Aufgabe 2) Riemannscher Krümmungstensor des Torus

a) $\vec{r} = \begin{pmatrix} \cos(\vartheta)(a+r\cos(\varphi)) \\ \sin(\vartheta)(a+r\cos(\varphi)) \\ r\sin(\varphi) \end{pmatrix}$, $\frac{\partial \vec{r}}{\partial \varphi} = \begin{pmatrix} -r\cos(\vartheta)\sin(\varphi) \\ -r\sin(\vartheta)\sin(\varphi) \\ r\cos(\varphi) \end{pmatrix}$ ✓

$\frac{\partial \vec{r}}{\partial \vartheta} = \begin{pmatrix} -\sin(\vartheta)(a+r\cos(\varphi)) \\ \cos(\vartheta)(a+r\cos(\varphi)) \\ 0 \end{pmatrix}$ ✓

Eigentl. Diff-Geom. ohne Minkowski

$g_{\mu\nu} = \frac{\partial x^\alpha}{\partial \lambda^\mu} \frac{\partial x^\beta}{\partial \lambda^\nu} \eta_{\alpha\beta}$ mit $\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$?

hat eigentlich nix mit Minkowski zu tun

$g_{11} = \frac{\partial x^1}{\partial \varphi}\frac{\partial x^1}{\partial \varphi}(-1) + \frac{\partial x^2}{\partial \varphi}\frac{\partial x^2}{\partial \varphi}(-1) + \frac{\partial x^3}{\partial \varphi}\frac{\partial x^3}{\partial \varphi}(-1)$

∼ → nicht nötig

$= -r^2\cos^2(\vartheta)\sin^2(\varphi) - r^2\sin^2(\vartheta)\sin^2(\varphi) - r^2\cos^2(\varphi)$

$= -r^2(\sin^2(\varphi) + \cos^2(\varphi)) = -r^2$

$g_{22} = \frac{\partial x^1}{\partial \vartheta}\frac{\partial x^1}{\partial \vartheta}(-1) + \frac{\partial x^2}{\partial \vartheta}\frac{\partial x^2}{\partial \vartheta}(-1) + \frac{\partial x^3}{\partial \vartheta}\frac{\partial x^3}{\partial \vartheta}(-1)$

$= -\sin^2(\vartheta)(a+r\cos(\varphi))^2 - \cos^2(\vartheta)(a+r\cos(\varphi))^2$

$= -(a+r\cos(\varphi))^2$

$g_{12} = g_{21} = 0$, weil $\eta_{\alpha\beta} = \mathrm{diag}(-1,-1,-1)$

$\Rightarrow g_{\mu\nu} = \begin{pmatrix} -r^2 & 0 \\ 0 & -(a+r\cos(\varphi))^2 \end{pmatrix}$ (✓)

b) $\frac{\partial^2 \vec{r}}{\partial \varphi^2} = \begin{pmatrix} -r\cos(\vartheta)\cos(\varphi) \\ -r\sin(\vartheta)\cos(\varphi) \\ -r\sin(\varphi) \end{pmatrix}$ ✓ , $\frac{\partial^2 \vec{r}}{\partial \vartheta^2} = \begin{pmatrix} -\cos(\vartheta)(a+r\cos(\varphi)) \\ -\sin(\vartheta)(a+r\cos(\varphi)) \\ 0 \end{pmatrix}$ ✓

$\frac{\partial^2 \vec{r}}{\partial \varphi \partial \vartheta} = \begin{pmatrix} r\sin(\vartheta)\sin(\varphi) \\ -r\cos(\vartheta)\sin(\varphi) \\ 0 \end{pmatrix}$ ✓

$L_{ij} = \hat{n}\, \frac{\partial \vec{e}_i}{\partial u_j}$

$\hat{n} = \frac{1}{\left|\frac{\partial \vec{r}}{\partial \vartheta} \times \frac{\partial \vec{r}}{\partial \varphi}\right|} \frac{\partial \vec{r}}{\partial \vartheta} \times \frac{\partial \vec{r}}{\partial \varphi} = \begin{pmatrix} \cos(\varphi)\cos(\vartheta) \\ \cos(\varphi)\sin(\vartheta) \\ \sin(\varphi) \end{pmatrix}$ ✓

$$\hat{n} = \begin{pmatrix} \cos(\varphi)\cos(\vartheta) \\ \cos(\varphi)\sin(\vartheta) \\ \sin(\varphi) \end{pmatrix}, \quad L_{ij} = \hat{n} \cdot \frac{\partial \vec{e}_i}{\partial u^j}$$

$$L_{11} = \hat{n}\,\frac{\partial \vec{e}_\varphi}{\partial u^1} = \begin{pmatrix} \cos(\varphi)\cos(\vartheta) \\ \cos(\varphi)\sin(\vartheta) \\ \sin(\varphi) \end{pmatrix} \cdot \frac{\partial}{\partial \varphi} \begin{pmatrix} -r\cos(\vartheta)\cos(\varphi) \\ -r\sin(\vartheta)\cos(\varphi) \\ -r\sin(\varphi) \end{pmatrix}$$

$$= -r\cos^2(\vartheta)\cos^2(\varphi) - r\cos^2(\varphi)\sin^2(\vartheta) - r\sin^2(\varphi)$$

