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Consensus of heterogeneous multi-agent systems

Y. Zheng[1] *Y. Zhu*[1] *L. Wang*[2]

[1]*Center for Complex Systems, School of Mechano-electronic Engineering, Xidian University, Xi'an 710071, People's Republic of China*
[2]*Center for Systems and Control, College of Engineering and Key Laboratory of Machine Perception (Ministry of Education), Peking University, Beijing 100871, People's Republic of China*
E-mail: zhengyuanshi2005@163.com

Abstract: In this study, the consensus problem of heterogeneous multi-agent system is considered. First, the heterogeneous multi-agent system is proposed which is composed of first-order and second-order integrator agents in two aspects. Then, the consensus problem of heterogeneous multi-agent system is discussed with the linear consensus protocol and the saturated consensus protocol, respectively. By applying the graph theory and Lyapunov direct method, some sufficient conditions for consensus are established when the communication topologies are undirected connected graphs and leader-following networks. Finally, some examples are presented to illustrate the theoretical results.

1 Introduction

In recent years, distributed coordination of multi-agent systems has attracted more and more attention in a wide range including system control theory, applied mathematics, statistical physics, biology, communication, computer science etc. Consensus problem, which is fundamental to distributed coordination, has been studied as an active research area in many fields. Consensus means that a group of agents reaches an agreement on a common value by negotiating with their neighbours asymptotically or in a finite time. Roughly speaking, the main objective of the consensus problem is to design an appropriate control input such that a group of agents converges to a consistent quantity of interest. The control input is usually called consensus protocol, and the consistent quantity that depends on the initial state is usually called consensus state. Up to now, by using the matrix theory, the graph theory, the frequency-domain analysis method, the Lyapunov direct method etc., consensus problem of multi-agent systems has been studied in detail, and the consensus criterions have been obtained under first-order, second-order or high-order multi-agent systems [1].

The consensus problem of first-order multi-agent systems is primarily studied. Vicsek *et al.* [2] proposed a discrete-time model of n agents all moving in the plane with the same speed and demonstrated by simulation that all agents move to one direction asymptotically. Jabdabaie *et al.* [3] provided a theoretical explanation of the consensus behaviour in the Vecsek model, and analysed the alignment of a network of agents with switching topologies that are periodically connected. Olfati-Saber and Murray [4] discussed the consensus problem for networks of dynamic agents with switching topologies and time delays in a continuous-time model by defining a disagreement function, and obtained some useful results for solving the average-consensus problem. Ren and Beard [5] presented some more relaxable conditions for consensus of states under dynamically changing interaction topologies. With the development of issue, a lot of new consensus results were given out with different models and protocols by a single integrator. Xiao and Wang studied the consensus problem of discrete-time multi-gent systems with time-delays [6, 7], asynchronous consensus of multi-agent systems with switching topologies and time-varying delays [8]. The finite-time consensus problem of multi-agent systems for both the bidirectional and unidirectional interaction case was also considered by Wang and Xiao [9]. Sun *et al.* [10] discussed the average-consensus problem in undirected networks of multi-agent systems with fixed and switching topologies as well as multiple time-varying communication delays. Hui and Haddad [11] investigated the consensus problem for non-linear multi-agent systems with fixed and switching topologies, and Liu *et al.* [12] considered the consensus problem in directed networks via non-linear protocols. Li and Zhang [13] gave the necessary and sufficient condition of mean square average-consensus for multi-agent systems with noises. Hatano and Mesbahi [14] considered the consensus problem for multi-agent systems with random topology for the first time and Tahbaz-Salehi and Jadbabaie [15] gave a necessary and sufficient condition for consensus with random topology. Unlike the first-order case, a (directed) spanning tree is a necessary rather than a sufficient condition for consensus seeking with second-order dynamics. Therefore the extension of consensus algorithms from first order to second order is non-trivial [16]. Xie and Wang [17] investigated the consensus problem of second-order multi-agent systems with fixed and switching topologies. Ren [18] considered the consensus problem of multi-agent systems with double-integrator dynamics in four cases. Hong *et al.* [19] designed the distributed observers for the leader-following problem of multi-agents systems with switching topology. Zhu *et al.* [20] considered the general consensus protocol of multi-agent systems with second-order dynamics and obtained the necessary and sufficient conditions for solving the consensus problem. Tian and Liu [21] obtained the robust consensus of second-order multi-agent systems with diverse input delays and asymmetric interconnection perturbations based on the frequency-domain analysis method. Lin and Jia [22] investigated the consensus problem of second-order discrete-time multi-agent systems with non-uniform time delays and dynamically changing topologies. Yu *et al.* [23] studied the second-order consensus for non-linear multi-agent systems with directed topologies, and Song *et al.* [24] investigated second-order leader-following consensus of non-linear multi-agent via pinning control. Wang and Hong [25] considered the finite-time consensus problem for the second-order multi-agent systems.

