Foundations of Higher Mathematics
Homework 11

Problem 67

$f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 3x^2 + 2$.

a) $f([2,3])$
$f(2) = 3(2)^2 + 2 = 14$ and $f(3) = 3(3)^2 + 2 = 29$. So, $f([2,3]) = [14, 29]$.

b) $f^{-1}([55, 307]) = \{x \in \mathbb{R} | \; f(x) \in [55, 307]\}$

$$\begin{aligned} 55 &= 3x^2 + 2 \\ 53 &= 3x^2 \\ \frac{53}{3} &= x^2 \\ \pm\sqrt{\frac{53}{3}} &= x \end{aligned}$$

$$\begin{aligned} 307 &= 3x^2 + 2 \\ 305 &= 3x^2 \\ \pm\sqrt{\frac{305}{3}} &= x \end{aligned}$$

So, $f^{-1}([55, 307]) = \left[-\sqrt{\frac{53}{2}}, \sqrt{\frac{305}{5}}\;\right]$

c) $f^{-1}([1,2]) = \{x \in \mathbb{R} | \; f(x) \in [1,2]\}$

$$\begin{aligned} 1 \le 3x^2 + 2 &\le 2 \\ -1 \le 3x^2 &\le 0 \end{aligned}$$

Since $3x^2 = -1$ has no solutions in $\mathbb{R}$, $f^{-1}([1,2]) = \{0\}$.

d)

$$\begin{aligned} f^{-1}(f(3)) &= f^{-1}(3 \cdot (3)^2 + 2) \\ &= f^{-1}(29) \\ 29 &= 3x^2 + 2 \\ 27 &= 3x^2 \\ x &= \pm 3 \end{aligned}$$

e) $f(\{-1, -2, -3\}) = \{3(-1)^2 + 2, 3(-2)^2 + 2, 3(-2)^2 + 2\} = \{5, 14, 29\}$

f) $f(\{1, 2, 3\}) = \{5, 14, 29\}$, same work as last problem.

Problem 69

a) $f(A) = \{mn | \; (m, n) \in \mathbb{N} \times \mathbb{N} : m = 1$ and n is even$\}$
$= 1 \cdot n$ where n is even, so $f(A)$ is all even numbers.

b) $f(B) = \{mn | \; (m, n) \in \mathbb{N} \times \mathbb{N} : m$ and n are even$\}$
If m and n are even then, $m = 2j$ and $n = 2k$, so $mn = 4jk$. Thus, $f(B) = 4\mathbb{N}$.

c) $f(C) = \{mn | (m, n) \in \mathbb{N} \times \mathbb{N} : m \text{ is even or } n \text{ is even}\}$
Case 1: m even, b even
Same as part b, $mn = 4\mathbb{N}$.
Case 2: m even, b odd
$2k \cdot 2j + 1 = 2(2jk + k)$ so mn is even.
Case 3: m odd, b even
mn is even.

Since all $4\mathbb{N}$ are in $2\mathbb{N}$, $f(C) = 2\mathbb{N}$

d)

$$\begin{aligned} f^{-1}(D) &= \{m, n \in \mathbb{N} | \ f(m, n) \in D\} \\ &= \{m, n \in \mathbb{N} | \ f(m, n) \in \{x \in \mathbb{N} | \ x \text{ is odd}\} \\ &= \{m, n \in \mathbb{N} | \ m \cdot n \in \{x \in \mathbb{N} | \ x \text{ is odd}\} \end{aligned}$$

$m \cdot n$ is only odd when m is odd and n is odd. ($m = 2k + 1$, $n = 2j + 1$, $mn = 4jk + 2j + 2k + 1 = 2(2jk + j + k) + 1$). So, $f^{-1}(D) = (m, n)$ such that m is odd and n is odd.

e)

$$\begin{aligned} f^{-1}(E) &= \{m, n \in \mathbb{N} | \ f(m, n) \in \{n \in \mathbb{N} | \ x \text{ is even}\} \\ &= \mathbb{N} \times \mathbb{N} \end{aligned}$$

Because with even m and odd n, $m \cdot n$, similarly for odd m and even n. So, all pairs of $\mathbb{N}$ are in the set.

f)

$$\begin{aligned} f^{-1}(F) &= \{m, n \in \mathbb{N} | \ f(m, n) = 14\} \\ &= \{(1, 14), (14, 1), (2, 7), (7, 2)\} \end{aligned}$$

Problem 70

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \lfloor x \rfloor$, and let $A = [3, 5] \cup (7, 9) \cup (11, 15)$.