$$= -r\cos^2(\varphi) - r\sin^2(\varphi) = \underline{\underline{-r}}$$

$$L_{12} = \begin{pmatrix} \cos(\varphi)\cos(\vartheta) \\ \cos(\varphi)\sin(\vartheta) \\ \sin(\varphi) \end{pmatrix} \begin{pmatrix} r\sin(\vartheta)\sin(\varphi) \\ -r\cos(\vartheta)\sin(\varphi) \\ 0 \end{pmatrix} = \underline{\underline{0}} \underset{\text{S.v.S.}}{=} L_{21}$$

$$L_{22} = \begin{pmatrix} \cos(\varphi)\cos(\vartheta) \\ \cos(\varphi)\sin(\vartheta) \\ \sin(\varphi) \end{pmatrix} \begin{pmatrix} -\cos(\vartheta)(a + r\cos(\varphi)) \\ -\sin(\vartheta)(a + r\cos(\varphi)) \\ 0 \end{pmatrix} = \underline{\underline{-\cos(\varphi)(a + r\cos(\varphi))}}$$

$$L_{ij} = \begin{pmatrix} -r & 0 \\ 0 & -\cos(\varphi)(a + r\cos(\varphi)) \end{pmatrix}$$

c) $\cancel{K = \lambda_1\lambda_2 = (-r)\cdot(-\cos(\varphi)(a + r\cos(\varphi))) = r\cos(\varphi)(a + r\cos(\varphi))}$

~~Eigenwerte von L_{ij}~~

$$K = \frac{\det(L_{ij})}{\det(g_{\mu\nu})} = \frac{r\cos(\varphi)(a + r\cos(\varphi))}{r^2(a + r\cos(\varphi))^2} = \frac{\cos(\varphi)}{r(a + r\cos(\varphi))}$$

d) $R_{1212} = -K \cdot \det(g) = -\det(L_{ij}) = -r\cos(\varphi)(a + r\cos(\varphi))$

Unter den gegebenen Symmetrien besitzt der Riemann-Tensor lediglich eine unabh. Komponente.

e) $R^1_{212} = \frac{\partial \Gamma^1_{21}}{\partial x^2} - \frac{\partial \Gamma^1_{22}}{\partial x^1} + \Gamma^2_{21}\Gamma^1_{22} - \Gamma^2_{22}\Gamma^1_{12}$

$$\Gamma^1_{21} = \Gamma^1_{12} = \frac{1}{2} g^{1l}\left(\partial_1 g_{2l} + \partial_2 g_{1l} - \partial_l g_{12}\right) = \frac{1}{2} g^{11}\left(\partial_1 g_{21} + \partial_2 g_{11} - \partial_1 g_{12}\right) = 0$$

$$\Gamma^1_{22} = \frac{1}{2} g^{1l}\left(\partial_2 g_{2l} + \partial_2 g_{2l} - \partial_l g_{22}\right) = \frac{1}{2} g^{11}\left(-\partial_1 g_{22}\right) = \frac{1}{2}(-r^2)^{-1}\left(-2r\sin(\varphi)(a+r\cos(\varphi))\right)$$

$$= \frac{1}{r}\sin(\varphi)(a + r\cos(\varphi))$$

$$\Gamma_{11} = \frac{1}{2} g^{11}\Big(\underbrace{\partial_1 g_{11}}_{=\frac{\partial}{\partial\varphi}(-r^2)=0} + \underbrace{\partial_1 g_{11} - \partial_1 g_{11}}_{=0}\Big) = 0$$

$$\Gamma^2_{11} = \frac{1}{2} g^{22}\Big(\underbrace{\partial_1 g_{12}}_{=0} + \partial_1 \underbrace{g_{12}}_{=0} - \underbrace{\partial_2 g_{11}}_{=\frac{\partial}{\partial\vartheta}(-r^2)=0}\Big) = 0$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2} g^{22}\Big(\partial_1 g_{22} + \partial_2 \underbrace{g_{12}}_{=0} - \partial_2 \underbrace{g_{12}}_{=0}\Big) = \frac{-1}{2(a+r\cos(\varphi))^2} \cdot 2r\sin(\varphi)(a+r\cos(\varphi))$$

$$= -\frac{r\sin(\varphi)}{(a+r\cos(\varphi))}$$

$$\Gamma_{22} = \frac{1}{2} g^{22}\Big(\underbrace{\partial_2 g_{22}}_{\frac{\partial}{\partial\vartheta}(a+r\cos(\varphi))=0} + \underbrace{\partial_2 g_{22} - \partial_2 g_{22}}_{=0}\Big)$$

$$= 0$$

$$R^1_{212} = 0 - \left(\frac{1}{r}\cos(\varphi)(a+r\cos(\varphi)) + \frac{1}{r}(-r)\sin^2(\varphi)\right) - \sin^2(\varphi)$$

$$= -\frac{1}{r}\cos(\varphi)(a+r\cos(\varphi))$$

$$R_{1212} = g_{11} R^1_{212} = (-r^2)\left(-\frac{1}{r}\cos(\varphi)(a+r\cos(\varphi))\right) = r\cos(\varphi)(a+r\cos(\varphi))$$

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