To the best of our knowledge, however, the existing results of consensus analysis are on the multi-agent systems with the same-order integrators, that is, all the agents have the same dynamics behaviours. In the practical systems, the dynamics of the agents coupled with each others are different because of various restrictions or the common goals with mixed agents, but there is little attention to the consensus problem of heterogeneous multi-agent systems, in which the agents have the different dynamics. Different from the previous multi-agent systems, a kind of consensus problem in dynamics networks with different-order integrator agents is considered in this paper, which are first-order and second-order integrator agents, respectively. The main contribution of this paper is threefold. First, we propose the heterogeneous multi-agent system and give the definition of consensus. Second, we obtain the consensus criterions for heterogeneous multi-agent system with the linear consensus protocol when the communication topologies are undirected connected graphs and leader-following networks. Finally, we discuss the consensus problem of heterogeneous multi-agent system with the saturated consensus protocol.

An outline of this paper is shown as follows. In Section 2, we present some concepts in graph theory and formulate the model to be studied. The consensus problem of heterogeneous multi-agent system is discussed with the linear consensus protocol and the saturated consensus protocol in Sections 3 and 4, respectively. The simulation results are presented to illustrate the effectiveness of the theoretical results in Section 5. Finally, we present the conclusion in Section 6.

Notation: Throughout this paper, we let $\mathbb{R}$, $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ be the set of real number, positive real number and non-negative real number, $\mathbb{R}^n$ be the n-dimensional real vector space. $\mathcal{I}_n = \{1, 2, \ldots, n\}$. For a given vector or matrix X, X^{T} denotes its transpose, $\|X\|$ denotes the Euclidean norm of a vector X. $\mathbf{1}_n$ is a vector with all elements being one. Matrix $A = [a_{ij}]$ is said to be non-negative (resp. positive) if all entries a_{ij} are non-negative (resp. positive), denoted by $A \geq 0$ (resp. $A > 0$).

2 Preliminaries

2.1 Graph theory

In this subsection, we shall review the graph theory [26] which is fundamental to the later development. In the multi-agent system, each agent can communicate with other agents that are defined as its neighbours. Let $G(A) = (V, E, A)$ be a weighted undirected graph of order n with a vertex set $V = \{s_1, s_2, \ldots, s_n\}$, an edge set $E = \{e_{ij} = (s_i, s_j)\} \subset V \times V$ and a non-negative symmetric matrix $A = [a_{ij}]$. $(s_j, s_i) \in E \Leftrightarrow a_{ij} > 0 \Leftrightarrow$ agents i and j can communicate with each other, namely, they are adjacent. Moreover, we assume $a_{ii} = 0$. A is called the weighted matrix and a_{ij} is the weigh of $e_{ij} = (s_i, s_j)$. The set of neighbours of s_i is denoted by $N_i = \{s_j\colon e_{ji} = (s_j, s_i) \in E\}$. A path that connects s_i and s_j in the graph G is a sequence of distinct vertices $s_{i_0}, s_{i_1}, s_{i_2}, \ldots, s_{i_m}$, where $s_{i_0} = s_i, s_{i_m} = s_j$ and $(s_{i_r}, s_{i_{r+1}}) \in E$, $0 \leq r \leq m-1$. An undirected graph is said to be connected if there exists a path between any two distinct vertices of the graph. It is easy to see that adjacency matrix A is symmetric if G is an undirected graph. In the multi-agent system, we refer to the agent as the leader if it only sends the information to other agents and cannot receive any information from other agents, that is, $a_{n1} = a_{n2} = \cdots = a_{nn} = 0$ and $\bar{a} = [a_{1n}, a_{2n}, \ldots, a_{(n-1)n}]^{\mathrm{T}} \geq 0$ if the agent n is the leader.

2.2 Heterogeneous multi-agent systems

In this subsection, we propose the heterogeneous multi-agent system and define the concept of consensus.

Suppose that the heterogeneous multi-agent system consists of first-order and second-order integrator agents. The number of agents is n, labelled 1 through n, where the number of second-order integrator agents is m $(m < n)$. Each second-order agent dynamics is given as follows

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = u_i(t), \quad i \in \mathcal{I}_m \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}$, $v_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the position-like, velocity-like and control input, respectively, of agent i. The initial conditions are $x_i(0) = x_{i0}$, $v_i(0) = v_{i0}$. Each first-order agent dynamics is given as follows

$$\dot{x}_i(t) = u_i(t), \quad i \in \{m+1, \ldots, n\} \tag{2}$$

where $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the position-like and control input, respectively, of agent i. The initial condition is $x_i(0) = x_{i0}$. Let $x(0) = [x_{10}, x_{20}, \ldots, x_{n0}]$, $v(0) = [v_{10}, v_{20}, \ldots, v_{m0}]$.

Definition 1: The heterogeneous multi-agent system (1–2) is said to reach consensus if for any initial condition we have

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \quad \text{for } i, j \in \mathcal{I}_n$$
$$\lim_{t\to\infty} \|v_i(t) - v_j(t)\| = 0, \quad \text{for } i, j \in \mathcal{I}_m$$

3 Consensus with a linear consensus protocol

In this section, we first give a linear consensus protocol (control input) for the heterogeneous multi-agent system (1–2). Then, we obtain the consensus criterions for the heterogeneous multi-agent system (1–2) when communication topology is undirected and connected by using Lasalle's invariance principle. Finally, we obtain the consensus criterions for the heterogeneous multi-agent system (1–2) when the agents have a leader and the

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communication topology of followers is undirected and connected (leader-following network in short).

