Pape
Becher
~~Grisard~~

Aufgabe 1 ART Blatt 2

$a = \frac{d}{dt}(\gamma v)$, $at = \gamma v \Leftrightarrow v = \frac{at}{\gamma}$, $v = \frac{at}{\sqrt{1+\frac{a^2t^2}{c^2}}}$ (1)

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$$r = \int_0^T v\, dt = \int_0^T \frac{at}{\gamma}\, dt$$

$$= \int_0^T \frac{at}{\sqrt{1+\frac{a^2t^2}{c^2}}}\, dt \underset{\text{Sub.}}{\overset{\text{Lineare}}{=}} \left[\frac{c^2}{a}\sqrt{1+\frac{a^2}{c^2}t^2}\right]_0^T$$

$$r = \frac{c^2}{a}\left[\sqrt{1+\frac{a^2}{c^2}T^2} - 1\right]$$

$r_g = r_1 + r_2 + r_3 + r_4 = 4r_1$, $T = 2a = 60 \cdot 60 \cdot 24 \cdot 365\, s$, $a = 10\, \frac{m}{s^2}$

$r_1 = 1.194 \cdot 10^{16}\, m \Rightarrow r_g = 4 \cdot 1.194 \cdot 10^{16}\, m$

$$x = \frac{c^2}{\ddot{x}}\left[\sqrt{1+\frac{\ddot{x}^2}{c^2}T^2} - 1\right] \quad \text{(DGL)}$$

$$\tau = \int \frac{1}{\gamma}\, dt = \int \sqrt{1-\frac{v^2}{c^2}}\, dt \overset{(1)}{=} \int \frac{1}{\sqrt{1+\frac{a^2t^2}{c^2}}}\, dt$$

$= \frac{c}{a}\,\mathrm{arsh}\left(\frac{a}{c}t\right)$, $\tau \hat{=}$ Zeit im Raumschiff , $t \hat{=}$ Zeit auf der Erde

$$\Leftrightarrow \frac{a\tau}{c} = \mathrm{arsh}\left(\frac{a}{c}t\right) \Leftrightarrow t = \sinh\left(\frac{a\tau}{c}\right)\frac{c}{a}$$

$\tau = 60 \cdot 60 \cdot 24 \cdot 365\, s$

$\tau_G = \tau_1 + \tau_2 + \tau_3 + \tau_4 = 4\tau_1$

$\tau_1 = 3.84 \cdot a \Rightarrow \tau_G = 4 \cdot 3.84 = 15.35\, a$

-> Gretel ist ca. 7,354 Jahre älter.

Auflösung:

Wir haben ein beschleunigtes System. (kein Inertialsystem)

Entfernung?

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Aufgabe 2

a) $F^{\mu\nu}$ ist ein Tensor, daher ist $F_{\mu\nu}F^{\mu\nu}$ ein Lorentzskalar. $\Rightarrow \mathcal{L}$ Lorentzinvariant

b) Vorüberlegung: • $\dfrac{\partial[\partial_\mu A_\nu]}{\partial[\partial_\alpha A_\beta]} = \delta_\mu{}^\alpha \, \delta_\nu{}^\beta$

• $\dfrac{\partial\mathcal{L}}{\partial A_\mu} = 0$ ($\mathcal{L}$ enthält nur Ableitungen)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= -\frac{1}{4}\Big[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \underbrace{\partial_\nu A_\mu \partial^\mu A^\nu}_{\mu\leftrightarrow\nu} + \underbrace{\partial_\nu A_\mu \partial^\nu A^\mu}_{\mu\leftrightarrow\nu}\Big]$$

Indices umbenannt $\hookrightarrow$ $$= -\frac{1}{2}\left[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right]$$

Nebenrechnungen:

$$\frac{\partial[\partial_\mu A_\nu \partial^\mu A^\nu]}{\partial(\partial_\alpha A_\beta)} = \left[\frac{\partial[\partial_\mu A_\nu]}{\partial[\partial_\alpha A_\beta]}\right]\partial^\mu A^\nu + \left[\frac{\partial[\partial^\mu A^\nu]}{\partial[\partial_\alpha A_\beta]}\right]\partial_\mu A_\nu$$

$$= \delta_\mu{}^\alpha \, \delta_\nu{}^\beta \, \partial^\mu A^\nu + \partial_\mu A_\nu \left[\frac{\eta^{\sigma\mu}\eta^{\rho\nu}\partial[\partial_\sigma A_\rho]}{\partial[\partial_\alpha A_\beta]}\right]$$

