

Course Notes for Math 320: Fundamentals of Mathematics Chapter 3: Induction.

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1 Proof by Induction

Definition 1.1. A subset S of the natural numbers is said to be inductive if $\forall n \in S$ we have $n + 1 \in S$.

Example 1.2. Which sets are inductive?

1. Natural Numbers: $\mathbf{N} = \{1, 2, 3, 4, \dots\}$
2. Even Natural numbers: $2\mathbf{N} = \{m \in \mathbf{N} \mid \exists a \in \mathbf{N} \text{ such that } m = 2a\}$
3. Natural numbers that are perfect squares
4. All natural numbers greater than or equal to 2591
5. $\{n \in \mathbf{N} \mid 4^n > n^4\}$
6. $\{n \in \mathbf{N} \mid 2^{n+1} \leq 3^n\}$
7. $\{n \in \mathbf{N} \mid 3 \mid (5^n - 2^n)\}$
8. $\{n \in \mathbf{N} \mid 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}\}$

Principle 1.3. Principle of Mathematical Induction:

If a subset S of \mathbf{N} is inductive and $1 \in S$ then $S = \mathbf{N}$.

We can use this to prove stuff when we feel ourselves saying... “and you continue in this way” or “repeating this process over and over”...

Proposition 1.4. (Gauss, age 10)

For all natural numbers n we have that $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$

Proposition 1.5. *For all natural numbers n we have that $3|(5^n - 2^n)$.*

Proposition 1.6. *For all natural numbers n we have that $(x^n - y^n)$ is divisible by $x - y$, where x and y are any two distinct integers.*

Proposition 1.7. *For all natural numbers n we have that $1^2 + 2^3 + 3^3 + \dots + n^3 = \frac{n(n+1)(2n+1)}{6}$*

Proposition 1.8. *For all natural numbers n we have that if A is a finite set with $|A| = n$ then $|P(A)| = 2^n$.*

Remark 1.9. *To prove such results we follow a proof by induction:*

1. *Define a set $S = \{n \in \mathbf{N} \mid \text{stuff is true} \}$*
2. *Show that $1 \in S$*
3. *Show that it is inductive.*

Conlcusion: $S = \mathbf{N}$.

More proofs

Example 1.10. (F&P3.9) What is wrong with the following proof that all members of the human race are of the same sex?

Proof: Let

$$S = \{n \in \mathbb{N} \mid \text{in any set of } n \text{ people, all members of the set are the same sex}\}.$$

If we have a set consisting of one person, then clearly all the members of the set are of the same sex, so $1 \in S$. Suppose that $n \in S$. Then in any set of n people, all the members of the set are of the same sex. In order to show that $n + 1 \in S$, we need to show that in any set of $n + 1$ people, all the members of the set are of the same sex. Let A be a set consisting of $n + 1$ people. Let's call these people $a_1, a_2, \dots, a_n, a_{n+1}$. If we send one person out of the room, say a_1 , then we have a set of n people left in the room, so by the assumption that $n \in S$, they are all of the same sex. Now let's bring a_1 back into the room and set a_2 out. Again there is a set of n people left in the room, so by the assumption that $n \in S$ they are all of the same sex. Now observe that everyone in the original set of $n + 1$ people is of the same sex as a_3 , so they are all of the same sex. Therefore $n + 1 \in S$. By the Principle of Mathematical Induction, $S = \mathbb{N}$.

2 Variants on Principle of Mathematical Induction

Example 2.1. Recall the set $S = \{n \in \mathbf{N} \mid 4^n > n^4\}$ which we believed to be inductive after $n = 5$. In fact we think that it contains $\{n \in \mathbf{N} \mid n \geq 5\}$. Can we use PMI for this?

Principle 2.2. Extended Principle of Mathematical Induction

Let $k \in \mathbf{N}$ and let S be a subset of \mathbf{N} such that

- a. $k \in S$, and
- b. if $n \geq k$ and $n \in S$ then $n + 1 \in S$.

Then $\{n \in \mathbf{N} \mid n \geq k\} \subset S$.

Theorem 2.3. $PMI \Rightarrow EPMI$

Proposition 2.4. *For all natural numbers $n \geq 2$ we have that $2^{n+1} \leq 3^n$. (Hint: EPMI)*

Principle 2.5. Second Principle of Mathematical Induction

Let S be a subset of \mathbf{N} such that

- a. $1 \in S$, and*
- b. For each $n \geq k$ if $\{1, 2, 3, \dots, n\} \subset S$ then $n + 1 \in S$.*

Then $S = \mathbf{N}$.