The linear consensus protocol has been widely applied for multi-agent systems with the same-order integrators. We present the linear consensus protocol for the heterogeneous multi-agent system (1–2) as follows

$$u_i(t) = \begin{cases} \sum_{j=1}^{n} a_{ij}(x_j - x_i) - k_1 v_i, & i \in \mathcal{I}_m \\ k_2 \sum_{j=1}^{n} a_{ij}(x_j - x_i), & i \in \{m+1, \ldots, n\} \end{cases} \tag{3}$$

where $A = [a_{ij}]_{n\times n}$ is the weighted adjacency matrix, $k_1 > 0$, $k_2 > 0$ are the feedback gains.

Theorem 1: Suppose the communication network *G(A)* is undirected and connected, that is, $a_{ij} = a_{ji}$ for all $i, j \in \mathcal{I}_n$. Then the heterogeneous multi-agent system (1–2) can solve the consensus problem with consensus protocol (3).

Proof: The heterogeneous multi-agent system (1–2) with consensus protocol (3) can be written as follows

$$\begin{cases} \dot{x}_i(t) = v_i(t), & i \in \mathcal{I}_m \\ \dot{v}_i(t) = \sum_{j=1}^{n} a_{ij}(x_j - x_i) - k_1 v_i, & i \in \mathcal{I}_m \\ \dot{x}_i(t) = k_2 \sum_{j=1}^{n} a_{ij}(x_j - x_i), & i \in \{m+1, \ldots, n\} \end{cases} \tag{4}$$

Take a Lyapunov function for (4) as

$$V_1(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{(x_i(t) - x_j(t))^2}{2} + \sum_{i=1}^{m}(v_i(t))^2$$

which is positive definite with respect to $x_i(t) - x_j(t)$ $(\forall i \neq j, i, j \in \mathcal{I}_n)$ and $v_i(t)$ $(i \in \mathcal{I}_m)$. Differentiating $V_1(t)$, gives

$$\begin{aligned} \dot{V}_1(t) &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_j - x_i)(\dot{x}_j - \dot{x}_i) + \sum_{i=1}^{m} 2v_i\dot{v}_i \\ &= \sum_{i=1}^{m} 2v_i\left(\sum_{j=1}^{n} a_{ij}(x_j - x_i) - k_1 v_i\right) \\ &\quad + \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}(x_j - x_i)(v_j - v_i) \\ &\quad + \sum_{i=m+1}^{n}\sum_{j=1}^{m} a_{ij}(x_j - x_i)(v_j - \dot{x}_i) \\ &\quad + \sum_{i=1}^{m}\sum_{j=m+1}^{n} a_{ij}(x_j - x_i)(\dot{x}_j - v_i) \\ &\quad + \sum_{i=m+1}^{n}\sum_{j=m+1}^{n} a_{ij}(x_j - x_i)(\dot{x}_j - \dot{x}_i) \end{aligned}$$

As $A = [a_{ij}]_{n\times n}$ is a symmetric matrix, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}(x_j - x_i)(v_j - v_i) = -2\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}(x_j - x_i)v_i$$

$$\sum_{i=m+1}^{n}\sum_{j=1}^{m} a_{ij}(x_j - x_i)(v_j - \dot{x}_i) = \sum_{i=1}^{m}\sum_{j=m+1}^{n} a_{ij}(x_j - x_i)(\dot{x}_j - v_i)$$

and

$$\sum_{i=m+1}^{n}\sum_{j=m+1}^{n} a_{ij}(x_j - x_i)(\dot{x}_j - \dot{x}_i) = 2\sum_{i=m+1}^{n}\sum_{j=m+1}^{n} a_{ij}(x_j - x_i)\dot{x}_j$$

Hence

$$\begin{aligned} \dot{V}_1(t) &= -2k_1\sum_{i=1}^{m} v_i^2 + 2\sum_{i=1}^{m}\sum_{j=m+1}^{n} a_{ij}(x_j - x_i)\dot{x}_j \\ &\quad + 2\sum_{i=m+1}^{n}\sum_{j=m+1}^{n} a_{ij}(x_j - x_i)\dot{x}_j \\ &= -2k_1\sum_{i=1}^{m} v_i^2 + 2\sum_{i=1}^{n}\sum_{j=m+1}^{n} a_{ij}(x_j - x_i)\dot{x}_j \\ &= -2k_1\sum_{i=1}^{m} v_i^2 - 2\sum_{i=m+1}^{n}\dot{x}_i\sum_{j=1}^{n} a_{ij}(x_j - x_i) \\ &= -2k_1\sum_{i=1}^{m} v_i^2 - \frac{2}{k_2}\sum_{i=m+1}^{n}\dot{x}_i^2 \leq 0 \end{aligned}$$

Then, we employ Lasalle's invariance principle. Denote the invariant set $S = \{(x_1, v_1, \ldots, x_m, v_m, x_{m+1}, \ldots, x_m) | \dot{V}_1 \equiv 0\}$. Note that $\dot{V}_1 \equiv 0$ implies that $v_i = 0 (i \in \mathcal{I}_m)$ and $\dot{x}_i = 0 (i \in \{m+1, \ldots, n\})$, which in turn implies that $\sum_{j=1}^{n} a_{ij}(x_j - x_i) = 0$ for all $i \in \mathcal{I}_n$. Then we obtain

$$\sum_{i=1}^{n} x_i\sum_{j=1}^{n} a_{ij}(x_j - x_i) = 0$$

Since the undirected graph $G(A)$ is connected, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_j - x_i)^2 = 0$$

which implies that $x_i = x_j$ for all $i, j \in \mathcal{I}_n$. It follows from Lasalle's invariance principle that

$$\begin{aligned} &\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0, \quad \text{for } i, j \in \mathcal{I}_n \\ &\lim_{t\to\infty}\|v_i(t)\| = 0, \quad \text{for } i \in \mathcal{I}_m \end{aligned}$$

Theorem 1 is proved. □

The network studied in Theorem 1 is the undirected connected graph. As an extension, we consider the leader-following network.

