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Foundations of Higher Mathematics
Homework 8

Problem 35

Proof. **Reflexive.** Suppose we have an a such that, $f(a) = a^2$. Then:

$$\begin{aligned} f(a) &= a^2 \\ &= f(a) \end{aligned}$$

So, $a \simeq a$ and thus $\simeq$ is reflexive.
Symmetric. If $a \simeq b$ then $f(a) = a^2$ and $f(b) = b^2$ such that $f(a) = f(b)$. It follows that:

$$\begin{aligned} f(a) &= f(b) \\ a^2 &= b^2 \\ b^2 &= a^2 \\ f(b) &= f(a) \end{aligned}$$

Thus $b \simeq a$. So $\simeq$ is symmetric.
Transitive. If $a \simeq b$ and $b \simeq c$ then $f(a) = a^2$, $f(b) = b^2$, $f(c) = c^2$ and $f(a) = f(b)$, $f(b) = f(c)$.

$$\begin{aligned} f(a) &= f(b) \\ &= f(c) \end{aligned}$$

■

$[7] = \{-7, 7\}$

Problem 37

Set $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y - x \text{ is an integer.}\}$

a) *Proof.* **Reflexive.** If $x \in \mathbb{R}$, then $x - x = 0 \in \mathbb{Z}$. Thus, $(x, x) \in S$.
Symmetric. If $(x, y) \in S$, then $y - x \in \mathbb{Z}$. It follows that $x - y \in -\mathbb{Z}$ and since $\mathbb{Z} = -\mathbb{Z}$, so $x - y \in \mathbb{Z}$. Thus, $(y, x) \in S$.
Transitive. Let $(x, y) \in S$ and $(y, z) \in S$. Thus, $y - x = a \in \mathbb{Z}$ and $z - y = b \in \mathbb{Z}$. It follows that $y - x + z - y = a + b$, and thus $z - x = a + b$. Since $a + b \in \mathbb{Z}$, $(x, z) \in S$. ■

b) $[\pi] = \{\pi, \pi + 1, \pi + 2, \pi + 3\}$

c) Any integer belongs to $[-17]$

Problem 40

a) *Proof.* **Reflexive.** If $f \in S$ then f is differentiable function $f(x)$. We can see that $f'(x) = 1 \cdot f'(x)$ and since 1 is non-zero, $f \simeq f$. Therefore, $\simeq$ is reflexive.
Symmetric. Let $f \simeq g$. Then $f'(x) = kg'(x)$ where k is a non-zero number. It follows that $g'(x) = \frac{1}{k} f'(x)$, so $g \simeq f$. Therefore, $\simeq$ is symmetric.
Transitive. If $f \simeq g$ and $g \simeq h$ then $f'(x) = kg'(x)$ and $g'(x) = jh'(x)$, where k and j are non-zero. It follows that $f'(x) = kjh'(x)$, from substituting $g'(x)$. Since k and j were non-zero, jk is non-zero, so $f \simeq h$. Therefore, $\simeq$ is transitive. Thus, proving that $\simeq$ is an equivalence relation on S. ■

b) When $f(x) = x^2 + 17x + 11$
$[f] = \{x^2 + 17x + 12, x^2 + 17x + 13, x^2 + 17x + 14, x^2 + 17x + 15, \ldots\}$

Problem 42

a) *Proof.* **Reflexive.** Let $x \in X$. Then $(x, x) \in R$ because R is an equivalence relation on X and $(x, x) \in S$ because S is an equivalence relation on X. So, $(x, y) \in R \cap S$.
Symmetric. Let $(x, y) \in R \cap S$. Then $(x, y) \in R$ and $(x, y) \in S$. Since R and S are both equivalence relations on X, $(y, x) \in S$ and $(y, x) \in R$. Therefore, $(y, x) \in R \cap S$.
Transitive. Let $(x, y) \in R \cap S$ and $(y, z) \in R \cap S$. It follows that $(x, y), (y, z) \in R$ and $(x, y), (y, z) \in S$. Since R and S are equivalence relations on X, they are both transitive and thus, $(x, z) \in R$ and $(x, z) \in S$. Therefore, $(x, z) \in R \cap S$.

■

b) $x \in X, (R \cap S)[x] = R[x] \cap S[x]$

Proof. ($\subseteq$) Let $y \in (R \cap S)[x]$. Then $(x, y) \in (R \cap S)$. It follows that $(x, y) \in R$ and $(x, y) \in S$. Thus, $y \in R[x]$ and $y \in S[x]$. Therefore, $y \in R[x] \cap S[x]$.
($\supseteq$) Let $y \in R[x] \cap S[x]$. Then $y \in R[x] \cap S[x]$. It follows that $y \in R[x]$ and $y \in S[x]$. Thus, $(x, y) \in R$ and $(x, y) \in S$. So, $(x, y) \in R \cap S$. Therefore, $(R \cap S)[x]$

■

Problem 65

For the equivalence relation $a \equiv b \pmod 9$ we have that for each natural number n, $[10^n] = [1]$.

Proof. We will show this by induction. If $n = 1$, then $[10^1] = [1]$. It follows from the definition of addition for congruence classes that, $[10] = [9] \oplus [1]$. We can see that $9 \equiv 0 \pmod 9$, because $9 \big| 9 - 9$. So, $[9] = 0$ and thus,

$$\begin{aligned} [10] &= [9] \oplus [1] \\ &= [0] \oplus [1] \\ &= [1] \end{aligned}$$

Thus proving the $n = 1$ case. Now assume n, so $[10^n] = [1]$. We want to show that $n + 1$ is true. The results follow from the definitions of addition and multiplication for the congruence classes:

$$\begin{aligned} [10^{n+1}] &= [10 \cdot 10^n] \\ &= [10] \odot [10^n] \\ &= [1] \odot [10^n] && \text{As shown in the n=1 case} \\ &= [1] \odot [1] && \text{From the } n \text{ assumption} \\ &= [1] \end{aligned}$$

Thus, $n + 1$ is true. Therefore, by PMI, $[10^n] = [1]$ for $a \equiv b \pmod 9$. ■