A Class of Bounded Distributed Control Strategies for Connectivity Preservation in Multi-Agent Systems

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***Abstract*—This technical note proposes a general class of distributed potential-based control laws with the connectivity preserving property for single-integrator agents. The potential functions are designed in such a way that when an edge in the information flow graph is about to lose connectivity, the gradient of the potential function lies in the direction of that edge, aiming to shrink it. The results are developed for a static information flow graph first, and then are extended to the case of dynamic edge addition. Connectivity preservation for problems involving static leaders is covered as well. The potential functions are chosen to be smooth, resulting in bounded control inputs. Other constraints may also be imposed on the potential functions to satisfy various design criteria such as consensus, containment, and formation convergence. The effectiveness of the proposed control strategy is illustrated by simulation for examples of consensus and containment.**

***Index Terms*—Autonomous agents, cooperative control, distributed control, graph connectivity.**

I. Introduction

Cooperative control of a group of autonomous agents has been extensively studied in the past few years [1]–[3]. This relatively new line of research has been motivated by the increasing application of multi-agent systems such as mobile robots, formation flying of UAVs, automated highway systems, air traffic control, and mobile sensor networks [1], [4], [5]. The main goal in such applications is to find distributed control paradigms satisfying a global objective defined over the entire network. Examples of such an objective include flocking, consensus, containment, and formation [2], [6]–[9].

One of the main assumptions in the distributed control of multi-agent systems is the connectivity of the corresponding network. Therefore, regardless of the overall objective, the designed control laws should preserve the network connectivity, which is usually distance-dependent. This issue has been investigated in several recent papers (e.g., see [4], [10]–[20]). A localized notion of connectedness is introduced in [10], where it is shown that under certain conditions the global connectedness of the network is also guaranteed. In [11] and [12], the problem is tackled by maximizing the second largest eigenvalue of the state-dependent Laplacian of the network graph. [13] presents a leader to follower ratio that ensures connectivity preservation in a leader-follower multi-agent network. In order to maintain the existing links in the network, the papers [4], [14]–[16] use some potential fields that "blow up" whenever a link in the network is losing connectivity. In [17]–[19], appropriate nonlinear weights are designed for the edges of the interaction graph to ensure network connectivity. However, these weights tend to infinity when a pair of agents forming an edge approach a critical distance at which they lose connectivity. These techniques may not be effective in practice since the actuators of the agents can only handle finite forces or torques. To the best of the authors' knowledge, the only bounded control law reported in the literature so far is the one proposed in [20], where connectivity is claimed to maintain for a distributed navigation function which was used earlier in [21]–[23] for collision avoidance concerning robot navigation, and in [24] for formation stabilization.

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In this work, a general class of distributed potential functions is introduced with the connectivity preserving property for single-integrator agents. The main idea of the proposed approach is to design the potential functions in such a way that when an edge belonging to the information flow graph is about to lose connectivity, the gradient of the potential function lies in the direction of that edge, aiming to shrink it. The results are presented for a static information flow graph first, and are then extended to the case of dynamic edge addition. The topology of the agents that may stay fixed under the proposed control strategy is properly characterized with the purpose of extending the strategy to problems involving static leaders in which the agents assigned as leaders are to stay fixed. This is another advantage of the control scheme presented here over existing connectivity preserving approaches. The potential functions are chosen to be smooth, resulting in bounded control inputs. Additional constraints may be imposed on the potential functions to meet other design specifications such as consensus, containment, and formation convergence. It is to be noted that although the connectivity preserving control law proposed in [20] is also bounded, the corresponding framework can be regarded as a subcase of the one in the present technical note. Furthermore, [20] does not consider the case where some of the edges of the information flow graph start exactly at the critical distance. Consequently, [20] cannot be used in the case of static leaders.

The remainder of this technical note is organized as follows. The problem statement is provided in Section II. The proposed connectivity preserving control design is elaborated in Section III. The extension of the results to the case of dynamic information flow graph and problems involving static leaders is presented in Sections IV and V. Simulation results for the examples of consensus and containment are provided in Section VI, and finally concluding remarks are drawn in Section VII.

II. Problem Formulation

Definition 1: For a real or vector-valued function $f(t)$, the index of f at time t, denoted by $\rho(f(t))$, is defined to be the smallest natural number n for which $f^{(n)}(t) \neq 0$, where $f^{(n)}(t)$ is the n-th derivative of f at time t.

Definition 2: The function f is said to be of class C^k if the derivatives $f^{(1)}, \ldots, f^{(k)}$ exist and are continuous. A function f of class C^∞ is referred to as a smooth function.

Definition 3: Multinomial coefficients are defined by

$$\binom{k}{r_1, r_2, \ldots, r_\mu} := \frac{k!}{r_1! r_2! \ldots r_\mu!}$$

where $r_1, r_2, \ldots, r_\mu$ are nonnegative integers, and $k = r_1 + r_2 + \cdots + r_\mu$. In the special case of $\mu = 2$, the corresponding coefficients are called the binomial coefficients, and are shown by $\binom{k}{r_1, r_2} = \binom{k}{r_1} = \binom{k}{r_2}$.