Theorem 2: Suppose that the heterogeneous multi-agent system (1–2) has a leader and $n-1$ followers, and the network among the followers is undirected and connected. Then the heterogeneous multi-agent system (1–2) can solve

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the consensus problem with consensus protocol (3) if the leader is a first-order integrator agent.

Proof: Without loss of generality, we assume that the agents $1, 2, \ldots, n-1$ are the follower and n is the leader. Thus, we have $\bar{A} = [a_{ij}]_{1\le i,j\le n-1} = \bar{A}^{\mathrm{T}}$, $a_{n1} = a_{n2} = \cdots = a_{nn} = 0$, $\bar{a} \ge 0$, where $\bar{a} = [a_{1n}, a_{2n}, \ldots, a_{(n-1)n}]^{\mathrm{T}}$. We rewrite the protocol (3) as follows (see (5))

Let $y_i(t) = x_i(t) - x_n(t)$, $i \in \mathcal{I}_{n-1}$. Without loss of generality, we assume $(n-m) > 1$. Then we have

$$\begin{cases} \dot{y}_i(t) = v_i(t), & i \in \mathcal{I}_m \\ \dot{v}_i(t) = \sum_{j=1}^{n-1} a_{ij}(y_j - y_i) - a_{in}y_i - k_1 v_i, & i \in \mathcal{I}_m \\ \dot{y}_i(t) = k_2 \sum_{j=1}^{n-1} a_{ij}(y_j - y_i) - k_2 a_{in} y_i, & i \in \{m+1, \ldots, n-1\} \end{cases} \tag{6}$$

Take a Lyapunov function for (8) as

$$V_2(t) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} a_{ij}\frac{(y_i(t) - y_j(t))^2}{2} + \sum_{i=1}^{n-1} a_{in}(y_i(t))^2 + \sum_{i=1}^{m}(v_i(t))^2$$

Differentiating $V_2(t)$, yields that

$$\dot{V}_2(t) = -2k_1\sum_{i=1}^{m} v_i^2 - \frac{2}{k_2}\sum_{i=m+1}^{n-1}\dot{y}_i^2 \le 0$$

Denote the invariant set $S = \{(y_1, y_2, \ldots, y_{n-1}, v_1, \ldots, v_m) | \dot{V}_2 \equiv 0\}$. Note that $\dot{V}_2 \equiv 0$ implies that $v_i = 0 (i \in \mathcal{I}_m)$ and $\dot{y}_i = 0 (i \in \{m+1, \ldots, n-1\})$, which in turn implies that $\sum_{j=1}^{n-1} a_{ij}(y_j - y_i) - a_{in}y_i = 0$ for all $i \in \mathcal{I}_{n-1}$. Then we obtain

$$\sum_{i=1}^{n-1} y_i\left(\sum_{j=1}^{n-1} a_{ij}(y_j - y_i) - a_{in}y_i\right) = 0$$

Since $\bar{A} = [a_{ij}]_{1\le i,j\le n-1} = \bar{A}^{\mathrm{T}}$, we have $y_i(t) = 0$ for all $i \in \mathcal{I}_{n-1}$. It follows from Lasalle's invariance principle that

$$\lim_{t\to\infty}\|x_i(t) - x_n(t)\| = 0, \quad \text{for } i \in \mathcal{I}_n$$
$$\lim_{t\to\infty}\|v_i(t)\| = 0, \quad \text{for } i \in \mathcal{I}_m$$

Theorem 2 is proved. □

We know that the linear consensus protocol of a second-order integrator agent can also be given as

$$u_i(t) = \sum_{j=1}^{n} a_{ij}(x_j - x_i) + k_1\sum_{j=1}^{n} b_{ij}(v_j - v_i)$$

if the number of agents is n. Therefore the linear consensus protocol of a heterogeneous multi-agent system can also be presented as follows

$$u_i(t) = \begin{cases} \sum_{j=1}^{n} a_{ij}(x_j - x_i) + k_1\sum_{j=1}^{m} b_{ij}(v_j - v_i), & i \in \mathcal{I}_m \\ k_2\sum_{j=1}^{n} a_{ij}(x_j - x_i), & i \in \{m+1, \ldots, n\} \end{cases} \tag{7}$$

where $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{m\times m}$ are the weighted adjacency matrix, $k_1 > 0$, $k_2 > 0$ are the feedback gains.

Corollary 1: Suppose the communication networks *G(A)* and *G(B)* are undirected connected graphs (or leader-following networks and the leader is a first-order integrator agent). Then the heterogeneous multi-agent system (1–2) can solve a consensus problem with consensus protocol (7).

