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Dynamic consensus of linear multi-agent systems

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Abstract: This study concerns the consensus of a network of agents with general linear or linearised dynamics, whose communication topology contains a directed spanning tree. An observer-type consensus protocol based on the relative outputs of the neighbouring agents is adopted. The notion of consensus region is introduced, as a measure for the robustness of the protocol and as a basis for the protocol design. For neutrally stable agents, it is shown that there exists a protocol achieving consensus together with a consensus region that is the entire open right-half plane if and only if each agent is stabilisable and detectable. An algorithm is further presented for constructing such a protocol. For consensus with a prescribed convergence speed, a multi-step protocol design procedure is given, which yields an unbounded consensus region and at the same time maintains a favourable decoupling property. Finally, the consensus algorithms are extended to solve the formation control problems.

1 Introduction

In recent years, the consensus problem of multi-agent systems has received increasing attention from scientific communities, for its broad applications in such areas as satellite formation flying, cooperative unmanned air vehicles, scheduling of automated highway systems and air traffic control.

A large volume of research works on consensus has emerged in the past few years. In [1], a simple model is proposed for phase transition of a group of self-driven particles with numerical analysis on its complexity. In [2], a theoretical explanation is provided for the behaviour observed in [1] based on graph theory. In [3], a general framework of consensus problem for networks of dynamic agents with fixed or switching topologies is established. The conditions given by Olfati-Saber and Murray [3] are further relaxed in [4]. In [5], tracking control for multi-agent consensus with an active leader is studied and a local controller is designed together with a neighbour-based state-estimation rule. In [6], the consensus problem is addressed for directed networks of agents with external disturbances and model uncertainties. The consensus problem of networks with double-integrator agents is studied in [7–11]. In [12], a distributed algorithm is developed, which asymptotically achieves consensus in finite time. Average consensus of multi-agent systems under communication constraints and control of networked systems with limited channels are studied in [13–16]. The controlled agreement problem for multi-agent networks is considered from a graph-theoretic perspective in [17]. In most existing works on consensus, the agent dynamics are restricted to be first-, second- and, sometimes, high-order integrators, and the proposed consensus protocols are based on information of relative states among neighbouring agents, which in many cases are not available.

This study deals with the consensus of networks of identical agents with linear or linearised dynamics whose communication topology contains a directed spanning tree. Previous works along this line include [18–21]. A framework was introduced in [18] to address, in a unified way, the consensus of multi-agent systems and the synchronisation of complex networks studied in the same line as [22–24]. In [19–21], and in most existing works, static consensus protocols based on relative states of the neighbouring agents were used. The dynamic consensus protocol in [25] requires the absolute output measurement of each agent to be available, which is impractical in many cases, for example, in deep-space formation flying [26]. In contrast, an observer-type protocol proposed in [18] using only the relative outputs of the neighbouring agents is adopted here. A decomposition approach is utilised to cast the consensus problem of the multi-agent system into the stability of a set of matrices that have the same dimension as that of a single agent. Compared to the related works, a distinct feature of this paper is that the notion of consensus region is used as a measure for the robustness of the consensus protocol with respect to the communication topology and as the basis for protocol design.

The main contributions of the present paper are three-fold. First, it is shown that, for neutrally stable agents, there exists a protocol achieving consensus and having a consensus region that is the entire open right-half plane if and only if each agent is stabilisable and detectable. An algorithm is then presented to construct such a protocol. The main results in [20] can be easily derived as special cases here. Second, the consensus problem with a prescribed convergence speed is investigated. The convergence speed of consensus is analysed and can be estimated for networks of integrators [3, 27, 28]. To the best knowledge of the authors, this is the first attempt to design a protocol to reach consensus with a given convergence speed for linear multi-agent systems. A necessary and sufficient condition is then derived for the existence of a protocol that reaches consensus with a convergence speed larger than a given positive value and meanwhile yields an unbounded consensus region which means good robustness to the communication topology. A multi-step protocol design procedure is then proposed. Such a design procedure maintains a decoupling property, which is quite attractive especially for cases where the number of agents is large. Third, the consensus algorithms are extended to solve formation control problems of multi-agent systems. It is pointed out that the formation structures must satisfy certain constraints in order to be achievable. A sufficient condition is further given for the existence of a protocol to achieve a specified achievable formation structure for the multi-agent network, which generalises the results in [7, 29–31]; especially the agent dynamics in [7, 30, 31] are (generalised) double integrators and the protocol in [29] is based on the relative states of the neighbouring agents.

The rest of this paper is organised as follows. Section 2 introduces some basic concepts and notations in matrix theory and algebraic graph theory. Section 3 presents the main results on dynamic consensus with an observer-type protocol. Section 4 discusses the consensus with a static protocol. The consensus algorithms are extended to solve the formation control problem in Section 5. Section 6 concludes the paper.

2 Concepts and notation

Let $\mathbf{R}^{n\times n}$ and $\mathbf{C}^{n\times n}$ be the sets of $n\times n$ real matrices and complex matrices, respectively. The superscript T means transpose for real matrices and H means conjugate transpose for complex matrices. $\|\cdot\|$ denotes the induced two-norm. I_N is the identity matrix of dimension N. Let $\mathbf{1}\in\mathbf{R}^p$ denote the vector with all entries equal to one. Matrices, if not explicitly stated, are assumed to have compatible dimensions. $\otimes$ denotes the Kronecker product. For $\zeta\in\mathbf{C}$, denote its real part by $\mathrm{Re}(\zeta)$ and its imaginary part by $\mathrm{Im}(\zeta)$. A matrix $A\in\mathbf{C}^{n\times n}$ is neutrally stable if it has no eigenvalue with positive real part and the Jordan block corresponding to any eigenvalue on the imaginary axis is of size one, while A is Hurwitz (or stable) if all of its eigenvalues have strictly negative real parts. Matrix A is critically unstable if it is not neutrally stable. Matrix $S\in\mathbf{R}^{n\times n}$ is skew-symmetric if $S+S^{\mathrm{T}}=0$.

