3. $x^3 + y^3 = z^3$ *where none of x, y and z are zero.*

  4. *Let the universe be the set of all CSUN students. Let X be one such student, X got a*
     C or better in math 150B.

Notation:

Quantifiers to Propositional functions:

Example 2.5 *Are the following **Propositions** true or false?*

  1. **There exist** *real numbers x such that $x + 3 > 5$*

  2. **For all** *pairs of real number (x, y), $x^2 + y^2 = 4$.*

3. **There are no** *integer solutions to $x^3 + y^3 = z^3$ where none of x, y and z are zero.*

4. *Let the universe be the set of all CSUN.* **There exists** X *who got a C or better in math 150B.*

Caution: What is the difference between "there exists" and "for all?"

Example 2.6 *What are the differences between the following propositions?*

1. *All students in math 320 get an A.*

2. *There exists a student in math 320 who gets an A.*

3. *For all real numbers x, $x^2 > 0$.*

4. *There exists a real number x, such that $x^2 > 0$.*

   For 5-8, recall that a prime number is a positive integer $p > 1$ such that the only positive integers that divide it are 1 and p. For example $2, 3, 5, 7$ and 11 are primes, but $9 = (3)(3)$ is not.

5. *For any prime number p, the number $p + 2$ is a prime number.*

6. *For some prime number p, the number $p + 2$ is a prime number.*

7. *There exists a prime number p such that $p + 1$ is prime.*

8. *There exists an odd prime number p such that $p + 1$ is prime.*

Definition 2.7 *Quantifiers*

1. *In a proposition any expression that indicates the existence of at least one element lying in the truth set of a propositional funtion is called an* **existential quantifier** *and it can be represented by* $\exists$. **(Rmk: $\exists x \in U$ is always followed by the expression (maybe just implied) "such that".)**

2. *Any expression that indicates that all elements in the given universe lie in the truth set of a propositional function is called a* **universal quantifier** *and can be represented by* $\forall$.

Example 2.8 *Translate either into the $\forall$ or $\exists$ notation or out of it. Which are true? For those that are false state the negation.*

1. $\forall x \in \mathbf{R}, x^4 > x$.

2. *Let* $\mathbf{P}$ *be the set of all prime numbers.* $\forall p \in \mathbf{P}$, $p + 2 \in \mathbf{P}$.

3. *For some prime numbers p, p + 2 is a prime number.*

4. *There exists a prime number p such that p + 1 is prime.*

Example 2.9 *The quantifiers may be hidden and these propositions can be more compli-cated. Two of the statements below are false... which ones, and can you prove it?*

1. *If x is a real number then $e^x > x$.*

2. *For a positive integer to be prime it is sufficient that it is odd.*

3. *Some Mammals can fly.*

4. *Some mammals have gills.*

5. *(Continuity of the function e^x at $x = 2$.) $\forall \epsilon > 0, \exists \delta > 0$, such that if $|x - 2| < \delta$ then $|e^x - e^2| < \epsilon$.*

6. *Given a set of linearly independent vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n \in \mathbf{R}^n$. $\forall \vec{v} \in \mathrm{Span}(\vec{v}_1, \vec{v}_2, ..., \vec{v}_n)$, $\exists$ unique $a_1, ..., a_n \in \mathbf{R}$ such that $\vec{v} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + ... + a_n \vec{v}_n$.*

Proposition 2.10 *If $P(x)$ is a propositional function with variable x, then the following indicates how to negate an $\forall$ or $\exists$ expression.*

1. *$Not\,((\forall x)P(x)) \Leftrightarrow (\exists x)\,(Not P(x))$ (counter-example);*

2. *$Not\,((\exists x)P(x)) \Leftrightarrow (\forall x)\,(Not P(x))$.*

Example 2.11 *Negate the following:*

1. *For some prime numbers p, p + 2 is a prime number.*

2. *Let $\mathbf{P}$ be the set of all prime numbers. $\forall p \in \mathbf{P}$, $p + 2 \in \mathbf{P}$.*

Quantifiers and Defintions.

Recall (?) the following definitions:

Def: A function f is said to be continuous at a value $x = x_0$ if $\forall \epsilon > 0, \exists \delta > 0$, such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$.

Def: Given a set of vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n \in \mathbf{R}^n$, another vector $\vec{v}$ lies in $\mathrm{Span}(\vec{v}_1, \vec{v}_2, ..., \vec{v}_n)$ if there exist $a_1, a_2, ..., a_n \in \mathbf{R}$ such that $\vec{v} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + ... + a_n \vec{v}_n$.