Principle 2.6. Extended Second Principle of Mathematical Induction

Let $k \in \mathbf{N}$ and let S be a subset of \mathbf{N} such that

a. $k \in S$, and

b. if $n \geq k$ and $\{k, k+1, \dots, n\} \subset S$ then $n+1 \in S$.

Then $\{n \in \mathbf{N} | n \geq k\} \subset S$.

Principle 2.7. Well Ordering Principle or Least Natural Number Principle

Every nonempty set of the natural numbers has a least member.

Theorem 2.8. $PMI \Leftrightarrow LNNP \Leftrightarrow SPMI$

Proposition 2.9. *Every natural number greater than 1 is either a prime number or the product of prime numbers. (Hint: SPMI)*

Theorem 2.10. The Division Algorithm for Integers

Let a be an integer and let b be a natural number. Then there exist integers q and r such that $a = bq + r$ and $0 \leq r < b$. Furthermore, the numbers q and r exist uniquely in the sense that if q' and r' are integers such that $a = bq' + r'$ and $0 \leq r' < b$ then $q = q'$ and $r = r'$.

Proof uses LNNP.

Hints and group work for 3.1 Homework:

1. Hint for problem 3.1e

- (a) Consider the following sum of a sequence of numbers:

$$2 + 5 + 8 + \dots + (3n - 1).$$

What would be the next term (the $(n + 1)^{st}$ term) of this sum?

- (b) Replace n by $n + 1$ in the following expression

$$\frac{n(3n + 1)}{2}.$$

- (c) Use PMI to prove: For every natural number n ,

$$2 + 5 + 8 + \dots + (3n - 1) = \frac{n(3n + 1)}{2}.$$

2. Hint for 3.5: Prove that the product of any three consecutive natural numbers is divisible by 6. When you get to $(n + 1)(n + 2)(n + 3)$ remember that for any three numbers $AB(C + D) = ABC + ABD$. Applying this to $(n + 1)(n + 2)(n + 3)$ should get you through the induction step.

3. Hint for problem 3.6

- (a) Find the sum of the first 4 odd numbers.
- (b) Find the sum of the first 5 odd numbers.
- (c) Find the sum of the first 6 odd numbers.
- (d) Make a conjecture (guess) as to what the sum of the first n odd numbers is.
- (e) Use PMI to prove your conjecture.

4. Hint of 3.8: Check carefully that the assumptions in the induction step apply to *all* $n \geq 1$.

5. Hint for problem 3.15: After you show that $1 \in S$. You will assume that $n \in S$ and want to show that $n + 1 \in S$. As $n \in S$ we know that $(1 + x)^n \geq 1 + nx$. Now consider $(x + 1)^{n+1} = (x + 1)(x + 1)^n$. Since $x > -1$ (by hypothesis) we have $x + 1 > 0$ (why?). Use that and the fact $n \in S$ so show that $(x + 1)(x + 1)^n \geq (x + 1)(1 + nx)$ (why do you need $x + 1 > 0$?). Now factor this out and use that $nx^2 > 0$ to finish the proof.

Group Work and Homework Hints for 3.2 Homework:

1. Prove that for each natural number $n \geq 6$ we have $2n + 3 \leq 2^n$.
2. (FP #20) Use the above to prove that for each natural number $n \geq 6$ we have $(n+1)^2 \leq 2^n$. [Hint: $n^2 + 4n + 4 = (n^2 + 2n + 1) + (2n + 3)$.]
3. (FP # 28) Consider the set $T = \{n \in \mathbf{N} | n > 1, n \text{ not prime and } n \text{ not the product of primes}\}$. You want to show that $T = \emptyset$ (why?). Well proceed by contradiction. Suppose that $T \neq \emptyset$. Then T has a least element (why?). Now call that least element d . Since $d \in T$ we know that:
 - (a) $d \neq 1$,
 - (b) d is not a prime, and
 - (c) d is not a product of primes.

The second of these properties shows that there exists $a, b \in \mathbf{Z}$ such that $d = ab$ where neither of a and b are equal to d nor are they equal to 1 (why?). Notice also that $a, b < d$ and thus they are NOT in T (why?). What does this tell you about a and b . From this info on a and b and what we know about d draw a contradiction.