Consider a set of n single-integrator agents in a plane with a control law of the form

$$\dot{q}_i(t) = u_i = -\frac{\partial h_i}{\partial q_i} \tag{1}$$

where $q_i(t)$ denotes the position of agent i in the plane at time t, and h_i's are distributed potential functions. Denote with $G = (V, E)$ the information flow graph, with $V = \{1, \ldots, n\}$ its vertices, and with $E \subset V \times V$ its edges. It is assumed that the information flow graph

G is connected and undirected, and that each agent can only use the relative position of its neighbors in its control law. Denote the set of the neighbors of agent i in G with $N_i(G)$, and the degree of agent i in G with $d_i(G)$. Two agents i and j are said to be in the connectivity range if $\|q_i - q_j\| \leq d$, for a pre-specified positive real number d, where $\|\cdot\|$ denotes the Euclidean norm. It is assumed that all agents in $N_i(G)$ are initially located in the connectivity range of agent i. The goal is to design a class of distributed potential functions that preserve connectivity. More precisely, it is desired to find a control scheme such that if $\|q_i(0)-q_j(0)\| \leq d$ for all $(i,j) \in E$, then $\|q_i(t)-q_j(t)\| \leq d$, for all $(i,j) \in E$ and all $t \geq 0$.

III. Connectivity Preserving Controller Design

For every agent i, define

$$\sigma_i(t) := \frac{1}{2} \sum_{j \in N_i(G)} \|q_i(t) - q_j(t)\|^2 \tag{2}$$

$$\pi_i(t) := \frac{1}{2} \prod_{j \in N_i(G)} (d^2 - \|q_i(t) - q_j(t)\|^2) \tag{3}$$

$$\pi_{ij}(t) := \prod_{\substack{k \in N_i(G) \\ k \neq j}} (d^2 - \|q_i(t) - q_k(t)\|^2). \tag{4}$$

Consider a set of distributed smooth potential functions of the form $h_i(\sigma_i, \pi_i)$ with the following properties

$$\frac{\partial h_i}{\partial \sigma_i}(\sigma_i, 0) = 0,\ \frac{\partial h_i}{\partial \pi_i}(\sigma_i, 0) < 0, \qquad \forall \sigma_i \in \mathbb{R}^+. \tag{5}$$

Intuitively, under these conditions when agent i is about to lose connectivity ($\pi_i = 0$), any change in h_i results only from a change in π_i, and if the agents move in a direction that decreases h_i, then the connectivity will improve (i.e., π_i will increase). On the other hand, when π_i becomes zero, any change in it results only from a change in q_i and q_j, where j is the agent which is exactly at distance d from agent i; therefore, only q_i and q_j can influence h_i. Agent i is clearly moving in a direction which tends to decrease h_i, according to (1). It can be shown that agent j also moves in a direction which tends to decrease h_i (although its corresponding potential function is different from h_i). This argument is valid for the case when agent i is at distance d from only one neighbor. For the general case, one should look at higher-order derivatives (not just the gradient).

It is desired now to show that using this type of potential function, the control law (1) is connectivity preserving. Using the equality $\partial h_i/\partial q_i = (\partial h_i/\partial \sigma_i)(\partial \sigma_i/\partial q_i) + (\partial h_i/\partial \pi_i)(\partial \pi_i/\partial q_i)$, one can rewrite the control law (1) as

$$\dot{q}_i = -\sum_{j \in N_i(G)} (q_i - q_j) \left(\frac{\partial h_i}{\partial \sigma_i} - \frac{\partial h_i}{\partial \pi_i} \pi_{ij} \right). \tag{6}$$

Define $\mathcal{T}$ to be the set of those time instants $t \geq 0$ at which $\|q_i(t) - q_j(t)\| \leq d$, for all $(i,j) \in E$. For any $t \in \mathcal{T}$, construct a graph $G_d(t) = (V_d(t), E_d(t))$ as the union of those edges $(i,j) \in E$ for which $\|q_i(t) - q_j(t)\| = d$. Define $s_{ij}(t) = \|q_i(t) - q_j(t)\|^2$, for $(i,j) \in E_d$. The following lemmas are key to the proof of the main results.

Lemma 1: Consider a real or vector-valued function f for which $f^{\rho(f(t))}(t) < 0$, for some t; then f is monotonically decreasing in the interval $[t, t+\epsilon]$, for some $\epsilon > 0$.

Proof: The proof is straightforward and is omitted here. ■

Lemma 2: Suppose that $q_i^{(r)}(t) = q_j^{(r)}(t) = 0$, for all $r \in \{1, \ldots, k-1\}$ and some t; then

$$s_{ij}^{(k)}(t) = 2(q_i(t) - q_j(t))^T (q_i^{(k)}(t) - q_j^{(k)}(t)). \tag{7}$$

Proof: The proof is straightforward and is omitted here. ■

Lemma 3: Consider agent i in $G_d(t)$ for some $t \in \mathcal{T}$, and assume that $\eta = \min_{j \in N_i(G)} \{\rho(\pi_{ij})\}$. Assume also that $d_i(G_d) \geq 2$; then the following statements hold:

i) $\pi_{ij}^{(r)} = 0$, for $0 \leq r \leq \eta - 1$, and $j \in N_i(G)$.
ii) $\pi_i^{(r)} = 0$, for $0 \leq r \leq \eta - 1$.
iii) $(\partial h_i/\partial \sigma_i)^{(r)} = 0$, for $0 \leq r \leq \eta - 1$.
iv) $\rho(q_i) \geq \eta + 1$.

Proof:

Part (i): Since $d_i(G_d) \geq 2$, one can easily verify that $\pi_{ij} = 0$. The rest of the proof follows immediately from Definition 1.

