A Container Model of Type Theory

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Motivation:
modelling inductive types

Functorial semantics, for ordinary inductive types

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data ℕ : Set where
  zero : ℕ
  succ : ℕ → ℕ
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$F_C : \mathbf{Set} \to \mathbf{Set}$

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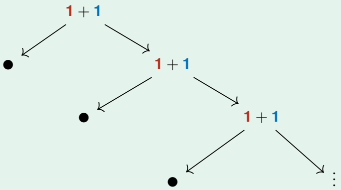

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$\mathbb{N}$'s container representation

$$\mathrm{S} := \mathbf{1} + \mathbf{1}$$

$$\mathrm{P}(\mathrm{inl}\,\star) := \mathbf{0}$$

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$$[\![S \lhd P]\!] : \mathbf{Set} \to \mathbf{Set}$$

$$\begin{aligned} [\![S \lhd P]\!]\, X &= \sum (s : \mathbf{1} + \mathbf{1})((\lambda\, (\mathrm{inl}\,\star).\mathbf{0};\, (\mathrm{inr}\,\star).\mathbf{1}) \to X) \\ &\cong \mathbf{1} + X \end{aligned}$$

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Containers enforce strict positivity semantically.

An overview of inductive types

Class of types	**Functor type**	**Category theory semantics**	**Type theoretic normal form**	**Universal type**
ordinary inductive types e.g. $\mathbb{N}$: Set	**Set** $\rightarrow$ **Set**	initial algebras of endofunctors on **Set**	containers	W-type

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QIITs e.g. $\mathtt{Con} : \mathrm{Set}$, $\mathtt{Ty} : \mathtt{Con} \to \mathrm{Set}$	?	?	?	?

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Example 1.1: (Simplified) intrinsic syntax of type theory

```
data Con : Set
data Ty : Con → Set

data Con where
    ◇ : Con
    _,_ : (Γ : Con) (A : Ty Γ) → Con
    eq : (Γ : Con) (A : Ty Γ) (B : Ty (Γ , A)) →
          ((Γ , A) , B) ≡ (Γ , Σ Γ A B)

data Ty where
    ι : (Γ : Con) → Ty Γ
    Σ : (Γ : Con) (A : Ty Γ) → Ty (Γ , A) → Ty Γ
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Functorial semantics, for QIITs [Altenkirch et al., 2018]

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2. Constructor specification. The n^{th} constructor is specified by two functors

$$L_n : \mathbf{A_n} \to \mathbf{Set},$$

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$$L_n \colon \mathbf{A_n} \to \mathbf{Set},$$
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3. Category of algebras. $\mathbf{A_{n+1}}$ is the category having objects of type $\sum(A : |\mathbf{A_n}|)(c \colon (x : L_n\, A) \to R_n(A, x))$.

Containerification

Goal: restrict

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Definition

Given category $\mathbf{C}$, a *generalised container* is a pair S : Set, $P\colon S \to |\mathbf{C}|$.

The *extension functor* $[\![S \triangleleft P]\!]\colon \mathbf{C} \to \mathbf{Set}$ is defined by

$$[\![S \triangleleft P]\!]\, X := \sum(s : S)(\mathbf{C}(P\, s, X)).$$

An overview of inductive types, revisited

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QIITs e.g. $\mathtt{Con} : \mathrm{Set}$, $\mathtt{Ty} : \mathtt{Con} \to \mathrm{Set}$	sequence of functors L_n and R_n and sequence of categories of dialgebras	initial object in last constructed category of dialgebras $\mathbf{A_n}$	representations constructed via generalised containers	? (QW-type)

The container model

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- A functor $Ty\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$.
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- For every $\Gamma : |\mathbf{C}|$ and $A : Ty(\Gamma)$,
  - an object $\Gamma.A : |\mathbf{C}|$
  - a morphism $p\colon \Gamma.A \to \Gamma$ in $\mathbf{C}$
  - and a term $q : Tm(\Gamma.A, A[p])$,

  with a certain universal property.

($-[f]$ denotes the action of Ty and Tm on a morphism f.)

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- Terms in context Γ of type A are dependent natural transformations from Γ to A.

$$Tm \colon (\textstyle\int Ty)^{\mathrm{op}} \to \mathbf{Set}$$
$$Tm\,(\Gamma, A) :\equiv \int_{X:\mathrm{Set}} (\gamma : \Gamma\, X) \to A(X, \gamma)$$

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- Context extension
  $(\Gamma.A)\,X = \sum (\Gamma.S)(A.SA) \triangleleft \lambda\,(s\Gamma, sA).\,(A.PA)\,s\Gamma\,sA$

Presheaf model & container model

Presheaf model

- Contexts: $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$
- Substitutions: natural transformations
- Types: $(\int \Gamma)^{\mathrm{op}} \to \mathbf{Set}$
- Terms:

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data U : Set where
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  - Strictify pullbacks and pushouts (e.g. when proving $A[f \circ g] \equiv A[f][g]$).
- For QIIT semantics, we need **Con** to be the category of generalised containers (as opposed to set-containers).

Related work

- Thorsten Altenkirch and Ambrus Kaposi's TYPES abstract 'A container model of type theory'.

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- Tamara von Glehn's polynomial functor model using comprehension categories.
- Bob Atkey and András Kovács's implementation of the same model as a CwF.

Summary

- QIITs combine set-truncated **equalities** with **induction-induction**.
- We can represent **QIITs semantically** as initial objects in a category of algebras.
- Containerification of QIIT semantics requires as a prerequisite the ability to **express any statement in type theory as a container**. This can be achieved by a **container model** of type theory.
- The container model is a **restricted version of the presheaf model**.

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Thank you!

References I

Altenkirch, T., Capriotti, P., Dijkstra, G., Kraus, N., and Nordvall Forsberg, F. (2018).
Quotient inductive-inductive types.
In Baier, C. and Dal Lago, U., editors, *FoSSACS*, pages 293–310. Springer.

Altenkirch, T. and Kaposi, A. (2021).
A container model of type theory.
In *TYPES 2021*.

Atkey, R. (2020).
Interpreting dependent types with containers.
Talk at the MSP101 seminar, University of Strathclyde, slides at `https://bentnib.org/docs/tt-in-containers.pdf`, code at `https://gist.github.com/bobatkey/0d1f04057939905d35699f1b1c323736`.

References II

Kovács, A. (2020).
Construction of the containers / polynomials cwf in agda.
Code at `https://gist.github.com/AndrasKovacs/cf7cc88f667c8f0087a4981f6be4eef8`.

von Glehn, T. (2015).
Polynomials and models of type theory.
PhD thesis, University of Cambridge.