A directed graph $\mathcal{G}$ is a pair of $(\mathcal{V},\mathcal{E})$, where $\mathcal{V}$ is a finite non-empty node set and $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$ is an edge set in which an edge is represented by an ordered pair of distinct nodes. For an edge (i,j), node i is called the parent node, j the child node and j is neighbouring to i. A path on $\mathcal{G}$ from node i_1 to node i_l is a sequence of ordered edges of the form (i_k,i_{k+1}), $k=1,\ldots,l-1$. A directed graph contains a directed spanning tree if there exists a node called root such that there exists a directed path from this node to every other node.

Suppose that there are m nodes in the graph. The adjacency matrix $\mathcal{A}\in\mathbf{R}^{m\times m}$ is defined by $a_{ii}=0$, and $a_{ij}>0$ if $(j,i)\in\mathcal{E}$ but 0 otherwise. The Laplacian matrix $\mathcal{L}\in\mathbf{R}^{m\times m}$ is defined as $\mathcal{L}_{ii}=\sum_{j\neq i}a_{ij}$, $\mathcal{L}_{ij}=-a_{ij}$ for $i\neq j$. It follows immediately that 0 is an eigenvalues of $\mathcal{L}$ with $\mathbf{1}$ as the corresponding right eigenvector and all the non-zero eigenvalues have positive real parts. For a directed graph, 0 is a simple eigenvalue of $\mathcal{L}$ if and only if the graph has a directed spanning tree [4].

3 Dynamic consensus with observer-type protocol

Consider a network of N identical agents with linear or linearised dynamics, where the dynamics of the ith agent are described by

$$\begin{aligned}\dot{x}_i&=Ax_i+Bu_i\\ y_i&=Cx_i\end{aligned}\tag{1}$$

where $x_i\in\mathbf{R}^n$ is the state, $u_i\in\mathbf{R}^p$ is the control input, $y_i\in\mathbf{R}^q$ is the measured output and A, B and C are constant matrices with compatible dimensions.

The communication topology among agents is represented by a directed graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$, where $\mathcal{V}=\{1,\ldots,N\}$ is the set of nodes (i.e. agents), and $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$ is the set of edges. An edge (i,j) in graph $\mathcal{G}$ means that agent j can obtain information from agent i, but not conversely.

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At each time instant, the information available to agent i is the relative measurements of other agents with respect to i itself, given by

$$\zeta_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j) \tag{2}$$

where $(a_{ii})_{N\times N}$ is the adjacency matrix of graph $\mathcal{G}$. The consensus protocol takes the following observer-type form [18]

$$\begin{aligned}\dot{v}_i &= (A+BK)v_i + cL\left(\sum_{j=1}^{N} a_{ij}C(v_i - v_j) - \zeta_i\right)\\ u_i &= Kv_i\end{aligned} \tag{3}$$

where $v_i \in \mathbf{R}^n$ is the protocol state, $i = 1, 2, \ldots, N$, $c > 0$ denotes the coupling strength, $L \in \mathbf{R}^{q\times n}$ and $K \in \mathbf{R}^{p\times n}$ are feedback gain matrices to be determined. In (3), the term $\sum_{j=1}^{N} a_{ij}C(v_i - v_j)$ denotes the information exchanges between the protocol of agent i and those of its neighbouring agents. It is observed that the protocol in (3) maintains the same communication topology of the agents in (1).

Let $z_i = [x_i^{\mathrm{T}}, v_i^{\mathrm{T}}]^{\mathrm{T}}$. Then, systems (1) and (3) together can be written as

$$\dot{z}_i = \mathcal{A}z_i + c\sum_{j=1}^{N} \mathcal{L}_{ij}\mathcal{H}z_j, \quad i = 1, 2, \ldots, N \tag{4}$$

where $\mathcal{L} = (\mathcal{L}_{ij})_{N\times N}$ is the Laplacian matrix of graph $\mathcal{G}$, and

$$\mathcal{A} = \begin{bmatrix} A & BK \\ 0 & A+BK \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} 0 & 0 \\ -LC & LC \end{bmatrix}$$

3.1 Dynamic consensus

The concept of dynamic consensus is introduced first

Definition 1: Given agents (1), protocol (3) is said to solve the dynamic consensus problem if the states of system (4) satisfy

$$\lim_{t\to\infty} \|z_i(t) - z_j(t)\| = 0, \quad \forall\, i, j = 1, 2, \ldots, N \tag{5}$$