Part (ii): This can be easily deduced from part (i), on noting that $\pi_i = \frac{1}{2}\pi_{ij} \times (d^2 - \|q_i - q_j\|^2)$ for any j in $N_i(G)$.

Part (iii): From (5), $(\partial h_i/\partial \sigma_i)(\sigma_i, 0) = 0$. It can be shown recursively that $\partial^r h_i/\partial \sigma_i^r = 0$ for all $r \in \mathbb{N}$. Using induction on r, one can express $(\partial h_i/\partial \sigma_i)^{(r)}$ in the form of

$$\left(\frac{\partial h_i}{\partial \sigma_i}\right)^{(r)} = \sum_{m \leq r+1} \frac{\partial^m h_i}{\partial \sigma_i^m} a_m(\sigma_i) + \sum_{m \leq r} b_m(\sigma_i, \pi_i) \pi_i^{(m)}. \tag{8}$$

The first term in the right side of (8) is zero as noted above. Hence, the proof is completed by noting that $\pi_i^{(m)} = 0$ for $m \leq r$ (from the result of part (ii)).

Part (iv): By differentiating k times both sides of (6), one arrives at

$$q_i^{(k+1)} = -\sum_{j \in N_i(G)} \sum_{r=0}^{k} (q_i - q_j)^{(r)} \left(\frac{\partial h_i}{\partial \sigma_i} - \frac{\partial h_i}{\partial \pi_i} \pi_{ij} \right)^{(k-r)} \binom{k}{r}. \tag{9}$$

The right side of the above equation is equal to zero for all $k \in \{0, \ldots, \eta - 1\}$, as a consequence of parts (i)-(iii). This implies that $\rho(q_i) \geq \eta + 1$. ■

Remark 1: In the case when $d_i(G_d) = 1$, it is straightforward to show that $\dot{q}_i = (\partial h_i/\partial \pi_i)\pi_{ij}(q_i - q_j)$, where j is the neighbor for which $\|q_i - q_j\| = d$.

Remark 2: If $\rho(\pi_{ij})$ is not the same for all $j \in N_i(G_d)$, then part (ii) of Lemma 3 also holds for $r = \eta$. Consequently, part (iii) also holds for $r = \eta$.

Lemma 4: Consider agent i in $G_d(t)$, $t \in \mathcal{T}$, and let ν be one of the (possibly multiple) neighbors of i in $G_d(t)$ for which $\rho(q_\nu) = \max_{j \in N_i(G_d)} \{\rho(q_j)\}$. Then

$$\rho(q_i) \geq 1 + \sum_{\substack{j \in N_i(G_d) \\ j \neq \nu}} \rho(q_j). \tag{10}$$

Proof: The proof is trivial for the case when $d_i(G_d) = 1$. Hence, consider the case $d_i(G_d) \geq 2$; for any $j \in N_i(G)$, by differentiating (4) k times, one can show that

$$\pi_{ij}^{(k)} = \sum_{\substack{r_1 + \cdots + r_\mu = k \\ r_1, \ldots, r_\mu \geq 0}} \binom{k}{r_1, \ldots, r_\mu} \prod_{s=1}^{\mu} (d^2 - \|q_i - q_{i_s}\|^2)^{(r_s)} \tag{11}$$

where $\{i_1, \ldots, i_\mu\} = N_i(G) - \{j\}$. Let $k \leq \eta$; then, on using Lemma 2 and noting (from Lemma 3) that $\rho(q_i) > k$, one can easily verify that the term corresponding to $(r_1, \ldots, r_\mu)$ in the above summation is nonzero only if $r_s \geq \rho(q_{i_s})$ for every $i_s \in N_i(G_d) - \{j\}$. On the other hand

$$k = \sum_{s=1}^{\mu} r_s \geq \sum_{\substack{i_s \in N_i(G_d) \\ i_s \neq j}} r_s. \tag{12}$$

Therefore, a necessary condition for $\pi_{ij}^{(k)}$ to be nonzero can be obtained as

$$k \geq \sum_{\substack{i_s \in N_i(G_d) \\ i_s \neq j}} \rho(q_{i_s}). \quad (13)$$

Now, choose $k = \eta$; since $\eta = \min_{j \in N_i(G)}\{\rho(\pi_{ij})\}$, thus $\pi_{ij}^{(\eta)} \neq 0$ for at least one $j \in N_i(G)$. Hence, (13) should hold for $k = \eta$ and at least one $j \in N_i(G)$. Clearly, the right side of (13) is minimized when $\rho(q_j)$ is maximized (i.e., when $j = \nu$). This fact along with part (iv) of Lemma 3 results in (10). ■

Lemma 5: Let $\rho_l(q_i)$ be the lower bound for $\rho(q_i)$ given in Lemma 4, i.e.

$$\rho_l(q_i) = 1 + \sum_{\substack{j \in N_i(G_d) \\ j \neq \nu}} \rho(q_j) \quad (14)$$

where $\nu = \operatorname{argmax}_{j \in N_i(G_d)}\{\rho(q_j)\}$. If ν is unique, then

i) $\pi_{i\nu}^{(\rho_l(q_i)-1)} = \tilde{\pi}_{i\nu} \prod_{\substack{j \in N_i(G_d) \\ j \neq \nu}} (q_i - q_j)^T q_j^{(\rho(q_j))}$, where $\tilde{\pi}_{i\nu} > 0$.

ii) $q_i^{(\rho_l(q_i))} = (\partial h_i/\partial \pi_i)\tilde{\pi}_{i\nu}(q_i - q_\nu) \prod_{\substack{j \in N_i(G_d) \\ j \neq \nu}} (q_i - q_j)^T \backslash q_j^{(\rho(q_j))}$.

