A23

a) $x_1 = a z_1 + b$ $\quad \begin{pmatrix} z_1 & 1 \\ z_2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\vec{x}$

$x_2 = a z_2 + b$

$$A^T = \begin{pmatrix} z_1 & z_2 \\ 1 & 1 \end{pmatrix} \qquad A^T A = \begin{pmatrix} z_1^2 + z_2^2 & z_1 + z_2 \\ z_1 + z_2 & 2 \end{pmatrix}$$

$$\det(A^T A) = 2(z_1^2 + z_2^2) - (z_1 + z_2)^2 = (z_1 - z_2)^2$$

$$(A^T A)^{-1} = \frac{1}{\det(A^T A)} \begin{pmatrix} 2 & -(z_1+z_2) \\ -(z_1+z_2) & z_1^2 + z_2^2 \end{pmatrix}$$

$$(A^T A)^{-1} A^T = \frac{1}{\det(A^T A)} \begin{pmatrix} 2z_1 - (z_1+z_2) & 2z_2 - (z_1+z_2) \\ -z_1(z_1+z_2) + z_1^2 + z_2^2 & -z_2(z_1+z_2) + z_1^2 + z_2^2 \end{pmatrix}$$

$$= \frac{1}{\det(A^T A)} \begin{pmatrix} z_1 - z_2 & z_2 - z_1 \\ z_2^2 - z_1 z_2 & z_1^2 - z_2 z_1 \end{pmatrix}$$

$$(A^T A)^{-1} A^T \vec{x} = \frac{1}{\det(A^T A)} \begin{pmatrix} x_1(z_1 - z_2) + x_2(z_2 - z_1) \\ x_1(z_2^2 - z_1 z_2) + x_2(z_1^2 - z_2 z_1) \end{pmatrix}$$

$$a = \frac{x_1 - x_2}{z_1 - z_2} \qquad b = x_2 \frac{z_1}{z_1 - z_2} - x_1 \frac{z_2}{z_1 - z_2}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{z_1 - z_2} & -\frac{1}{z_1 - z_2} \\ -\frac{z_2}{z_1 - z_2} & \frac{z_1}{z_1 - z_2} \end{pmatrix}}_{B} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad V(x) = \begin{pmatrix} \sigma_{x_1}^2 & 0 \\ 0 & \sigma_{x_2}^2 \end{pmatrix}$$

$$\sigma_{x_1}^2 = \sigma_1^2 \; ; \; \sigma_{x_2}^2 = \sigma_2^2$$

$$V(a,b) = B V B^T = \begin{pmatrix} \frac{1}{z_1 - z_2} & -\frac{1}{z_1 - z_2} \\ -\frac{z_2}{z_1 - z_2} & \frac{z_1}{z_1 - z_2} \end{pmatrix} \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \begin{pmatrix} \frac{1}{z_1 - z_2} & -\frac{z_2}{z_1 - z_2} \\ -\frac{1}{z_1 - z_2} & \frac{z_1}{z_1 - z_2} \end{pmatrix}$$

$$= \begin{pmatrix} 1/(z_1 - z_2) & -1/(z_1 - z_2) \\ -z_2/(z_1 - z_2) & z_1/(z_1 - z_2) \end{pmatrix} \begin{pmatrix} \sigma_1^2 \left(\frac{1}{z_1 - z_2}\right) & -\sigma_1^2 \left(\frac{z_2}{z_1 - z_2}\right) \\ -\sigma_2^2 \left(\frac{1}{z_1 - z_2}\right) & \sigma_2^2 \left(\frac{z_1}{z_1 - z_2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_1^2 \frac{1}{(z_1 - z_2)^2} + \sigma_2^2 \frac{1}{(z_1 - z_2)^2} & -\sigma_1^2 \frac{z_2}{(z_1 - z_2)^2} - \sigma_2^2 \frac{z_1}{(z_1 - z_2)^2} \\ -\sigma_1^2 \frac{z_2}{(z_1 - z_2)^2} - \sigma_2^2 \frac{z_1}{(z_1 - z_2)^2} & \frac{\sigma_1^2 z_2^2}{(z_1 - z_2)^2} + \frac{\sigma_2^2 z_1^2}{(z_1 - z_2)^2} \end{pmatrix}$$

$$\rightarrow \sigma_A = \frac{1}{(z_1 - z_2)} \sqrt{\sigma_1^2 + \sigma_2^2} \qquad \sigma_B = \frac{1}{z_1 - z_2} \sqrt{\sigma_1^2 z_2^2 + \sigma_2^2 z_1^2}$$

$$\mathrm{Cov}(a,b) = \frac{1}{(z_1 - z_2)^2} \left(-\sigma_1^2 z_2 - \sigma_2^2 z_1 \right)$$

$$\rho = \frac{Cov(a,b)}{\sigma_A \sigma_B} = \frac{(-\sigma_1^2 z_2 - \sigma_2^2 z_1)}{\sqrt{(\sigma_1^2+\sigma_2^2)(\sigma_1^2 z_2^2 + \sigma_2^2 z_1^2)}}$$

$$= \frac{(-\sigma_1^2 z_2 - \sigma_2^2 z_1)}{\sqrt{z_2^2(\sigma_1^4+\sigma_1^2\sigma_2^2) + z_1^2(\sigma_2^4+\sigma_1^2\sigma_2^2)}}$$

b)

$$x_3 = z_3 \frac{x_1 - x_2}{z_1 - z_2} + \frac{x_2 z_1}{z_1 - z_2} - \frac{x_1 z_2}{z_1 - z_2}$$

$$= \frac{1}{z_1 - z_2}\left[z_3 x_1 - z_3 x_2 + x_2 z_1 - x_1 z_2\right]$$

$$= \frac{1}{z_1 - z_2}\left[x_1[z_3 - z_2] + x_2[z_1 - z_3]\right]$$

$$\sigma_{x_3} = \sqrt{(z_3\sigma_a)^2 + \sigma_b^2 + 2 z_3 \, Cov(a,b)},$$

da $\frac{\partial x_3}{\partial a} = z_3$ $\quad \frac{\partial x_3}{\partial b} = 1$

c) $\sigma_{x_3} = \sqrt{(z_3\sigma_a)^2 + \sigma_b^2}$ ← wäre größer als in b), da $Cov(a,b) < 0$

$$= \frac{1}{z_1 - z_2}\sqrt{z_3^2(\sigma_1^2+\sigma_2^2) + z_2^2\sigma_1^2 + \sigma_2^2 z_1^2}$$