Proof: The heterogeneous multi-agent system (1–2) with consensus protocol (7) can be written as follows

$$\begin{cases} \dot{x}_i(t) = v_i(t), & i \in \mathcal{I}_m \\ \dot{v}_i(t) = \sum_{j=1}^{n} a_{ij}(x_j - x_i) + k_1\sum_{j=1}^{m} b_{ij}(v_j - v_i), & i \in \mathcal{I}_m \\ \dot{x}_i(t) = k_2\sum_{j=1}^{n} a_{ij}(x_j - x_i), & i \in \{m+1, \ldots, n\} \end{cases} \tag{8}$$

If the communication networks $G(A)$ and $G(B)$ are undirected connected graphs, similar to the analysis of Theorem 1, differentiating $V_1(t)$, gives

$$\begin{aligned} \dot{V}_1(t) &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_j - x_i)(\dot{x}_j - \dot{x}_i) + \sum_{i=1}^{m} 2v_i\dot{v}_i \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}(x_j - x_i)(v_j - v_i) \\ &\quad + \sum_{i=m+1}^{n}\sum_{j=1}^{m} a_{ij}(x_j - x_i)(v_j - \dot{x}_i) \end{aligned}$$

$$u_i(t) = \begin{cases} \sum_{j=1}^{n-1} a_{ij}(x_j - x_i) + a_{in}(x_n - x_i) - k_1 v_i, & i \in \mathcal{I}_m \\ k_2\sum_{j=1}^{n-1} a_{ij}(x_j - x_i) + k_2 a_{in}(x_n - x_i), & i \in \{m+1, \ldots, n-1\} \\ 0, & i = n \end{cases} \tag{5}$$

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$$+\sum_{i=1}^{m}\sum_{j=m+1}^{n} a_{ij}(x_j - x_i)(\dot{x}_j - v_i)$$
$$+\sum_{i=m+1}^{n}\sum_{j=m+1}^{n} a_{ij}(x_j - x_i)(\dot{x}_j - \dot{x}_i)$$
$$+\sum_{i=1}^{m} 2v_i\left(\sum_{j=1}^{n} a_{ij}(x_j - x_i) + k_1\sum_{j=1}^{m} b_{ij}(v_j - v_i)\right)$$

As $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{m\times m}$ are symmetric matrices, we have

$$\dot{V}_1(t) = -2k_1\sum_{i=1}^{m} b_{ij}(v_j - v_i)^2 - \frac{2}{k_2}\sum_{i=m+1}^{n}\dot{x}_i^2 \le 0$$

Denote the invariant set $S = \{(x_1, v_1, \ldots, x_m, v_m, x_{m+1}, \ldots, x_m) | \dot{V}_1 \equiv 0\}$. Note that $\dot{V}_1 \equiv 0$ implies that $v_i = v_j (i, j \in \mathcal{I}_m)$ and $\dot{x}_i = 0 (i \in \{m+1, \ldots, n\})$, which in turn implies that $\sum_{k=1}^{n} a_{ik}(x_k - x_i) = \sum_{k=1}^{n} a_{jk}(x_k - x_j)(i, j \in \mathcal{I}_m)$ and $\sum_{k=1}^{n} a_{ik}(x_k - x_i) = 0 (i \in \{m+1, \ldots, n\})$. As $G(A)$ is an undirected connected graph, we have

$$\begin{aligned}0 &= \sum_{i=m+1}^{n}\sum_{k=1}^{n} a_{ik}(x_k - x_i)\\ &= \sum_{i=m+1}^{n}\sum_{k=1}^{m} a_{ik}(x_k - x_i) + \sum_{i=m+1}^{n}\sum_{k=m+1}^{n} a_{ik}(x_k - x_i)\\ &= \sum_{i=m+1}^{n}\sum_{k=1}^{m} a_{ik}(x_k - x_i)\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{m}\sum_{k=1}^{n} a_{ik}(x_k - x_i) &= \sum_{i=1}^{m}\sum_{k=1}^{m} a_{ik}(x_k - x_i)\\ &\quad + \sum_{i=1}^{m}\sum_{k=m+1}^{n} a_{ik}(x_k - x_i)\\ &= \sum_{i=1}^{m}\sum_{k=m+1}^{n} a_{ik}(x_k - x_i)\\ &= \sum_{i=m+1}^{n}\sum_{k=1}^{m} a_{ik}(x_i - x_k) = 0\end{aligned}$$

Hence, $\sum_{k=1}^{n} a_{ik}(x_k - x_i) = 0 (i \in \mathcal{I}_n)$, which implies that $x_i = x_j$ for all $i, j \in \mathcal{I}_n$. It follows from Lasalle's invariance principle that

$$\begin{aligned}&\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0, \quad \text{for } i, j \in \mathcal{I}_n\\ &\lim_{t\to\infty}\|v_i(t) - v_j(t)\| = 0, \quad \text{for } i, j \in \mathcal{I}_m\end{aligned}$$

that is, the heterogeneous multi-agent system (1–2) can solve the consensus problem with consensus protocol (7) when the communication networks $G(A)$ and $G(B)$ are undirected connected graphs.