Proof:

Part (i): The proof follows by revisiting (11) for $k = \rho_l(q_i) - 1$ and noting that (13) holds only for $j = \nu$. The details of the proof are omitted due to space restrictions.

Part (ii): The proof can be carried out by applying Lemma 3 to (9), using Remark 2, and noting that $\pi_{ij}^{(k)} = 0$ (for $j \neq \nu$) for $k = \rho_l(q_i) - 1$. ■

Lemma 6: Define the subgraph $G_d^{<\infty}(t)$ of $G_d(t)$ as the union of those edges $e = (i,j) \in E_d(t)$ for which $\min(\rho(q_i), \rho(q_j)) < \infty$, and denote its set of edges with $E_d^{<\infty}(t)$. Then, for any $(i,j) \in E_d^{<\infty}(t)$, the relations $\rho(s_{ij}) = \min\{\rho(q_i), \rho(q_j)\}$ and $s_{ij}^{(\rho(s_{ij}))} < 0$ hold.

Proof: One can prove this lemma by induction on $\min(\rho(q_i), \rho(q_j))$. The case of $\min(\rho(q_i), \rho(q_j)) = 1$ is straightforward using Remark 1. Now, suppose that the lemma holds for $\min(\rho(q_i), \rho(q_j)) < k$. To prove the lemma for $\min(\rho(q_i), \rho(q_j)) = k$, assume without loss of generality that $\rho(q_i) = k$. Using Lemma 4 one can easily show that $\operatorname{argmax}_{\omega \in N_i(G_d)}\{\rho(q_\omega)\}$ is unique, and is, in fact, equal to j. Also, $\rho(q_\omega) < \rho(q_i)$ for $\omega \in N_i(G_d)$, $\omega \neq j$. Therefore, $\min(\rho(q_i), \rho(q_\omega)) = \rho(q_\omega) < k$ and hence $\rho(s_{i\omega}) = \rho(q_\omega)$ and $s_{i\omega}^{(\rho(s_{i\omega}))} < 0$. This along with Lemmas 2 and 5 yields that

$$q_i^{(\rho_l(q_i))} = \frac{\partial h_i}{\partial \pi_i}\tilde{\pi}_{ij}(q_i - q_j) \prod_{\substack{\omega \in N_i(G_d) \\ \omega \neq j}} -\frac{1}{2} s_{i\omega}^{(\rho(s_{i\omega}))}. \quad (15)$$

Thus

$$(q_i - q_j)^T q_i^{(\rho_l(q_i))} = \frac{\partial h_i}{\partial \pi_i}\tilde{\pi}_{ij} d^2 \prod_{\substack{\omega \in N_i(G_d) \\ \omega \neq j}} -\frac{1}{2} s_{i\omega}^{(\rho(s_{i\omega}))} < 0 \quad (16)$$

from which one can conclude that $\rho(q_i) = \rho_l(q_i)$. The rest of the proof is quite straightforward on noting that

$$s_{ij}^{(\rho(q_i))} = 2(q_i - q_j)^T q_i^{(\rho(q_i))} + 2(q_j - q_i)^T q_j^{(\rho(q_i))}. \quad (17)$$

■

Remark 3: From the proof of Lemma 6, it can be easily seen that for every edge in $E_d^{<\infty}(t)$ the movement of the agent with lower (or equal) index is in the direction of the other agent, which results in shrinking of the edge.

Lemma 7: Consider a dynamic system of the form

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases} \quad (18)$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and f and g are C^1 functions. Assume that for some $y_0 \in \mathbb{R}^n$, $g(x, y_0)$ is equal to zero for every $x \in \mathbb{R}^m$. Now, suppose that $y(t_0) = y_0$ for some $t_0 \in \mathbb{R}$. Then, $y(t) = y_0$ for all $t \in \mathbb{R}$.

Proof: The proof is omitted due to space limitations. ■

Lemma 8: Consider the partition $E_d(t) = E_d^{\infty}(t) \cup E_d^{<\infty}(t)$. Then, for every $t \in \mathcal{T}$ and every $i \in V_d^{\infty}(t)$:

i) $d_i(G_d^{\infty}) \geq 2$;

ii) $q_i(\tau) = q_i(t)$, for $\tau \geq 0$.

Proof:

Part (i): If $d_i(G_d^{\infty}) = 1$, then there exists a unique $j \in V_d^{\infty}$ for which $\rho(q_i) = \rho(q_j) = \infty$. This implies that $\operatorname{argmax}_{\omega \in N_i(G_d)}\{\rho(q_\omega)\}$ is unique and is equal to j. Hence, one can use Lemma 5 to obtain

$$q_i^{(\rho_l(q_i))} = \frac{\partial h_i}{\partial \pi_i}\tilde{\pi}_{ij}(q_i - q_j) \prod_{\substack{\omega \in N_i(G_d) \\ \omega \neq j}} (q_i - q_\omega)^T q_\omega^{(\rho(q_\omega))} \quad (19)$$

which is nonzero according to Lemma 6. This yields that $\rho(q_i) = \rho_l(q_i) < \infty$, which is a contradiction; thus, $d_i(G_d^{\infty}) \geq 2$.