Let $r \in \mathbf{R}^N$ be such that $r^{\mathrm{T}}\mathcal{L} = 0$ and $r^{\mathrm{T}}\mathbf{1} = 1$. Introduce a new variable

$$\begin{aligned}\delta(t) &= z(t) - ((\mathbf{1}r^{\mathrm{T}}) \otimes I_{2n})z(t)\\ &= ((I_N - \mathbf{1}r^{\mathrm{T}}) \otimes I_{2n})z(t)\end{aligned} \tag{6}$$

where $z = [z_1^T, \ldots, z_N^{\mathrm{T}}]^{\mathrm{T}}$ and $\delta \in \mathbf{R}^{2Nn\times 2Nn}$ satisfies $(r^{\mathrm{T}} \otimes I_{2n})\delta = 0$. Similar to [3], δ is referred to as the disagreement vector. It is easy to see that 0 is a simple eigenvalue of $I_N - \mathbf{1}r^{\mathrm{T}}$ with $\mathbf{1}$ as the right eigenvector, and 1 is another eigenvalue with multiplicity $N-1$. Thus, it follows from (6) that $\delta = 0$ if and only if $z_1 = \ldots = z_N$, that is, the dynamic consensus problem can be recast into the asymptotical stability of vector δ, which evolves according to the following dynamics

$$\dot{\delta} = (I_N \otimes \mathcal{A} + c\mathcal{L} \otimes \mathcal{H})\delta \tag{7}$$

A decomposition approach to the dynamic consensus problem is now presented.

Theorem 1 [18]: For the communication topology $\mathcal{G}$ containing a directed spanning tree, protocol (3) solves the dynamic consensus problem if and only if all the matrices $A + BK$, $A + c\lambda_i LC$, $i = 2, 3, \ldots, N$, are Hurwitz, where λ_i, $i = 2, 3, \ldots, N$, are the non-zero eigenvalues of the Laplacian matrix $\mathcal{L}$.

Proof: The ideas behind the proof is sketched here for convenience of latter reference in the sections below. For a complete proof, refer to [18].

Let $Y_1 \in \mathbf{R}^{N\times(N-1)}$, $Y_2 \in \mathbf{R}^{(N-1)\times N}$, $T \in \mathbf{R}^{N\times N}$, and an upper-triangular matrix $\Delta \in \mathbf{R}^{(N-1)\times(N-1)}$ be such that

$$\begin{aligned}T &= \begin{bmatrix}\mathbf{1} & Y_1\end{bmatrix}, \quad T^{-1} = \begin{bmatrix} r^{\mathrm{T}} \\ Y_2 \end{bmatrix},\\ T^{-1}\mathcal{L}T &= \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & \Delta \end{bmatrix}\end{aligned} \tag{8}$$

where the diagonal entries of Δ are the non-zero eigenvalues of $\mathcal{L}$. Then, (7) can be rewritten in terms of ξ, where $\xi = (T^{-1} \otimes I_{2n})\delta$ with $\xi = [\xi_1^{\mathrm{T}}, \ldots, \xi_N^{\mathrm{T}}]^{\mathrm{T}}$, as follows

$$\dot{\xi} = (I_N \otimes \mathcal{A} + c\Lambda \otimes \mathcal{H})\xi \tag{9}$$

As to ξ_1, it can be seen from (6) that

$$\xi_1 = (r^{\mathrm{T}} \otimes I_{2n})\delta \equiv 0 \tag{10}$$

Note that the elements of the state matrix of (9) are either block diagonal or block upper triangular. Hence, ξ_i, $i = 2, \ldots, N$, converge asymptotically to zero, if and only if the $N-1$ subsystems

$$\dot{\xi}_i = (\mathcal{A} + c\lambda_i\mathcal{H})\xi_i, \quad i = 2, 3, \ldots, N \tag{11}$$

are asymptotically stable, which leads to the assertion by noting that matrices $\mathcal{A} + \lambda_i\mathcal{H}$ are similar to $\begin{bmatrix} A + \lambda_i LC & 0 \\ -\lambda_i LC & A + BK \end{bmatrix}$, $i = 2, \ldots, N$. $\square$

Remark 1: The importance of this theorem lies in the fact that it converts the consensus of the multiagent system under the dynamic protocol (3) to the stability of a set of matrices with the same low dimensions as a single agent. The effects of the communication topology on the consensus are characterised by the eigenvalues of the corresponding Laplacian matrix $\mathcal{L}$. Moreover, protocol (3) is based only

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on relative output measurements between neighbouring systems, in contrast to [19–21] where the relative-state coupling laws are required.

Lemma 1 [18]: Consider network (4) whose communication topology $\mathcal{G}$ has a directed spanning tree. If protocol (3) satisfies Theorem 1, then

$$x_i(t) \to \varpi(t) \triangleq (r^{\mathrm{T}} \otimes e^{At}) \begin{bmatrix} x_1(0) \\ \vdots \\ x_N(0) \end{bmatrix} \tag{12}$$
$$v_i(t) \to 0, \quad i = 1, 2, \ldots, N, \text{ as } t \to \infty$$

where $r \in \mathbf{R}^N$ is such that $r^{\mathrm{T}}\mathcal{L} = 0$ and $r^{\mathrm{T}}\mathbf{1} = 1$.

From (12), it follows that agents (1) are excluded from having poles in the open right-half plane; otherwise, the consensus value reached by the states of (1) will tend to infinity exponentially. On the other hand, if matrix A is Hurwitz, then the agents will reach consensus onto 0. Therefore it is critical for matrix A in (1) to have eigenvalues along the imaginary axis, so that the systems can reach consensus on a non-zero value, a special case of which is that matrix A is neutrally stable.

3.2 Consensus region

Given a protocol of the form (3), the dynamic consensus problem can be cast into analysing the system

$$\dot{\varsigma} = (\mathcal{A} + \sigma\mathcal{H})\varsigma \tag{13}$$

where $\varsigma \in \mathbf{R}^{2n}$, $\sigma \in \mathbf{C}$. The stability of system (13) depends on the parameter σ, based on which the notion of consensus region is now introduced.