If the communication networks $G(A)$ is a leader-following network and the leader is a first-order integrator agent, $G(B)$ is an undirected connected graph, the proof is similar to the analysis of Theorem 2 and is omitted here. □

Remark 1: In fact, the velocities of all second-order integrator agents converge to zero with both protocol (3) and (7). As a heterogeneous multi-agent system (1–2) can solve a consensus problem, we have $\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0$ for $i, j \in \mathcal{I}_n$, which implies that $\lim_{t\to\infty}\|\dot{x}_i(t) - \dot{x}_j(t)\| = 0 (i \in \mathcal{I}_m, j \in \{m+1, \ldots, n\})$ and $\lim_{t\to\infty}\|\sum_{j=1}^{n} a_{ij}(x_j - x_i)\| = 0 (i \in \{m+1, \ldots, n\})$, which in turn implies that $\lim_{t\to\infty}\|\dot{x}_i(t) - \dot{x}_j(t)\| = \lim_{t\to\infty}\|v_i(t)\| = 0 (i \in \mathcal{I}_m, j \in \{m+1, \ldots, n\})$. It means that the velocities of second-order integrator agents are decided by the control input of first-order integrator agents and all agents will stay in one place through cooperation.

4 Consensus with a saturated consensus protocol

In this section, we first give a saturated consensus protocol for the heterogeneous multi-agent system (1–2). Similar to the analysis in Section 3, we obtain the consensus criterions for the heterogeneous multi-agent system (1–2) under the undirected connected graphs and the leader-following networks, respectively.

Note that (3) does not explicitly take into account actuator saturation. We propose a consensus protocol for the heterogeneous multi-agent system (1–2) with a saturated consensus protocol as follows

$$u_i(t) = \begin{cases}\sum_{j=1}^{n} a_{ij}\tan h(x_j - x_i) - k_1 \tan h(v_i), & i \in \mathcal{I}_m\\ k_2\sum_{j=1}^{n} a_{ij}\tan h(x_j - x_i), & i \in \{m+1, \ldots, n\}\end{cases} \tag{9}$$

where $A = [a_{ij}]_{n\times n}$ is the weighted adjacency matrix, $k_1 > 0$, $k_2 > 0$ are the feedback gains. Note that

$$\|u_i\|_\infty \le \max\left\{\sum_{j=1}^{n} a_{ij} + k_1, k_2\sum_{j=1}^{n} a_{ij}\right\}$$

which is independent of the initial states of the agents.

Theorem 3: Suppose the communication network $G(A)$ is undirected and connected, that is, $a_{ij} = a_{ji}$ for all $i, j \in \mathcal{I}_n$. Then the heterogeneous multi-agent system (1–2) can solve the consensus problem with consensus protocol (9).

Proof: Analogous to the analysis of Theorem 1, the heterogeneous multi-agent system (1–2) with consensus protocol (9) can be written as follows

$$\begin{cases}\dot{x}_i(t) = v_i(t), & i \in \mathcal{I}_m\\ \dot{v}_i(t) = \sum_{j=1}^{n} a_{ij}\tan h(x_j - x_i) - k_1\tan h(v_i), & i \in \mathcal{I}_m\\ \dot{x}_i(t) = k_2\sum_{j=1}^{n} a_{ij}\tan h(x_j - x_i), & i \in \{m+1, \ldots, n\}\end{cases} \tag{10}$$

Take a Lyapunov function for (10) as

$$V_3(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\log(\cos h(x_i(t) - x_j(t))) + \sum_{i=1}^{m}(v_i(t))^2$$

which is positive definite with respect to $x_i(t) - x_j(t)(\forall i \neq j, i, j \in \mathcal{I}_n)$ and $v_i(t)(i \in \mathcal{I}_m)$. Similar to the proof of Theorem 1, we have

$$\begin{aligned} \dot{V}_3(t) &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \tan h(x_j - x_i)(\dot{x}_j - \dot{x}_i) + \sum_{i=1}^{m} 2v_i\dot{v}_i \\ &= -2k_1 \sum_{i=1}^{m} v_i \tan h(v_i) - \frac{2}{k_2}\sum_{i=m+1}^{n} \dot{x}_i^2 \leq 0 \end{aligned}$$

Denote the invariant set $S = \{(x_1, v_1, \ldots, x_m, v_m, x_{m+1}, \ldots, x_m) | \dot{V}_3 \equiv 0\}$. Note that $\dot{V}_3 \equiv 0$ implies that $v_i = 0 (i \in \mathcal{I}_m)$ and $\dot{x}_i = 0 (i \in \{m+1, \ldots, n\})$, which in turn implies that $x_i = x_j$ for all $i, j \in \mathcal{I}_n$. It follows from Lasalle's invariance principle that

$$\begin{aligned} &\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \quad &&\text{for } i, j \in \mathcal{I}_n \\ &\lim_{t\to\infty} \|v_i(t)\| = 0, &&\text{for } i \in \mathcal{I}_m \end{aligned}$$

Theorem 3 is proved. □

Theorem 4: Suppose that the heterogeneous multi-agent system (1–2) has a leader and $n - 1$ followers, and the network among the followers is undirected and connected. Then the heterogeneous multi-agent system (1–2) can solve the consensus problem with consensus protocol (9) if the leader is a first-order integrator agent.