Part (ii): Choose an arbitrary $t \in \mathcal{T}$. Let $y(\tau)$ represent the positions of the agents belonging to $V_d^{\infty}(t)$, and $x(\tau)$ represent the positions of all other agents. Since $d_i(G_d^{\infty}) \geq 2$, one can conclude that if $y(\tau) = y(t)$ for some $\tau \geq 0$, then $\pi_{ij}(\tau) = 0$, for any $i \in V_d^{\infty}(t)$ and $j \in N_i(G)$. Using this argument, it is easy to show that x and y satisfy the conditions of Lemma 7, and as a result $q_i(\tau) = q_i(t)$ for $\tau \geq 0$ and $i \in V_d^{\infty}(t)$. ■

Lemma 9: Under the conditions given in (5), the control law (1) is connectivity preserving.

Proof: Assume that $\|q_i(0) - q_j(0)\| \leq d$ for all $(i,j) \in E$ (i.e., $0 \in \mathcal{T}$), and let $t_0 = \inf\{t | \exists (i,j) \in E : \|q_i(t) - q_j(t)\| > d\}$. Clearly, any $t \leq t_0$ belongs to $\mathcal{T}$. Therefore, to prove the lemma it suffices to show that there is a neighborhood of t_0 in which for every $(i,j) \in E_d(t_0)$, s_{ij} is either decreasing or fixed. It follows from Lemmas 6 and 1 that s_{ij} is decreasing in a neighborhood of t_0 for any $(i,j) \in E_d^{<\infty}(t_0)$. Also, from Lemma 8, s_{ij} is fixed for any $(i,j) \in E_d^{\infty}(t_0)$. The proof is completed on noting that $E_d(t_0) = E_d^{\infty}(t_0) \cup E_d^{<\infty}(t_0)$. ■

Lemma 10: Suppose that $\|q_i(0) - q_j(0)\| \leq d$ for all $(i,j) \in E$ (i.e., $0 \in \mathcal{T}$). Then,

i) $G_d^{<\infty}(t) = \emptyset$ for $t > 0$.

ii) $G_d^{\infty}(t) = G_d^{\infty}(0)$ for $t \geq 0$.

iii) $G_d^{\infty}(0)$ is the maximal induced subgraph of $G_d(0)$ with the property that the degree of each vertex in it is at least 2.

Proof:

Part (i): Note that since $0 \in \mathcal{T}$, according to Lemma 9, $\mathcal{T} = \mathbb{R}^+ \cup \{0\}$. Hence, $G_d(t)$ is well-defined for $t > 0$, and so are $G_d^{<\infty}(t)$ and $G_d^{\infty}(t)$. Now, assume that $G_d^{<\infty}(t) \neq \emptyset$ for some $t > 0$, and let $u = \operatorname{argmin}_{i \in V_d^{<\infty}(t)}\{\rho(q_i(t))\}$. Lemma 4 implies that $d_u(G_d) = 1$, and consequently from Remark 1, $\rho(q_u(t)) = 1$. Let $v \in G_d(t)$ be the neighbor of u. According to Lemma 6, $\dot{s}_{uv}(t) < 0$, implying that $\|q_u - q_v\| > d$ in the interval $(t - \epsilon, t)$ for some $\epsilon > 0$, which contradicts Lemma 9.

Part (ii): This part is a straightforward consequence of part (ii) of Lemma 8.

Part (iii): Let $G_M = (V_M, E_M)$ be the maximal induced subgraph of $G_d(0)$ such that $d_i(G_M) \geq 2$ for $i \in V_M$. From part (i) of Lemma 8, $G_d^\infty(0) \subset G_M$. Clearly, $\pi_{ij} = 0$ for any $i \in V_M$ and $j \in N_i(G)$. Similar to Lemma 8, one can use Lemma 7 to deduce that $q_i(t) = q_i(0)$ for any $t \geq 0$ and $i \in V_M$. Thus, $\rho(q_i(0)) = \infty$ for $i \in V_M$, which implies $G_M \subset G_d^\infty(0)$. This completes the proof. ■

Theorem 1: Consider a set of n agents in the plane with the dynamics of the form (1), and assume the conditions given in (5) hold. Assume also that $\|q_i(0)-q_j(0)\| \leq d$ for all $(i,j) \in E$. Then, the control law (1) is connectivity preserving. Moreover, Let $G_M = (V_M, E_M)$ be the maximal induced subgraph of $G_d(0)$ such that $d_i(G_M) \geq 2$ for every $i \in V_M$. Then, at any time $t \geq 0$, $q_i(t) = q_i(0)$ for $i \in V_M$, and $\|q_i(t) - q_j(t)\| < d$ for $(i,j) \in E - E_M$.

Proof: The proof follows directly from Lemmas 9 and 10. ■

Remark 4: It results from Theorem 1 (as a special case of practical interest) that if $\|q_i(0) - q_j(0)\| < d$ for all $(i,j) \in E$, then connectivity preservation is strict, meaning that $\|q_i(t) - q_j(t)\| < d$, at all times $t > 0$, and for all $(i,j) \in E$.

IV. Dynamic Information Flow Graph

The results presented so far can be easily extended to the case of dynamic edge addition, where new edges may be added to the information flow graph once two agents enter the connectivity range. Suppose that new edges are added to the information flow graph at time instants t_k, $k = 1, 2, \ldots$, and denote with $G^{(k)}$ the resultant information flow graph at time t_k. Note that the two agents associated with a newly added edge to the information flow graph at time t_k should be in the connectivity range at the time of addition. Clearly, according to Theorem 1 the proposed control law preserves the connectivity of the agents connected in $G^{(k)}$ during the time interval $[t_k, t_{k+1}]$. This implies that for any edge added to the information flow graph, the connectivity of the corresponding agents will be preserved at all times, provided they are in the connectivity range at the time of addition.

