

add proof for  $\frac{1}{2}$  to appendix