Example 2.12 *Group Work: How propositions and definitions interact.*

1. *What would you have to show in order to say that a function $g(x)$ is continuous at a point x_0?*

2. *What would you have to show in order to say that a vector $\vec{v}$ lies in the $\mathrm{Span}(\vec{v}_1, \vec{v}_2, ..., \vec{v}_n)$?*

3. *State what it would mean for a function $g(x)$ to fail to be continuous at a point x_0.*

4. *State what it would mean a vector $\vec{v}$ to fail to lie in the $\mathrm{Span}(\vec{v}_1, \vec{v}_2, ..., \vec{v}_n)$?*

3 Direct, Contrapositive, Contradition Proofs

Direct Proof of $(P \Rightarrow Q)$	Contrapositive Proof $(P \Rightarrow Q)$	Contradiction Proof $(P \Rightarrow Q)$
Assume P	Assume *Not* Q	Assume P and *Not* Q
Logical sequence of steps	Logical sequence of steps	Logical sequence of steps
Conclude Q.	Conclude *Not* P	Conclude Propositions R and *Not* R

4 Group Work:

Definition 4.1 *Let a and b be integers.*

1. *We say that a divides b if $b = ac$ for some positive integer c. We sometimes denote this by $a|b$.*

2. *We say that b is even if there is an integer k such that $b = 2k$ (i.e. 2 divides b).*

3. *We say that b is odd if there is an integer k such that $b = 2k + 1$.*

From lecture and book:

Theorem 4.2 *An integer is even if and only if n^2 is even.*

Warm-up: Prove the following:

1. Lemma 1: Let m and n be odd integers, then mn is an odd integer.
   Hint: write m and n out as odd integers using the definition above.

2. Lemma 2: If b is odd then b^3 is odd.

3. Lemma 3: If b^3 is even then b is even.

Group Exercise:

1. (FP 1.74) In this exercise we give both a "proof" and a "counterexample" to the conjecture: If x, y, and z are natural numbers such that xz divides yz, then x divides y. Is the "proof" correct? If not, why not? Is the "counter-example correct? If not, why not?

   "Proof:" Suppose x, y, and z are natural number such that xz divides yz but x does not divide y. Then y divides x, so there is an integer p such that $x = py$. Thus $xz = (py)z = p(yz)$. There fore yz divides xz, and this is a contradiction because xz divides yz.

   "Counter-example:" Let $x = 4$, $y = 6$ and $z = 16$. Then xz and divides yz but x does not divide y.

2. (FP1.73) Let a and b be positive integers such that $a^2 = b^3$. Come up with some examples.

(a) Prove that: Given $a, b \in \mathbf{Z}$ if $a^2 = b^3$ and a is even, then 4 divides a.
   Hint:
   i. Assume that a is even and write $a = 2m$
   ii. Use the above theorem to show that a^2 is even.
   iii. Use this to show that b^3 is even.
   iv. Use this and the warm-up to show that b is even and write that $b = 2n$.
   v. Use this to show that this implies that 8 divides b^3.
   vi. Justify the equality $8n^3 = 4m^2$.
   vii. Use this to show that m^2 is even.
   viii. Use this and the warm-up to show that m is even.
   ix. Now show that this implies that 4 divides a.

(b) Given that 4 divides a and that $a^2 = b^3$, prove that 4 divides b.
   Hint:
   i. Write $a = 4k$. What is $b^3 = a^2$ in terms of k? Show that b^3 is even.
   ii. Use the warm-up to show that b is even and write $b = 2c$.
   iii. Justify the expression $8c^3 = 16k^2$.
   iv. Show that c^3 is even and conclude (how?) that c is even.
   v. Use this to see that 4 divides b.

(c) Given that 4 divides b and $a^2 = b^3$, prove that 8 divides a. Hint:
   i. Write $b = 4e$. What is b^3 in terms of e?
   ii. Justify the expression $a^2 = 4^3 e^3$.
   iii. Use the theorem to say that a is even and write $a = 2z$.
   iv. Justify the following $4z^2 = 4^3 e^3$.
   v. Use this to show that z is even, and let $z = 2q$.
   vi. Justify $4q^2 = 16e^3$.
   vii. Use this to show that q is even and write $q = 2d$.
   viii. Now $a = 2z = {}_{---}q = {}_{----}d$.

(d) Give an example of positive integers a and b such that a is even and $a^2 = b^3$, but b is not divisible by 8.