$$= \partial^\alpha A^\beta + \partial_\mu A_\nu \eta^{\sigma\mu}\eta^{\rho\nu}\delta_\sigma{}^\alpha \, \delta_\rho{}^\beta$$

$$= \partial^\alpha A^\beta + \partial^\sigma A^\rho \, \delta_\sigma{}^\alpha \, \delta_\rho{}^\beta = \partial^\alpha A^\beta + \partial^\alpha A^\beta$$

$$= 2\partial^\alpha A^\beta$$

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analog: $$\frac{\partial[\partial_\mu A_\nu \partial^\nu A^\mu]}{\partial(\partial_\alpha A_\beta)} = 2\partial^\beta A^\alpha$$

$$\Rightarrow 0 = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\mu F^{\mu\nu}$$

2x Maxwell im Isolator!

A.3.)

Transformationsvorschrift: $g_{\mu\nu} = \frac{\partial x^\alpha}{\partial \xi^\mu} \frac{\partial x^\beta}{\partial \xi^\nu} \eta_{\alpha\beta}$

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \longrightarrow \xi^\mu = \begin{pmatrix} ct \\ r \\ \vartheta \\ \varphi \end{pmatrix}$$

$$x^\mu = \begin{pmatrix} ct \\ r\cos\varphi\sin\vartheta \\ r\sin\varphi\sin\vartheta \\ r\cos\vartheta \end{pmatrix}, \quad \eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\frac{\partial x^\mu}{\partial \xi^0} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \frac{\partial x^\mu}{\partial \xi^1} = \begin{pmatrix} 0 \\ \cos\varphi\sin\vartheta \\ \sin\varphi\sin\vartheta \\ \cos\vartheta \end{pmatrix}$$

$$\frac{\partial x^\mu}{\partial \xi^2} = \begin{pmatrix} 0 \\ r\cos\varphi\cos\vartheta \\ r\sin\varphi\cos\vartheta \\ -r\sin\vartheta \end{pmatrix}, \quad \frac{\partial x^\mu}{\partial \xi^3} = \begin{pmatrix} 0 \\ -r\sin\varphi\sin\vartheta \\ r\cos\varphi\sin\vartheta \\ 0 \end{pmatrix}$$

$\frac{\partial x^\alpha}{\partial \xi^\mu} \frac{\partial x^\beta}{\partial \xi^\nu} = 0$, für $\mu \neq \nu$, da die Vektoren orthogonal zueinander sind.

$\frac{\partial x^\alpha}{\partial \xi^\mu} \frac{\partial x^\beta}{\partial \xi^\nu} \eta_{\alpha\beta} = 0$, für $\alpha \neq \beta$ wegen $\eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$

$$g_{00} = \frac{\partial x^\alpha}{\partial \xi^0} \frac{\partial x^\beta}{\partial \xi^0} \eta_{\alpha\beta} = 1$$

$$g_{11} = \frac{\partial x^\alpha}{\partial \xi^1} \frac{\partial x^\beta}{\partial \xi^1} \eta_{\alpha\beta} = -\frac{\partial x^1}{\partial \xi^1}\frac{\partial x^1}{\partial \xi^1} - \frac{\partial x^2}{\partial \xi^1}\frac{\partial x^2}{\partial \xi^1} - \frac{\partial x^3}{\partial \xi^1}\frac{\partial x^3}{\partial \xi^1}$$
$$= -\cos^2\varphi\sin^2\vartheta - \sin^2\varphi\sin^2\vartheta - \cos^2\vartheta = -1$$

$$g_{22} = \frac{\partial x^\alpha}{\partial \xi^2} \frac{\partial x^\beta}{\partial \xi^2} = -r^2\cos^2\varphi\cos^2\vartheta - r^2\sin^2\varphi\cos^2\vartheta - r^2\sin^2\vartheta = -r^2$$

$$g_{33} = \frac{\partial x^\alpha}{\partial \xi^3} \frac{\partial x^\beta}{\partial \xi^3} = -r^2\sin^2\varphi\sin^2\vartheta - r^2\cos^2\varphi\sin^2\vartheta = -r^2\sin^2\vartheta$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\vartheta \end{pmatrix}$$

$$ds^2 = dx^\mu dx^\nu g_{\mu\nu} = c^2dt^2 - dr^2 - r^2 d\vartheta^2 - r^2\sin^2\vartheta\, d\varphi^2$$

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