Definition 2 [18]: The region $\mathcal{S}$ of the complex parameter σ, such that (13) is asymptotically stable, is called the consensus region of network (4).

It follows from Theorem 1 that consensus is reached if and only if

$$c(\alpha_k + i\beta_k) \in \mathcal{S}, \quad k = 2, 3, \ldots, N$$

where $i = \sqrt{-1}$, $\alpha_k = \mathrm{Re}(\lambda_k)$ and $\beta_k = \mathrm{Im}(\lambda_k)$. For an undirected communication graph, its consensus region $\mathcal{S}$ is an interval or a union of several intervals on the real axis. However, for a directed graph, where the eigenvalues of $\mathcal{L}$ are generally complex numbers, its consensus region $\mathcal{S}$ is a region or a union of several regions on the complex plane, which can be bounded or unbounded.

For a detailed discussion on consensus regions, the reader is refereed to [18]. It should be noted that the consensus region serves in certain sense as a measure for the robustness of the protocol (3) to parametric uncertainties of its feedback gain matrix L and the communication topology. Given a consensus protocol, the consensus region should be large enough for the protocol to maintain a desirable robustness margin.

3.3 Consensus with matrix A neutrally stable

In this subsection, for the case when matrix A is neutrally stable, it is shown that an unbounded consensus region in the form of the open right-half plane can be achieved. A constructive design algorithm for protocol (3) is then presented.

Lemma 2 [32]: A complex matrix $A \in \mathbf{C}^{n\times n}$ is Hurwitz if and only if there exist a positive definite matrix $Q = Q^{\mathrm{H}}$ and a matrix $C \in \mathbf{C}^{m\times n}$ such that (A, C) is observable and $A^{\mathrm{H}}Q + QA = -C^{\mathrm{H}}C$.

Lemma 3: For matrices $S \in \mathbf{R}^{n\times n}$, $H \in \mathbf{R}^{m\times n}$, where S is skew-symmetric and (S, H) is observable, the matrix $S - (x + \mathrm{i}y)H^{\mathrm{T}}H$ is Hurwitz for any $x > 0$, $y \in \mathbf{R}$.

Proof: Let $\tilde{S} = S - (x + \mathrm{i}y)H^{\mathrm{T}}H$. Then

$$\begin{aligned} \tilde{S} + \tilde{S}^{\mathrm{H}} &= S - (x + \mathrm{i}y)H^{\mathrm{T}}H + S^{\mathrm{T}} - (x - \mathrm{i}y)H^{\mathrm{T}}H \\ &= -2xH^{\mathrm{T}}H \le 0, \quad \forall x > 0 \end{aligned} \tag{14}$$

Obviously, $(\tilde{S}, H)$ is observable, for (S, H) is observable. By Lemma 2, (14) directly leads to the assertion. □

Next, a constructive algorithm for protocol (3) is presented, which will be used later.

Algorithm 1: Given that $A \in \mathbf{R}^{n\times n}$ is neutrally stable and that the pair (A, B, C) is stabilisable and detectable, the dynamic protocol (3) can be constructed as follows:

1. Let K be such that $A + BK$ is Hurwitz.

2. Choose $U \in \mathbf{R}^{n\times n_1}$ and $W \in \mathbf{R}^{n\times(n-n_1)}$ such that

$$\begin{bmatrix} U & W \end{bmatrix}^{-1} A \begin{bmatrix} U & W \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & X \end{bmatrix} \tag{15}$$

where $S \in \mathbf{R}^{n_1\times n_1}$ is skew-symmetric and $X \in \mathbf{R}^{(n-n_1)\times(n-n_1)}$ is Hurwitz.

3. Let $L = -UU^{\mathrm{T}}C^{\mathrm{T}}$.

4. Select the coupling strength $c > 0$.

In the above algorithm, note that matrices U and W can be derived by rendering matrix A into the real Jordan canonical form [33].

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Theorem 2: Given that $A \in \mathbf{R}^{n\times n}$ is neutrally stable and that $\mathcal{G}$ has a directed spanning tree, there exists a distributed protocol in the form of (3) that solves the dynamic consensus problem and meanwhile yields an unbounded consensus region $(0, \infty) \times (-\infty, \infty)$, if and only if (A, B, C) is stabilisable and detectable.

Proof: (Necessity) It is trivial by Theorem 1.

(Sufficiency) Let the related variables be defined as in Algorithm 1. Construct the protocol (3) by Algorithm 1, and let $H = CU$. Then, (S, H) is observable for (A, C) is detectable. Let $U^{\dagger} \in \mathbf{R}^{n_1\times n}$ and $W^{\dagger} \in \mathbf{R}^{(n-n_1)\times n}$ be such that $\begin{bmatrix} U^{\dagger} \\ W^{\dagger} \end{bmatrix} = \begin{bmatrix} U & W \end{bmatrix}^{-1}$, where $U^{\dagger}U = I$, $W^{\dagger}W = I$, $U^{\dagger}W = 0$ and $W^{\dagger}U = 0$. It can be verified by some algebraic manipulations that

$$\begin{aligned} &\begin{bmatrix} U & W \end{bmatrix}^{-1}(A + (x+iy)LC)\begin{bmatrix} U & W \end{bmatrix} \\ &\quad = \begin{bmatrix} S + (x+iy)U^{\dagger}LCU & (x+iy)U^{\dagger}LCW \\ (x+iy)W^{\dagger}LCU & X + (x+iy)W^{\dagger}LCW \end{bmatrix} \\ &\quad = \begin{bmatrix} S - (x+iy)H^{\mathrm{T}}H & -(x+iy)H^{\mathrm{T}}SW \\ 0 & X \end{bmatrix} \end{aligned} \tag{16}$$

which implies that matrix $A + (x+iy)LC$ is Hurwitz for all $x > 0$ and $y \in \mathbf{R}$, because by Lemma 1 matrix $S - (x+iy)H^{\mathrm{T}}H$ is Hurwitz for any $x > 0$ and $y \in \mathbf{R}$. Hence, by Theorem 1, the protocol given by Algorithm 1 solves the dynamic consensus problem with an unbounded consensus region $(0, \infty) \times (-\infty, \infty)$. □

