Stefan Eng
Dr. Katherine Stevenson
(1.4-1.5) 63, 66, 70, 72, 75 Math 320
(2.2) 24, 29a, c, d, 30, 31 9/18/13

Foundations of Higher Mathematics
Homework 3

Chapter 1

Problem 63

Let A be an even integer and B be an odd integer. Prove that A + B is odd and AB is even.

Proof. If A is even and B is odd, then there exists integers k, n, such that $A = 2q$ and $B = 2k + 1$. It follows that,

$$\begin{aligned} A + B &= 2q + (2k + 1) \\ &= (2q + 2k) + 1 \end{aligned}$$

Thus, since $2q + 2k \in \mathbb{Z}$, A + B is odd. Similarly for AB,

$$\begin{aligned} AB &= 2q(2k + 1) \\ &= 4qk + 2q \\ &= 2(2qk + q) \end{aligned}$$

it follows that since $2qk + q$ is an integer, AB is even. ■

Problem 66

Let x be a real number. Prove $x = -1$ if and only if $x^3 + x^2 + x + 1 = 0$

Proof. $(\rightarrow)$
Assume $x = -1$. Then

$$\begin{aligned} x^3 + x^2 + x + 1 &= 0 \\ (-1)^3 + (-1)^2 + (-1) + 1 &= 0 \\ -1 + 1 - 1 + 1 &= 0 \\ 0 &= 0 \end{aligned}$$

Thus, $x = -1$ implies $x^3 + x^2 + x + 1 = 0$.

Proof. $(\leftarrow)$ Assume $x^3 + x^2 + x + 1 = 0$. Then

$$\begin{aligned} x^3 + x^2 + x + 1 &= 0 \\ (x^3 + x^2) + (x + 1) &= 0 \\ x^2(x + 1) + (x + 1) &= 0 \\ (x + 1)(x^2 + 1) &= 0 \end{aligned}$$

Since $x \in \mathbb{R}$ the only solution is $x = -1$. Therefore, $x^3 + x^2 + x + 1 = 0$ implies $x = -1$.

Proving both directions, we can conclude that $x = -1$ if and only if $x^3 + x^2 + x + 1 = 0$ ■

Problem 70

Let n be an integer such that n^2 is even. Prove that n^2 is divisible by 4.

Proof. Assume n^2 is even. Then by theorem 1.12 in the book, n is even. Since n is even, there exists a k such that, $n = 2k$. It follows that,

$$\begin{aligned} n^2 &= 2k \times 2k \\ &= 4k^2 \\ &= 4(k^2) \end{aligned}$$

Since k^2 is an integer, 4 divides n^2. ■

Problem 72

The student was

Problem 75

Prove, by contradiction, that the sum of two even integers is even.

Proof. (Contradiction): Assume the sum of two even integers is odd. Then there exist integers k, q such that, 2k + 2q = 2(k + q). This is a contradiction, since we said the sum was odd, so therefore the sum of two even integers is even. ■

Chapter 2

Problem 24

Prove that $A \cap B = A$ if and only if $A \subseteq B$.

Proof. ($\rightarrow$) Assume $A \cap B = A$. Then $A \subseteq A \cap B$. If we take an arbitrary element x of A, then x is in A and B. It follows that every element in A is also in B. Thus, $A \subseteq B$. Therefore, $A \cap B = A$ implies $A \subseteq B$.

Proof. ($\leftarrow$) Assume $A \subseteq B$. Suppose x is an arbitrary element of A. Then x is in B. So if x is in A and x is in B, then for all x that are in A and B, x is in A. Thus, $A \cap B \subseteq A$. Similarly, $A \subseteq A \cap B$ which implies that $A \cap B = A$. Therefore, $A \subseteq B$ implies $A \cap B = A$.

Therefore, $A \cap B = A$ if and only if $A \subseteq B$ ■

Problem 29a

$\varnothing \cap A = \varnothing$ and $\varnothing \cup A = A$

Proof. ($\varnothing \cap A = \varnothing$) Assume an arbitrary x in $\varnothing$ and A. There are no elements that are in $\varnothing$ and A so it is (vacuously) true that x $\in \varnothing$. The statement $\varnothing \subseteq \varnothing \cap A$ follows directly from the fact the empty set is a subset of every set. Therefore $\varnothing \cap A = \varnothing$. ■

Proof. ($\varnothing \cup A = A$) Assume x is in $\varnothing$ or A. If x in A, $A \subseteq A$ from *proposition 2.2* in the book. If x in A, $\varnothing \subseteq A$ since $\varnothing$ is a subset of every set. Also, if x in A, then x in A or $\varnothing$. Thus, $\varnothing \cup A \subseteq A$ and $A \subseteq \varnothing \cup A$. Therefore, $\varnothing \cup A = A$. ■

Problem 29c

$A \subseteq A \cup B$

Proof. Assume x is an element of A. Then x is in A or x is in B. Therefore, $A \subseteq A \cup B$. ■

Problem 29d

$A \cup B = B \cup A$ and $A \cap B = B \cap A$

Proof. $(A \cup B = B \cup A)$
Suppose x is and element of $A \cup B$. Then x is in A or x is in B. Thus x is in B or x is in A. Hence, $A \cup B \subseteq B \cup A$ and similarly $B \cup A \subseteq A \cup B$. Therefore, $A \cup B = B \cup A$. ■

Proof. $(A \cap B = B \cap A)$
The proof is the same as above, replacing the words *or* for *and*. ■

Problem 30

Counterexample: $P = \varnothing, Q = \varnothing, R = \{1\}$

$$\begin{gathered}(P \cap Q) \cup R = P \cap (Q \cup R) \\ (\varnothing \cap \varnothing) \cup \{1\} = \varnothing \cap (\varnothing \cup \{1\}) \\ \varnothing \cup \{1\} = \varnothing \cap \{1\} \\ \{1\} \neq \varnothing\end{gathered}$$

Problem 31

The student's proof was **wrong.** The mistake he or she made was in the step, $x \in A$ and $x \notin B$ or $x \notin C$. The statement should look like, $x \in A$ and $x \notin B$ **and** $x \notin C$.