Adding new edges to the information flow graph may result in more fixed agents since it may change the structure of G_M defined in Theorem 1. To avoid this problem, an additional constraint is imposed that at the time of adding a new edge, the corresponding agents should be in the strict connectivity range. Under this condition, the addition of new edges will not affect G_M, and hence the structure of the fixed agents can be determined from G_M.

V. Connectivity Preservationfor Problems Involving Static Leaders

Consider the case in which some of the agents, called static leaders, are required to stay fixed. In this case, even if conditions given in (5) hold for the rest of the agents, called followers, one cannot directly deduce connectivity preservation from Theorem 1. In this section, it is shown how by using a simple trick connectivity preservation can be guaranteed assuming conditions (5) hold for the followers. Denote the set of static leaders with $\mathcal{L} \subset V(G)$; thus, $\dot{q}_i(t) = 0$ for every $i \in \mathcal{L}$ and $t \geq 0$. Assume that control laws of the form (1) are applied to the followers, where h_i's satisfy conditions given in (5). Construct a new graph $\bar{G}$ from G as follows. For any $i \in \mathcal{L}$, consider two *virtual agents* i_1 and i_2, initially located at distance d from each other and from i. Add the two new vertices i_1 and i_2 to $V(G)$, and all the possible edges between i, i_1, and i_2 to $E(G)$. Choose any h_i, h_{i_1}, and h_{i_2} satisfying conditions (5); then connectivity preservation is guaranteed for $\bar{G}$ according to Theorem 1. Clearly $i, i_1, i_2 \in \bar{G}_M$, and hence the corresponding agents remain fixed as desired. Therefore, connectivity preservation for the case of static leaders is deduced.

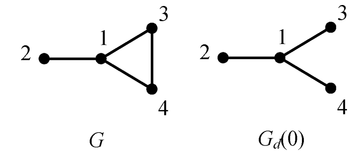

Fig. 1. Information flow graph G and the graph $G_d(0)$ for the consensus example.

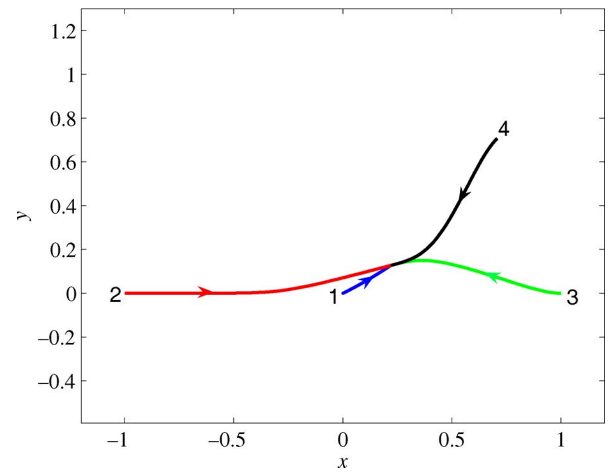

Fig. 2. Agents' planar motion in the consensus example.

VI. Simulation Results

A. Consensus Example

Consider 4 single-integrator agents moving in a two-dimensional space with the information flow graph G depicted in Fig. 1. The agents are to aggregate while preserving connectivity. This can be achieved by using the control law (1) with an appropriate choice of h_i's. Assume that in addition to the conditions in (5), h_i's also satisfy the following constraints:

$$\frac{\partial h_i}{\partial \sigma_i}(\sigma_i, \pi_i) > 0,\ \frac{\partial h_i}{\partial \pi_i}(\sigma_i, \pi_i) \leq 0, \quad \forall \sigma_i \geq 0,\ \forall \pi_i > 0. \tag{20}$$

Let d be equal to 1, and the initial position of each agent be marked by its label as shown in Fig. 2. As depicted in Fig. 1, $G_d(0)$ is a tree and hence $G_M = \emptyset$. Therefore, it results from Theorem 1 that $\|q_i(t) - q_j(t)\| < d$ for all $(i,j) \in E(G)$ and $t > 0$. Now, (20) yields that for any $i \in V(G)$ and $j \in N_i(G)$

$$\beta_{ij} := \frac{\partial h_i}{\partial \sigma_i} - \frac{\partial h_i}{\partial \pi_i}\pi_{ij} > 0, \qquad \forall t > 0. \tag{21}$$

The above inequality along with (6) implies that the velocity of each agent lies in the relative interior of the tangent cone to the convex hull of agent i and its neighbors at q_i. Thus, the strict sub-tangentiality condition of [25] holds, which leads to the convergence of the agents to a single point according to [25].