Remark 2: The consensus region $(0, \infty) \times (-\infty, \infty)$ achieved by the protocol constructed by Algorithm 1 means that such a protocol can reach consensus for any communication topology containing a directed spanning tree and for any positive coupling strength. However, the communication topology and the coupling strength do affect the performances of consensus, for example, the convergence speed, which is the topic of the next subsection. It should be pointed out that similar results were reported in [20]. Nevertheless, a general case of dynamic consensus protocol based on relative measurements of the neighbouring agents are investigated here, containing the static protocol study of [20] as a special case. Moreover, the method leading to Theorem 2 here is quite different from that in [20], and indeed is comparatively much simpler.

3.4 Consensus with prescribed convergence speed

For the general case where matrix A has no eigenvalues in the open right-half plane, the protocol (3) is designed in this subsection to achieve consensus with a prescribed convergence speed. The results given in [18] can be regarded as a special case without specification on the convergence speed. Previous works along this line include [3, 27, 28], where the convergence speed of consensus for networks of integrators was analysed.

From the proof of Theorem 1, it is easy to see that the convergence speed of N agents in (1) reaching consensus under protocol (3) is equal to the minimal decay rate of the $N-1$ systems in (11). The decay rate of system $\dot{x} = Ax$ is defined as the maximum of negative real parts of the eigenvalues of matrix A [34]. Thus, the convergence speed of agents (1) reaching consensus can be manipulated by properly assigning the eigenvalues of matrices $A + BK$, $A + c\lambda_i LC$, $i = 2, 3, \ldots, N$.

Lemma 4 [34]: The decay rate of system $\dot{x} = Ax$ is larger than $\alpha > 0$, if and only if there exists a matrix $Q > 0$ such that

$$A^{\mathrm{T}}Q + QA + 2\alpha Q < 0$$

Proposition 1: Given the agents (1), there exists a matrix L such that $A + (x+iy)LC$ is Hurwitz with a decay rate larger than α for all $x \in [1, \infty)$, $y \in (-\infty, \infty)$, if and only if there exists a matrix $Q = Q^{\mathrm{T}} > 0$ such that

$$A^{\mathrm{T}}Q + QA - 2C^{\mathrm{T}}C + 2\alpha Q < 0 \tag{17}$$

Proof: (Sufficiency) By Lemma 4, there exists a matrix L such that $A + LC$ is Hurwitz with a decay rate larger than α if and only if there exists a matrix $Q = Q^{\mathrm{T}} > 0$ such that

$$(A + LC)^{\mathrm{T}}Q + Q(A + LC) + 2\alpha Q < 0$$

Let $QL = V$. Then, the above inequality becomes

$$A^{\mathrm{T}}Q + QA + VC + C^{\mathrm{T}}V^{\mathrm{T}} + 2\alpha Q < 0$$

By Finsler's Lemma [35], there exists a matrix V satisfying the above inequality if and only if there exists a scalar $\tau > 0$ such that

$$A^{\mathrm{T}}Q + QA - \tau C^{\mathrm{T}}C + 2\alpha Q < 0 \tag{18}$$

Without loss of generality, letting $\tau = 2$ in (18) leads to (17). Take $V = -C^{\mathrm{T}}$, that is, $L = -Q^{-1}C^{\mathrm{T}}$. By the above inequalities, one has

$$\begin{aligned} &(A + (x+iy)LC)^{\mathrm{H}}Q + Q(A + (x+iy)LC) + 2\alpha Q \\ &\quad = (A + (x-iy)LC)^{\mathrm{T}}Q + Q(A + (x+iy)LC) + 2\alpha Q \\ &\quad = AQ + QA^{\mathrm{T}} - 2xC^{\mathrm{T}}C + 2\alpha Q < 0 \end{aligned}$$

for all $x \geq 1$, that is, $A + (x+iy)LC$ is Hurwitz with a decay rate larger than α for all $x \in [1, \infty)$, $y \in (-\infty, \infty)$.

(Necessity) It is trivial by letting $x = 1$, $y = 0$. □

The above proposition and Theorem 1 lead to the following result.

Theorem 3: For network (4) with $\mathcal{G}$ containing a directed spanning tree, there exists a protocol (3) that solves the consensus problem with a convergence rate larger than α and yields an unbounded consensus region $[1, \infty) \times (-\infty, \infty)$, if and only if there exist matrices K and L such that both $A + BK$ and $A + LC$ are Hurwitz with a decay rate larger than α.

Algorithm 2: For graph $\mathcal{G}$ containing a directed spanning tree, a protocol (3) solving the dynamic consensus problem with a convergence speed larger than α can be constructed as follows:

1. Obtain the feedback gain matrix K, for example, by using the Ackermann's formula, such that the poles of matrix $A + BK$ lie in the left-half plane of $x = -\alpha$.

