

add proof for $\frac{1}{2}$ to appendix