- $f(A) = \{f(a) | \ a \in A\} = [3, 5] \cup [7, 8] \cup [11, 14]$
- $f^{-1}(A) = \{x \in \mathbb{R} | \ f(x) \in A\} = [3, 6) \cup [8, 9) \cup [12, 15)$.
- $f(f^{-1}(A)) = f([3, 6) \cup [8, 9) \cup [12, 15)) = [3, 5] \cup \{8\} \cup [12, 14)$
- $f^{-1}(f(A)) = \{x \in \mathbb{R} | \ f(x) \in f(A)\} = [3, 6] \cup [7, 9) \cup [11, 15)$

Problem 72

Proof. Assume $A, B \in \mathcal{P}(X)$ such that $A \neq B$. That is, there exists an $x \in A$ and $x \notin B$, or $x \notin A$ and $x \in B$. Assume $x \in A$ and $a \notin B$. (Identical argument follows for $x \notin A$, $x \in B$) Since, $x \in A$, $x \notin X - A$. It follows that since $x \in A$ and $A \subseteq X$, that $x \in X$. Now take $C_X(B) = X - B$. Since $x \in X$, and $x \notin B$, $x \in X - B$. Since there exists $x \in X - A$ and $x \notin X - B$, then $X - A \neq X - B$ and thus, $C_x(A) \neq C_x(B)$. Therefore, we have shown that C_x is injective.

■

$$\begin{aligned} C_X^{-1}(A) &= \{B \in \mathcal{P}(X) |\ C_x(B) = A\} \\ &= \{B \in \mathcal{P}(X) |\ X - B = A\} \end{aligned}$$

So C_X^{-1} is just the set that is the complement of A.

Problem 75

($\Leftarrow$) Assume $f(A) \cap f(B) = f(A \cap B)$. Let $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. Want to show that $x_1 = x_2$. Since A and B are sets, represent x_1 and x_2 as one element sets, $\{x_1\}$ and $\{x_2\}$. It follows that $f(\{x_1\}) \cap f(\{x_2\}) = f(\{x_1\} \cap \{x_2\})$. Since $f(x_1) = f(x_2)$, and the intersection of two identical sets is itself, $f(\{x_1\}) = f(\{x_1\} \cap \{x_2\})$. It follows that:

$$f(\{x_1\}) = \{f(x) |\ x \in \{x_1\} \text{ and } x \in \{x_2\}\}$$

It follows that $x_1 = x_2$. Therefore, f is injective.
($\Rightarrow$) Assume f is *injective*. We want to show, $f(A) \cap f(B) = f(A \cap B)$.
($\subseteq$) Assume an arbitrary $y \in f(A) \cap f(B)$. It follows that $y \in f(A)$ and $y \in f(B)$. By definition, $y \in \{f(a) |\ a \in A\}$ and $y \in \{f(b) |\ b \in B\}$. We can see that $y \in Im(f)$ because A and B are subsets of X. It follows that there is an unique element, call it x, such that $f(x) = y$. Since $y \in f(A)$ and $y \in f(B)$, this unique $f(x) \in f(A)$ and $f(x) \in f(B)$. Since x is unique, $x \in A$ and $x \in B$. So, $x \in A \cap B$. It follows that $f(x) \in f(A \cap B)$. Finally, $y \in f(A \cap B)$.
($\supseteq$) Assume an arbitrary $y \in f(A \cap B)$. Since $y \in Im(f)$, there is a unique element, x, such that $f(x) = y$. So, $x \in A \cap B$. It follows that $x \in A$, and $x \in B$. Thus, $f(x) \in f(A)$ and $f(x) \in f(B)$. Therefore, since $f(x) = y$, $y \in f(A) \cap f(B)$.
Proving both subsets, we can conclude that $f(A) \cap f(B) = f(A \cap B)$.