2. Choose the feedback gain matrix $L = -Q^{-1}C^{\mathrm{T}}$, where $Q > 0$ is one solution to (17).

3. Select the coupling strength c larger than the threshold value c_{th} given by

$$c_{\mathrm{th}} = \frac{1}{\min_{i=2,\ldots,N} \mathrm{Re}(\lambda_i)} \tag{19}$$

where λ_i, $i = 2, \ldots, N$, are the non-zero eigenvalues of matrix $\mathcal{L}$.

Remark 3: Algorithm 2 has a favourable decoupling feature. To be specific, steps 1 and 2 deal only with the agent dynamics and the feedback gain matrices of the consensus protocol, leaving the communication topology of the multi-agent network to be handled in step 3 by manipulating the coupling strength. The protocol designed by Algorithm 2 for one communication graph is applicable to any other graph with larger minimum real parts of eigenvalues, thereby is robust in this sense to the communication topology. For the case where the agent number N is large, for which the eigenvalues of the corresponding Laplacian matrix are hard to determine or even troublesome to estimate, one only needs to choose the coupling strength to be large enough.

Remark 4: Compared to the consensus when A is neutrally stable, where the coupling strength can be chosen as any positive scalar, for the case where A is critically unstable or a prescribed convergence speed is desired, the coupling strength generally has to be larger than a threshold value, which is related to the specific communication topology. This is consistent with the intuition that unstable behaviours are more difficult to synchronise than stable behaviours.

Remark 5: One sufficient condition satisfying Theorem 3 is that (A, B, C) is controllable and observable. Under such a condition, the protocol achieving consensus with a convergence speed larger than an arbitrary given positive value can be constructed by Algorithm 2. However, larger α implies higher feedback gains in protocol (3). Thus, a trade-off has to be made between the convergence speed and the cost of the consensus protocol.

Example 1: The agent dynamics are given by (1), with

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Obviously, matrix A is neutrally stable, and (A, B, C) is controllable and observable. A third-order consensus protocol is in the form of (3).

By the function place of Matlab, the feedback gain matrix K of (3) is given as $K = \begin{bmatrix} -4.5 & -5.5 & -3 \end{bmatrix}$ such that the poles of $A + BK$ are -1, -1.5, -2. The matrix U such that $U^{-1}AU = J$ is of the real Jordan canonical form is

$$U = \begin{bmatrix} 0 & 0.5774 & 0 \\ 0 & 0 & -0.5774 \\ 1 & -0.5774 & 0 \end{bmatrix}, \quad \text{with}$$

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Then, by Algorithm 1, the feedback gain L of (3) is obtained as $L = \begin{bmatrix} 0.3333 & 0 & -1.3333 \end{bmatrix}^{\mathrm{T}}$. By Theorem 2, the agents under this protocol can reach consensus with respect to any communication graph containing a spanning tree and for any positive coupling strength. One such graph with 6 nodes is shown in Fig. 1, whose non-zero eigenvalues are 3 and 1 with multiplicity 4. Select the coupling strength $c = 1$ for simplicity. It can be verified that the convergence speed in this case equals 0.0303.

Next, protocol (3) is redesigned to achieve consensus with a specified convergence speed larger than 1. The feedback gain K is chosen the same as above. Solving linear matrix

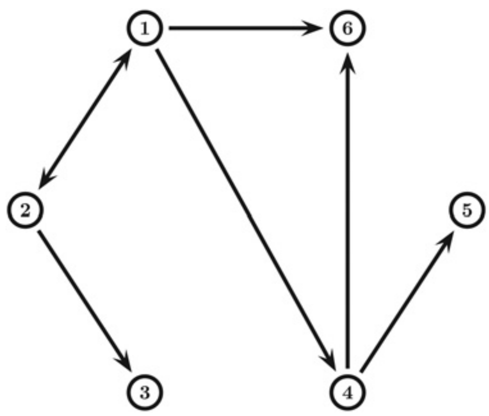


Figure 1 *Communication graph*

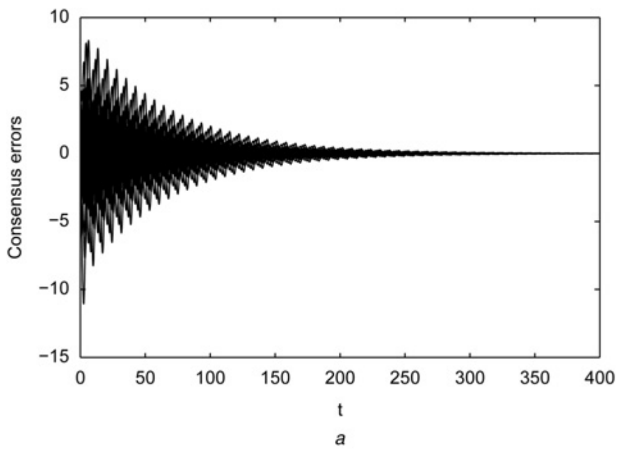
Inequality (LMI) (17) with $\alpha = 1$ by using Sedumi toolbox [36] gives $L = [-8.5763 \quad -15.2128 \quad -5.6107]^{\mathrm{T}}$. For the graph in Fig. 1, the threshold value for the coupling strength is $c_{\mathrm{th}} = 1$ by (19). Select $c = 1$, the same as before. The consensus errors $x_i - x_1$, $i = 2, \ldots, 6$, for the graph in Fig. 1 under the protocols generated by Algorithms 1 and by Algorithm 2 with $\alpha = 1$, are depicted in Figs. 2a and b, respectively. It can be observed that the consensus process of the former case is indeed much slower than the latter.

