

# Coherences for the Container Model of Type Theory

Stefania Damato Thorsten Altenkirch

University of Nottingham, UK

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# Why do we need a container model of type theory?

For categorical semantics of (Q)IITs using containers!

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## Example 1: (Simplified) intrinsic syntax of type theory

```
data Con : Set
data Ty  : Con → Set

data Con where
  ◇ : Con
  _ , _ : (Γ : Con) (A : Ty Γ) → Con
  eq : (Γ : Con) (A : Ty Γ) (B : Ty (Γ , A)) →
        ((Γ , A) , B) ≡ (Γ , Σ Γ A B)

data Ty where
  ι : (Γ : Con) → Ty Γ
  Σ : (Γ : Con) (A : Ty Γ) → Ty (Γ , A) → Ty Γ
```

# Why do we need a container model of type theory?

For categorical semantics of (Q)IITs using containers!

Constructors are expressed via argument and target functors  
[Altenkirch et al., 2018].

E.g. For  $\_,\_ : (\Gamma : \mathbf{Con}) (A : \mathbf{Ty} \ \Gamma) \rightarrow \mathbf{Con}$  we have:

$$L : \mathbf{A}_1 \rightarrow \mathbf{Set}$$

$$L(C, T, e) := \sum (\Gamma : C) (T \ \Gamma)$$

$$R : \int L \rightarrow \mathbf{Set}$$

$$= (C : \mathbf{Set}) (T : C \rightarrow \mathbf{Set}) (e : C) (L(C, T, e)) \rightarrow \mathbf{Set}$$

$$R(C, T, e, \Gamma, A) := C$$

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Conjecture:

Restricting ourselves to **container functors** ensures we can only construct **strictly positive** (Q)IITs.

Prerequisite:

A general way to **express type contexts as containers**, i.e. a container model of type theory.

# The container model (outlined in [Altenkirch and Kaposi, 2021])

- Contexts are set-containers  $S_\Gamma: \mathbf{Set} \triangleleft P_\Gamma: S_\Gamma \rightarrow \mathbf{Set}$  with extension functor

$$\llbracket S_\Gamma \triangleleft P_\Gamma \rrbracket: \mathbf{Set} \rightarrow \mathbf{Set}$$

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$$[[S_\Gamma \triangleleft P_\Gamma]]: \mathbf{Set} \rightarrow \mathbf{Set}$$

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- Terms of type  $A$  in context  $\Gamma$  are dependent natural transformations from  $\llbracket \Gamma \rrbracket$  to  $\llbracket A \rrbracket$ :

$$\int_{X: \mathbf{Set}} (\gamma: \llbracket \Gamma \rrbracket X) \rightarrow \llbracket A \rrbracket(X, \gamma)$$

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- Context extension is given by  $\Gamma.A = S_A \triangleleft P_A^X$

## Presheaf model

- Contexts:  $\mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$
- Substitutions: natural transformations
- Types:  $(\int \Gamma)^{\text{op}} \rightarrow \mathbf{Set}$
- Terms:  
$$\int_{X:\mathbf{Set}} (\gamma : \Gamma X) \rightarrow A(X, \gamma)$$
- Context extension:  $\Gamma.A X$   
$$= \sum (\rho : \Gamma X) (A(X, \rho))$$

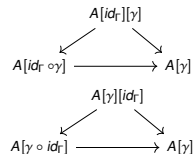
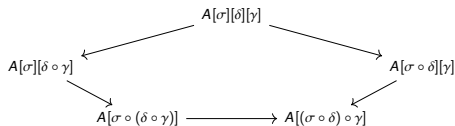
## Container model

- Contexts:  $\mathbf{Set} \rightarrow \mathbf{Set}$
- Substitutions: container morphisms
- Types:  $(\int \llbracket \Gamma \rrbracket) \rightarrow \mathbf{Set}$
- Terms:  
$$\int_{X:\mathbf{Set}} (\gamma : \llbracket \Gamma \rrbracket X) \rightarrow \llbracket A \rrbracket(X, \gamma)$$
- Context extension:  $\llbracket \Gamma.A \rrbracket X$   
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# Coherence issues in the absence of UIP

1  $Ty: \mathbf{Con}^{\text{op}} \rightarrow \mathbf{Set}$  ~~Gpd~~

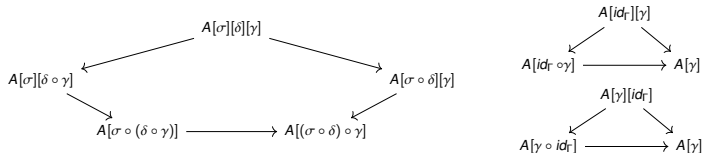
The **collection of types** is a **groupoid**, not an h-set. Additional coherence laws need to be checked.