Proof: Analogous to the proof of Theorems 2 and 3, it is easy to establish this theorem. □

From the proof of the consensus problem with a saturated consensus protocol for heterogeneous multi-agent system, we propose the non-linear consensus protocol as follows

$$u_i(t) = \begin{cases} \sum_{j=1}^{n} a_{ij} f_1(x_j - x_i) - k_1 f_2(v_i), & i \in \mathcal{I}_m \\ k_2 \sum_{j=1}^{n} a_{ij} f_1(x_j - x_i), & i \in \{m+1, \ldots, n\} \end{cases} \tag{11}$$

where $A = [a_{ij}]_{n\times n}$ is the weighted adjacency matrix, $k_1 > 0$, $k_2 > 0$ are the feedback gains. Suppose that function f_i: $\mathbb{R} \to \mathbb{R}$, $(i = 1, 2)$ satisfies the following assumptions

1. $f_i(\cdot)$ is continuous;
2. $f_i(0) = 0$ and $xf_i(x) > 0$ for $x \neq 0$;
3. $f_i(\cdot)$ is an odd function.

Corollary 2: Suppose the communication network $G(A)$ is undirected and connected (or a leader-following network and the leader is a first-order integrator agent), and the assumptions (1)–(3) are established. Then the heterogeneous multi-agent system (1–2) can solve the consensus problem with consensus protocol (11).

Proof: Take a Lyapunov function as follows

$$V(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \int_0^{(x_i(t)-x_j(t))} f_1(s)\mathrm{d}s + \sum_{i=1}^{m} (v_i(t))^2$$

Then, similar to the proof of Theorems 1–4, the heterogeneous multi-agent system (1–2) can solve the consensus problem with the non-linear consensus protocol (11). □

The saturated consensus protocol of a heterogeneous multi-agent system can also been presented as follows (see (12))

$$u_i(t) = \begin{cases} \sum_{j=1}^{n} a_{ij} \tan h(x_j - x_i) + k_1 \sum_{j=1}^{m} b_{ij} \tan h(v_j - v_i), & i \in \mathcal{I}_m \\ k_2 \sum_{j=1}^{n} a_{ij} \tan h(x_j - x_i), & i \in \{m+1, \ldots, n\} \end{cases} \tag{12}$$

where $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{m\times m}$ are the weighted adjacency matrices, $k_1 > 0$, $k_2 > 0$ are the feedback gains. Note that

$$\|u_i\|_\infty \leq \max\left\{\sum_{j=1}^{n} a_{ij} + k_1 \sum_{j=1}^{m} b_{ij}, k_2 \sum_{j=1}^{n} a_{ij}\right\}$$

which is independent of the initial states of the agents.

Corollary 3: Suppose the communication network $G(A)$ and $G(B)$ are undirected connected graphs (or leader-following networks and the leader is a first-order integrator agent). Then the heterogeneous multi-agent system (1–2) can solve the consensus problem with consensus protocol (12).

Proof: The proof is similar to the analysis of Corollary 3 and Theorem 3 (Theorem 4), and is omitted here. □

5 Simulations

In this section, we provide simulations to demonstrate the effectiveness of the theoretical results in this paper.

Example 1: Fig. 1 shows an undirected connected graph with six vertices. Suppose that the vertices 1–4 denote the second-order integrator agents and the vertices 5–6 denote the first-order integrator agents, and $a_{ij} = 1$ if $(s_j, s_i) \in E$, otherwise $a_{ij} = 0$, where $i, j \in \mathcal{I}_6$. We further assume that $k_1 = k_2 = 1$ and $x(0) = [8, 5, 2, -4, 1, -5]$, $v(0) = [1, -5\ 5\ 3]$. Then Figs. 2 and 3 show the simulation results of the

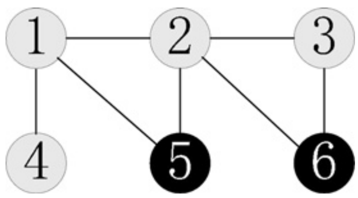

Fig. 1 *Undirected connected graph*

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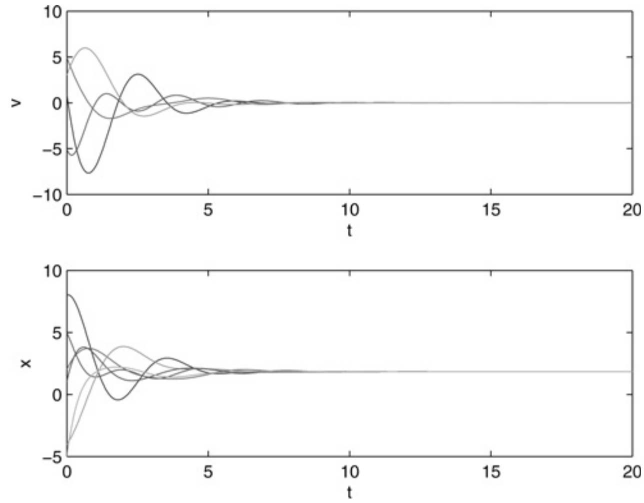

Fig. 2 *Simulation results with the network depicted in Fig. 1 and consensus protocol (3)*