4 Consensus with static protocols

In this section, a special case where the relative states between neighbouring agents are available is considered. For this case, a distributed static protocol is proposed as

$$u_i = cF \sum_{j=1}^{N} a_{ij}(x_i - x_j) \tag{20}$$

where $c > 0$ and a_{ij} are the same as those defined in (3), and $F \in \mathbf{R}^{p\times n}$ is the feedback gain matrix to be determined. For protocol (20), the dynamic consensus problem studied in Section 3 reduces to the following static consensus problem.



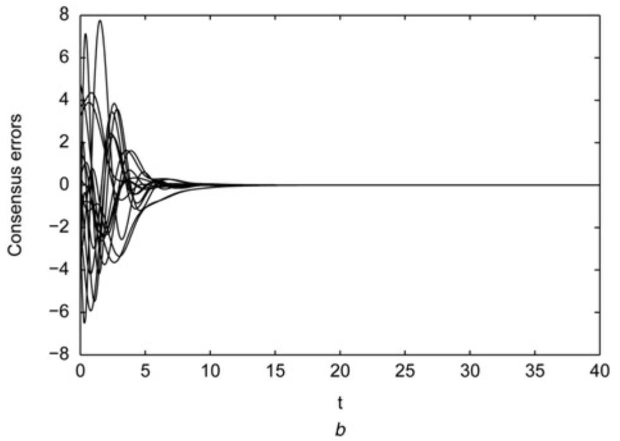


Figure 2 *Consensus errors*

a Algorithm 1
b Algorithm 2 with $\alpha = 1$

Definition 3: Protocol (20) is said to solve the (static) consensus problem if the states of agents (1) with (20) satisfy

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \quad \forall\, i, j = 1, 2, \ldots, N \tag{21}$$

The following result is a direct consequence of Theorem 1 in the static case.

Corollary 1: For graph $\mathcal{G}$ containing a directed spanning tree, there exist a protocol (20) solving the consensus problem if and only if all the matrices $A + c\lambda_i BF$, $i = 2, 3, \ldots, N$, are Hurwitz, where λ_i, $i = 2, 3, \ldots, N$, are the same as in Theorem 1.

For the case where matrix A is neutrally stable, one has the following.

Corollary 2: Given that $A \in \mathbf{R}^{n\times n}$ is neutrally stable and that $\mathcal{G}$ has a directed spanning tree, there exists a protocol (20) solving the consensus problem and yielding a consensus region $(0, \infty) \times (-\infty, \infty)$, if and only if (A, B) is stabilisable.

Contrary to the sufficient condition given in [20], a necessary and sufficient condition is obtained here as a consequence of Theorem 2. The method for constructing the protocol (20) is similar to Algorithm 1, therefore is omitted for brevity.

The design procedure for the protocol (20) solving consensus with a specification on the convergence speed is now presented.

Algorithm 3: For a controllable pair (A, B), a protocol (20) solving the consensus problem with a convergence speed larger than α can be constructed as follows:

1. Choose the feedback gain matrix $F = -B^{\mathrm{T}}P^{-1}$, where $P > 0$ is a solution to

$$AP + PA^{\mathrm{T}} - 2BB^{\mathrm{T}} + 2\alpha P < 0$$

2. Select the coupling strength $c \geq c_{\mathrm{th}}$, with c_{th} given by (19).

5 Extension to formation control

In this section, the consensus algorithms are modified to solve formation control problems of multi-agent systems.

Let $\widetilde{H} = (h_1, h_2, \ldots, h_N) \in \mathbf{R}^{n\times N}$ describe a constant formation structure of the agent network in a reference coordinate frame, where $h_i \in \mathbf{R}^n$, is the formation variable corresponding to agent i. Then, variable $h_i - h_j$ can be used to denote the relative formation vector between agents i and j, which is independent of the reference coordinate. For the agents (1), a distributed formation protocol is proposed as

$$v_i^+ = (A + BK)v_i + cL\left(\sum_{j=1}^{N} d_{ij}C(v_i - v_j) - \sum_{j=1}^{N} d_{ij}(y_i - y_j - C(h_i - h_j))\right) \quad (22)$$
$$u_i = Kv_i$$

where the variables are the same as those in (3). It should be noted that (22) reduces to the consensus protocol (3), when $h_i - h_j = 0, \forall\, i, j = 1, 2, \ldots, N$.

Definition 4: The agents (1) under protocol (22) achieve a given formation $\widetilde{H} = (h_1, h_2, \ldots, h_N)$, if

$$\lim_{t\to\infty} \|(x_i(t) - h_i) - (x_j(t) - h_j)\| \to 0, \quad \forall\, i, j = 1, 2, \ldots, N \quad (23)$$

Theorem 4: For graph $\mathcal{G}$ containing a directed spanning tree, the agents (1) reach the formation $\widetilde{H}$ under protocol (22) if all the matrices $A + BK$, $A + c\lambda_i LC$, $i = 2, \ldots, N$, are Hurwitz, and $Ah_i = 0$, $\forall\, i = 1, 2, \ldots, N$, where λ_i, $i = 2, 3, \ldots, N$, are the non-zero eigenvalues of matrix $\mathcal{L}$.