The above discussion shows that if h_i's satisfy conditions given by (5) and (20), then the agents reach consensus while preserving connectivity. There are a variety of functions satisfying these conditions, including the one used in [20]. The following function will be used in the simulation:

$$h_i(\sigma_i, \pi_i) = \frac{\sigma_i}{\sigma_i + \pi_i + {\pi_i}^2}. \tag{22}$$

The planar motion of the agents is shown in Fig. 2. Denote the relative distance between agent i and its neighbor j with d_{ij} (i.e., $d_{ij} := \|q_i - q_j\|$). The relative distances d_{12}, d_{13}, and d_{34} are depicted in Fig. 3. Although d_{12}, d_{13} and d_{14} are initially equal to d ($d_{12} = d_{13} = d_{14} = 1$ at $t = 0$), the proposed controller ensures that $d_{ij} < d$ for all

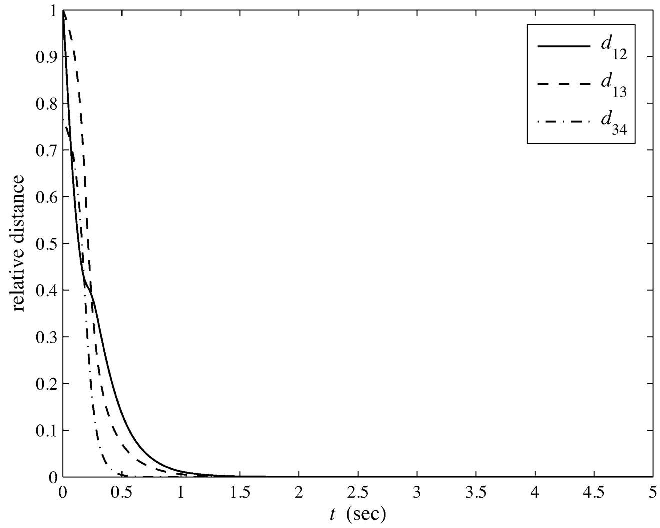

Fig. 3. Relative distances d_{12}, d_{13} and d_{34} in the consensus example.

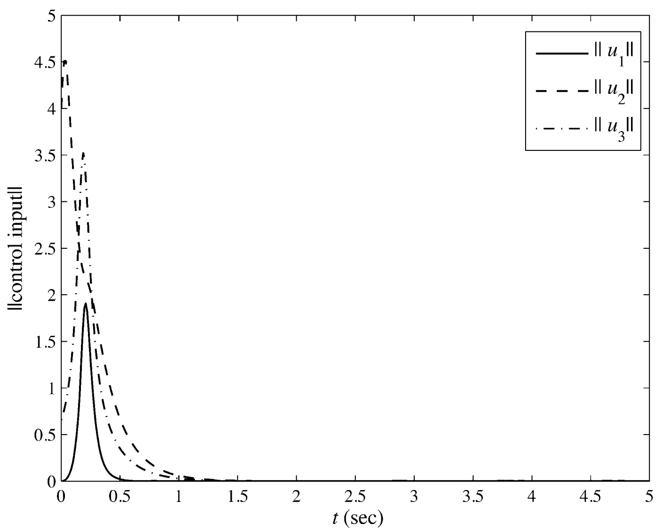

Fig. 4. Norms of the control inputs u_1, u_2 and u_3 in the consensus example.

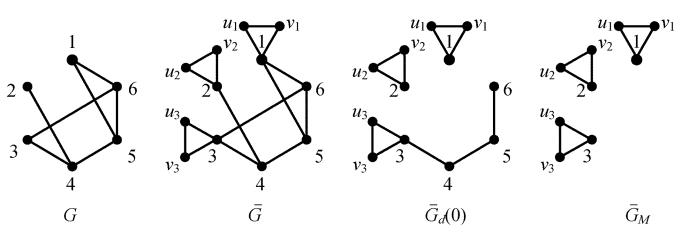

Fig. 5. Information flow graph G along with the graphs $\bar{G}$, $\bar{G}_d(0)$, and $\bar{G}_M$ for the containment example.

$(i,j) \in E(G)$, while the agents converge to consensus. Furthermore, the norms of the control inputs u_1, u_2 and u_3 are bounded, as depicted in Fig. 4. It is to be noted that in this example d_{13} and d_{14} are almost the same, and so are u_3 and u_4.

B. Containment Example

For this example, a team of 3 static leaders and 3 followers is considered, where the followers are desired to converge to the triangle of the leaders while preserving the connectivity of the information flow graph G given in Fig. 5. Consider the following potential function:

$$h_i(\sigma_i, \pi_i) = -\pi_i. \tag{23}$$

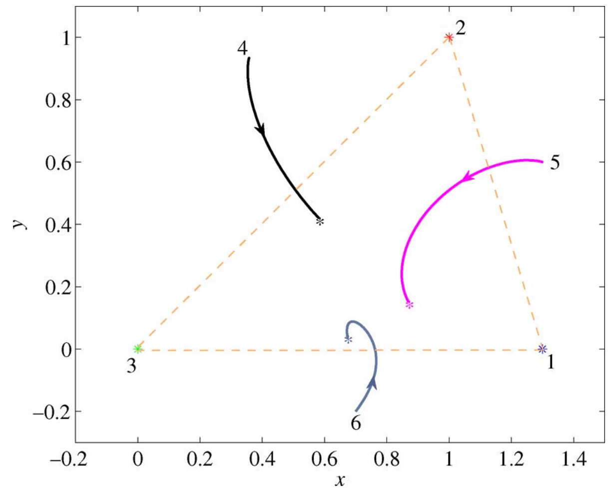

Fig. 6. Agents' planar motion in the containment example.