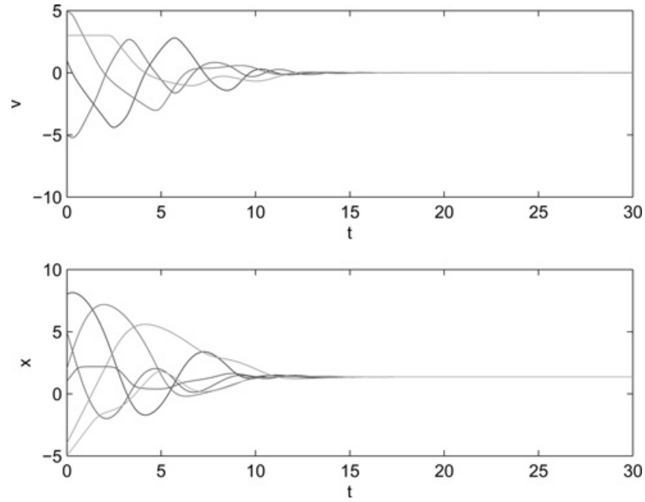

Fig. 3 *Simulation results with the network depicted in Fig. 1 and consensus protocol (9)*

heterogeneous multi-agent system (1–2) with consensus protocols (3) and (9), respectively. From Fig. 2, we know that the heterogeneous multi-agent system (1–2) with consensus protocol (3) can solve consensus problem under the undirected connected graph. From Fig. 3, we know that the heterogeneous multi-agent system (1–2) with consensus protocol (9) can solve consensus problem under the undirected connected graph.

Example 2: Fig. 4 shows a leader-following network with six vertices. With the same assumption of Example 1, Figs. 5 and 6 show the simulation results of the heterogeneous multi-

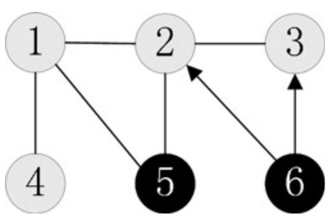

Fig. 4 *Leader-following network*

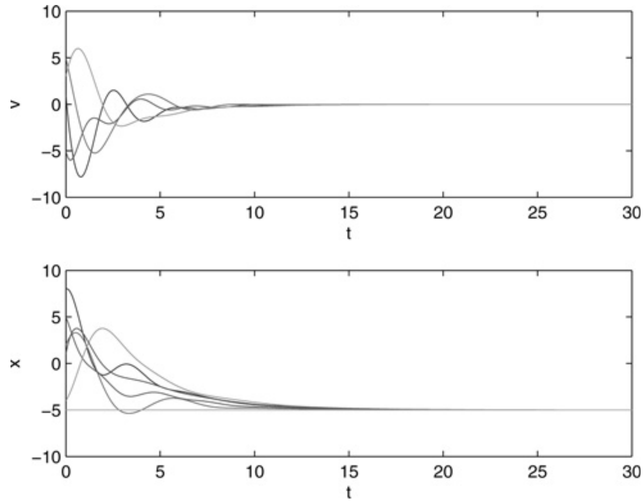

Fig. 5 *Simulation results with the network depicted in Fig. 4 and consensus protocol (3)*

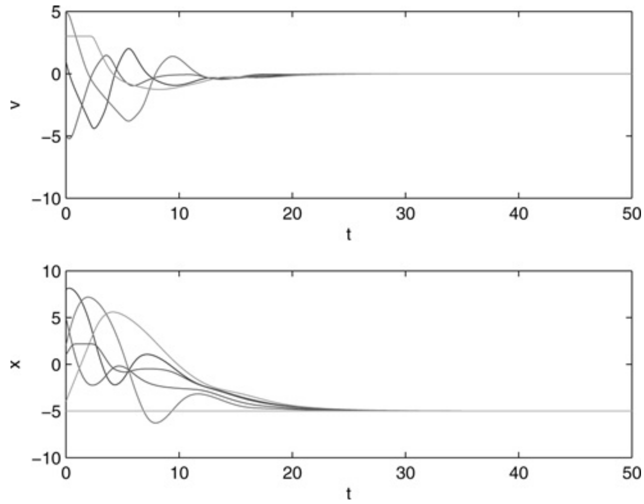

Fig. 6 *Simulation results with the network depicted in Fig. 4 and consensus protocol (9)*

agent system (1–2) with consensus protocols (3) and (9), respectively. From Fig. 5, we know that the heterogeneous multi-agent system (1–2) with consensus protocol (3) can solve the consensus problem under the leader-following network. From Fig. 6, we know that the heterogeneous multi-agent system (1–2) with consensus protocol (9) can solve the consensus problem under the leader-following network.

6 Conclusion

In this paper, we consider the consensus problem of heterogeneous multi-agent system with linear consensus protocol and saturated consensus protocol, respectively. Through using Lyapunov function method and Lasalle's invariance principle, the heterogeneous multi-agent system can solve the consensus problem when the communication topologies are undirected connected graphs and leader-following networks, respectively. Some examples are given to illustrate the effectiveness of theoretical results in the last. The future work will focus on the more complex consensus problem of heterogeneous multi-agent systems, for example, heterogeneous multi-agent systems with delays, heterogeneous multi-agent systems under directed graphs/switching topologies/random networks, discrete-time heterogeneous multi-agent systems etc.

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