Proof: Let $\tilde{x}_i = x_i - h_i$ and $\tilde{z}_i = [\tilde{x}_i^{\mathrm{T}}, v_i^{\mathrm{T}}]^{\mathrm{T}}$, $i = 1, 2, \ldots, N$. Then, systems (1) and (22) together can be written as

$$\dot{\tilde{z}}_i = \mathcal{A}\tilde{z}_i + c\sum_{j=1}^{N} \mathcal{L}_{ij}\mathcal{H}\tilde{z}_j + \begin{bmatrix} Ah_i \\ 0 \end{bmatrix}, \quad i = 1, 2, \ldots, N \quad (24)$$

where matrices $\mathcal{A}$ and $\mathcal{H}$ are defined in (4). Note that the formation $\widetilde{H}$ is achieved if system (24) reaches consensus, which by (6) implies that $Ah_i = 0$, $i = 1, 2, \ldots, N$. The rest is similar to the proof of Theorem 1, thus is omitted for brevity. □

Remark 6: Note that not all kinds of formation structure can be achieved for the agents (1) by using protocol (22). The achievable formation structures have to satisfy the constraints $Ah_i = 0$, $\forall\, i = 1, 2, \ldots, N$. Note that h_i can be replaced by $h_i - h_1$, $i = 2, \ldots, N$, in order to be independent of the reference coordinate, by simply choosing h_1 corresponding to agent 1 as the origin. The formation protocol (22) satisfying Theorem 4 can be constructed by using Algorithm 1 or 2. Theorem 4 generalises the results given in [7, 29, 30, 31], where the agent dynamics in [7, 30, 31] are (generalised) double integrators and the protocol in [29] is static.

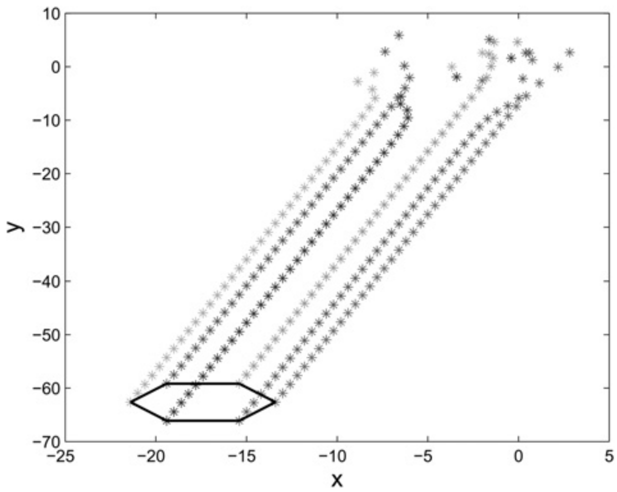


Figure 3 *Six agents form a hexagon*

Example 2: Consider a network of six double integrators, described by

$$\dot{x}_i = v_i$$
$$\dot{\tilde{v}}_i = u_i$$
$$y_i = x_i, \quad i = 1, 2, \ldots, 6$$

where $x_i \in \mathbf{R}^2$, $\tilde{v}_i \in \mathbf{R}^2$, $y_i \in \mathbf{R}^2$ and $u_i \in \mathbf{R}^2$ are the position, velocity, measured output and acceleration input of agent i, respectively.

The objective is to design a dynamic protocol (22) such that the agents will evolve to form a regular hexagon with edge length 4. In [7], a different dynamic protocol is proposed to study formation control of double-integrator agents, which however, is applicable only to undirected communication topologies. As pointed out in [26], the spacecraft's dynamics in deep space can be modelled as a double integrator. Therefore this example may have possible applications in deep-space formation flying mission.

In this case, choose $h_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$, $h_2 = \begin{bmatrix} 4 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$, $h_3 = \begin{bmatrix} 6 & 2\sqrt{3} & 0 & 0 \end{bmatrix}^{\mathrm{T}}$, $h_4 = \begin{bmatrix} 4 & 4\sqrt{3} & 0 & 0 \end{bmatrix}^{\mathrm{T}}$, $h_5 = \begin{bmatrix} 0 & 4\sqrt{3} & 0 & 0 \end{bmatrix}^{\mathrm{T}}$ and $h_6 = \begin{bmatrix} -2 & 2\sqrt{3} & 0 & 0 \end{bmatrix}^{\mathrm{T}}$. Take $K = \begin{bmatrix} -1.5 & -2.5 \end{bmatrix} \otimes I_2$ in (22) such that matrix $A + BK$ has eigenvalues -1 and -1.5. By solving LMI (17) with $\alpha = 1$, one obtains $L = \begin{bmatrix} -3.6606 & -4.8221 \end{bmatrix}^{\mathrm{T}} \otimes I_2$. By Theorem 4 and Algorithm 2, the six agents under protocol (22) with K, L given as above, and $c = 1$ will form a regular hexagon with a convergence rate larger than 1 for the communication topology given in Fig. 1. The state trajectories of the six agents are depicted in Fig. 3.

6 Conclusions

This paper has studied the dynamic consensus of a linear or linearised multi-agent system whose communication

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topology has a directed spanning tree. An observer-type protocol based on the relative outputs of the neighbouring agents has been adopted. For neutrally stable agents, it has been shown that there exists a protocol achieving consensus over a consensus region that is the entire open right-half plane if and only if each agent is stabilisable and detectable. Moreover, algorithms have been proposed to derive protocols to achieve consensus with or without convergence speed specification. These design procedures have a favourable decoupling property, very desirable from a computational point of view. The consensus algorithms have been further extended to solve formation control problems.

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