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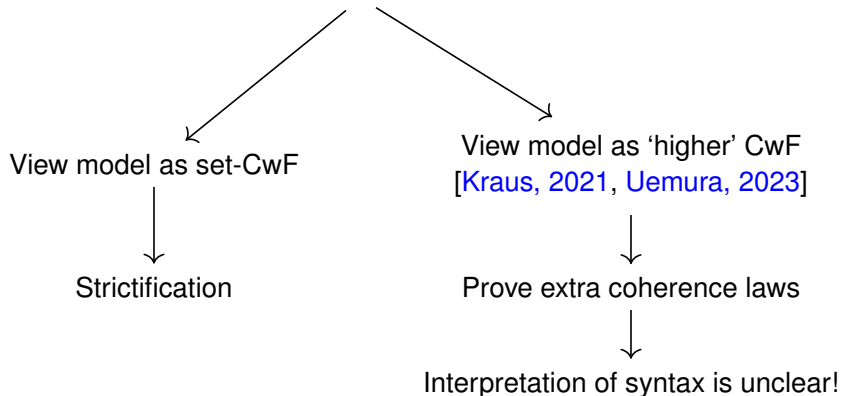
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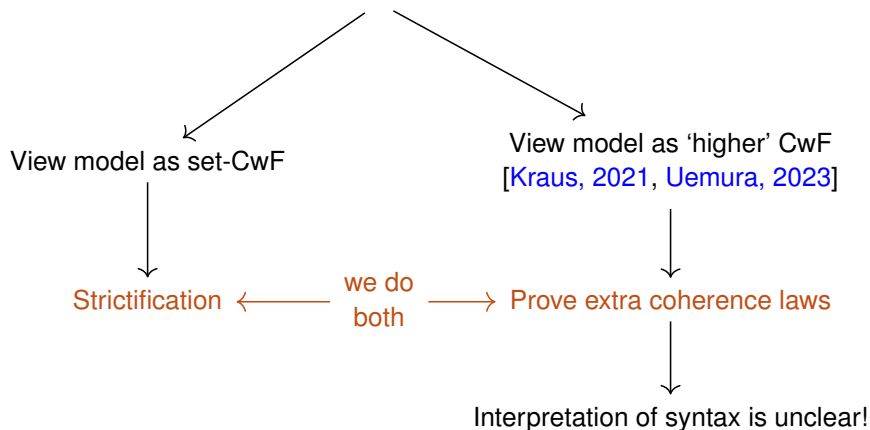
## 2 **Functor laws do not hold strictly**, but only up to isomorphism, due to definition of type substitution $A[\gamma]$ :

$$\begin{array}{ccc}
 S_{A[\gamma]} & \xrightarrow{snd} & S_A \\
 \text{fst} \downarrow & \lrcorner & \downarrow P_A^s \\
 S_\Delta & \xrightarrow{\gamma_s} & S_\Gamma
 \end{array}
 \triangleleft
 \begin{array}{ccc}
 P_\Gamma(\gamma_s(fst\ s)) & \xrightarrow{P_A^f(snd\ s)} & P_A^X(snd\ s) \\
 \gamma_p(fst\ s) \downarrow & & \downarrow \text{inr} \\
 P_\Delta(fst\ s) & \xrightarrow{\text{inl}} & P_{A[\gamma]}^X s
 \end{array}$$

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# Strictified container model

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- Types in context  $\Gamma$  are codes for generalised containers, together with a substitution—we delay substitution

$$(\Gamma \xrightarrow{\delta} \Delta, S_B^U : U \triangleleft P_B^U : El\ S_B^U \rightarrow |\int[\![\Delta]\!]|^U)$$



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- Type substitution can now be defined as

$$(B[\delta])[\gamma] := B[\delta \circ \gamma]$$

Now, the collection of types is an h-set, and functor laws hold strictly.





# Contributions & ongoing work

- Worked out details of ‘higher’ container model outlined in [[Altenkirch and Kaposi, 2021](#)], including extra coherence laws
- Finalizing details of strictified container model
- Constructing  $\Pi$ -types,  $\Sigma$ -types, universe in both versions
- Started a formalisation of the ‘higher’ container model in Cubical Agda

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Thank you!

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