It can be easily verified that the function given above satisfies the conditions in (5), which means that the corresponding control law is connectivity preserving. Let d in this example also be equal to 1, and the initial position of each agent be marked by its label, as shown in Fig. 6. The graphs $\bar{G}$ (obtained by adding the virtual agents to G), $\bar{G}_d(0)$, and $\bar{G}_M$ are depicted in Fig. 5. According to Theorem 1, for all $(i,j) \in E(\bar{G}) - E(\bar{G}_M) = E(G)$, the inequality $\|q_i(t) - q_j(t)\| < d$ holds for any $t > 0$. To prove the convergence of the followers to the convex hull of the leaders, consider the function $\pi(t)$ defined by

$$\pi(t) = \prod_{\substack{(i,j)\in E(G) \\ i<j}} (1 - \|q_i(t) - q_j(t)\|^2). \tag{24}$$

Note that $\dot{\pi} = \sum_{i=4}^{6} \dot{q}_i{}^T (\partial \pi / \partial q_i) = \sum_{i=4}^{6} \dot{q}_i{}^T (\partial \pi_i / \partial q_i) \bar{\pi}_i = \sum_{i=4}^{6} \bar{\pi}_i \|\dot{q}_i\|^2$, where $\bar{\pi}_i$ is the product of those terms in π which do not appear in π_i (i.e., $\pi = \pi_i \bar{\pi}_i$). It results from strict connectivity preservation that $\bar{\pi}_i > 0$ for $t > 0$, and hence $\dot{\pi} \geq 0$ for $t > 0$. On the other hand, $0 < \pi < 1$ for $t > 0$; therefore, using LaSalle's invariance principle one can conclude the convergence of the agents to the largest invariant set in $\dot{\pi} = 0$, which is $\dot{q}_i = 0$ for $i = 4, 5, 6$, i.e., an equilibrium of (1). Moreover, it yields from (6) that $\sum_{j \in N_i(G)} \pi_{ij}(q_i - q_j) = 0$ for each follower i. Therefore, $q_i = \sum_{j \in N_i(G)} \alpha_{ij} q_j$, where $\alpha_{ij} = \pi_{ij} / (\sum_{j \in N_i(G)} \pi_{ij})$. Clearly, $0 < \alpha_{ij} < 1$ and $\sum_{j \in N_i(G)} \alpha_{ij} = 1$. This means that at equilibrium each follower is in the convex hull of its neighbors and hence cannot be on a vertex of the convex hull of the team. This concludes the convergence of the followers to the convex hull of the leaders.

The motion of the agents is depicted in Fig. 6, and the relative distances are sketched in Fig. 7. The control input norms $\|u_4\|$, $\|u_5\|$ and $\|u_6\|$ are plotted in Fig. 8. This figure shows the boundedness of the control inputs, although some of the agents are initially about to lose connectivity.

VII. Conclusion

This work presents sufficient conditions for a class of distributed potential functions which guarantee the connectivity preservation of the resultant control laws for the single-integrator agents. The main idea behind the proposed technique is that when two agents are about to lose connectivity, the gradients of their corresponding potential fields lie in the direction of the edge connecting the two agents, aiming to shrink it. When an agent is at a critical distance from more than one agent, this gradient vanishes. To handle the problem in this case, the lowest order nonzero derivative of the agent's position at any given time (referred to as *index* of the function) is used in the analysis. Shrinking of

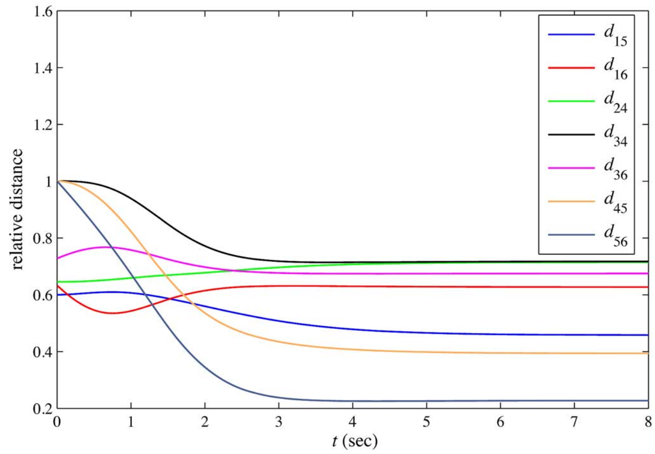

Fig. 7. Relative distances in the containment example.

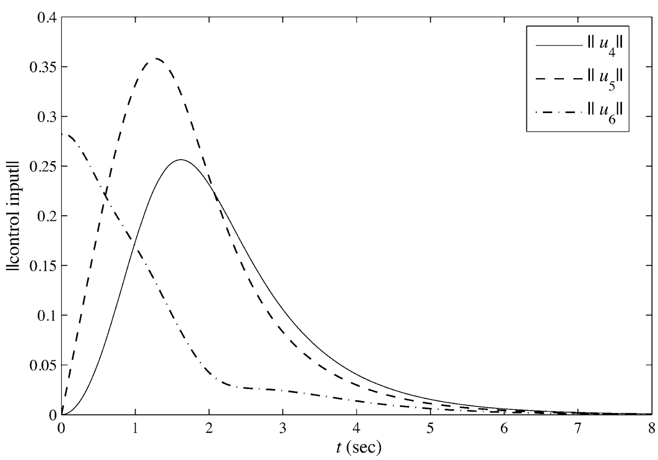

Fig. 8. Norms of the control inputs in the containment example.

the edge is performed by moving the agent with lower index towards the agent with higher index. The results are valid for both static and dynamic information flow graphs, and are also extended to cover the problems involving static leaders. Unlike many existing connectivity preserving control strategies proposed in the literature, the potential functions here are designed in such a way that the corresponding control inputs are bounded, making them more practical (as far as the actuators are concerned).

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